Doubly periodic patterns of modulated hydrodynamic waves: Exact solutions of the Davey – Stewartson system

J. H. Li[(1)], S. Y. Lou[(2)], K. W. Chow[(1)]*

(1) = Department of Mechanical Engineering, University of Hong Kong,

Pokfulam Road, Hong Kong

(2) = Department of Physics, Shanghai Jiao Tong University,

Shanghai 200030, People's Republic of China

* = Corresponding author

Phone: (852) 2859 2641

Fax: (852) 2858 5415

Email: kwchow@hku.hk

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Abstract

Exact, doubly periodic standing wave patterns of the Davey – Stewartson equations (DS) are derived in terms of rational expressions of elliptic functions. In fluid mechanics, DS govern the evolution of weakly nonlinear, free surface wave packets when long wavelength modulations in two mutually perpendicular, horizontal directions are incorporated. Elliptic functions with two different moduli (periods) are necessary in the two directions. The relation between the moduli and the wave numbers constitutes the 'dispersion relation' of such waves. In the long wave limit, localized pulses are recovered.

1 Introduction

The properties of hydrodynamic wave packets under the competing influences of dispersion, long wavelength modulations and weak nonlinearity have been studied intensively. The relevant fluid physics is the evolution of the wave envelope in water of finite depth. The governing model describing the propagation of wave envelope to leading order is the nonlinear Schrödinger equation [1, 2]. Extensions to higher spatial dimensions, where modulations in two mutually perpendicular, horizontal directions are permitted, have also been conducted by many investigators. The generalized evolution equations have commonly been termed the 'Davey – Stewartson' equations (DS) [3] (although preliminary forms of such equations actually appeared slightly earlier [1]):

$$i\frac{\partial A}{\partial t}-\frac{\partial^2 A}{\partial x^2}-\sigma^2\frac{\partial^2 A}{\partial y^2}+\lambda A^2 A^* = 2QA\,, \tag{1}$$

$$\frac{\partial^2 Q}{\partial x^2}-\sigma^2\frac{\partial^2 Q}{\partial y^2}=\lambda\frac{\partial^2 (AA^*)}{\partial x^2}\,. \tag{2}$$

Here A is the (complex valued) envelope of the wave packet associated with the fast oscillations, and Q is the induced mean flow, x and y are the slow, horizontal scales parallel and perpendicular to the fast oscillations respectively, while t is the slow time in the group velocity frame. As usual, irrotational flow of an inviscid fluid is studied, and A is connected with the velocity potential. The coefficients in system (1, 2) have been normalized, and the original values will depend on the physical properties of the fluid configuration [1 – 3]. The constant λ measures the cubic nonlinearity. Of particular significance is the parameter σ, and $\sigma^2 = +1$ ($\sigma = 1$), and $\sigma^2 = -1$ ($\sigma = i$) are termed Davey – Stewartson I equations (DSI) and DSII respectively. DSI and DSII correspond to the situations where the governing equation of the mean flow is hyperbolic / elliptic respectively.

The main objective of the present work is to deduce further exact solutions of (1, 2), and thus remarks on the background and development of DS are in order:

(A) *Significance in fluid dynamics*: Since the derivation of DS in the period of 1969 – 1974, additional fluid physics and effects have been incorporated. Surface tension can be taken into account [4], and perspectives in terms of wave collapse have been considered [5]. This coupling of the envelope with the mean flow has also been studied for internal waves [6]. For an infinite or semi–infinite fluid, the induced mean flow becomes a delicate issue, and the evolution equations for interfacial waves in the presence of uniform shear currents (i.e. the Kelvin Helmholtz instability) have been investigated [7].

Modern applications of DS are often closely related to oceanic waves. One example is the formation of freak waves [8], or large surface displacements from an otherwise tranquil background. Here modulation instabilities, or Benjamin –

Feir instabilities, will lead to periodic focusing of waves, which is associated with the existence of ‘breathers’, or oscillating modes. The long term evolution is often dictated by the Fermi – Pasta – Ulam recurrence. The presence of bottom friction might lead to DS system with energy gain and loss [9].

In a two–layer fluid, DS system is a good starting point for considering amplification of nonlinear surface waves by wind [10]. Quite remarkably, DS system is also relevant in sound propagating in bubbly fluids [11].

