

ON HOLOMORPHIC ISOMETRIC EMBEDDINGS OF THE UNIT DISK INTO POLYDISKS

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ABSTRACT. We study the classification of holomorphic isometric embeddings of the unit disk into polydisks. As a corollary of our results, we can give a complete classification when the target is the 2-disk and the 3-disk. We also prove that the holomorphic isometric embeddings between polydisks are induced by those of the unit disk into polydisks.

1. INTRODUCTION

The study of holomorphic isometries between complex manifolds is a classical topic in complex geometry. For Kähler manifolds, the early work dates back to Calabi's seminal work in 1953 [E.53]. In 2003 [LE03], for the purpose of characterizing modular correspondences among other correspondences in arithmetic geometry, Clozel and Ullmo were led to study holomorphic isometric embeddings of an irreducible bounded symmetric domain Ω into its product spaces Ω^p , $p \geq 2$, up to scalar constants (the isometric constants). Here all bounded symmetric domains are equipped with their Bergman metrics. As explained in their paper, if the rank of Ω is at least 2, then it follows from Mok's work on Hermitian metric rigidity that the isometric embeddings are necessarily totally geodesic (i.e. the diagonal embeddings). When the rank is equal to 1, i.e. $\Omega = \mathbb{B}^n$, the methods in Hermitian metric rigidity do not apply. Nevertheless, for $n \geq 2$, Mok in [N.b] (see the remark in this section) proved that the embeddings are also totally geodesic. Thus, there remains the case with $n = 1$, i.e. $\Omega = \Delta$, when the domain is the unit disk.

For their purpose, Clozel and Ullmo in [LE03] showed that an isometric embedding of Δ into Δ^p is totally geodesic under the assumption that the image is invariant under certain automorphisms of the target polydisk. They then conjectured that the invariance property is not needed (Conjecture 2.2 in [LE03]). However, Mok [N.ng] constructed a non-totally geodesic isometric embedding of Δ into Δ^p for each $p \geq 2$, which we will call the *p-th root embedding*. Among the *p*-th root embeddings, together with the diagonal embeddings, we can form compositions or reparametrize them by Möbius transformations to obtain more isometric embeddings of the unit disk into polydisks. The question of whether all isometric embeddings can be obtained in such a way is certainly very interesting.

The purpose of this paper is to show that under certain restrictions on the sheeting number of their algebraic extensions, all isometric embeddings can be

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constructed in this way. As a corollary, we can determine all isometric embeddings of Δ into Δ^2 and Δ^3 . They are the diagonal embeddings and those embeddings constructed from the square root embedding and the cube root embedding. In the final part of the paper, we will also prove a “splitting theorem”, which essentially says that the study of isometric embeddings of Δ^q into Δ^p reduces to the study of isometric embeddings of Δ into Δ^p .

Our work is based on the functional equation

$$(1.1) \quad \prod_{i=1}^p (1 - |f_i|^2) = (1 - |z|^2)^k,$$

where $z \mapsto (f_1(z), \dots, f_p(z))$ is an isometric embedding of Δ into Δ^p with the isometric constant equal to k . It is derived from the isometry condition on the Kähler potentials of Δ and Δ^p . From the functional equation it can then be proven that the graph of the embedding extends algebraically and the extension is a finite branched cover over the domain. After establishing the algebraic extension, some general properties of the embedding such as the distribution of poles and the branchings of the component functions can be obtained. These properties then enable us to give classifications under certain restrictions on the sheeting number.

Remark. In [N.b], the main theorem was proven only for dimensions greater than 1, while it was stated for all dimensions in the article. For the details, please refer to the erratum [N.a] of the article.

2. THE FUNCTIONAL EQUATION

Let λ be a positive real number and $(M, g_M), (N, g_N)$ be Kähler manifolds. A map $F : M \rightarrow N$ is said to be an isometric embedding with the *isometric constant* λ if $F : (M, \lambda g_M) \rightarrow (N, g_N)$ is an isometric embedding.

Let $F : \Delta \rightarrow \Delta^p$, $F(z) = (f_1(z), \dots, f_p(z))$ be an isometric embedding with the isometric constant λ . So we will be studying isometric embeddings of Δ into Δ^p up to scalar constants.

Without loss of generality, we may assume that $F(0) = \mathbf{0}$. The Bergman metric on Δ is given by $ds_\Delta^2 = 2\operatorname{Re}(g dz \otimes d\bar{z})$, where $g = -2 \frac{\partial^2}{\partial z \partial \bar{z}} \log(1 - |z|^2)$. We can take as Kähler potentials for ds_Δ^2 and $ds_{\Delta^p}^2$ the real analytic functions $-2 \log(1 - |z|^2)$ and $-2 \sum_{i=1}^p \log(1 - |z_i|^2)$ respectively. By the assumption $F^* ds_{\Delta^p}^2 = \lambda ds_\Delta^2$ it follows that

$$-2\sqrt{-1} \partial \bar{\partial} \sum_{i=1}^p \log(1 - |f_i|^2) = -2\lambda \sqrt{-1} \partial \bar{\partial} \log(1 - |z|^2);$$

hence

$$-\sum_{i=1}^p \log(1 - |f_i|^2) = -\lambda \log(1 - |z|^2) + \operatorname{Re} h$$

for some holomorphic function h on Δ . Since $F(0) = \mathbf{0}$, comparing Taylor expansions we conclude as in Clozel-Ullmo [LE03] that $h \equiv 0$. Therefore we obtain

$$-\sum_{i=1}^p \log(1 - |f_i|^2) = -\lambda \log(1 - |z|^2),$$

i.e.

$$\prod_{i=1}^p (1 - |f_i|^2) = (1 - |z|^2)^\lambda.$$

By the standard polarization technique, we have the polarized functional equation:

$$(2.1) \quad \prod_{i=1}^p (1 - f_i(z) \overline{f_i(w)}) = (1 - z\overline{w})^\lambda,$$

where $(1 - z\overline{w})^\lambda \equiv e^{\lambda \log(1 - z\overline{w})}$.

For the unit disk, the isometric constant λ is necessarily an integer. This can be obtained quite easily once the existence of an algebraic extension is established. The following is proven in Mok [N.ng].

