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H_∞ Model Reduction for Positive Systems

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***Abstract*— This paper is concerned with the model reduction of positive systems. For a given stable positive system, our attention is focused on the construction of a reduced-order model in such a way that the positivity of the original system is preserved and the error system is stable with a prescribed H_∞ performance. Based upon a system augmentation approach, a novel characterization on the stability with H_∞ performance of the error system is first obtained in terms of linear matrix inequality (LMI). Then, a necessary and sufficient condition for the existence of a desired reduced-order model is derived accordingly. A significance of the proposed approach is that the reduced-order system matrices can be parametrized by a positive definite matrix with flexible structure, which is fully independent of the Lyapunov matrix; thus, the positivity constraint on the reduced-order system can be readily embedded in the model reduction problem. Finally, a numerical example is provided to show the effectiveness of the proposed techniques.**

I. Introduction

In many practical systems, there is such a kind of systems whose state variables are confined to be positive. Such systems are frequently encountered in various fields, for instance, biomedicine, pharmacokinetics, chemical reactions, industrial engineering, social science and economics. These systems belong to the class of positive systems, whose state variable and output are always positive (at least nonnegative) whenever the initial state and the input are positive [1] [2]. *Positivity* of the system state for all times will bring about many new issues, which cannot be solved in general by using well-established methods for general linear systems, mainly due to the fact that positive systems are defined on cones rather than linear spaces. Therefore, the study on this kind of systems has drawn the attention of many researchers in recent years [3] [4] [5] [6].

Mathematical modeling of positive systems, such as molecular dynamics, industrial wastewater treatment, and chemical reactors, often results in complex high-order models, which will bring serious difficulties to analysis and synthesis of positive systems, irrespective of the computational resources available [7]. Therefore, in practical applications, it is necessary to replace high-order models by reduced ones with respect to some given criterion. In fact, such a topic is actually a model reduction problem in control area, and has received considerable attention in the past decades [8] [9] [10] [11] [12] [13] [14]. Amongst the many optimality criteria for approximation, one is based on the H_∞ norm of the associated error system. The characterization of H_∞ model reduction solution was first proposed in [15], and many important results have been reported for various kinds of systems, such as stochastic systems [16] and switched systems [17] [18]. Very recently, based on the methods of balanced truncation and matrices inequalities, the model reduction problem for positive systems has been investigated in [19] and [20], respectively. It should be pointed out that traditional approaches developed for general linear systems, including the widely adopted projection approach and similarity transformation [16] [21], are no longer applicable for positive systems in general, since they cannot guarantee the positivity of the reduced-order system. This indicates that conventional approaches, if used to construct a reduced-order system, may generate a meaningless approximation for the actual system whose state is always positive all the time. Indeed, the introduction of positivity of the reduced-order system will lead to new difficulties, which cannot be easily dealt with by existing approaches. Therefore, it is necessary to develop new approaches to the H_∞ model reduction problem for positive systems with positivity preserved. However, such a problem has not been well studied in the literature, and still remains as a challenging open issue.

In the present work, we are concerned with the H_∞ model reduction problem for positive systems. More specifically, for a given positive linear continuous-time system, the aim is to construct a *positive* lower-order system such that the H_∞ norm of the difference between the original system and the desired lower-order one satisfies a prescribed H_∞ norm bound constraint. Based on a system augmentation approach, the associated error system is first represented as a singular system form, and a novel characterization on the stability of the error system under the H_∞ performance is derived in the form of LMI. Then, a necessary and sufficient condition for the existence of a desired reduced-order system is proposed, and an iterative LMI approach is developed to compute the reduced-order system matrices. It is well worth pointing out that the approach developed in this paper has the advantage that the reduced-order system matrices can be parametrized by a positive definite matrix with flexible structure, which is fully independent of the Lyapunov matrix. Such a characterization will greatly facilitate the parametrization with positivity constraints.