(B) *Exact solutions of DS from the viewpoint of nonlinear dynamics*: Given the importance of DS, it is not surprising that exact solutions have been documented over the years. In fact DS constitute one of the simplest (2+1) (2 spatial and 1 temporal) dimensional extensions of the well known nonlinear Schrödinger equation (NLS), just like the role the Kadomtsev – Petviashvili equation serves for the classical Korteweg – de Vries equation. ‘Solitons’, localized pulses of permanent form, have been obtained by standard techniques of the modern theory of nonlinear waves, e.g. scattering formulation [12] and the Hirota bilinear method [13].

Singly or doubly periodic solutions of DS can be obtained from soliton solutions by employing a pair of complex conjugate wave numbers [14].

The main objective here is to utilize theta and elliptic functions to deduce further exact solutions of DS. Elliptic functions have been employed frequently in solid mechanics [15] and the present work would be an illustrative example in fluid mechanics. Although elliptic functions have been applied earlier to a large variety of envelope equations in many works [16], mainly one dimensional modes are found, whereas surface wave profiles with independent variations in both x and y are presented here.

The motivation comes from the discovery of a class of 'doubly periodic' solutions of the nonlinear Schrödinger equation in the early 1990s [17]. Although these solutions come from a special 'first order relation' between the real and imaginary parts of NLS, these doubly periodic waves are readily re–derived by the Hirota bilinear method. Since most integrable equations have known bilinear forms, the corresponding formulation for extracting doubly periodic patterns can then be extended in a straightforward manner [18, 19].

The structure of the paper can now be explained. The basic ingredients, the bilinear transformation and simple representative doubly periodic solutions are first elucidated (Section 2). Two new, special classes of solutions of DS are presented (Sections 3, 4). The solitary wave, or long wave, limits are discussed (Section 5) and conclusions follow (Section 6).

2 Background

The nonlinear Schrödinger equation

$$iA_t + A_{xx} - 2\,A^2 A^* = 0$$

possesses a doubly periodic (periodic in both x and t) solution

$$A = \frac{rk_1}{\sqrt{1+k_1}}\left[\frac{\mathrm{cn}(st,k_1)+i\sqrt{1+k_1}\,\mathrm{sn}(st,k_1)\mathrm{dn}(rx,k)}{\sqrt{1+k_1}\,\mathrm{dn}(rx,k)+\mathrm{dn}(st,k_1)}\right]\exp\left(-\frac{2ir^2}{1+k_1}t\right),$$

where the wave number in the t direction, 's', and the modulus of the elliptic function there, k_1, are related to their counterparts in the x component, r and k respectively, as

$$s = \frac{2r^2}{1+k_1}, \qquad k^2 = \frac{2k_1}{1+k_1}.$$

Although the solution is expressed in a more compact form here using Jacobi elliptic functions, the intermediate calculations are actually performed using theta functions [20]. The Hirota bilinear transform and Hirota derivatives of theta functions have been discussed in the Ref. [21] and [19] respectively. The bilinear form of DS is (D_x, D_t, D_x^2 and D_y^2 are the Hirota operators)

$$A = G/f,\ \ Q = 2\,(\log f)_{xx},\ \ f \text{ real},\ \sigma^2 = \pm 1,\ \lambda,\ N_0 = \text{constant},$$

$$D_x^m D_t^n g\cdot f = \left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n g(x,t)f(x',t')\Big|_{x=x',t=t'},$$

$$(iD_t - D_x^2 - \sigma^2 D_y^2 - N_0)G\cdot f = 0, \tag{3}$$

$$(D_x^2 - \sigma^2 D_y^2 - N_0)f\cdot f = \lambda GG^*. \tag{4}$$

Several families of such solutions are given in earlier works [18, 19], and here two more families of solutions are presented. As the methods of calculations are now known in the literature, only the final form of the two new solutions will be given in the main text. Nevertheless, for completeness, a brief outline of the calculations is given in the Appendix.