Theorem 2.1 (Mok). *Let $\Omega \Subset \mathbb{C}^n$ and $\Omega' \Subset \mathbb{C}^N$ be bounded symmetric domains in their Harish-Chandra realizations. Let λ be any positive real number and $f : (\Omega, \lambda ds_\Omega^2) \rightarrow (\Omega', ds_{\Omega'}^2)$ be a germ of holomorphic isometry at $0 \in \Omega$ with $f(0) = 0$. Then, the germ of the graph of f extends to an affine algebraic variety $S^\# \subset \mathbb{C}^n \times \mathbb{C}^N$ such that $S = S^\# \cap (\Omega \times \Omega')$ is the graph of a holomorphic isometric embedding $F : \Omega \rightarrow \Omega'$ extending the germ of the holomorphic map f .*

From the existence of algebraic extensions, we can prove

Proposition 2.2. *Let $(\Delta, \lambda ds_\Delta^2) \rightarrow (\Delta^p, ds_{\Delta^p}^2)$ be an isometric embedding. Then λ is a positive integer.*

Proof. By Theorem 2.1, we know that the embedding can be extended across a general point on the unit circle. Let z_0 be a point on the unit circle at which the embedding can be extended across in a neighborhood. If we consider (2.1) and substitute $w = z_0$, then because each factor on the left-hand side can only vanish with an integral order at $z = z_0$, λ on the right-hand side must be a positive integer. \square

Remark. The integrality also follows from a slight variation of the simpler arguments of Mok [N.b] on algebraic extensions.

We will henceforth denote the isometric constant of an embedding of the unit disk into a polydisk by k as in (1.1), suggesting its integrality. It follows easily from Schwarz's lemma that if the target is Δ^p , then $k \leq p$ and the embedding is the diagonal embedding when $k = p$. Thus, for isometric embeddings of Δ into Δ^p , the non-trivial cases are those with $k \in \{1, \dots, p-1\}$.

3. EXAMPLES

For each positive integer $p \geq 2$, Mok [N.ng] constructed a non-totally geodesic isometric embedding of Δ into Δ^p with $k = 1$. The examples are best expressed when we transform the unit disk into the upper half-plane \mathcal{H} . For $\tau \in \mathcal{H}$, write $\tau = re^{i\theta}$ with $r > 0$ and $0 < \theta < \pi$. We denote the Poincaré metrics of \mathcal{H} and \mathcal{H}^p by $ds_{\mathcal{H}}^2$ and $ds_{\mathcal{H}^p}^2$, respectively.

Proposition 3.1 (Mok). *Let $p \geq 2$ be a positive integer and $\gamma = e^{i\pi/p}$. Then the map $F : (\mathcal{H}, ds_{\mathcal{H}}^2) \rightarrow (\mathcal{H}^p, ds_{\mathcal{H}^p}^2)$ defined by $F(\tau) = (\tau^{\frac{1}{p}}, \gamma\tau^{\frac{1}{p}}, \dots, \gamma^{p-1}\tau^{\frac{1}{p}})$ is an isometric embedding, where $\tau^{\frac{1}{p}} = r^{\frac{1}{p}} e^{\frac{i\theta}{p}}$.*

This embedding will be called the p -th root embedding.

Given an isometric embedding $F : \Delta \rightarrow \Delta^p$, we can construct a family of isometric embeddings parametrized by $\alpha \in \Delta$ and defined by

$$F^\alpha(w) = T \left(F \left(\frac{w + \alpha}{1 + \overline{\alpha}w} \right) \right),$$

where $T \in \text{Aut}(\Delta^p)$ with $T(F(\alpha)) = \mathbf{0} \in \Delta^p$ such that we have $F^\alpha(0) = \mathbf{0}$. F^α will be called a *reparametrization* of F .

Besides reparametrizations, we can also form “compositions” among isometric embeddings. More explicitly, for any two isometric embeddings, $F : \Delta \rightarrow \Delta^p$ and $G : \Delta \rightarrow \Delta^q$, they induce an isometric embedding $H : \Delta \rightarrow \Delta^p \rightarrow \Delta^{p+q-1}$ given by $H = G^\# \circ F$, where $G^\# : \Delta^p \rightarrow \Delta^{p+q-1}$ and $G^\#(z_1, \dots, z_p) = (z_1, \dots, z_{p-1}, G(z_p))$. Thus, we have already acquired a great bunch of isometric embeddings by the above constructions from the p -th root embeddings. We will prove that these constructions exhaust all isometric embeddings of Δ into Δ^2 and Δ^3 .

4. ALGEBRAIC EXTENSION

Although the algebraic extension has already been established in Theorem 2.1, we here give a more elementary proof under the assumption that the isometric constant is an integer. Let $F : U \rightarrow \Delta^p$ be a germ of an isometric embedding with the isometric constant k , where $U \subset \Delta$ is an open neighborhood containing the origin and $F(0) = \mathbf{0}$. We will assume that every component function is non-constant. We regard Δ as a domain in \mathbb{P}^1 . We identify \mathbb{P}^1 as $\mathbb{C} \cup \{\infty\}$ and use the inhomogeneous coordinate.

Lemma 4.1. *Let $f, g : \Omega \subset \mathbb{C}^m \rightarrow \mathbb{C}^n$, $f = (f_1, \dots, f_n)$, $g = (g_1, \dots, g_n)$ be holomorphic maps such that $\|f\|^2 = \|g\|^2$. Then there exists a unitary transformation U in \mathbb{C}^n such that $U \circ f = g$.*

Proof. Let $f_j(Z) = \sum_{|I|=0}^{\infty} f_{jI} Z^I$ and $g_j(Z) = \sum_{|I|=0}^{\infty} g_{jI} Z^I$, where I is a multi-index.

Suppose

$$|f_1|^2 + \dots + |f_n|^2 = |g_1|^2 + \dots + |g_n|^2.$$

Polarizing the power series expansion of the equation, we get

$$f_{1I} \overline{f_{1J}} + \dots + f_{nI} \overline{f_{nJ}} = g_{1I} \overline{g_{1J}} + \dots + g_{nI} \overline{g_{nJ}}$$

for every $I, J \in \mathbb{N}^m$.

Let $\mathbf{f}_I = (f_{1I}, \dots, f_{nI})$ and $\mathbf{g}_I = (g_{1I}, \dots, g_{nI})$. Then for the two families $\{\mathbf{f}_I\}_{I \in \mathbb{N}^m}$ and $\{\mathbf{g}_I\}_{I \in \mathbb{N}^m}$ of vectors in \mathbb{C}^n , we know that there is an $n \times n$ unitary matrix U such that $U\mathbf{f}_I = \mathbf{g}_I$, $\forall I \in \mathbb{N}^m$. Hence,

$$U \sum_{|I|=0}^{\infty} \mathbf{f}_I z^I = \sum_{|I|=0}^{\infty} \mathbf{g}_I z^I \iff U \circ f = g. \quad \square$$

Proposition 4.2. *There exists an irreducible one-dimensional projective algebraic subvariety V in $\mathbb{P}^1 \times (\mathbb{P}^1)^p$ extending the graph of F . In addition, by considering the projection map from V onto the first factor, V is a finite branched cover over \mathbb{P}^1 .*

