Markowitz's Mean-Variance Asset-liability Management with Regime Switching: A Multi-period Model

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Abstract

This paper considers an optimal portfolio selection problem under Markowitz's mean-variance portfolio selection problem in a multi-period regime switching model. We assume that there are $n+1$ securities in the market. Given an economic state which is modeled by a finite state Markov chain, the return of each security at a fixed time point is a random variable. The return random variables may be different if the economic state is changed even for the same security at the same time point. We start our analysis from the no-liability case, in the spirit of Li and Ng (2000), both the optimal investment strategy and the efficient frontier are derived. Then we add uncontrollable liability into the model. By direct comparison with the no-liability case, the optimal strategy can be derived explicitly.

Key Words: discrete-time, multi-period, regime switching, Markov chain, asset-liability management, portfolio selection, efficient frontier.

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1 Introduction

Due to his seminal work in 1952 (see Markowitz (1952)), Markowitz is regarded as the pioneer of modern portfolio theory. Markowitz's mean-variance model is a single period model. There are many papers which extend the Markowitz's model in various ways. Samuelson (1969) extended the work of Markowitz to a dynamic model and considered a discrete time consumption investment model with objective of maximizing the overall expected consumption. Merton (1969, 1971) used stochastic optimal control to obtain the optimal portfolio strategy under specific assumptions about asset returns and investor preferences in a continuous time model. Li and Ng (2000) extended Markowitz's model to a dynamic setting, by using some techniques in optimization, optimal strategy and efficient frontier are obtained.

In recent years, regime switching models have become popular in finance and related fields. This type of model is motivated by the intension to reflect the state of the financial market. For example, the state of the market can be roughly divided into 'bullish' and 'bearish' two regimes, in which the price movements of the stocks are quite different. Generally, in a regime-switching model, the value of market modes are divided into a finite number of regimes. The key parameters, such as the bank interest rate, stocks appreciation rates, and volatility rates, will change according to the value of different market modes. Since the market state may change from one regime to another, both the nature of the regime and the change point should be estimated. If the market state process is modeled by a continuous time Markov chain with finite states, regime switching models are also referred to as Markov switching or Markov-modulated models in some literatures.

With time-varying parameters, regime switching models are obviously more realistic than constant parameter model to reflect the random market environment. As discussed in Neftci (1984), an appealing ability of these models is to account for the accumulating evidence that business cycles are asymmetric. Most of the studies indicate that regime-switching models perform well in some sense, for example, Hardy (2001) used monthly data from the Standard and Poor's 500 and the Toronto Stock Exchange 300 indices to fit a regime switching lognormal model. In her paper, the fit of the regime switching model to the data was compared with other econometric models and she found that regime-switching models provided a significant improvement over all the other models in the sense of maximizing the likelihood function. In a special case, if the data is in lognormal setting, the software "regime switching equity model workbook" developed by Hardy and her group which can be found on SOA web site can be applied directly, which greatly simplifies the application procedure of regime-switching model.

Regime-switching models are not new in statistics and economics, dating back to at least Henderson and Quandt (1958), where regime regression models were investigated. Kim and Nelson (1999) gave a brief review of Markov switching models and presented comprehensive exposition of statistical methods for these models as well as many empirical studies. One influential work on the application of regime switching models is Hamilton (1989), where dynamic models with Markov switching between regimes were introduced as a tool for dealing with endogenous structural breaks. And after that, enormous empirical works about regime switching structure were done in many economical aspects, such as business cycle asymmetry, see Hamilton (1989), Lam (1990); the effects of oil prices on U.S. GDP growth, see Raymond and Rich (1997); labor market recruitment, see Storer and Van Audenrode (1995); government expenditure, see Rugemurcia (1995); and the level of merger and acquisition activity, see Town (1992).

However, it is recent years for the application of regime switching models in finance. Early works are done on option pricing, see Di Masi *et al.* (1994), Buffington and Elliott (2002), Boyle and Draviam (2007). After that, regime switching models are applied to many other aspects, such as Equity-linked life insurance pricing, see Hardy (2003); Bond Pricing, see Elliott and Siu (2009a); Portfolio selection, see Zhou and Yin (2003), Guidolin and Timmermann (2007), Chen *et al.* (2008), Elliott and Siu (2009b); Optimal dividend, Li and Lu (2006, 2007), etc.

Nowadays, asset-liability management (ALM) problem is of both theoretical interest and practical importance in risk management and insurance. In ALM, the main concern is the surplus which is the difference of asset value and liability value. Accordingly, ALM is also known as surplus management. Similarly, during the whole portfolio selection process, the liability value of the company should also need to be considered. One feature of our optimal portfolio selection model is that we adopt uncontrollable liabilities. Sharpe and Tint (1990), Kell and Müller (1995) and Leippold *et al.* (2004) suggested that the dynamics of liability should not be affected by the asset trading strategy. That is, the liabilities are not controllable. Problems with this feature have been investigated in many literatures, for example, Norberg (1999), Browne (1997), Taksar and Zhou (1998), Decamps *et al.* (2006).

When both regime-switching and liabilities are presented in the model, we are concerned with the explicit solutions of the optimal portfolio under mean-variance criteria. This paper considers an optimal portfolio selection problem under Markowitz's mean-variance portfolio selection problem in a multi-period regime switching model. We assume that there are $n+1$ securities in the market. Given an economic state which is modeled by a finite state Markov chain, the return of each security at a fixed time point is a random variable. The return random variables may be different if the economic state is changed even for the same security at the same time point. We start our analysis from the no-liability case, in the spirit of Li and Ng (2000), both the optimal investment strategy and the efficient frontier are derived. Then follow the similar sprite of Leippold *et al.* (2004), we add uncontrollable liability into the model. The optimal strategy can be directly written out by comparison to the no-liability case.

The rest of the paper is organized as follows. Section 2 provides the formulation of the problem in the no-liability case and transforms the original problems into an unconstrained one. Section 3 and Section 4 gives the analytical solution to our problem. In Section 5, uncontrollable liability is added into the model and the optimal strategy is derived. Section 6 gives some numerical results and the last section gives certain concluding remarks.

2 Problem formulation in the case of no-liability

Consider a discrete-time model with T periods. Let $\mathcal{M} = \{1, 2, ..., m\}$ denote the collection of different market modes. Throughout the paper, let $(\Omega, \mathcal{F}, P)$ be a fixed complete probability space on which a discrete-time Markov chain $\{\alpha_t, t = 0, 1, ..., T-1\}$ taking value in $\mathcal{M}$ is defined. The Markov chain has transition probabilities

$$P\{\alpha_{t+1} = j | \alpha_t = i\} = p_{ij} \geq 0, \quad \sum_{j=1}^{m} p_{ij} = 1, \tag{1}$$

for $t = 0, 1, ..., T-1, i, j \in \mathcal{M}$. We assume that there are $n+1$ risky securities with random rates of returns in the capital market. An investor joins the market at time 0 with an initial wealth x_0. The investor is allowed to reallocate his wealth among $n+1$ assets over T consecutive time periods at dates $0, 1, ..., T-1$. The rates of return of the risky securities at time period t within the planning horizon are denoted by a vector $e_t(i) = [e_t^0(i), e_t^1(i), ..., e_t^n(i)]'$, where $e_t^k(i)$ is the random return for security k at time period t corresponding to market mode $i \in \mathcal{M}$. By the Markov property of α_t, vectors $e_t(\alpha_t)$, $t = 0, 1, ..., T-1$, are no longer independent which is quite different from the model in Li and Ng (2000). However, vectors $e_t(i)$ for $t = 0, 1, ..., T-1$ and fixed market mode $i \in \mathcal{M}$, are assumed independent. The corresponding mean and covariance are assumed as follows

$$\mathbb{E}(e_t(i)) = \begin{pmatrix} \mathbb{E}(e_t^0(i)) \\ \vdots \\ \mathbb{E}(e_t^n(i)) \end{pmatrix}, \quad \mathrm{cov}(e_t(i)) = \begin{pmatrix} \sigma_t^{00}(i) & \cdots & \sigma_t^{0n}(i) \\ \vdots & \ddots & \vdots \\ \sigma_t^{0n}(i) & \cdots & \sigma_t^{nn}(i) \end{pmatrix} \tag{2}$$

Denote by x_t the asset value of the investor at the beginning of the tth period. Let $u_t^k, k = 1, 2, ..., n$, be the amount invested in the kth risky asset at the beginning of the time period, then the amount investigated in the 0th risky asset at the beginning of the tth time period is equal to $x_t - \sum_{k=1}^n u_t^k$. The asset value dynamics can be written as

$$\begin{aligned} x_{t+1} &= \sum_{k=1}^n e_t^k(\alpha_t) u_t^k + \Big(x_t - \sum_{k=1}^n u_t^k \Big) e_t^0(\alpha_t) \\ &= e_t^0(\alpha_t) x_t + P_t(\alpha_t)' u_t \end{aligned} \tag{3}$$

where $u_t = [u_t^1, ..., u_t^n]'$, and for $\alpha_t = 1, 2, ..., m$,

$$P_t(i) := [P_t^1(i), ..., P_t^n(i)]' = \big[\big(e_t^1(i) - e_t^0(i)\big), ..., \big(e_t^n(i) - e_t^0(i)\big) \big]'. \tag{4}$$

We assume that the investor can observe the present asset value and the regime of the market directly. Thus we introduce $\mathcal{F}_t := \sigma\{x_s, \alpha_s | 0 \le s \le t\}$ to denote the information available to the investor up to time t.