The rest of this paper is organized as follows. Section II gives some notations and preliminaries. In Section III, a novel characterization on the stability and the H_∞ analysis of the error system is developed, and an iterative LMI algorithm is formulated to construct a reduced-order system.

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A numerical example is given in Section IV to show the applicability of the results obtained. Finally, we summarize our results in Section V.

II. System Description and Preliminaries

Notation: Let $\mathbb{R}$ be the set of real numbers; $\mathbb{R}^n$ denotes the n-column real vectors; $\mathbb{R}^{n\times m}$ is the set of all real matrices of dimension $n\times m$. $\mathbb{R}_+^{n\times m}$ represents the $n\times m$ dimensional matrices with nonnegative components and $\mathbb{R}_+^n \triangleq \mathbb{R}_+^{n\times 1}$. For a matrix $A\in\mathbb{R}^{m\times n}$, a_{ij} denotes the element located at the ith row and the jth column. Matrix A is said to be nonnegative if $\forall(i,j)\ a_{ij}\geqslant 0$; it is said to be positive, if $\forall(i,j)\ a_{ij}\geqslant 0$, $\exists(i,j)\ a_{ij}>0$. In view of the fact that the definitions of nonnegative and positive matrices are equivalent, except when a nonnegative matrix is identically zero which is the degenerate case and is of no interest, we do not distinguish these two throughout this paper, that is, we consider that these two conditions are equivalent in general cases. A matrix $A\in\mathbb{R}^{n\times n}$ is called Metzler, if all its off-diagonal elements are positive, that is, $\forall\,(i,j)$, $i\neq j$, $a_{ij}\geqslant 0$. I represents the identity matrix with appropriate dimension; For any real symmetric matrices P, Q, the notation $P\geq Q$ (respectively, $P>Q$) means that the matrix $P-Q$ is positive semi-definite (respectively, positive definite).

The notation $L_2[0,\infty)$ represents the space of square Lebesgue integrable functions over $[0,\infty)$ with the usual norm $\|\cdot\|_2$. For a transfer function matrix $\mathcal{G}(s)$, $\|\mathcal{G}\|_\infty$ represents the H_∞ norm of $\mathcal{G}(s)$. In addition, $\mathrm{Her}(M)\triangleq M^T+M$ is defined for any matrix $M\in\mathbb{R}^{n\times n}$; associated with a set of matrices $A_i\in\mathbb{R}^{n\times n}$, $i=1,2,\ldots,N$, $\mathrm{diag}\begin{pmatrix}A_1 & A_2 & \ldots & A_N\end{pmatrix}$ is defined as

$$\mathrm{diag}\begin{pmatrix}A_1 & A_2 & \ldots & A_N\end{pmatrix}\triangleq\begin{bmatrix}A_1 & \ldots & 0\\ \vdots & \ddots & \vdots\\ 0 & \ldots & A_N\end{bmatrix}.$$

The superscript "T" denotes matrix transpose and the symbol # is used to represent a matrix which can be inferred by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

Consider the following linear asymptotically stable system:

$$\left\{\begin{array}{rcl}\dot{x}(t) &=& Ax(t)+Bu(t),\\ y(t) &=& Cx(t)+Du(t),\\ x(0) &=& x_0,\end{array}\right. \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $u(t)\in\mathbb{R}^m$ is the input vector which belongs to $L_2[0,\infty)$, $y(t)\in\mathbb{R}^q$ is the output or measurement vector. Furthermore, A, B, C and D are real constant matrices with appropriate dimensions. System (1) is said to be a positive linear system if for all $x_0\in\mathbb{R}_+^n$ and all input $u(t)\in\mathbb{R}_+^m$, we have $x(t)\in\mathbb{R}_+^n$ and $y(t)\in\mathbb{R}_+^q$ for $t>0$.

The following lemma provides a well-known characterization of positive linear systems.

Lemma 1 ([1]): The system in (1) is positive if and only if A is Metzler, B, C and D are positive.