Proof. f_1, \dots, f_p satisfy the functional equation

$$\prod_{i=1}^p (1 - |f_i|^2) = (1 - |z|^2)^k.$$

Expand the brackets and rearrange the above equation to a form

$$\sum_{j=0}^d |A_j|^2 = \sum_{j=0}^e |B_j|^2,$$

where A_j, B_j are analytic functions and $A_i = f_i$ for $1 \leq i \leq p$. Here $e + 1 = d = 2^{p-1} + \frac{k}{2}$ if k is even and $d = e = 2^{p-1} + \frac{k-1}{2}$ if k is odd. If k is even, we add a term $B_d := 0$ to the right-hand side making both sides have the same number of terms. Then for each k , Lemma 4.1 says that there exists a $d \times d$ unitary matrix \mathbf{U} such that

$$\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_d \end{pmatrix} = \mathbf{U} \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_d \end{pmatrix}.$$

Among the above e equations, the first p of them are of the form

$$f_i = Q_i(z, f_1, \dots, f_p), \quad i = 1, \dots, p,$$

where the Q_i are polynomials in which every term is either an odd power of z or a product of an even number of the factors f_i . Applying the Implicit Function Theorem at the point $\mathbf{0} \in \mathbb{P} \times (\mathbb{P}^1)^p$, we see that the projective algebraic subvariety defined by these p equations is smooth at $\mathbf{0}$. If we let V be the irreducible component of this subvariety containing $\mathbf{0}$, then V is one-dimensional. It is clear that the germ of the graph of F is contained in V . Denote by π the projection map from $\mathbb{P}^1 \times (\mathbb{P}^1)^p$ to the first factor. By the Proper Mapping Theorem, $\pi(V) = \mathbb{P}^1$ and since V is connected, there exists an integer n such that for a generic point $z \in \mathbb{P}^1$, $\#(\pi^{-1}(z)) = n$. \square

Let the sheeting number of V relative to π be n . Consider the projection maps $P_i : V \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ defined by $P_i(z, \xi_1, \dots, \xi_p) = (z, \xi_i)$. Then $V_i = P_i(V) \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a one-dimensional projective algebraic subvariety containing the germ of the graph of the component function $f_i : U \rightarrow \Delta$. If we also define $\pi_i : V_i \rightarrow \mathbb{P}^1$ to be the projection map to the first factor, then π can be factorized as $\pi = \pi_i \circ P_i$, and therefore we see that V_i is a finite branched cover over \mathbb{P}^1 with the sheeting number being a factor of n .

We are going to show that the defining equation of every V_i takes a very specific form. At first, note that there exists a discrete subset $A \subset \mathbb{P}$ such that when restricting to $V \setminus \pi^{-1}(A)$, π is a holomorphic covering map. This is obvious if V is smooth and in the general case we can pass to its normalization.

Lemma 4.3. *Let $(z, a_1, \dots, a_p), (w, b_1, \dots, b_p)$ be any two points on V . Then*

$$(4.1) \quad \prod_{i=1}^p (1 - a_i \overline{b_i}) = (1 - z \overline{w})^k.$$

Proof. We first consider the polarized functional equation

$$\prod_{i=1}^p (1 - f_i(z) \overline{f_i(w)}) = (1 - z\overline{w})^k$$

in a small neighborhood $U \subset \Delta$ of the origin, where $z, w \in U$. We can fix z and do analytic continuation on the variable w and vice versa. Therefore (4.1) is true for all the points on $V \setminus \pi^{-1}(A)$. But $V \setminus \pi^{-1}(A)$ is an open dense subset of V and therefore (4.1) is true for any two points on V . \square

Lemma 4.4. *Let (z, a_1, \dots, a_p) be a point on V . Then a_i is finite for all i if z is finite, i.e. $z \in \mathbb{C} \subset \mathbb{P}^1 \implies (a_1, \dots, a_p) \in \mathbb{C}^p \subset (\mathbb{P}^1)^p$.*

Proof. Suppose (z, a_1, \dots, a_p) is a point on V with $z \in \mathbb{C}$ and $a_1 = \infty$. Let $U \subset \mathbb{C} \setminus A$ be an open set such that $\pi^{-1}(U) = \bigcup_{l=1}^n W_l$ and the W_l are disjoint. Taking any of the W_l , say W_1 , for all $(w, f_1, \dots, f_p) \in W_1$, we have from (4.1),

$$\prod_{i=1}^p (1 - a_i \overline{f_i}) = (1 - z\overline{w})^k.$$

Now the right-hand side is always finite, and because $a_1 = \infty$, we have either $f_1(w) \equiv 0$ or $(1 - a_j \overline{f_j}) \equiv 0$ for some j . Both will imply that one component function is constant and hence a contradiction. \square

We fix an $i \in \{1, \dots, p\}$ and write $h = f_i$. We also denote V_i by V_h and π_i by π_z .

Proposition 4.5. *Let the sheeting number of V_h relative to π_z be q . Then $V_h \subset \mathbb{P}^1 \times \mathbb{P}^1$ is defined by the equation*

$$(4.2) \quad h^q + P_{q-1}(z)h^{q-1} + P_{q-2}(z)h^{q-2} + \dots + P_1(z)h + P_0(z) = 0,$$

where $\forall j$, $P_j(z) = a_j z + b_j$, $a_j, b_j \in \mathbb{C}$ and $b_0 = 0$.

Proof. There exists a discrete subset $A \subset \mathbb{P}^1$ such that π_z is a covering map when restricting to $\pi_z^{-1}(\mathbb{P}^1 \setminus A)$. We consider the elementary symmetric functions of h relative to the covering map π_z , and they will be meromorphic functions on $\mathbb{P}^1 \setminus A$. Since h is just a coordinate in $\mathbb{P}^1 \times \mathbb{P}^1$, we see that these elementary symmetric functions can be extended to the whole \mathbb{P}^1 , and hence they are rational functions. Therefore the defining equation of V_h is of the form

$$(4.3) \quad h^q + P_{q-1}(z)h^{q-1} + \dots + P_1(z)h + P_0(z) = 0,$$

where P_1, \dots, P_{q-1} are rational functions. We are now going to show that every P_i must in fact be a linear polynomial in z . First of all, by the Implicit Function Theorem, for a generic choice of $(z_0, h_0) \in V_h$ with $|h_0| = 1$, there exists a neighborhood $U_0 \subset \mathbb{C}$ containing h_0 such that z can be expressed as a holomorphic function of h . From the functional equation, we see that for $h \in U_0$ with $|h| = 1$, we have $|z| = 1$. Therefore the Schwarz reflection principle implies that