3 The first new exact solution

The first new exact solution for DS has envelope A given by,

$$A = A_0 \left[\frac{S + iDC_1}{1 + CC_1} \right] \exp(ipx - i\Omega t), \tag{5}$$

and the mean flow Q is

$$Q = 2r^2 \left(1 - \frac{E}{K} \right) - 2r^2 k^2 + 2r^2 \left[\frac{k\sqrt{1-k^2}(C^2 - C_1^2) + (2k^2 - 1)C_1 C}{(1 + C_1 C)^2} \right], \tag{6}$$

where parameters S, C, D and C_1 are defined in terms of the three Jacobi elliptic functions, sn, cn, dn:

$$S = \sqrt{k}\,\mathrm{sn}(rx,k), \quad C = \frac{\sqrt{k}\,\mathrm{cn}(rx,k)}{\left(1-k^2\right)^{1/4}}, \quad D = \frac{\mathrm{dn}(rx,k)}{\left(1-k^2\right)^{1/4}}, \quad C_1 = \frac{\sqrt{k_1}\,\mathrm{cn}(sy,k_1)}{\left(1-k_1^2\right)^{1/4}} \tag{7}$$

with the corresponding wave numbers r, s and elliptic moduli k, k_1 in the x, y directions respectively. The wave number (p) and frequency (Ω) in the phase of A are related to r, s, k and k_1 by

$$p = \frac{\sigma^2 s^2}{r} \frac{k_1\sqrt{1-k_1^2}}{\sqrt{1-k^2}}, \tag{8}$$

$$\Omega = -p^2 + r^2 \left(2 - k^2 - \frac{4E}{K} \right) + \sigma^2 s^2 (1 - 2k_1^2), \tag{9}$$

where K and E are the complete elliptic integrals of the first and second kind respectively:

$$K = \int_0^{\pi/2} \frac{d\xi}{\sqrt{1-k^2 \sin^2 \xi}}, \quad E = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \xi}\, d\xi \ . \tag{10}$$

The amplitude A_0 is

$$\lambda A_0^2 = \frac{1}{k}[r^2(1-2k^2) - \sigma^2 s^2 (1-2k_1^2)]. \tag{11}$$

The most important expression is the connection among the wave numbers r, s, and moduli of the elliptic functions, k, k_1 (closely connected to the periods of the Jacobi elliptic functions), roughly a 'dispersion relation' in the present context:

$$r^2 = \sigma^2 s^2 \left[1 - 2k_1^2 - \frac{2kk_1\sqrt{1-k_1^2}}{\sqrt{1-k^2}} \right]. \tag{12}$$

The intermediate calculations are outlined in the Appendix.

The remarkable difference from the previous families of solutions is that the necessary sign of σ^2, i.e. $\sigma = 1$ or $\sigma = i$, is *not* determined yet from (12). Indeed, depending on the choice of k and k_1, σ can be ± 1 or $\pm i$, and thus the expressions (5 – 12) can be a valid solution for both DSI and DSII. From the structural form of (5), this solution for the envelope A represents *standing waves*. The doubly periodic nature of the intensity $|A|^2$ versus x and y is clearly illustrated in Figure 1. The validity of the new solution is also verified independently by direct differentiation of (5 – 12) with the computer algebra software MATHEMATICA and substitution into the original system (1, 2).

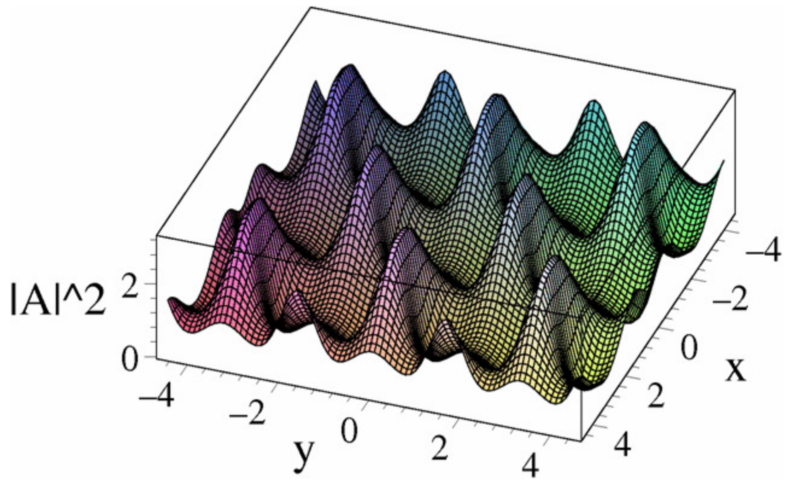

Fig. 1 Intensity $|A|^2$ of the complex envelope for the first exact solution, Equation 5, $r = 1.2$, $s = 2$, $k = 0.4$, $k_1 = 0.4$, $\lambda = -2$, $\sigma = 1$ (DSI).