Notice that $\mathbb{E}(e_t(i)e_t(i)') = \mathrm{cov}(e_t(i)) + \mathbb{E}(e_t(i))\mathbb{E}(e_t(i)')$. It is reasonable to assume that $\mathbb{E}(e_t(i)e_t(i)')$ is positive definite for all time periods and for all $i \in \mathcal{M}$. Otherwise, there is no need to invest in the risky assets. That is, for all $t = 0, 1, ..., T-1$, $i = 1, 2, ..., m$,

$$\mathbb{E}(e_t(i)e_t(i)') = \begin{pmatrix} \mathbb{E}((e_t^0(i))^2) & \mathbb{E}(e_t^0(i)e_t^1(i)') & \cdots & \mathbb{E}(e_t^0(i)e_t^n(i)') \\ \mathbb{E}(e_t^1(i)e_t^0(i)') & \mathbb{E}((e_t^1(i))^2) & \cdots & \mathbb{E}(e_t^1(i)e_t^n(i)') \\ \vdots & \vdots & \vdots & \vdots \\ \mathbb{E}(e_t^n(i)e_t^0(i)') & \mathbb{E}(e_t^n(i)e_t^1(i)') & \cdots & \mathbb{E}((e_t^n(i))^2) \end{pmatrix} > 0. \tag{5}$$

Then the following is true from equation (5):

$$\begin{aligned} &\begin{pmatrix} \mathbb{E}((e_t^0(i))^2) & \mathbb{E}(e_t^0(i)P_t(i)') \\ \mathbb{E}(e_t^0(i)P_t(i)) & \mathbb{E}(P_t(i)P_t(i)') \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & \cdots & 1 \end{pmatrix} \mathbb{E}(e_t(i)e_t(i)') \begin{pmatrix} 1 & -1 & \cdots & -1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} > 0 \end{aligned} \tag{6}$$

Furthermore, we have the following from equation (6):

$$\mathbb{E}(P_t(i)P_t(i)') > 0 \tag{7}$$

and

$$\mathbb{E}((e_t^0(i))^2) - \mathbb{E}(e_t^0(i)P_t(i)')\mathbb{E}^{-1}(P_t(i)P_t(i)')\mathbb{E}(e_t^0(i)P_t(i)) > 0 \tag{8}$$

for all $i \in \mathcal{M}$, $t = 0, 1, ..., T-1$.

Then the portfolio selection problem of our investor can be formulated in the following three forms:

$$\begin{aligned} (P1(\sigma)): \qquad & \max \mathbb{E}(x_T) \\ & \text{subject to } \mathrm{Var}(x_T) \leq \sigma \text{ and } (3), \end{aligned} \tag{9}$$

where $\sigma > 0$ and

$$\begin{aligned} (P2(\epsilon)): \qquad & \min \mathrm{Var}(x_T) \\ & \text{subject to } \mathbb{E}(x_T) \geq \epsilon \text{ and } (2.3), \end{aligned} \tag{10}$$

where $\epsilon \geq 0$ and

$$\begin{aligned} (P3(\omega)): \qquad & \max \mathbb{E}(x_T) - \omega \mathrm{Var}(x_T) \\ & \text{subject to } (2.3), \end{aligned} \tag{11}$$

for some strictly positive risk aversion parameter ω.

A multi-period portfolio policy is an investment sequence,

$$
\begin{aligned}
u &= \{u_0, u_1, ..., u_{T-1}\} \\
&= \left\{ \begin{pmatrix} u_0^1 \\ u_0^2 \\ \vdots \\ u_0^n \end{pmatrix}, \begin{pmatrix} u_1^1 \\ u_1^2 \\ \vdots \\ u_1^n \end{pmatrix}, ..., \begin{pmatrix} u_{T-1}^1 \\ u_{T-1}^2 \\ \vdots \\ u_{T-1}^n . \end{pmatrix} \right\}
\end{aligned} \tag{12}
$$

where $u_t \in \mathcal{F}_t$ for $t = 0, 1, ..., T-1$.

A multi-period portfolio policy, u^*, is said to be efficient if there exists no other multi-period portfolio policy u that is 'better' than u^*. The 'better' here means $\mathbb{E}(x_T)|_u \geq \mathbb{E}(x_T)|_{u^*}$ and $\mathrm{Var}(x_T)|_u \leq \mathrm{Var}(x_T)|_{u^*}$ with at least one inequality strict. By varying the value of σ, ϵ and ω in $(P1(\sigma))$, $(P2(\epsilon))$ and $(P3(\omega))$ under the efficient portfolio u^* respectively, a set of $(\mathbb{E}(x_T)|_{u^*}, \mathrm{Var}(x_T)|_{u^*})$ can be generated, which is called the efficient frontier according to problems $(P1(\sigma))$, $(P2(\epsilon))$ and $(P3(\omega))$ respectively.

It is well known that the above three formulations are equivalent in the sense that they have the same efficient frontier for some specific parameters σ, ϵ and ω. More specifically, If u^* solves $(P1(\sigma))$ or $(P2(\epsilon))$, then it solves $(P3(\omega))$ for some ω by the classical Lagrange method (e.g. Luenberger (1969)). On the other hand, if u^* solves $(P3(\omega))$, then it solves $(P1(\sigma))$ with $\sigma = \mathrm{Var}(x_T|_{u^*})$ and solves $(P2(\epsilon))$ with $\epsilon = \mathbb{E}(x_T|u^*)$. Further, as is shown by Li and Ng (2000), the identity $\omega = \frac{\partial \mathbb{E}(x_T)}{\partial \mathrm{Var}(x_T)}$ holds for the solution of $(P3(\omega))$. From a mathematical point of view, problem $(P3(\omega))$ is preferable to be adopted in investment situations where the parameter ω can be seen as a trade-off between the expected terminal wealth and the associated risk, which prevents the troublesome constrain in $(P1(\sigma))$ and $(P2(\epsilon))$.

However, $(P3(\omega))$ is still difficult to solve due to $(\mathbb{E}(x_T))^2$ part in the expression of $\mathbb{E}(x_T) - \omega \mathrm{Var}(x_T)$, which results in the non-separability in the sense of dynamic programming. Li and Ng (2000) shows that $(P3(\omega))$ can be solved by embedding it into the following auxiliary problem that is separable,

$$
\begin{aligned}
(P4(\lambda,\omega)): \qquad & \max \mathbb{E}\{-\omega x_T^2 + \lambda x_T\} \\
& \text{subject to (2.3).}
\end{aligned} \tag{13}
$$

Moreover, if u^* solves $(P3(\omega))$, then it solves $(P4(\lambda,\omega))$ for $\lambda = 1 + 2\omega \mathbb{E}(x_T)|_{u^*}$. On the other hand, if u^* solves $(P4(\lambda^*,\omega))$, then a necessary condition for u^* solves $(P3(\omega))$ is

$$
\lambda^* = 1 + 2\omega \mathbb{E}(x_T)|_{u^*}. \tag{14}
$$

Note that the objective function of $(P4(\lambda,\omega))$ is of a quadratic form while the system dynamic (3) is of a linear form, which leads to the tractability of $(P4(\lambda,\omega))$ by standard methods in dynamic programming.

3 Analytical solution to the unconstrained problem $(P4(\lambda,\omega))$

In this section, the optimal multi-period portfolio policy for problem $(P4(\lambda,\omega))$ will be derived. According to this optimal policy, we also derive the efficient frontier of the problem.