$$z\left(\frac{1}{\bar{h}}\right) = \frac{1}{\overline{z(h)}}$$

for $h, 1/\bar{h} \in U_0$. This means that (z, h) satisfies (4.3) if and only if $(1/\bar{z}, 1/\bar{h})$ does. Let r be the maximum of the degrees of all the P_i . Suppose $r > 1$. Then for a generic choice of $h^\# \in U_0$ we can solve (4.3) (when $h = h^\#$) with r distinct solutions of z_1, \dots, z_r such that $(z_1, h^\#), \dots, (z_r, h^\#)$ do not lie in the singular part

of V_h . Due to the Schwarz reflection principle, when $h = 1/\overline{h^\#}$, the solutions of (4.3) are $1/\overline{z_1}, \dots, 1/\overline{z_r}$ and $(1/\overline{z_1}, 1/\overline{h^\#}), \dots, (1/\overline{z_r}, 1/\overline{h^\#})$ are regular points of V_h . Now we can make a reparametrization of the isometric embedding such that $(z_1, h^\#) \mapsto (0, 0)$. Then $1/\overline{h^\#} \mapsto \infty$ and that means h takes ∞ on r distinct points on \mathbb{P}^1 , which contradicts Lemma 4.4. Hence r must be equal to 1. Furthermore, due to Lemma 4.4 again, $P_j(z)$ cannot have poles in \mathbb{C} and therefore they are linear polynomials in z . Finally, $b_0 = 0$ follows from $h(0) = 0$ and the proof is complete. \square

Corollary 4.6. *For every i , there exists a rational function R_i such that $z = R_i(f_i(z))$. Furthermore, $R_i : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ maps the unit circle into the unit circle.*

Corollary 4.7. *(4.2) is invariant under the transformation $(z, h) \mapsto (1/\overline{z}, 1/\overline{h})$. In particular, for $|z| = 1$, if h_j , $1 \leq j \leq q$, are the q roots of (4.2), then $1/\overline{h_j}$, $1 \leq j \leq q$, is a permutation of the roots.*

Definition 4.8. The sheeting number of a component function f_i is defined to be the degree of R_i in Corollary 4.6.

From the results of Clozel-Ullmo [LE03] or Mok [N.ng], every germ of an isometric embedding of the unit disk into a polydisk can be extended over the whole unit disk. But $\mathbb{P}^1 \setminus \overline{\Delta}$ is biholomorphic to the unit disk, so we conclude that the branch points of h must lie on the unit circle. Let z_1, \dots, z_m be the branch points of h and b_i the branching order at z_i . Then from the Riemann-Hurwitz formula, we have

$$(4.4) \quad \sum_{i=1}^m b_i = 2(q-1).$$

The branching orders b_i will be important for classifying isometric embeddings. For example, we will use the following lemma later.

Lemma 4.9. *Let h be a component function of an isometric embedding of Δ into Δ^p and let its sheeting number be q . If h has exactly two branch points, then up to reparametrizations, h is a component function of the q -th root embedding.*

Proof. Since the sheeting number of h is q , the branching order at each branch point is $q-1$. Also, we know that the branch points of h lie on the unit circle. Now, by Cayley transformations, we can consider h as a map from the upper half-plane into the upper half-plane. By Corollary 4.6, there is a rational function R of degree q such that $\tau = R(h(\tau))$. We can assume that the branch points of h are 0 and ∞ by doing some reparametrization. Then we have $R(h) = \alpha h^q$, where $\alpha \in \mathbb{C}$. By Corollary 4.6 again, R maps the real axis into the real axis; therefore $\alpha \in \mathbb{R}$. We can further make the coordinate transformation $h \mapsto |\alpha|^{1/q} h$ and now we have $\tau = h^q$ or $\tau = -h^q$. These two equations define $2q$ branches of an analytic function from the upper half-plane into the complex plane. Among them there are q of them mapping the upper half-plane into the upper half-plane, and they are exactly the q component functions of the q -th root embedding. \square

5. SHEETING NUMBERS

We now return to the isometric embedding itself. For all i , let s_i be the sheeting number of f_i . Let us introduce the notation $\Delta^+ = \Delta$ and $\Delta^- = \mathbb{P}^1 \setminus \overline{\Delta}$. By Corollary 4.6, for each i , there exists a rational function R_i of degree s_i such that

$z = R_i(f_i(z))$ and R_i maps the unit circle into the unit circle. As just mentioned, the branch points of the algebraic extension of the map must lie on the unit circle. Therefore we have $\pi_i^{-1}(\Delta^+) = \bigcup_{j=1}^{s_i} U_{j;i} \subset \mathbb{P}^1 \times \mathbb{P}^1$, $U_{j;i}$ are disjoint and each lies entirely in either $\Delta^+ \times \Delta^+$ or $\Delta^+ \times \Delta^-$.

Similarly we also have $\pi^{-1}(\Delta^+) = \bigcup_{j=1}^n U_j \subset \mathbb{P}^1 \times (\mathbb{P}^1)^p$, where the U_j are disjoint and n is the sheeting number of V . We can associate a “parity” to each U_j , defined as $(\chi_{j;1}, \dots, \chi_{j;p})$, $\chi_{j;i} \in \{+, -\}$ when $P_i(U_j) \subset \Delta^+ \times \Delta^{\chi_{j;i}}$.

Lemma 5.1. *If U_i and U_j have the same parity, then $U_i = U_j$.*

Proof. Let the parity of U_i and U_j be $(+, \dots, +)$. We can consider U_i and U_j as the graphs of two isometric embeddings of Δ into Δ^p , denoted by F_i and F_j . From the functional equation, we see that $F_i(0) = F_j(0) = \mathbf{0} \in (\Delta^+)^p$. Thus, F_i and F_j agree at the origin. However, every point in the unit disk can become the origin by reparametrization. More explicitly, for every $\alpha \in \Delta^+$, we consider the reparametrization of F_i given by $F_i^\alpha(w) = T(F_i(\frac{w+\alpha}{1+\bar{\alpha}w}))$, where T is an automorphism of $(\Delta^+)^p$ such that $T(F_i(\alpha)) = \mathbf{0}$. We can find a path $\gamma : [0, 1] \rightarrow \mathbb{P}^1$ with $\gamma(0) = \gamma(1) = 0$ such that when we perform the analytic continuation F_i along γ , we will come up with F_j at the end. We can correspondingly perform the analytic continuation of F_i^α along γ^α , where γ^α is the image of γ under the transformation $z = \frac{w+\alpha}{1+\bar{\alpha}w}$. Denote by \widehat{F}_i^α the resulting extension. Then we have $\widehat{F}_i^\alpha(w) = T(F_j(\frac{w+\alpha}{1+\bar{\alpha}w}))$. Observe that $\widehat{F}_i^\alpha(0) = \mathbf{0}$ because $\widehat{F}_i^\alpha(\Delta^+) \subset (\Delta^+)^p$ and \widehat{F}_i^α satisfies the functional equation. Hence $T(F_j(\alpha)) = \mathbf{0} \implies F_i(\alpha) = F_j(\alpha)$. Since α is arbitrary, this implies that $U_i = U_j$. By doing coordinate transformations ($z \mapsto 1/w$), the same is true for other parities. \square