4 The second new exact solution

Along the same line of reasoning, another extensive search leads to the second new exact solution, with symbols bearing similar interpretations as the previous section (Jacobi functions again given by (7)):

$$A = A_0\left[\frac{SD_1 + iC}{D + D_1}\right]\exp(ipx - i\Omega t)\,, \tag{13}$$

$$Q = -\frac{2r^2E}{K} + 2r^2\left[\frac{\sqrt{1-k^2}(1 + D^2D_1^2) + (2-k^2)DD_1}{(D + D_1)^2}\right], \tag{14}$$

where D_1 is defined by

$$D_1 = \frac{\mathrm{dn}(sy, k_1)}{(1-k_1^2)^{1/4}} \ . \tag{15}$$

The other parameters p, Ω and A_0 are given by

$$p = -\frac{\sigma^2 s^2}{r}\frac{\sqrt{1-k_1^2}}{\sqrt{1-k^2}}\,, \tag{16}$$

$$\Omega = -p^2 + r^2\left(3 - 2k^2 - \frac{4E}{K}\right) - \sigma^2 s^2 (2 - k_1^2)\,, \tag{17}$$

$$\lambda A_0^2 = k\left[\frac{2\sigma^2 s^2 \sqrt{1-k_1^2}}{\sqrt{1-k^2}} - 2r^2\right], \tag{18}$$

and the 'dispersion relation' is

$$r^2 k^2 = \sigma^2 s^2\left[\frac{2\sqrt{1-k_1^2}}{\sqrt{1-k^2}} + k_1^2 - 2\right]. \tag{19}$$

The doubly periodic nature of this standing wave pattern is again clearly depicted in a graph of intensity $|A|^2$ versus the horizontal coordinates (Figure 2). This set of solutions, (13 – 19), may apply to either the DSI ($\sigma = 1$) or the DSII ($\sigma = i$) regime, depending on the values of k and k_1. The validity of (13 – 19) is also verified by direct differentiation with a computer algebra software and direct substitution into the system (1, 2).

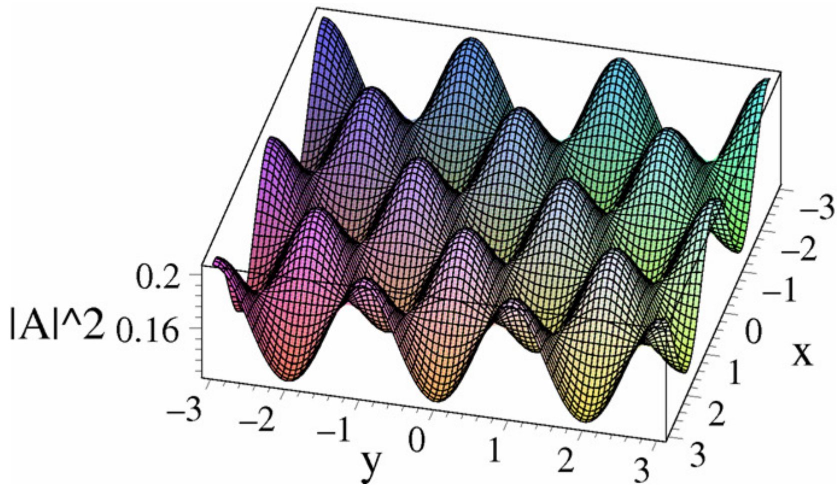

Fig. 2 Intensity $|A|^2$ of the complex envelope for the second exact solution, Equation 13, $r = 1.126$, $s = 2$, $k = 0.4$, $k_1 = 0.8$, $\lambda = -2$, $\sigma = i$ (DSII).

5 Long wave limits

In the long wavelength regime, these complicated expressions of elliptic functions reduce to the elementary functions under the scheme, (sn x, cn x, dn x) $\rightarrow$ (tanh x, sech x, sech x) as $k \rightarrow 1$. Physically, solitary or localized pulses are expected to be recovered.