In this paper, we aim at approximating system (1) by a reduced-order stable system described by

$$\left\{\begin{array}{rcl}\dot{x}_r(t) &=& A_rx_r(t)+B_ru(t),\\ y_r(t) &=& C_rx_r(t)+D_ru(t),\\ x_r(0) &=& x_{r0},\end{array}\right. \tag{2}$$

where $x_r(t)\in\mathbb{R}^{n_r}$ is the state vector of the reduced-order system (2) with $0<n_r<n$, and $y_r(t)\in\mathbb{R}^q$. A_r, B_r, C_r and D_r are matrices to be determined later.

For the stable system in (1), the transfer function from input $u(t)$ to output $y(t)$ is given by

$$\mathcal{G}_{uy}(s)=C\,(sI-A)^{-1}B+D. \tag{3}$$

Traditionally, the H_∞ model reduction problem was formulated by finding a reduced-order system (2), such that

$$\|\mathcal{G}_{uy}-\mathcal{G}_{uy_r}\|_\infty<\gamma, \tag{4}$$

where

$$\mathcal{G}_{uy_r}(s)=C_r\,(sI-A_r)^{-1}B_r+D_r \tag{5}$$

is the transfer function of system (2) from $u(t)$ to $y_r(t)$, and $\gamma>0$ is a prescribed scalar.

However, such a specification is not sufficient for positive systems, since as an approximation of system (1), it is naturally desirable that system (2) should also be positive, like system (1) itself. That is, in addition to the H_∞ performance in (4), the positivity should also be preserved when considering the model reduction problem for the positive system in (1). To ensure the positivity of system (2), if follows from Lemma 1 that A_r should be Metzler, B_r, C_r and D_r should be positive.

For convenience, denote set $\mathbb{S}\triangleq\{(A_r,B_r,C_r,D_r)$: A_r is Metzler, B_r, C_r and D_r are positive$\}$.

Let $\hat{x}(t)=\left[x^T(t),x_r^T(t)\right]^T$ and $e(t)=y(t)-y_r(t)$. Then, from (1) and (2), we obtain the associated error system as

$$\left\{\begin{array}{rcl}\dot{\hat{x}}(t) &=& \hat{A}\hat{x}(t)+\hat{B}u(t),\\ e(t) &=& \hat{C}\hat{x}(t)+\hat{D}u(t),\end{array}\right. \tag{6}$$

where

$$\begin{array}{rcl}\hat{A} &=& \begin{bmatrix}A & 0\\ 0 & A_r\end{bmatrix},\ \hat{B}=\begin{bmatrix}B\\ B_r\end{bmatrix},\\ \hat{C} &=& \begin{bmatrix}C & -C_r\end{bmatrix},\ \hat{D}=D-D_r.\end{array}$$

Obviously, condition in (4) is equivalent to

$$\|\mathcal{G}_{ue}(s)\|_\infty<\gamma, \tag{7}$$

where

$$\mathcal{G}_{ue}(s)=\hat{C}\left(sI-\hat{A}\right)^{-1}\hat{B}+\hat{D} \tag{8}$$

is the transfer function from $u(t)$ to $e(t)$. In addition, the stability of system (1) and (2) is naturally equivalent to that of system (6).

Based on the above discussion, the problem of positivity-preserving H_∞ model reduction for positive systems in (1) to be addressed in this paper is formulated as follows.

Problem PP-H_∞-MR (Positivity-Preserving H_∞ Model Reduction): Given a disturbance attenuation level $\gamma > 0$, construct a system (2) such that the following two requirements are fulfilled simultaneously.

(1) $(A_r, B_r, C_r, D_r) \in \mathbb{S}$.

(2) The error system in (6) is asymptotically stable and satisfies the H_∞ performance $\|\mathcal{G}_{ue}\|_\infty < \gamma$.

The following result gives a fundamental characterization on the stability of (6) with H_∞ performance, which will be used later.