Among the 2^p different parities, we see that there are only 2^{p-1} of them consistent with the functional equation. Hence we have

Proposition 5.2. *The sheeting number of V relative to π is at most 2^{p-1} , i.e. $n \leq 2^{p-1}$.*

Remark. The maximum 2^{p-1} is attainable by composing the square root embedding. More precisely, consider the isometric embedding of Δ into Δ^p given by $F = F_{p-1} \circ F_{p-2} \circ \dots \circ F_1$, where $F_1 : \Delta \rightarrow \Delta^2$, $F(z) = (\alpha(z), \beta(z))$ is the square root embedding and $F_i : \Delta^i \rightarrow \Delta^{i+1}$ is defined by $F(z_1, \dots, z_i) = (z_1, \dots, z_{i-1}, \alpha(z_i), \beta(z_i))$. Then the sheeting number of the graph of F is 2^{p-1} .

It follows from Proposition 4.5 that each f_i takes ∞ on one and only one branch among its s_i branches. This observation leads to the following.

Proposition 5.3. *The sheeting numbers s_i satisfy*

$$(5.1) \quad \sum_{i=1}^p \frac{1}{s_i} = k, \quad s_i | n, \forall i.$$

Proof. As noted after Proposition 4.2, s_i is a factor of n . Within the n branches of F , each coordinate function f_i has s_i distinct branches and each distinct branch repeats itself n/s_i times. Among the s_i distinct branches, there is only one branch such that $f_i(\infty) = \infty$, and hence f_i takes an infinite value n/s_i times at infinity within the n branches of F . It is easy to deduce from the polarized functional equation that there are exactly k component functions taking infinite value at infinity on each branch of F . Therefore the sum of n/s_i over i is equal to kn . \square

From Proposition 5.3, we obtain a lower bound for n .

Corollary 5.4. *The sheeting number of V relative to π is greater than or equal to p/k , i.e. $n \geq p/k$.*

Proof. Write (5.1) as

$$\sum_{i=1}^p \frac{n}{s_i} = nk, \quad s_i | n, \forall i.$$

Since $n/s_i \geq 1$, we have $nk \geq p$. \square

6. TWO THEOREMS

In this section, we will prove two theorems about the classification of isometric embeddings of the unit disk into polydisks with the isometric constant $k = 1$. We will continue to use V to denote the irreducible projective algebraic subvariety in $\mathbb{P}^1 \times (\mathbb{P}^1)^p$ extending the germ of an isometric embedding of Δ into Δ^p . Denote also by π the projection map onto the first factor and n the sheeting number of V relative to π . By Proposition 5.2 and Corollary 5.4, we have $p \leq n \leq 2^{p-1}$. The two theorems are on the extremal cases.

6.1. On the minimal case. Let p be an odd number. Let $F : \Delta \rightarrow \Delta^p$, $F(z) = (f_1(z), \dots, f_p(z))$ be an isometric embedding with the isometric constant $k = 1$ and $n = p$. From Proposition 5.3, we deduce that the sheeting number of every f_i is p . In what follows, a complex number of modulus 1 will be called unimodular.

If $z_0 \in \partial\Delta$ is not a branch point of any f_i , then there exists a small neighborhood U of z_0 containing no branch points and such that $\pi^{-1}(U) = \bigcup_{j=1}^p U_j$, where U_j are disjoint and each U_j can be regarded as the graph of a branch of the embedding over U . Let us denote the p branches of the embedding by $F^j(z) = (f_1^j(z), \dots, f_p^j(z))$, $1 \leq j \leq p$.

Lemma 6.1. *Let $z_0 \in \partial\Delta$ such that it is not a branch point of any component function. For each j , among the set $\{f_i^j(z_0) : 1 \leq i \leq p\}$, there is exactly one unimodular value.*

Proof. Consider the polarized functional equation

$$\prod_{i=1}^p (1 - f_i^j(z) \overline{f_i^j(w)}) = 1 - z\overline{w},$$

where $z, w \in U$. Substituting $w = z_0$, we get

$$\prod_{i=1}^p (1 - f_i^j(z) \overline{f_i^j(z_0)}) = 1 - z\overline{z_0}.$$

Since the order of zero at $z = z_0$ is 1 on the right-hand side and hence on the left-hand side, we therefore obtain the desired result. \square

Lemma 6.2. *Let $z_0 \in \partial\Delta$ such that it is not a branch point of any component function. For each i , among the set $\{f_i^j(z_0) : 1 \leq j \leq p\}$, there is exactly one unimodular value. In particular, the rational function $R_i : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ in Corollary 4.6 satisfying $z = R_i(f_i(z))$ maps the unit circle bijectively onto the unit circle.*

Proof. By Corollary 4.7, $r \in \{f_i^j(z_0) : 1 \leq j \leq p\}$ if and only if $1/\bar{r} \in \{f_i^j(z_0) : 1 \leq j \leq p\}$. Since p is odd, at least one element of $\{f_i^j(z_0) : 1 \leq j \leq p\}$ is unimodular. Combining with Lemma 6.1, we see that exactly one of them is unimodular. \square

Now we take an arbitrary point $z' \in \partial\Delta$. If it happens that z' is a branch point of some f_i , then we cannot define branches of the embedding as above. On the other hand, since each component function f_i satisfies an equation of the form of (4.2), so we can still talk about the p “values” of $f_i(z')$ by taking them as the p roots of (4.2) when $z = z'$. For each i , let $\beta_{l,i}$, $1 \leq l \leq r_i$, be the distinct values of $f_i(z')$ and $n_{l,i}$, $1 \leq l \leq r_i$, be the corresponding multiplicities.