Theorem 3.1. *Given the optimization problem (13), the optimal policy for* $t = 0, 1, ..., T-$

1 *under the market mode* $i \in \mathcal{M} = \{1, 2, \dots, m\}$, *is of the following form,*

$$u_t^*(i) = -L_t(i)\Big[M_t(i)x_t - \frac{\lambda}{2\omega}N_t(i)\Big], \tag{15}$$

where $L_t(i)$, $M_t(i)$ *and* $N_t(i)$ *are derived by the following iteration procedure,*

$$\begin{aligned}
L_{T-1}(i) &= [\mathbb{E}\{P_{T-1}(i)P_{T-1}(i)'|\mathcal{F}_{T-1}\}]^{-1}, && (16)\\
M_{T-1}(i) &= \mathbb{E}\{e^0_{T-1}(i)P_{T-1}(i)|\mathcal{F}_{T-1}\}, && (17)\\
N_{T-1}(i) &= \mathbb{E}\{P_{T-1}(i)|\mathcal{F}_{T-1}\}, && (18)\\
A^1_{T-1}(i) &= \mathbb{E}\{e^0_{T-1}(i)|\mathcal{F}_{T-1}\} - M_{T-1}(i)'L_{T-1}(i)N_{T-1}(i), && (19)\\
A^2_{T-1}(i) &= \mathbb{E}\{[e^0_{T-1}(i)]^2|\mathcal{F}_{T-1}\} - M_{T-1}(i)'L_{T-1}(i)M_{T-1}(i) > 0, && (20)\\
L_t(i) &= [\mathbb{E}\{P_t(i)P_t(i)'|\mathcal{F}_t\}\sum_{j=1}^m p_{ij}A^2_{t+1}(j)]^{-1}, t = 0, 1, ..., T-2, && (21)\\
M_t(i) &= \mathbb{E}\{e^0_t(i)P_t(i)|\mathcal{F}_t\}\sum_{j=1}^m p_{ij}A^2_{t+1}(j), t = 0, 1, ..., T-2, && (22)\\
N_t(i) &= \mathbb{E}\{P_t(i)|\mathcal{F}_t\}\sum_{j=1}^m p_{ij}A^1_{t+1}(j), t = 0, 1, ..., T-2, && (23)\\
A^1_t(i) &= \mathbb{E}e^0_t(i)\sum_{j=1}^m p_{ij}A^1_{t+1}(j) - M_t(i)'L_t(i)N_t(i)\\
&= \Big\{\mathbb{E}\{e^0_t(i)|\mathcal{F}_t\} - \mathbb{E}\{e^0_t(i)P_t(i)'|\mathcal{F}_t\}[\mathbb{E}\{P_t(i)P_t(i)'|\mathcal{F}_t\}]^{-1}\mathbb{E}\{P_t(i)|\mathcal{F}_t\}\Big\}\\
&\quad \sum_{j=1}^m p_{ij}A^1_{t+1}(j), \qquad t = 0, 1, ..., T-2, && (24)\\
A^2_t(i) &= \mathbb{E}\{[e^0_t(i)]^2|\mathcal{F}_t\}\sum_{j=1}^m p_{ij}A^2_{t+1}(j) - M_t(i)'L_t(i)M_t(i)\\
&= \Big\{\mathbb{E}\{[e^0_t(i)]^2|\mathcal{F}_t\} - \mathbb{E}\{e^0_t(i)P_t(i)'|\mathcal{F}_t\}[\mathbb{E}\{P_t(i)P_t(i)'|\mathcal{F}_t\}]^{-1}\mathbb{E}\{e^0_t(i)P_t(i)|\mathcal{F}_t\}\Big\}\\
&\quad \sum_{j=1}^m p_{ij}A^2_{t+1}(j) > 0, \qquad t = 0, 1, ..., T-2. && (25)
\end{aligned}$$

Proof. By (7) and (8), the positivity of $A^2_t(i)$ for $t = 0, 1, ..., T-1$ and $i \in \mathcal{M}$ can be easily seen, which also guarantees the feasibility of the iteration procedure. By Bellman's optimization principle, the global optimization implies the optimality at any current time $t = 0, 1, ..., T-1$. Notice that when $T-1$ is the current time, then the value of α_{T-1} and x_{T-1} can be treated as known, we have

$$\begin{aligned}
& \mathbb{E}\{-\omega x_T^2 + \lambda x_T\} \\
= \;& \mathbb{E}\big\{\mathbb{E}\{-\omega x_T^2 + \lambda x_T | \mathcal{F}_{T-1}\}\big\} \\
= \;& \mathbb{E}\big\{\mathbb{E}\{-\omega[e_{T-1}^0(\alpha_{T-1})x_{T-1} + P_{T-1}(\alpha_{T-1})'u_{T-1}]^2 + \lambda[e_{T-1}^0(\alpha_{T-1})x_{T-1} + P_{T-1}(\alpha_{T-1})'u_{T-1}]|\mathcal{F}_{T-1}\}\big\} \\
= \;& \mathbb{E}\big\{-\omega[u_{T-1} - u_{T-1}^*(\alpha_{T-1})]'[L_{T-1}(\alpha_{T-1})]^{-1}[u_{T-1} - u_{T-1}^*(\alpha_{T-1})] \\
& +\omega[M_{T-1}(\alpha_{T-1})x_{T-1} - \frac{\lambda}{2\omega}N_{T-1}(\alpha_{T-1})]'L_{T-1}(\alpha_{T-1})[M_{T-1}(\alpha_{T-1})x_{T-1} \\
& -\frac{\lambda}{2\omega}N_{T-1}(\alpha_{T-1})] - \omega\mathbb{E}\{[e_{T-1}^0(\alpha_{T-1})]^2|\mathcal{F}_{T-1}\}x_{T-1}^2 + \lambda\mathbb{E}\{e_{T-1}^0(\alpha_{T-1})|\mathcal{F}_{T-1}\}x_{T-1}\} \\
= \;& \mathbb{E}\big\{-\omega[u_{T-1} - u_{T-1}^*(\alpha_{T-1})]'[L_{T-1}(\alpha_{T-1})]^{-1}[u_{T-1} - u_{T-1}^*(\alpha_{T-1})] - \omega A_{T-1}^2(\alpha_{T-1})x_{T-1}^2 \\
& +\lambda A_{T-1}^1(\alpha_{T-1})x_{T-1} + \frac{\lambda^2}{4\omega}N_{T-1}(\alpha_{T-1})'L_{T-1}(\alpha_{T-1})N_{T-1}(\alpha_{T-1})\big\}
\end{aligned} \tag{26}$$

where $u_{T-1}^*(i)$ is defined in (15), $L_{T-1}(i)$, $M_{T-1}(i)$, $N_{T-1}(i)$, $A_{T-1}^1(i)$ and $A_{T-1}^2(i)$ are defined in (16)–(20) when $\alpha_{T-1} = i \in \mathcal{M}$. Note that (26) is in quadratic form with respect to u_{T-1}, which leads to the optimality of $u_{T-1}^*(i)$ when $\alpha_{T-1} = i \in \mathcal{M}$. By adopting this optimal allocation $u_{T-1}^*(i)$, we move back to $T-2$ as the current time, then