(A) *Long wave limit of the first solution*

The only nontrivial balance is the combination of $k \rightarrow 1$, $k_1 \rightarrow 0$, and an order one parameter m such that

$$k_1 = m^2\sqrt{1-k^2}\ ,$$

and this results in the long wave limit of

$$A = A_0\left[\frac{\tanh(rx) + im\ \mathrm{sech}(rx)\cos(sy)}{1 + m\ \mathrm{sech}(rx)\cos(sy)}\right]\exp(ipx - i\Omega t)\,, \qquad (20)$$

$$Q = \frac{2r^2\ \mathrm{sech}(rx)[\mathrm{sech}(rx) + m\ \cos(sy)]}{[1 + m\ \mathrm{sech}(rx)\cos(sy)]^2}\,, \qquad (21)$$

$$r^2 = \sigma^2 s^2 (1 - 2m^2)\,, \qquad (22)$$

$$\lambda A_0^2 = 2\sigma^2 s^2 (m^2 - 1)\,, \qquad (23)$$

$$pr = m^2 \sigma^2 s^2\,, \qquad (24)$$

$$\Omega = -p^2 + r^2 + \sigma^2 s^2\,. \qquad (25)$$

Hence, with proper selections of the values of m and λ, this solution may apply to either DSI or DSII. Figure 3 shows a very peculiar behavior for some typical values of the parameters. The intensity of the envelope, $|A|^2$, displays the features of a 'dark soliton' [22, 23] in the x direction, i.e. approaching a constant in the far field ($|x|\to\infty$), but possessing local minima at certain fixed locations. In the y direction, however, the cosine function causes periodic modulation and generates many turning points within the 'valley' or 'trough' of the 'dark soliton'.

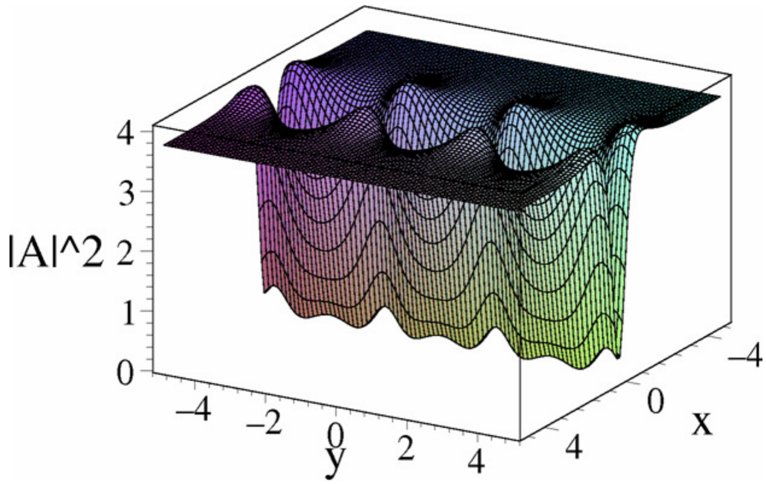

Fig. 3 Intensity $|A|^2$ of the complex envelope for the first long wave limit, Equation 20, $r = 1.871$, $s = 2$, $m = 0.25$, $\lambda = -2$, $\sigma = 1$ (DSI).

(B) *Long wave limit of the second solution*

Similarly the nontrivial limit here is the combination of k, $k_1 \to 1$, with a parameter m such that

$$(1-k_1^2)^{1/4} = m(1-k^2)^{1/4}, \tag{26}$$

and the corresponding long wave limit is

$$A = A_0\left[\frac{\tanh(rx)\,\mathrm{sech}(sy) + im\;\mathrm{sech}(rx)}{m\,\mathrm{sech}(rx) + \mathrm{sech}(sy)}\right]\exp(ipx - i\Omega t)\,, \tag{27}$$

$$Q = \frac{2r^2\;\mathrm{sech}(rx)\,\mathrm{sech}(sy)[\mathrm{sech}(rx)\,\mathrm{sech}(sy) + im]}{[m\;\mathrm{sech}(rx) + \mathrm{sech}(sy)]^2}\,, \tag{28}$$

$$r^2 = \sigma^2 s^2 (2m^2 - 1)\,, \tag{29}$$

$$\lambda A_0^2 = 2\sigma^2 s^2 (1 - m^2)\,, \tag{30}$$

$$pr = -m^2\sigma^2 s^2, \tag{31}$$

$$\Omega = -p^2 + r^2 - \sigma^2 s^2. \tag{32}$$

The symmetry among the parameters displayed in the sets (22 – 25) and (29 – 32) is striking. Figure 4 shows that the expressions (26 – 32) just constitute an ordinary 'dark' 2–soliton solution (collision of two dark solitons). These calculations confirm that the two families of doubly periodic solutions (Sections 3, 4) are indeed different, as they have different long wave limits.