Lemma 2 ([21]): The error system in (6) is asymptotically stable and satisfies $\|\mathcal{G}_{ue}\|_\infty < \gamma$, if and only if there exists a matrix $\hat{P} > 0$, such that

$$\begin{bmatrix} \mathrm{Her}\left(\hat{A}^T\hat{P}\right) & \hat{P}\hat{B} & \hat{C}^T \\ \# & -\gamma I & \hat{D}^T \\ \# & \# & -\gamma I \end{bmatrix} < 0, \tag{9}$$

where $\hat{P}$ is usually referred to as the Lyapunov matrix.

III. Main Result

In this section, we aim to construct a positive lower-order system such that the H_∞ norm of the difference between the original positive system and the desired lower-order one satisfies a prescribed H_∞ norm bound constraint. To achieve this, we first present a novel characterization on the stability and the H_∞ performance of (6) by means of a system augmentation. Then, a necessary and sufficient condition for the existence of a desired reduced-order system is proposed, and an iterative LMI approach is developed to compute the reduced-order system matrices.

A. Novel Stability and H_∞ Performance Characterization

In this subsection, we first represent system (6) by means of a system augmentation approach, which will facilitate the parametrization on the positivity constraint. Then, a novel characterization on the stability and the H_∞ performance of (6) is developed in terms of linear matrix inequality, which will play a key role for the computation of the reduced-order system matrices.

Define

$$G_r = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix},$$

which collects the representation for the system matrices in (2) into one matrix. We further make the following definitions:

$$\begin{aligned} \bar{A} &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \ \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \ \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \ \bar{D} = D, \\ \bar{F} &= \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}, \ \bar{M} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \\ \bar{N} &= \begin{bmatrix} 0 \\ I \end{bmatrix}, \ \bar{H} = \begin{bmatrix} 0 & -I \end{bmatrix}, \end{aligned}$$

which are entirely in terms of the state space matrices for system (1), then we have

$$\begin{aligned} \hat{A} &= \bar{A} + \bar{F}G_r\bar{M}, \ \hat{B} = \bar{B} + \bar{F}G_r\bar{N}, \\ \hat{C} &= \bar{C} + \bar{H}G_r\bar{M}, \ \hat{D} = \bar{D} + \bar{H}G_r\bar{N}. \end{aligned}$$

Although the system matrices in (2) are encapsulated into G_r, one can see that it is still embedded with two other matrices. In addition, when applying Lemma 2, we have that G_r is still coupled with the Lyapunov matrix $\hat{P}$, which makes them hard to solve. More significantly, such a problem will become more difficult and arduous, in particular when additional constraints on G_r are taken into account.

To overcome these difficulties, we introduce an auxiliary variable $\hat{\vartheta}(t) = G_r\bar{M}\hat{x}(t) + G_r\bar{N}u(t)$ as a state component, and choose $\boldsymbol{x}(t) = \begin{bmatrix} \hat{x}^T(t) & \hat{\vartheta}^T(t) \end{bmatrix}^T$ as a new state variable. Then the error system in (6) can be equivalently described by the following descriptor system:

$$\left\{ \begin{aligned} \boldsymbol{E}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t), \\ e(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}u(t), \end{aligned} \right. \tag{10}$$

where

$$\begin{aligned} \boldsymbol{E} &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{A} = \begin{bmatrix} \bar{A} & \bar{F} \\ G_r\bar{M} & -I \end{bmatrix}, \\ \boldsymbol{B} &= \begin{bmatrix} \bar{B} \\ G_r\bar{N} \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} \bar{C} & \bar{H} \end{bmatrix}, \boldsymbol{D} = \bar{D}. \end{aligned}$$

Remark 1: A major obstacle for the construction of the reduced-order system in (2) is that it should be positive, which results in the additional constraints on the system matrices A_r, B_r, C_r, and D_r. Focusing on this, one can see that the advantage of the above manipulations lies in the following aspects. First, these system matrices are assembled to a single matrix G_r, which will be convenient for the synthesis consideration. Second, by means of system augmentation approach in (10), G_r is successfully extracted from the middle of two matrices, and can be further parametrized by a free positive definite matrix, which will be shown later. Such an approach will introduce the flexibility to the construction of G_r, in particular when G_r has some certain constraints.