α_{T-2} and x_{T-2} can be seen as known, by (26) we have

$$
\begin{aligned}
& \mathbb{E}\{-\omega x_T^2 + \lambda x_T\} \\
= \ & \mathbb{E}\Big\{ -\omega A^2_{T-1}(\alpha_{T-1}) x^2_{T-1} + \lambda A^1_{T-1}(\alpha_{T-1}) x_{T-1} \\
& \qquad\qquad + \frac{\lambda^2}{4\omega} N_{T-1}(\alpha_{T-1})' L_{T-1}(\alpha_{T-1}) N_{T-1}(\alpha_{T-1}) \Big\} \\
= \ & \mathbb{E}\Big\{ \mathbb{E}\{ -\omega A^2_{T-1}(\alpha_{T-1}) x^2_{T-1} + \lambda A^1_{T-1}(\alpha_{T-1}) x_{T-1} \\
& \qquad\qquad + \frac{\lambda^2}{4\omega} N_{T-1}(\alpha_{T-1})' L_{T-1}(\alpha_{T-1}) N_{T-1}(\alpha_{T-1}) | \mathcal{F}_{T-2} \} \Big\} \\
= \ & \mathbb{E}\Big\{ \mathbb{E}\{ -\omega A^2_{T-1}(\alpha_{T-1}) [e^0_{T-2}(\alpha_{T-2}) x_{T-2} + P_{T-2}(\alpha_{T-2})' u_{T-2}]^2 \\
& + \lambda A^1_{T-1}(\alpha_{T-1}) [e^0_{T-2}(\alpha_{T-2}) x_{T-2} + P_{T-2}(\alpha_{T-2})' u_{T-2}] \\
& + \frac{\lambda^2}{4\omega} N_{T-1}(\alpha_{T-1})' L_{T-1}(\alpha_{T-1}) N_{T-1}(\alpha_{T-1}) | \mathcal{F}_{T-2} \} \Big\} \\
= \ & \mathbb{E}\Big\{ -\omega [u_{T-2} - u^*_{T-2}(\alpha_{T-2})]' [L_{T-2}(\alpha_{T-2})]^{-1} [u_{T-2} - u^*_{T-2}(\alpha_{T-2})] \\
& + \omega [M_{T-2}(\alpha_{T-2}) x_{T-2} - \frac{\lambda}{2\omega} N_{T-2}(\alpha_{T-2})]' L_{T-2}(\alpha_{T-2}) [M_{T-2}(\alpha_{T-2}) x_{T-2} - \frac{\lambda}{2\omega} N_{T-2}(\alpha_{T-2})] \\
& - \omega \mathbb{E}\{ A^2_{T-1}(\alpha_{T-1}) [e^0_{T-2}(\alpha_{T-2})]^2 | \mathcal{F}_{T-2} \} x^2_{T-2} + \lambda \mathbb{E}\{ A^1_{T-1}(\alpha_{T-1}) e^0_{T-2}(\alpha_{T-2}) | \mathcal{F}_{T-2} \} x_{T-2} \\
& + \mathbb{E}\{ \frac{\lambda^2}{4\omega} N_{T-1}(\alpha_{T-1})' L_{T-1}(\alpha_{T-1}) N_{T-1}(\alpha_{T-1}) | \mathcal{F}_{T-2} \} \Big\} \\
= \ & \mathbb{E}\Big\{ -\omega [u_{T-2} - u^*_{T-2}(\alpha_{T-2})]' [L_{T-2}(\alpha_{T-2})]^{-1} [u_{T-2} - u^*_{T-2}(\alpha_{T-2})] \\
& - \omega A^2_{T-2}(\alpha_{T-2}) x^2_{T-2} + \lambda A^1_{T-2}(\alpha_{T-2}) x_{T-2} + \frac{\lambda^2}{4\omega} N_{T-2}(\alpha_{T-2})' L_{T-2}(\alpha_{T-2}) N_{T-2}(\alpha_{T-2}) \\
& + \mathbb{E}\{ \frac{\lambda^2}{4\omega} N_{T-1}(\alpha_{T-1})' L_{T-1}(\alpha_{T-1}) N_{T-1}(\alpha_{T-1}) | \mathcal{F}_{T-2} \} \Big\} \qquad (27)
\end{aligned}
$$

where $u^*_{T-2}(i)$ is defined in (3.1), $L_{T-2}(i)$, $M_{T-2}(i)$, $N_{T-2}(i)$, $A^1_{T-2}(i)$ and $A^2_{T-2}(i)$ are defined in (21)–(25) when $t = T-2$ and $\alpha_{T-2} = i \in \mathcal{M}$. Note that (27) is in quadratic form with respect to u_{T-2}, which leads to the optimality of $u^*_{T-2}(i)$ when $\alpha_{T-2} = i \in \mathcal{M}$. We also note that equation (27) is in similar form with (26), then we can follow the same procedure of deriving $u^*_{T-2}(i)$ to derive $u^*_{T-3}(i)$, we can continue this process until the optimal original allocation $u^*_0(i)$ being derived. Accordingly, all the $u^*_t(i)$ for

$t = 0, 1, ..., T-1$ follow the dynamic (15). $\square$

Remark 3.1. The proof of Theorem 3.1 is nothing else than a direct application of Bellman's optimal principle in dynamic programming, but the optimal strategy which is revealed by expression (15) is interesting. From a mathematical point of view, all the coefficients $L_t(i)$, $M_t(i)$ and $N_t(i)$ can be calculated at the beginning of the investment period, which is very similar to the situation in multi-period mean variance model without regime switching. The difference here is the optimal allocation among wealth for the latter model has only one choice which can be calculated in advance, whereas the optimal strategy for the former one has a set of candidates at each time point $t = 0, 1, ..., T-1$ which can only be decided until that time. Practically speaking, at the very beginning of the investment, we make a set of plan $\{u_t^*(i), i \in \mathcal{M}\}$ at each allocation time t according to different markets modes. When times goes by, the markets changes. The final choice of the plan at time t depends on the markets modes α_t. That is to say, we have to wait until time t to make a right choice $u_t^*(i)$ when we know that the markets modes α_t is i. This greatly enhances the flexibility of multi-period model.

Remark 3.2. When there is no regime-switching phenomenon, that is, $p_{i_0 j} = 0$ for $j \neq i_0$, it is not difficult to verify that our expression for the optimal investment strategy (15) is the same as Li, Chan and Ng (1998), which is cited by Li and Ng (2000) to illustrate their results.

Having addressed the issue of analytical solution for $(P4(\lambda, \omega))$, we proceed with the study of efficient frontier. Under the optimal strategy (15) and by (3), we have

$$
\begin{aligned}
x_T &= e^0_{T-1}(\alpha_{T-1})x_{T-1} + P_{T-1}(\alpha_{T-1})'u^*_{T-1}(\alpha_{T-1}) \\
&= e^0_{T-1}(\alpha_{T-1})x_{T-1} - P_{T-1}(\alpha_{T-1})'L_{T-1}(\alpha_{T-1})M_{T-1}(\alpha_{T-1})x_{T-1} \\
&\quad + \frac{\lambda}{2\omega}P_{T-1}(\alpha_{T-1})'L_{T-1}(\alpha_{T-1})N_{T-1}(\alpha_{T-1}) \\
&= B_{T-1}(\alpha_{T-1})x_{T-1} + \frac{\lambda}{2\omega}C_{T-1}(\alpha_{T-1}) \\
&= B_{T-1}(\alpha_{T-1})[B_{T-2}(\alpha_{T-2})x_{T-2} + \frac{\lambda}{2\omega}C_{T-2}(\alpha_{T-2})] + \frac{\lambda}{2\omega}C_{T-1}(\alpha_{T-1}) \\
&= B_{T-1}(\alpha_{T-1})B_{T-2}(\alpha_{T-2})x_{T-2} + \frac{\lambda}{2\omega}[B_{T-1}(\alpha_{T-1})C_{T-2}(\alpha_{T-2}) + C_{T-1}(\alpha_{T-1})] \\
&= \cdots = B_{T-1}(\alpha_{T-1})B_{T-2}(\alpha_{T-2})\cdots B_0(\alpha_0)x_0 \\
&\quad + \frac{\lambda}{2\omega}[B_{T-1}(\alpha_{T-1})\cdots B_1(\alpha_1)C_0(\alpha_0) + B_{T-1}(\alpha_{T-1})\cdots B_2(\alpha_2)C_1(\alpha_1) \\
&\quad + \cdots + B_{T-1}(\alpha_{T-1})C_{T-2}(\alpha_{T-2}) + C_{T-1}(\alpha_{T-1})] \\
&= \mu(\alpha_0,\alpha_1,...,\alpha_{T-1})x_0 + \frac{\lambda}{2\omega}\nu(\alpha_0,\alpha_1,...,\alpha_{T-1}) \qquad (28)
\end{aligned}
$$

where for $t = 0, 1, ...T-1$ and $i \in \mathcal{M}$,

$$
\begin{aligned}
B_t(i) &= e^0_t(i) - P_t(i)'L_t(i)M_t(i), \qquad (29) \\
C_t(i) &= P_t(i)'L_t(i)N_t(i), \qquad (30)
\end{aligned}
$$

and for $(\alpha_0, \alpha_1, ..., \alpha_{T-1}) = (i_0, i_1, ..., i_{T-1}) \in \mathcal{M}^T$,

$$
\begin{aligned}
\mu(i_0,i_1,...,i_{T-1}) &= B_{T-1}(i_{T-1})B_{T-2}(i_{T-2})\cdots B_0(i_0), \qquad (31) \\
\nu(i_0,i_1,...,i_{T-1}) &= B_{T-1}(i_{T-1})\cdots B_1(i_1)C_0(i_0) + B_{T-1}(i_{T-1})\cdots B_2(i_2)C_1(i_1) \\
&\quad + \cdots + B_{T-1}(i_{T-1})C_{T-2}(i_{T-2}) + C_{T-1}(i_{T-1}). \qquad (32)
\end{aligned}
$$