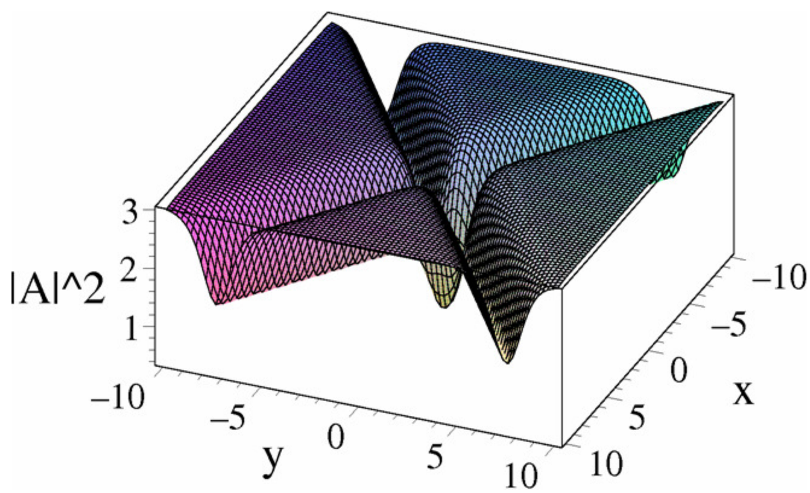

Fig. 4 Intensity $|A|^2$ of the complex envelope for the second long wave limit, Equation 27, r = 1.414, s = 2, m = 0.25, λ = –2, σ = i (DSII).

6 Conclusions

Solitons, i.e. permanent, nonlinear localized modes of dispersive systems, have received tremendous attention both in optics [22, 23] and hydrodynamics

[24 – 26]. Solitary waves for generalized nonlinear long wave (Whitham – Broer – Kaup) models [25] and two–layer fluids [26] have been studied intensively. Here we focus on the finite depth fluid configuration, and allow for modulations in two mutually perpendicular, horizontal directions, i.e. the Davey – Stewartson equations (DS). Quite remarkably, DS can also arise in optics as well [27]. DS possess a variety of exact solutions, e.g. solitons [13], exponentially localized (dromions) [28] and periodic [18, 19] solutions, and we focus on the periodic ones in this paper.

Exact, doubly periodic standing wave patterns are obtained in terms of rational functions of elliptic functions. Moduli of the elliptic functions employed in the two horizontal directions are different. The relation among the moduli and the wave numbers constitutes a 'dispersion relation', as it measures the constraints on the spatial periods. Two families of exact solutions are derived, and they can be applied to both the DSI (hyperbolic governing equation for the induced mean flow) and DSII (elliptic governing equation for the induced mean flow) regimes.

The long wave limit will yield 'dark' localized modes (employing terminology borrowed from optics), which approach nonzero asymptotic states in the far field. The dark localized mode for the first family of exact solutions is especially novel, as periodic maxima and minima occur along the 'trough' or ‘valley’ of the wave profiles. The long wave limit for the second family of exact solutions just yields the collision of two (ordinary) dark solitons.

There are many possible further directions for fruitful research. In the Korteweg – de Vries / Boussinesq regimes, viscosity and bottom friction can be very important [29]. The same idea can be studied here. Mathematically, further searches might yield still further solutions. Given the tremendous scope and range of applications of DS (Section 1), these periodic and localized modes will prove to be beneficial and instructive in modeling and understanding oceanic phenomenon.

Appendix: Derivation for the first solution (Equations (5 – 12))

The classical Jacobi elliptic functions can be expressed as ratios of the theta functions, which are Fourier expansions with exponentially decaying coefficients. Theta functions have a huge variety of identities, and one such class of identities will be used to express the Hirota derivatives of theta functions in terms of theta functions themselves. As an illustrative example, by differentiating

$$\theta_3(x+y)\theta_3(x-y)\theta_2^2(0) = \theta_4^2(x)\theta_1^2(y) + \theta_3^2(x)\theta_2^2(y) \ ,$$

with respect to y twice and setting $y = 0$, one obtains

$$D_x^2\theta_3(x)\cdot\theta_3(x) = \frac{2\theta_2''(0)\theta_3^2(x)}{\theta_2(0)} + 2\theta_3^2(0)\theta_4^2(0)\theta_4^2(x) \ ,$$

where the auxiliary result $\theta_1'(0) = \theta_2(0)\theta_3(0)\theta_4(0)$ has been used. Many such identities have been derived in our earlier works [18, 19].