Lemma 6.3. *The number r_i and the set $\{n_{l,i} : 1 \leq l \leq r_i\}$ are independent of i .*

Proof. We fix an f_i and take a sufficiently small neighborhood U' of z' such that U' does not contain any branch point of any component function other than z' . Recall the notation that $\pi_i : V_i \rightarrow \mathbb{P}^1$ is the projection map onto the first factor for the graph of f_i . Since f_i is the inverse of a rational function, by taking U' sufficiently small we have $\pi_i^{-1}(U') = \bigcup_{l=1}^{r_i} W_i^l$, where W_i^l are the connected components of $\pi_i^{-1}(U')$ and each W_i^l is biholomorphic to some open neighborhood of the graph of $\zeta = \xi^{n_{l,i}}$ at the origin in \mathbb{C}^2 , where (ζ, ξ) are the coordinates in \mathbb{C}^2 . Now take two component functions f_i and f_j . Since V , V_i and V_j are all p -sheeted covers over \mathbb{P}^1 , the projection maps $P_i : V \rightarrow V_i$ and $P_j : V \rightarrow V_j$ are both generically one to one and fibre preserving. Hence, there is also a generically one to one and fibre preserving map between V_i and V_j , in particular, between $\pi_i^{-1}(U')$ and $\pi_j^{-1}(U')$. We therefore conclude that $r_i = r_j$ and after rearranging the index l if necessary, $n_{l,i} = n_{l,j}$ for every l . \square

In other words, the branching loci of all the component functions are the same, and we do not need to refer to a specific component function when we speak of branch points. Furthermore, if for each i , we let R_i be the rational function such that $z = R_i(f_i(z))$, then for any point (z, a_1, \dots, a_p) on V , the ramification orders of R_i at a_i , $1 \leq i \leq p$, are the same. Thus, we can define the ramification order of π at a point $(z, a_1, \dots, a_p) \in V$ to be the ramification order of any R_i at a_i .

Lemma 6.4. *Let $z' \in \partial\Delta$ be a branch point. At each point of $\pi^{-1}(z')$, the ramification order of π is at least 3.*

Proof. We first prove that there is no point in $\pi^{-1}(z')$ having the ramification order 1. Suppose there are r such points, denoted by v_1, \dots, v_r . We have $r < p$ because z' is a branch point. At each v_j , there is a small open neighborhood U_j in V such that U_j is the graph of some branch of the holomorphic map $z \mapsto (f_1(z), \dots, f_p(z))$. At each v_j , there is one and only one component function taking a unimodular value (cf. Lemma 6.1). Therefore at most r component functions can take unimodular values at the points v_1, \dots, v_r . On the other hand, for each fixed i , f_i has r simple roots for its defining equation (4.2) when $z = z'$. If r is odd, Corollary 4.7 implies that at least one of these roots is unimodular and that means that every f_i takes a unimodular value at some v_j and this is a contradiction. If r is even, Corollary 4.7 then implies that an even number of these roots are unimodular. But from Lemma 6.2, over any z' , each f_i can only take one unimodular value; therefore all the simple roots are non-unimodular and that means at each v_j no component function takes a unimodular value. This is again a contradiction. Thus, we conclude that there is no point in $\pi^{-1}(z')$ having the ramification order 1.

Next, we argue that there is no point in $\pi^{-1}(z')$ having the ramification order 2. Suppose $v \in \pi^{-1}(z')$, $v = (z', a_1, \dots, a_p)$ is a point with a ramification order 2. By definition, the ramification of R_i at a_i is 2 for every i . We claim that this implies that a_i is non-unimodular for every i . For if R_i has a ramification order 2 at a point on the unit circle, then the unit circle will not be mapped injectively by R_i and it contradicts Lemma 6.2. Thus, all f_i are non-unimodular at v and hence non-unimodular in a neighborhood W of v . We can always find a point $w \in W$ such that $\pi(w)$ is not a branch point, and the fact that no f_i takes a unimodular value at w contradicts Lemma 6.1. \square

Theorem 6.5. *Let p be an odd number. If $F : \Delta \rightarrow \Delta^p$ is an isometric embedding with the isometric constant $k = 1$ and with the sheeting number n equal to p , then up to reparametrizations, F is the p -th root embedding.*

Proof. Fix a component function, say f_1 . Let $\{z_1, \dots, z_m\}$ be the branch points and $\{b_1, \dots, b_m\}$ be their branching orders respectively. Then

$$\sum_{j=1}^m b_j = 2(p-1).$$

But $b_j = p - r_j$, where r_j is the number of distinct values of $f_1(z_j)$. Since the multiplicity of each of these distinct values is at least 3, we must have $r_j \leq p/3$. Then

$$\begin{aligned} 2(p-1) &= \sum_{j=1}^m b_j = \sum_{j=1}^m (p - r_j) \geq \sum_{j=1}^m \frac{2p}{3} = \frac{2mp}{3} \\ &\implies \frac{3(p-1)}{p} \geq m. \end{aligned}$$

So $m = 1, 2$. But m is at least 2, so by Lemma 4.9, up to some reparametrization, f_1 is a component function of the p -th root embedding. This is true for every f_i and since all f_i have the same branch points it is clear from the proof of Lemma 4.9 that there exists a single reparametrization such that every f_i will become one of the component functions of the p -th root embedding. Note that if $i \neq j$, then $f_i \neq f_j$ because on each branch of F , only one component function can take the value ∞ at infinity. So the component functions of F are precisely the p component functions of the p -th root embedding. \square

6.2. On the maximal case. Let $p \in \mathbb{N}^+$ and $F : \Delta \rightarrow \Delta^p$, $F(z) = (f_1(z), \dots, f_p(z))$ be an isometric embedding with the isometric constant $k = 1$.

Lemma 6.6. *Let r, s be two positive integers and $r \leq s$. Then the equation*

$$\sum_{l=1}^r 2^{t_l} = \sum_{l=1}^s 2^{l-1}, \quad 0 \leq t_1 \leq t_2 \leq \dots \leq t_r$$

has an integral solution if and only if $r = s$ and the solution is unique, i.e. $t_l = l-1$.

Proof. By induction on s . \square

Lemma 6.7. *Let $g : \Delta \rightarrow \Delta$ be a component function of an isometric embedding of Δ into Δ^p . If the sheeting number of g is 2, then there exists an analytic function $h : \Delta \rightarrow \Delta$ such that $F = (g, h)$ is an isometric embedding of Δ into Δ^2 with the isometric constant $k = 1$.*

Proof. If the sheeting number of g is 2, then the Riemann-Hurwitz formula implies that g has exactly two branch points; and therefore after some reparametrization $g \rightarrow g^\alpha$, g^α is one of two component functions of the square-root embedding and the lemma follows. \square

Theorem 6.8. *Let $F : \Delta \rightarrow \Delta^p$, $F(z) = (f_1(z), \dots, f_p(z))$ be an isometric embedding with the isometric constant $k = 1$. For all i , let s_i be the sheeting number of f_i . Assume that $s_1 \leq s_2 \leq \dots \leq s_p$. If $s_p = 2^{p-1}$, then the embedding can be factorized as $F = F_{p-1} \circ F_{p-2} \circ \dots \circ F_1$, where each intermediate map $F_i : \Delta^i \rightarrow \Delta^{i+1}$ is given by $F_i(z_1, \dots, z_i) = (z_1, \dots, z_{i-1}, \alpha_i(z_i), \beta_i(z_i))$ and $z \mapsto (\alpha_i(z), \beta_i(z))$ is some isometric embedding of Δ into Δ^2 with the isometric constant equal to 1. In other words, we can write*

$$\begin{aligned} f_1 &= \alpha_1, \\ f_2 &= \alpha_2 \circ \beta_1, \\ f_3 &= \alpha_3 \circ \beta_2 \circ \beta_1, \\ &\vdots \\ f_{p-1} &= \alpha_{p-1} \circ \beta_{p-2} \circ \dots \circ \beta_1, \\ f_p &= \beta_{p-1} \circ \beta_{p-2} \circ \dots \circ \beta_1, \end{aligned}$$

where for all i , $z \mapsto (\alpha_i(z), \beta_i(z))$ is an isometric embedding of Δ into Δ^2 .