Taking square on both sides of (28), we have

$$\begin{aligned} x_T^2 &= \mu^2(\alpha_0,\alpha_1,...,\alpha_{T-1})x_0^2 \\ &\quad +\frac{\lambda}{\omega}x_0\mu(\alpha_0,\alpha_1,...,\alpha_{T-1})\nu(\alpha_0,\alpha_1,...,\alpha_{T-1})+\frac{\lambda^2}{4\omega^2}\nu^2(\alpha_0,\alpha_1,...,\alpha_{T-1}). \end{aligned} \tag{33}$$

Taking expectation on both sides of (28) and (33), we have

$$\mathbb{E}x_T = \mu^{(1)}x_0+\frac{\lambda}{2\omega}\nu^{(1)} \tag{34}$$

$$\mathbb{E}x_T^2 = \mu^{(2)}x_0^2+\frac{\lambda}{\omega}\theta x_0+\frac{\lambda^2}{4\omega^2}\nu^{(2)} \tag{35}$$

where

$$\begin{aligned} \mu^{(1)} &= \mathbb{E}\big\{\mu(\alpha_0,\alpha_1,...,\alpha_{T-1})\big\} \\ &= \sum_{i_1,i_2,\cdots,i_{T-1}\in\mathcal{M}} p_{i_0i_1}p_{i_1i_2}\cdots p_{i_{T-2}i_{T-1}}\mathbb{E}\{\mu(i_0,i_1,...,i_{T-1})\} && (36) \\ \mu^{(2)} &= \mathbb{E}\big\{\mu^2(\alpha_0,\alpha_1,...,\alpha_{T-1})\big\} \\ &= \sum_{i_1,i_2,\cdots,i_{T-1}\in\mathcal{M}} p_{i_0i_1}p_{i_1i_2}\cdots p_{i_{T-2}i_{T-1}}\mathbb{E}\{\mu^2(i_0,i_1,...,i_{T-1})\} && (37) \\ \theta &= \mathbb{E}\big\{\mu(\alpha_0,\alpha_1,...,\alpha_{T-1})\nu(\alpha_0,\alpha_1,...,\alpha_{T-1})\big\} && (38) \\ &= \sum_{i_1,i_2,\cdots,i_{T-1}\in\mathcal{M}} p_{i_0i_1}p_{i_1i_2}\cdots p_{i_{T-2}i_{T-1}}\mathbb{E}\{\mu(i_0,i_1,...,i_{T-1})\nu(i_0,i_1,...,i_{T-1})\} \\ \nu^{(1)} &= \mathbb{E}\big\{\nu(\alpha_0,\alpha_1,...,\alpha_{T-1})\big\} \\ &= \sum_{i_1,i_2,\cdots,i_{T-1}\in\mathcal{M}} p_{i_0i_1}p_{i_1i_2}\cdots p_{i_{T-2}i_{T-1}}\mathbb{E}\{\nu(i_0,i_1,...,i_{T-1})\} && (39) \\ \nu^{(2)} &= \mathbb{E}\big\{\nu^2(\alpha_0,\alpha_1,...,\alpha_{T-1})\big\} \\ &= \sum_{i_1,i_2,\cdots,i_{T-1}\in\mathcal{M}} p_{i_0i_1}p_{i_1i_2}\cdots p_{i_{T-2}i_{T-1}}\mathbb{E}\{\nu^2(i_0,i_1,...,i_{T-1})\}. && (40) \end{aligned}$$

By noticing $\mathbb{E}[B_t(i)C_t(i)] = 0$ for $t = 0,1,...,T-1$ and recalling the independence structure for different time periods under fixed market mode i, we have $\theta = 0$.

By (34) and (35), we have

$$\begin{aligned}\mathrm{Var}x_T &= \mathbb{E}x_T^2 - (\mathbb{E}x_T)^2 \\ &= [\mu^{(2)} - (\mu^{(1)})^2]x_0^2 - \frac{\lambda}{\omega}\mu^{(1)}\nu^{(1)}x_0 + \frac{\lambda^2}{4\omega^2}[\nu^{(2)} - (\nu^{(1)})^2]. \end{aligned} \tag{41}$$

By (34) and (41), we eliminate the parameter $\frac{\lambda}{\omega}$ to have the efficient frontier of problem $(P4(\lambda, \omega))$ in the following

$$\begin{aligned}\mathrm{Var}x_T &= \frac{\nu^{(2)} - (\nu^{(1)})^2}{(\nu^{(1)})^2}\Big[\mathbb{E}x_T - \frac{\mu^{(1)}\nu^{(2)}x_0}{\nu^{(2)} - (\nu^{(1)})^2}\Big]^2 + \Big[\mu^{(2)} - (\mu^{(1)})^2 - \frac{(\mu^{(1)}\nu^{(1)})^2}{\nu^{(2)} - (\nu^{(1)})^2}\Big]x_0^2 \\ &\qquad \text{for} \quad \mathbb{E}x_T \geq \frac{\mu^{(1)}\nu^{(2)}x_0}{\nu^{(2)} - (\nu^{(1)})^2}. \end{aligned} \tag{42}$$

Remark 3.3. By noting $\frac{\nu^{(2)} - (\nu^{(1)})^2}{(\nu^{(1)})^2} \geq 0$, the efficient frontier (42) reveals explicitly the tradeoff between the mean and variance at the terminal time, that is, more risk should be taken if we want a higher expected return.

Remark 3.4. We observe that (42) is no longer a perfect square (or, equivalently, the efficient frontier in the mean-standard deviation diagram is no longer a straight line), the investor is not able to achieve a risk-free investment. This is consistent with the paper by Zhou and Yin (2003). In Li and Ng (2000), they showed that when risk-free asset is present in the portfolio panel, a perfect square mean-variance efficient frontier is obtained. Whereas in regime-switching case, risk-free asset depends on the regimes of the market. Zhou and Yin (2003) found in their regime-switching continuous-time model that perfect square is obtained only in the case of risk-free asset that is independent with the regimes of the market. In fact, due to regime-switching phenomenon, risk-free asset is not 'risk free' if it depends on the regimes when the multiple investment periods are taken as a whole. Only asset with fixed return that is independent with the market regime is a 'true' risk free asset. We will show in the following that perfect square efficient frontier is also obtained in our discrete model when the risk free asset is independent with market regime.

Let the 0th security be risk-free. In other words, our portfolio panel is composed of n risky assets and a risk-free asset. Let $e_t^0(i)$ be constant s_t and $\mathrm{cov}(e_t^0(i), e_t^k(i)) = 0$, $i \in \mathcal{M}$, $t = 0, 1, ..., T-1$, $k = 1, 2, ..., n$. Let $A_t(i) = \mathbb{E}P_t'(\mathbb{E}P_tP_t')^{-1}\mathbb{E}P_t$, $i = 0, ..., T-1$. By the definition of parameters (29)-(30) and simple calculation, we have

$$\begin{aligned}
\mathbb{E}B_t(i) &= s_t(1 - A_t(i)), \text{ for } t = 0, 1, ..., T-1, \\
\mathbb{E}[B_t(i)]^2 &= s_t^2(1 - A_t(i)), \text{ for } t = 0, 1, ..., T-1, \\
\mathbb{E}C_{T-1}(i) &= \mathbb{E}[C_{T-1}(i)]^2 = A_{T-1}(i), \\
\mathbb{E}C_t(i) &= \frac{A_t(i)}{\prod_{t+1}^{T-1} s_k}, \text{ for } t = 0, 1, ..., T-2, \\
\mathbb{E}[C_t(i)]^2 &= \frac{A_t(i)}{\prod_{t+1}^{T-1} s_k^2}, \text{ for } t = 0, 1, ..., T-2.
\end{aligned}$$