We now search for solutions of the bilinear DS, i.e. Eqs. (3) and (4) by taking G and f respectively as

$$G = A_0\left[\theta_1(\alpha x,\tau)\theta_4(\beta y,\tau_1) + i\theta_3(\alpha x,\tau)\theta_2(\beta y,\tau_1)\right]\exp(ipx - i\Omega t)\,, \quad \text{(A1)}$$

$$f = \theta_4(\alpha x,\tau)\theta_4(\beta y,\tau_1) + \theta_2(\alpha x,\tau)\theta_2(\beta y,\tau_1)\,, \quad \text{(A2)}$$

where parameters α, β, p, Ω and A_0 will be determined below.

Substituting (A1), (A2) into (3), and employing identities for Hirota derivatives of theta functions, one now proceeds to a two-step decomposition in x and y. On collecting terms of $\theta_2^2(\beta y,\tau_1)$, $\theta_4^2(\beta y,\tau_1)$ and $\theta_2(\beta y,\tau_1)\theta_4(\beta y,\tau_1)$, the coefficients must each vanish separately as these three theta functions are linearly independent, and thus one obtains

$$2A_0C_1\theta_1(\alpha x,\tau)\theta_4(\alpha x,\tau) - iA_0C_2\theta_2(\alpha x,\tau)\theta_3(\alpha x,\tau) = 0\,, \quad \text{(A3)}$$

$$-2iA_0C_1\theta_2(\alpha x,\tau)\theta_3(\alpha x,\tau) - A_0C_2\theta_1(\alpha x,\tau)\theta_4(\alpha x,\tau) = 0\,, \quad \text{(A4)}$$

$$-A_0C_3\left[\theta_1(\alpha x,\tau)\theta_2(\alpha x,\tau) + i\theta_3(\alpha x,\tau)\theta_4(\alpha x,\tau)\right] = 0\,, \quad \text{(A5)}$$

where

$$C_1 = p\alpha\theta_4^2(0,\tau) - \sigma^2\beta^2\theta_2^2(0,\tau_1)\theta_4^2(0,\tau_1)\,, \quad \text{(A6)}$$

$$C_2 = \left(\frac{\theta_2''(0,\tau)}{\theta_2(0,\tau)} + \frac{\theta_3''(0,\tau)}{\theta_3(0,\tau)}\right)\alpha^2 + 2\left(\frac{\theta_2''(0,\tau_1)}{\theta_2(0,\tau_1)} + \theta_4^4(0,\tau_1)\right)\sigma^2\beta^2 + N_0 - \Omega - p^2\,, \quad \text{(A7)}$$

$$C_3 = \left(\frac{\theta_3''(0,\tau)}{\theta_3(0,\tau)} + \frac{\theta_4''(0,\tau)}{\theta_4(0,\tau)} \right)\alpha^2 + \left(\frac{\theta_2''(0,\tau_1)}{\theta_2(0,\tau_1)} + \frac{\theta_4''(0,\tau_1)}{\theta_4(0,\tau_1)} \right)\sigma^2\beta^2 + 2p\alpha\theta_2^2(0,\tau) + N_0 - \Omega - p^2 \quad .$$

(A8)

In arriving at Eqs. (A6 – A8), the identities for theta functions of

$\theta_3^2(0)\theta_1^2(x) = \theta_2^2(0)\theta_4^2(x) - \theta_4^2(0)\theta_2^2(x)$, $\theta_3^2(0)\theta_3^2(x) = \theta_2^2(0)\theta_2^2(x) + \theta_4^2(0)\theta_4^2(x)$,

have been used to eliminate $\theta_1^2(\beta y,\tau_1)$ and $\theta_3^2(\beta y,\tau_1)$ in terms of $\theta_2^2(\beta y,\tau_1)$ and $\theta_4^2(\beta y,\tau_1)$. On continuing to use the linear independence of theta functions and setting the coefficients of $\theta_1(\alpha x,\tau)\theta_4(\alpha x,\tau)$, $\theta_2(\alpha x,\tau)\theta_3(\alpha x,\tau)$, $\theta_1(\alpha x,\tau)\theta_2(\alpha x,\tau)$ and $\theta_3(\alpha x, \tau)\theta_4(\alpha x, \tau)$ in Eqs.(A3 – A5) to be zero respectively, one obtains

$$C_1 = 0, \quad C_2 = 0, \quad C_3 = 0,$$

i.e.