Proof. First note that since every s_i is a factor of n and $n \leq 2^{p-1}$, we must have $n = 2^{p-1}$ and every s_i is a power of 2 with a power less than or equal to $p-1$. Let $r_i = n/s_i = 2^{u_i}$. By assumption, $r_p = 1$ and from Proposition 5.3 we have $\sum_{i=1}^p n/s_i = n$; thus,

$$\sum_{i=1}^{p-1} 2^{u_i} = 2^{p-1} - 1 = \sum_{i=1}^{p-1} 2^{i-1}, \quad 0 \leq u_{p-1} \leq u_{p-2} \leq \dots \leq u_1.$$

Therefore by Lemma 6.6,

$$s_j = 2^i, \quad i = 1, \dots, p-1.$$

We prove the theorem by induction on p . It is trivial for $p = 2$. Suppose the theorem has been proven for Δ^{p-1} . Letting $\alpha_1 = f_1$, since the sheeting number of α_1 is 2, by Lemma 6.7, we can find an analytic function β_1 such that $z \mapsto (\alpha_1(z), \beta_1(z))$ is an isometric embedding of Δ into Δ^2 , satisfying $(1 - |\alpha_1|^2)(1 - |\beta_1|^2) = 1 - |z|^2$. Hence from the functional equation, we have

$$\prod_{i=2}^p (1 - |f_i|^2) = 1 - |\beta_1|^2.$$

Since 0 is not a branch point, there exists a local inverse $\beta_1^{-1} : U \subset \Delta \rightarrow \Delta$ of β_1 . Then

$$\prod_{i=2}^p (1 - |f_i \circ \beta_1^{-1}(z)|^2) = 1 - |z|^2$$

for $z \in U$. So we see that $(f_2 \circ \beta_1^{-1}, \dots, f_p \circ \beta_1^{-1})$ constitutes an isometric embedding of U into Δ^{p-1} . We already have known that such an embedding can always be extended to the whole unit disk and we let $G(z) = (g_2, \dots, g_p)$ be the extension. For $2 \leq i \leq p$, we have $f_i \circ \beta_1^{-1} = g_i$ and hence $f_i = g_i \circ \beta_1$ on some open set.

Thus, we have for the local inverses, $f_i^{-1} = \beta_1^{-1} \circ g_i^{-1}$. Recall that each of these local inverses is just the restriction of some rational function on an open set. Since the sheeting numbers of f_i^{-1}, β_1^{-1} are $2^i, 2$ respectively, we deduce that the sheeting number of g_i^{-1} is 2^{i-1} . Therefore by the induction assumption, we can write

$$\begin{aligned} g_2 &= \alpha_2, \\ g_3 &= \alpha_3 \circ \beta_2, \\ &\vdots \\ g_{p-1} &= \alpha_{p-1} \circ \beta_{p-2} \circ \cdots \circ \beta_2, \\ g_p &= \beta_{p-1} \circ \beta_{p-2} \circ \cdots \circ \beta_2, \end{aligned}$$

where $z \mapsto (\alpha_i(z), \beta_i(z))$, $2 \leq i \leq p$, are isometric embeddings of Δ into Δ^2 with the isometric constant equal to 1. Since $f_1 = \alpha_1$ and $f_i = g_i \circ \beta_1$ for $2 \leq i \leq p$, the proof is complete. \square

7. CLASSIFICATION FOR Δ^2

For isometric embeddings of Δ into Δ^2 , we have already known that there are only the diagonal embedding and its reparametrizations when the isometric constant $k = 2$. So the only interesting case is where $k = 1$.

Theorem 7.1. *Let $F : \Delta \rightarrow \Delta^2$ be an isometric embedding with the isometric constant $k = 1$. Then up to reparametrizations, F is the square root embedding.*

Proof. Let $F(z) = (f_1(z), f_2(z))$. By Proposition 5.2 and Corollary 5.4, the sheeting number must be 2. It then follows from Proposition 5.3 that the sheeting numbers of f_1 and f_2 are both 2. So from the Riemann-Hurwitz formula (cf. (4.4)), both f_1 and f_2 have two branch points. Then Lemma 4.9 says that after some reparametrization we can take f_1 to be one of the two component functions of the square root embedding. But once f_1 is known, f_2 is also uniquely determined (up to a constant factor of modulus 1) and therefore up to reparametrizations, F is the square root embedding. \square

8. CLASSIFICATION FOR Δ^3

The classification of isometric embeddings of Δ into Δ^3 can be obtained from Theorem 6.5 and Theorem 6.8. By Proposition 5.2 and Corollary 5.4, if the isometric constant is k , then $\frac{3}{k} \leq n \leq 4$, where n is the sheeting number of V relative to π . For $k = 3$, there are only the diagonal embedding and its reparametrizations. We first consider the case where $k = 1$.

Theorem 8.1. *Let $F : \Delta \rightarrow \Delta^3$ be an isometric embedding with the isometric constant $k = 1$. Then up to reparametrizations, F is either the cube root embedding or F can be factorized as $F = F_2 \circ F_1$, where $F_1 : \Delta \rightarrow \Delta^2$, $F_1(z) = (\alpha_1(z), \beta_1(z))$ is an isometric embedding with the isometric constant equal to 1 and $F_2 : \Delta^2 \rightarrow \Delta^3$, $F_2(z_1, z_2) = (z_1, \alpha_2(z_2), \beta_2(z_2))$, in which $z \mapsto (\alpha_2(z), \beta_2(z))$ is an isometric embedding of Δ into Δ^2 with the isometric constant equal to 1.*

Proof. As $k = 1$, we have $3 \leq n \leq 4$. For $n = 3$, the result follows from Theorem 6.5. For $n = 4$, by Proposition 5.3 there exists one f_i with a sheeting number equal to 4 because the only solution (up to permutations) to (5.1) is $\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$. The result then follows from Theorem 6.8. \square

Next, when $k = 2$, we have

Theorem 8.2. *Let $F : \Delta \rightarrow \Delta^3$ be an isometric embedding with the isometric constant $k = 2$. Then up to reparametrizations, $F(z) = (z, \alpha(z), \beta(z))$, where $z \mapsto (\alpha(z), \beta(z))$ is an isometric embedding of Δ into Δ^2 with the isometric constant equal to 1.*