Recall that parameters for different time periods under the same fixed market mode are independent, by (31) and (32) we have

$$\begin{aligned}
\mathbb{E}\{\mu(i_0, ..., i_{T-1})\} &= \prod_{t=0}^{T-1} s_t(1 - A_t(i_t)), \\
\mathbb{E}\{\mu^2(i_0, ..., i_{T-1})\} &= \prod_{t=0}^{T-1} s_t^2(1 - A_t(i_t)), \\
\mathbb{E}\{\nu(i_0, ..., i_{T-1})\} &= 1 - \prod_{t=0}^{T-1}(1 - A_t(i_t)), \\
\mathbb{E}\{\nu^2(i_0, ..., i_{T-1})\} &= 1 - \prod_{t=0}^{T-1}(1 - A_t(i_t)) = \mathbb{E}\{\nu(i_0, ..., i_{T-1})\},
\end{aligned}$$

for $i_0, ..., i_{T-1} \in \mathcal{M}$. Plugging the above into (36)-(40) we have

$$\begin{aligned}
\mu^{(1)} &= \prod_{t=0}^{T-1} s_t \sum_{i_1, i_2, \cdots, i_{T-1} \in \mathcal{M}} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{T-2} i_{T-1}} \prod_{t=0}^{T-1} (1 - A_t(i_t)) \\
\mu^{(2)} &= \prod_{t=0}^{T-1} s_t^2 \sum_{i_1, i_2, \cdots, i_{T-1} \in \mathcal{M}} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{T-2} i_{T-1}} \prod_{t=0}^{T-1} (1 - A_t(i_t)) \\
\nu^{(1)} &= \sum_{i_1, i_2, \cdots, i_{T-1} \in \mathcal{M}} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{T-2} i_{T-1}} (1 - \prod_{t=0}^{T-1} (1 - A_t(i_t))) \\
&= 1 - \sum_{i_1, i_2, \cdots, i_{T-1} \in \mathcal{M}} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{T-2} i_{T-1}} \prod_{t=0}^{T-1} (1 - A_t(i_t)) \\
\nu^{(2)} &= \nu^{(1)}.
\end{aligned}$$

Then it is easy to derive

$$\mu^{(2)} - (\mu^{(1)})^2 - \frac{(\mu^{(1)} \nu^{(1)})^2}{\nu^{(2)} - (\nu^{(1)})^2} = 0$$

which indicates a perfect square efficient frontier in (42).

4 Analytical solution to the problem $(P1(\sigma))$, $(P2(\epsilon))$ and $(P3(\omega))$ and their efficient frontiers

In this section, the optimal multi-period portfolio policy for $(P3(\omega))$ will first be derived based on the result in the last section and its relationship with $(P4(\lambda, \omega))$ will be discussed. The analytical solution to problems $(P1(\sigma))$ and $(P2(\epsilon))$ will then be obtained based on their relationship with $(P3(\omega))$.

By the relation (14) between $(P3(\omega))$ and $(P4(\lambda, \omega))$, we plug in (3.20) which is the explicit expression of terminal expectation under the optimal policy, we have

$$\lambda^* = 1 + 2\omega\mu^{(1)}x_0 + \lambda^*\nu^{(1)}. \tag{43}$$

If $\nu^{(1)} \neq 1$, we have

$$\lambda^* = \frac{1 + 2\omega\mu^{(1)}x_0}{1 - \nu^{(1)}}. \tag{44}$$

Plugging (44) into (15), we have the optimal policy for problem $(P3(\omega))$ for $t = 0, 1, ..., T-1$ under the market mode $i \in \mathcal{M} = \{1, 2, \ldots, m\}$, is of the following form,

$$u_t^*(i) = -L_t(i)\Big[M_t(i)x_t - \frac{1 + 2\omega\mu^{(1)}x_0}{2\omega(1 - \nu^{(1)})}N_t(i)\Big], \tag{45}$$

where $L_t(i)$, $M_t(i)$ and $N_t(i)$ are the same as in Theorem 3.1.

If $\nu^{(1)} = 1$, then we only have solution to problem $(P3(\omega))$ for special $\omega = -1/(2\mu^{(1)})$. The optimal policy is still expressed as (45).

Plugging (44) into (34) and (41), we have the terminal expectation and variance under the optimal policy for problem $(P3(\omega))$,

$$\begin{aligned} \mathbb{E}(x_T(\omega)) &= ax_0 + \frac{b}{\omega} && (46) \\ \mathrm{Var}(x_T(\omega)) &= fx_0^2 + \frac{gx_0}{\omega} + \frac{h}{\omega^2} && (47) \end{aligned}$$

where

$$
\begin{aligned}
a &= \frac{\mu^{(1)}}{1-\nu^{(1)}} && (48)\\
b &= \frac{\nu^{(1)}}{2(1-\nu^{(1)})} && (49)\\
f &= \mu^{(2)} + \frac{(\mu^{(1)})^2(\nu^{(2)}-1)}{(1-\nu^{(1)})^2} && (50)\\
g &= \frac{\mu^{(1)}(\nu^{(2)}-\nu^{(1)})}{(1-\nu^{(1)})^2} && (51)\\
h &= \frac{\nu^{(2)}-(\nu^{(1)})^2}{4(1-\nu^{(1)})^2}. && (52)
\end{aligned}
$$

To obtain the solution to problem $(P1(\sigma))$ and $(P2(\epsilon))$, we first calculate the associate ω in terms of σ or ϵ using (47) or (46) and then plug this ω into (45). Specifically, for problem $(P1(\sigma))$, we let the right hand side of (47) $= \sigma$, which derives

$$
\omega(\sigma) = \frac{2h}{\sqrt{(gx_0)^2 + 4h(\sigma - fx_0^2)} - gx_0}. \qquad (53)
$$

Then the analytical solution for $(P1(\sigma))$ is also given by (45) with $\omega = \omega(\sigma)$. Similarly, by letting the right hand of (46) $= \epsilon$, the solution for problem $(P2(\epsilon))$ is given by (45) with

$$
\omega = \omega(\epsilon) = \frac{b}{\epsilon - ax_0}. \qquad (54)
$$

Remark 4.1. To guarantee the positivity of ω, it is interesting to notice that in (53) we should have an attached condition

$$
\sigma > fx_0^2. \qquad (55)
$$

This can be seen as a necessary condition for problem $(P1(\omega))$ to admit a solution. Similar to the Markowitz model, fx_0^2 is the minimum variance. From the practical point of view, we may not have an optimal policy (or the optimal policy is not unique) if the variance σ is too small. That is if the variance is less that the minimum variance, then the solution of the problem becomes trivial, or any portfolio in the efficient frontier is optimal. The same constrain comes from the positivity of $\omega(\epsilon)$ in (54), which also leads to a constrain to the parameter ϵ. Again if ϵ is too small, the optimal strategy is not unique (or we say that there is no optimal strategy).

5 Extension to with uncontrollable liability case

When uncontrollable is considered, let l_t be the liability value of the investor at the beginning of the tth period. We follow the similar setting to Leippold *et al.* (2004) such that

$$l_{t+1} = Q_t(\alpha_t)l_t, \tag{56}$$

where $Q_t(i)$ is the liability return corresponding to $\alpha_t = i \in \mathcal{M}$. Then the information filtration up to time t becomes $\hat{\mathcal{F}}_t := \sigma\{x_s, l_s, \alpha_s | 0 \leq s \leq t\}$. The surplus process is defined as

$$\begin{aligned} S_{t+1} &= x_{t+1} - l_{t+1} \\ &= \sum_{k=1}^{n} e_t^k(\alpha_t)u_t^k + \Big(x_t - \sum_{k=1}^{n} u_t^k\Big)e_t^0(\alpha_t) - Q_t(\alpha_t)l_t \\ &= e_t^0(\alpha_t)x_t + P_t(\alpha_t)'u_t - Q_t(\alpha_t)l_t \end{aligned} \tag{57}$$

Correspondingly, the portfolio selection problem of our investor can be formulated in the following three forms:

$$
\begin{aligned}
(P5(\sigma)): \qquad & \max \mathbb{E}(S_T) \\
& \text{subject to } \mathrm{Var}(S_T) \leq \sigma \text{ and (57)},
\end{aligned}
\tag{58}
$$

where $\sigma > 0$ and

$$
\begin{aligned}
(P6(\epsilon)): \qquad & \min \mathrm{Var}(S_T) \\
& \text{subject to } E(S_T) \geq \epsilon \text{ and (57)},
\end{aligned}
\tag{59}
$$

where $\epsilon \geq 0$ and

$$
\begin{aligned}
(P7(\omega)): \qquad & \max \mathbb{E}(S_T) - \omega \mathrm{Var}(S_T) \\
& \text{subject to (57)},
\end{aligned}
\tag{60}
$$

for some strictly positive risk aversion parameter ω.