$$p\alpha\theta_4^2(0,\tau) - \sigma^2\beta^2\theta_2^2(0,\tau_1)\theta_4^2(0,\tau_1) = 0,$$

$$\left(\frac{\theta_2''(0,\tau)}{\theta_2(0,\tau)} + \frac{\theta_3''(0,\tau)}{\theta_3(0,\tau)} \right)\alpha^2 + 2\left(\frac{\theta_2''(0,\tau_1)}{\theta_2(0,\tau_1)} + \theta_4^4(0,\tau_1) \right)\sigma^2\beta^2 + N_0 - \Omega - p^2 = 0,$$

$$\left(\frac{\theta_3''(0,\tau)}{\theta_3(0,\tau)} + \frac{\theta_4''(0,\tau)}{\theta_4(0,\tau)} \right)\alpha^2 + \left(\frac{\theta_2''(0,\tau_1)}{\theta_2(0,\tau_1)} + \frac{\theta_4''(0,\tau_1)}{\theta_4(0,\tau_1)} \right)\sigma^2\beta^2 + 2p\alpha\theta_2^2(0,\tau) + N_0 - \Omega - p^2 = 0$$

(A9)

and the remaining job is to convert these theta formulas to the notations of the classical Jacobi functions. This can be accomplished via standard formulas from tables and thus details will not be pursued here. This completes the treatment of Eq. (3).

Along exactly the same line of reasoning, one first substitutes (A1, A2) into the bilinear equation (4). On using formulas for the Hirota derivatives of theta functions, one now performs a two-step decomposition. The coefficients of $\theta_2^2(\beta y, \tau_1)$, $\theta_4^2(\beta y, \tau_1)$ and $\theta_2(\beta y, \tau_1)\theta_4(\beta y, \tau_1)$, are collected and set to be zero. This procedure results in three equations involving theta functions in *x*. One now insists that the coefficients of

$$\theta_2^2(\alpha x, \tau),\ \theta_4^2(\alpha x, \tau) \text{ and } \theta_2(\alpha x, \tau)\theta_4(\alpha x, \tau)$$

must vanish too, as these theta functions are linearly independent of each other. Finally, one generates the governing system for the wave numbers and angular frequencies:

$$2\left(\frac{\theta_2''(0,\tau)}{\theta_2(0,\tau)} + \theta_4^4(0,\tau)\right)\alpha^2 - 2\left(\frac{\theta_2''(0,\tau_1)}{\theta_2(0,\tau_1)} + \theta_4^4(0,\tau_1)\right)\sigma^2\beta^2 - N_0 - \lambda A_0^2\frac{\theta_2^2(0,\tau)}{\theta_3^2(0,\tau)} = 0,$$

$$2\alpha^2\theta_2^2(0,\tau)\theta_4^2(0,\tau) + 2\sigma^2\beta^2\theta_2^2(0,\tau_1)\theta_4^2(0,\tau_1) + \lambda A_0^2\frac{\theta_4^2(0,\tau)}{\theta_3^2(0,\tau)} = 0,$$

$$\left(\frac{\theta_2''(0,\tau)}{\theta_2(0,\tau)} + \frac{\theta_4''(0,\tau)}{\theta_4(0,\tau)}\right)\alpha^2 - \left(\frac{\theta_2''(0,\tau_1)}{\theta_2(0,\tau_1)} + \frac{\theta_4''(0,\tau_1)}{\theta_4(0,\tau_1)}\right)\sigma^2\beta^2 - N_0 = 0.$$

The first new solution given in Section 3 is obtained by solving these algebraic equations, (A9, A10), and converting them back into notations of the Jacobi elliptic functions. The solution described in Section 4 can be obtained along the same line of reasoning.

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