Proof. As $k = 2$, we have $2 \leq n \leq 4$. For $n = 2$, the only solution (up to permutations) to (5.1) is $\frac{1}{1} + \frac{1}{2} + \frac{1}{2} = 2$. So the sheeting numbers of f_1, f_2, f_3 are 1, 2, 2 respectively. It is then clear that $f_1(z) = \gamma z$ for some $|\gamma| \in \mathbb{C}$ with $|\gamma| = 1$, and we may take $\gamma = 1$. Thus, the functional equation reduces to

$$(1 - |f_2|^2)(1 - |f_3|^2) = 1 - |z|^2,$$

and therefore $f_2(z), f_3(z)$ constitute an isometric embedding from Δ into Δ^2 with the isometric constant equal to 1. For $n = 3$, there is no solution for (5.1). For $n = 4$, the only solution (up to permutation) is $\frac{1}{1} + \frac{1}{2} + \frac{1}{2} = 2$ and the sheeting numbers of f_1, f_2, f_3 are 1, 2, 2 respectively and it is the same as the case when $n = 2$ and hence is not possible. \square

9. ISOMETRIC EMBEDDINGS BETWEEN POLYDISKS

In this section, we will consider isometric embeddings of Δ^q into Δ^p , where $q \leq p$. The polydisks will be equipped with product Poincaré metrics. We will show that all these embeddings “split”, in the sense that they are induced from isometric embeddings of the unit disk into polydisks.

Let $F : \Delta^q \rightarrow \Delta^p$, $F(z_1, \dots, z_q) = (f_1(z_1, \dots, z_q), \dots, f_p(z_1, \dots, z_q))$ be an isometric embedding with the isometric constant k . Assume that $F(\mathbf{0}) = \mathbf{0}$. Then it satisfies the functional equation

$$(9.1) \quad \prod_{i=1}^p (1 - |f_i(z_1, \dots, z_q)|^2) = \prod_{j=1}^q (1 - |z_j|^2)^k.$$

The isometric constant k is necessarily an integer. It can be argued in the same way as in the case of the unit disk (cf. Proposition 2.2).

We now fix z_2, \dots, z_q and consider (9.1) as a functional equation in z_1 . By taking $-\sqrt{-1}\partial\bar{\partial}\log$ on both sides (with respect to z_1 only), we see that $z_1 \mapsto F(z_1, \dots, z_q)$ is an isometric embedding of Δ into Δ^p for every $(z_2, \dots, z_q) \in \Delta^{q-1}$. Except when $z_2 = \dots = z_q = 0$, the embedding does not preserve the origin.

Lemma 9.1. *For each i , if $\frac{\partial f_i}{\partial z_1}$ is not constantly zero, then*

$$f_i(0, z_2, \dots, z_q) = 0$$

for all $(z_2, \dots, z_q) \in \Delta^{q-1}$.

Proof. Fixing $(z_2, \dots, z_q) \in \Delta^{q-1}$, by following the same reasoning leading to Lemma 4.4, we see that for each i , $f_i(z_1, z_2, \dots, z_q)$ (as a multi-valued function of z_1) can only take the value infinity ∞ at $z_1 = \infty$. It is clear from the functional equation that f_i as a function of z_1 satisfies the Schwarz reflection principle (cf. Proposition 4.5); therefore $f_i(0, z_2, \dots, z_q) = 0$. \square

Lemma 9.2. *Let $W \subset \mathbb{C}$ be an open set and $(f_{w;1}(z), f_{w;2}(z), \dots, f_{w;p}(z))$ be a complex analytic family of holomorphic functions on Δ with $w \in W$ being the parameter such that*

$$\prod_{i=1}^p (1 - |f_{w;i}|^2) = (1 - |z|^2)^k.$$

Then for all i , $\frac{\partial f_{w;i}}{\partial w} \equiv 0$.

Proof. We can determine the algebraic subvariety extending the embedding for every w in the same way as in Proposition 4.2. But from Lemma 4.1 we see that the unitary matrix required is independent of w and therefore the algebraic subvariety hence obtained is also independent of w . It then follows that the embedding does not depend on w . \square

Theorem 9.3. *Let $F : \Delta^q \rightarrow \Delta^p$, $F = (f_1, \dots, f_p)$ be an isometric embedding with the isometric constant k . Then after rearranging the component indices if necessary, there exist positive integers p_1, \dots, p_q with $p_1 + \dots + p_q = p$ such that*

$$\begin{array}{ll} f_1, \dots, f_{p_1} & \text{depend on } z_1 \text{ only;} \\ f_{p_1+1}, \dots, f_{p_1+p_2} & \text{depend on } z_2 \text{ only;} \\ \vdots & \vdots \\ f_{p_1+\dots+p_{q-1}+1}, \dots, f_p & \text{depend on } z_q \text{ only.} \end{array}$$

In particular, the i -th group of the above q groups of functions constitutes an isometric embedding of Δ into Δ^{p_i} with the isometric constant equal to k .

Proof. By Lemma 9.1, if $\frac{\partial f_i}{\partial z_1}$ is not constantly zero, then $f_i(0, z_2, \dots, z_q) = 0$, $\forall (z_2, \dots, z_q) \in \Delta^{q-1}$. From the functional equation (9.1), we see that this cannot happen to all f_i . Without loss of generality, we may assume that $\frac{\partial f_i}{\partial z_1} \equiv 0$, $p_1 + 1 \leq i \leq p$, for some positive integer p_1 . Then we can write $f_i(z_1, z_2, \dots, z_q) = f_i(z_2, \dots, z_q)$, for $p_1 + 1 \leq i \leq p$. If we substitute $z_1 = 0$ in (9.1), we get

$$(9.2) \quad \prod_{i=p_1+1}^p (1 - |f_i(z_2, \dots, z_q)|^2) = \prod_{j=2}^q (1 - |z_j|^2)^k.$$

Dividing (9.1) by (9.2), we obtain

$$\prod_{i=1}^{p_1} (1 - |f_i(z_1, z_2, \dots, z_q)|^2) = (1 - |z_1|^2)^k.$$

So we have a complex analytic family (with parameters z_2, \dots, z_q) of functions satisfying the above functional equation in z_1 and by Lemma 9.2, for $1 \leq i \leq p_1$, f_i is a function of z_1 only. Now we can argue in the same way for the variable z_2 in (9.2). By induction, the proof is complete. \square

Corollary 9.4. *If $F : \Delta^q \rightarrow \Delta^p$ is an isometric embedding with the isometric constant equal to k , then*

$$p \geq qk.$$

Proof. As $p_i \geq k$, we have

$$p = \sum_{i=1}^q p_i \geq qk. \quad \square$$

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