A multi-period portfolio policy is an investment sequence,

$$
\begin{aligned}
u &= \{u_0, u_1, ..., u_{T-1}\} \\
&= \left\{ \begin{pmatrix} u_0^1 \\ u_0^2 \\ \vdots \\ u_0^n \end{pmatrix}, \begin{pmatrix} u_1^1 \\ u_1^2 \\ \vdots \\ u_1^n \end{pmatrix}, ..., \begin{pmatrix} u_{T-1}^1 \\ u_{T-1}^2 \\ \vdots \\ u_{T-1}^n. \end{pmatrix} \right\}
\end{aligned}
\tag{61}
$$

where $u_t \in \mathcal{F}_t$ for $t = 0, 1, ..., T-1$.

A multi-period portfolio policy, u^*, is said to be efficient if there exists no other multi-period portfolio policy u that is 'better' than u^*. The 'better' here means $\mathbb{E}(S_T)|_u \geq \mathbb{E}(S_T)|_{u^*}$ and $\mathrm{Var}(S_T)|_u \leq \mathrm{Var}(S_T)|_{u^*}$ with at least one inequality strict. By varying the value of σ, ϵ and ω in $(P5(\sigma))$, $(P6(\epsilon))$ and $(P7(\omega))$ under the efficient portfolio u^* respectively, a set of $(\mathbb{E}(S_T)|_{u^*}, \mathrm{Var}(S_T)|_{u^*})$ can be generated, which is called the efficient frontier according to problems $(P5(\sigma))$, $(P6(\epsilon))$ and $(P7(\omega))$ respectively. The three formulations are still equivalent in the sense of the same efficient frontier for some specific parameters σ, ϵ and ω. Similarly, $(P7(\omega))$ is transformed into the separable auxiliary problem

$$\begin{aligned}(P8(\lambda,\omega)): \qquad & \max \mathbb{E}\{-\omega S_T^2 + \lambda S_T\} \\ & \text{subject to (57).}\end{aligned} \tag{62}$$

Moreover, if u^* solves $(P8(\omega))$, then it solves $(P7(\lambda,\omega))$ for $\lambda = 1 + 2\omega\mathbb{E}(S_T)|_{u^*}$. On the other hand, if u^* solves $(P8(\lambda^*,\omega))$, then a necessary condition for u^* solves $(P7(\omega))$ is

$$\lambda^* = 1 + 2\omega\mathbb{E}(S_T)|_{u^*}. \tag{63}$$

To derive the optimal strategy for Problem $P8(\lambda^*, \omega)$, we find that

$$
\begin{aligned}
& \mathbb{E}\{-\omega S_T^2 + \lambda S_T\} \\
= \;& \mathbb{E}\Big\{ -\omega(x_T - l_T)^2 + \lambda(x_T - l_T)\Big\} \\
= \;& \mathbb{E}\Big\{ -\omega x_T^2 + (\lambda + 2\omega l_T)x_T - \omega l_T^2 - \lambda l_T\Big\} \\
= \;& \mathbb{E}\Big\{ -\omega x_T^2 + (\lambda + 2\omega l_T)x_T + ...\}.
\end{aligned} \tag{64}
$$

Omitting all the terms with no relationship with u and comparing (64) with (26), we can directly write out the optimal investment strategy as follows.

Theorem 5.1. *Given the optimization problem (62), the optimal policy for* $t = 0, 1, ..., T-1$ *under the market mode* $i \in \mathcal{M} = \{1, 2, \ldots, m\}$*, is of the following form,*

$$
\hat{u}_t^*(i) = -\hat{L}_t(i)\Big[\hat{M}_t(i)x_t - \frac{\hat{N}_t(i, \lambda, \omega)}{2\omega}\Big], \tag{65}
$$

where $\hat{L}_t(i)$, $\hat{M}_t(i)$ and $\hat{N}_t(i)$ are derived by the following iteration procedure,

$$\begin{aligned}
\hat{L}_{T-1}(i) &= [\mathbb{E}\{P_{T-1}(i)P_{T-1}(i)'|\mathcal{F}_{T-1}\}]^{-1}, && (66)\\
\hat{M}_{T-1}(i) &= \mathbb{E}\{e^0_{T-1}(i)P_{T-1}(i)|\mathcal{F}_{T-1}\}, && (67)\\
\hat{N}_{T-1}(i,\lambda,\omega) &= \lambda\mathbb{E}\{P_{T-1}(i)|\mathcal{F}_{T-1}\} + 2\omega\mathbb{E}\{P_{T-1}(i)Q_{T-1}(i)|\mathcal{F}_{T-1}\}, && (68)\\
\hat{K}_{T-1}(i) &= \mathrm{E}\{P_{T-1}(i)Q_{T-1}(i)l_{T-1}(i)|\mathcal{F}_{T-1}\} && (69)\\
\hat{A}^1_{T-1}(i,\lambda,\omega) &= \frac{\lambda}{\omega}\big(\mathbb{E}\{e^0_{T-1}(i)|\mathcal{F}_{T-1}\} - \hat{M}_{T-1}(i)'\hat{L}_{T-1}(i)\hat{N}_{T-1}(i)\big) && (70)\\
&\qquad +2(\mathbb{E}\{Q_{T-1}(i)e^0_{T-1}(i)l_{T-1}(i)|\mathcal{F}_{T-1}\} - \hat{M}_{T-1}(i)'\hat{L}_{T-1}(i)\hat{K}_{T-1}(i)) &&\\
\hat{A}^2_{T-1}(i) &= \mathbb{E}\{[e^0_{T-1}(i)]^2|\mathcal{F}_{T-1}\} - \hat{M}_{T-1}(i)'\hat{L}_{T-1}(i)\hat{M}_{T-1}(i) > 0, && (71)\\
\hat{L}_t(i) &= [\mathbb{E}\{P_t(i)P_t(i)'|\mathcal{F}_t\}\sum_{j=1}^m p_{ij}\hat{A}^2_{t+1}(j)]^{-1}, t=0,1,...,T-2, && (72)\\
\hat{M}_t(i) &= \mathbb{E}\{e^0_t(i)P_t(i)|\mathcal{F}_t\}\sum_{j=1}^m p_{ij}\hat{A}^2_{t+1}(j), t=0,1,...,T-2, && (73)\\
\hat{N}_t(i,\lambda,\omega) &= \mathbb{E}\{P_t(i)|\mathcal{F}_t\}\sum_{j=1}^m p_{ij}\hat{A}^1_{t+1}(j,\lambda,\omega), t=0,1,...,T-2, && (74)\\
\hat{A}^1_t(i,\lambda,\omega) &= \mathbb{E}\{e^0_t(i)|\mathcal{F}_t\}\sum_{j=1}^m p_{ij}\hat{A}^1_{t+1}(j,\lambda,\omega) - \hat{M}_t(i)'\hat{L}_t(i)\hat{N}_t(i,\lambda,\omega) &&\\
&= \big\{\mathbb{E}\{e^0_t(i)|\mathcal{F}_t\} - \mathbb{E}\{e^0_t(i)P_t(i)'|\mathcal{F}_t\}[\mathbb{E}\{P_t(i)P_t(i)'|\mathcal{F}_t\}]^{-1}\mathbb{E}\{P_t(i)|\mathcal{F}_t\}\big\} &&\\
&\quad \sum_{j=1}^m p_{ij}\hat{A}^1_{t+1}(j,\lambda,\omega), \qquad t=0,1,...,T-2, && (75)\\
& && (76)
\end{aligned}$$

$$\begin{aligned}
\hat{A}^2_t(i) &= \mathbb{E}\{[e^0_t(i)]^2|\mathcal{F}_t\}\sum_{j=1}^m p_{ij}\hat{A}^2_{t+1}(j) - \hat{M}_t(i)'\hat{L}_t(i)\hat{M}_t(i)\\
&= \big\{\mathbb{E}\{[e^0_t(i)]^2|\mathcal{F}_t\} - \mathbb{E}\{e^0_t(i)P_t(i)'|\mathcal{F}_t\}[\mathbb{E}\{P_t(i)P_t(i)'|\mathcal{F}_t\}]^{-1}\mathbb{E}\{e^0_t(i)P_t(i)|\mathcal{F}_t\}\big\},\\
&\quad \sum_{j=1}^m p_{ij}\hat{A}^2_{t+1}(j) > 0, \qquad t=0,1,...,T-2. \qquad (77)
\end{aligned}$$

The proof is similar to Theorem 3.1, we do not go to detail here. With similar calculation to no-liability case, we can derive the efficient frontier of Problem $(P8(\omega))$ as well as the analytical solution to the $(P5(\omega))$, $(P6(\omega))$ and $(P7(\omega))$following procedure. Since the calculation procedure has already been given in Section 3 and 4, we do not go to detail here.

6 Numerical Examples

Many papers on discrete-time mean-variance portfolio selection problems have provided numerical results to demonstrate the optimal strategy as well as the corresponding efficient frontier. In no regime-switching case, Li and Ng (2000) gave detailed calculation in three examples to derive the explicit expression of optimal strategy and efficient frontiers. Leippold *et al.*(2004) studied the subperiod selection to make the discrete-time model more close to the continuous-time model. They also study the impact of the investment horizon on the optimal investment strategy as well as the determination of an optimal initial funding ratio for ALM portfolios.

In this section, we mainly focus on the impacts of regime-switching on the efficient frontier. By using two examples, the impact of different starting market modes i_0 and different levels of regime-switching are studied. For simplicity, we only consider a special case that the level of liability is zero, which nests the problem of Li and Ng (2000).

In our examples, the market modes are roughly divided into two regimes, $i = 1$ is bearish, and $i = 2$ is bullish. The investor with an initial asset value $x_0 = 1$ choose to invest on $n = 3$ risky assets in the stock market within $T = 3$ time periods. Parameters for different market regimes are as follows

$$\mathbb{E}(e_t(1)) = \begin{pmatrix} 1.03 \\ 1.14 \\ 1.19 \end{pmatrix}, \quad \text{cov}(e_t(1)) = \begin{pmatrix} 0.0459 & 0.0211 & 0.0112 \\ 0.0211 & 0.0312 & 0.0215 \\ 0.0112 & 0.0215 & 0.0179 \end{pmatrix} \text{ for } t = 0, 1, 2;$$

$$\mathbb{E}(e_t(2)) = \begin{pmatrix} 1.162 \\ 1.246 \\ 1.228 \end{pmatrix}, \quad \text{cov}(e_t(2)) = \begin{pmatrix} 0.0146 & 0.0107 & 0.0105 \\ 0.0107 & 0.0154 & 0.0104 \\ 0.0105 & 0.0104 & 0.0089 \end{pmatrix} \text{ for } t = 0, 1, 2.$$

It's not difficult to verify that $\mathbb{E}\{e_t(i)e_t(i)'\}$ for $i = 1, 2$ and $t = 0, 1, 2$ are positive definite.

Example 6.1. This example mainly focuses on the impact of different starting market modes. It reasonable to start our investment when the market is bullish. In other words, we will expect higher expected rate of return with lower risk when entering the market at $i_0 = 2$. Let $p_{11} = p_{22} = 0.5$, efficient frontiers can be calculated according to (42). Figure 1 below clearly shows that if we expect a higher return with some risk level, we would better enter the market at a bullish time. This observation is consistent with the continuous-time model in Chen and Yang (2008).

Example 6.2. In this example, we are interested in the impact of regime-switching phenomenon on the corresponding efficient frontiers. Assuming that we always enter the market at bearish time, that is, $i_0 = 1$. We list different transition probabilities in Table 1 to indicate that the market becomes 'more bearish' when p_{11} increases from 0 to 1. We see from Figure 2 that the corresponding efficient frontier moves to the right during this shift. This indicates an increasing investment risk for the same expected rate of return,

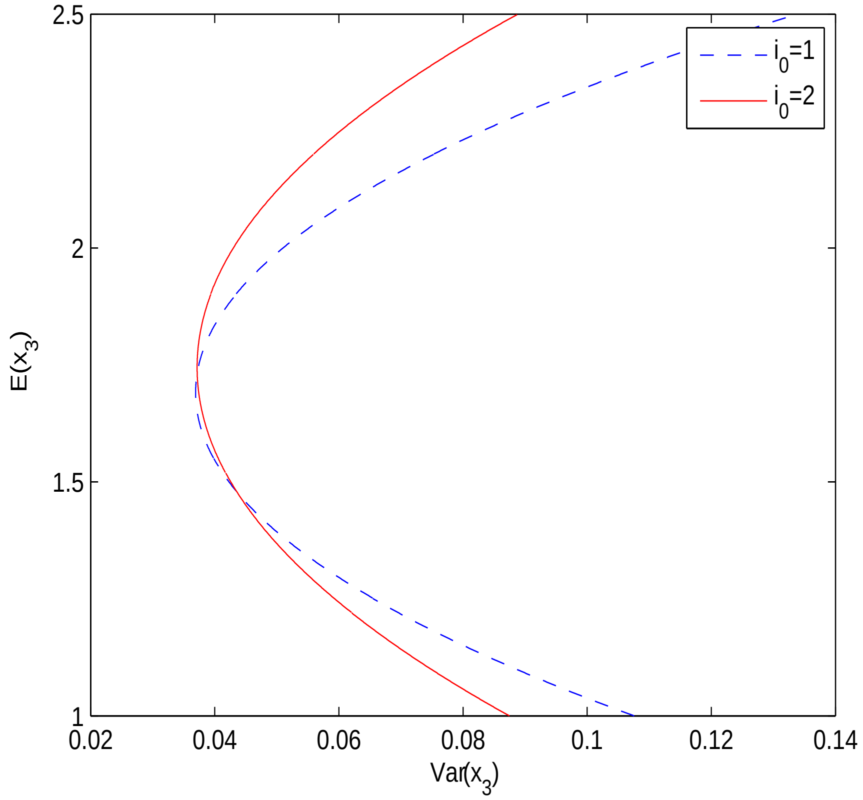

Figure 1: Mean-Variance efficient frontiers for Example 6.1 when $i_0 = 1, 2$.

which is reasonable.

Table 1: Different transition probabilities.

p_{11}	0	0.2	0.4	0.6	0.8	1
p_{22}	1	0.8	0.6	0.4	0.2	0

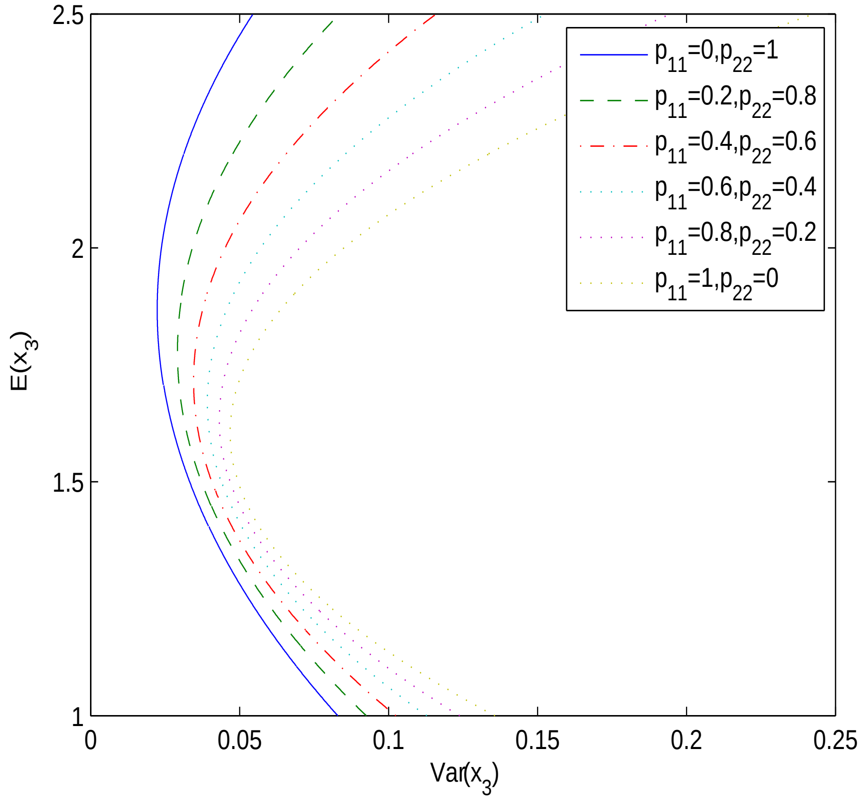

Figure 2: Mean-Variance efficient frontiers for Example 6.2 under different transition rates

7 Conclusion

In this paper, we study the analytical solutions of the optimal investment strategy under mean-variance criteria in a discrete model. We show that the solution for the ALM problem can be extended directly from the solution for the no-liability case. There are still a lot of interesting questions along this line. Such as the effects of regime-switching to the solution and the convergence property of discrete case to the continuous case.

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