Theorem 1: Given the system matrices A_r, B_r, C_r and D_r. Then the following statements are equivalent:

(i) The error system in (6) is asymptotically stable, and satisfies $\|\mathcal{G}_{ue}\|_\infty < \gamma$.

(ii) There exist matrices $\hat{P} > 0$, $X > 0$ such that

$$\Xi \triangleq \begin{bmatrix} \mathrm{Her}\left(\boldsymbol{A}^T\boldsymbol{P}\right) & \boldsymbol{P}^T(\boldsymbol{I}+\boldsymbol{J})\boldsymbol{B} & \boldsymbol{C}^T \\ \# & \digamma - \gamma I & \boldsymbol{D}^T \\ \# & \# & -\gamma I \end{bmatrix} < 0, \tag{11}$$

where

$$\digamma = -\boldsymbol{B}^T\boldsymbol{J}^T\left(\boldsymbol{P} + \boldsymbol{P}^T\right)\boldsymbol{J}\boldsymbol{B},$$

with

$$\boldsymbol{P} = \begin{bmatrix} \hat{P} & 0 \\ -\frac{1}{2}XG_r\bar{M} & \frac{1}{2}X \end{bmatrix},$$

$$\boldsymbol{I} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \ \boldsymbol{J} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

Proof: *(ii)⇒(i)*. Suppose there exist matrices $\hat{P} > 0$, $X > 0$ such that (11) holds. Define a nonsingular matrix

$$T \triangleq \begin{bmatrix} I & 0 & 0 & 0 \\ G_r\bar{M} & G_r\bar{N} & 0 & I \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}.$$

Pre- and post-multiplying (11) by T^T and T, respectively, we have

$$\bar{\Xi} \triangleq T^T \Xi T = \begin{bmatrix} \text{Her}\left(\hat{A}^T\hat{P}\right) & \hat{P}\hat{B} & \hat{C}^T & \hat{P}\bar{F} \\ \# & -\gamma I & \hat{D}^T & 0 \\ \# & \# & -\gamma I & \bar{H} \\ \# & \# & \# & -X \end{bmatrix} < 0. \tag{12}$$

Based on Lemma 2, the third leading principal submatrix of $\bar{\Xi}$ indicates that the error system in (6) is asymptotically stable, and satisfies $\|\mathcal{G}_{ue}\|_\infty < \gamma$, which completes this part of the proof.

(i)⇒(ii). If the error system in (6) is asymptotically stable, and satisfies $\|\mathcal{G}_{ue}\|_\infty < \gamma$, then it follows from Lemma 2 that there exists a matrix $\hat{P} > 0$, such that

$$\Theta = \begin{bmatrix} \text{Her}\left(\hat{A}^T\hat{P}\right) & \hat{P}\hat{B} & \hat{C}^T \\ \# & -\gamma I & \hat{D}^T \\ \# & \# & -\gamma I \end{bmatrix} < 0.$$

Then, for any matrix $S > 0$, there exists a sufficiently large scalar $\alpha > 0$ such that

$$-\alpha S - \begin{bmatrix} \hat{P}\bar{F} \\ 0 \\ \bar{H} \end{bmatrix}^T \Theta^{-1} \begin{bmatrix} \hat{P}\bar{F} \\ 0 \\ \bar{H} \end{bmatrix} < 0. \tag{13}$$

By choosing $X = \alpha S$ and applying Schur complement equivalence to (13), we have

$$\Xi = T^{-T}\bar{\Xi}T^{-1} < 0,$$

which completes the whole proof. □

Remark 2: Although the conditions in (9) and (11) are equivalent, it should be pointed out that the LMI formulation in (11) has some advantages over the one in (9). First, with the LMI characterization in (11), the reduced-order system matrices, or G_r equivalently, are not coupled with the Lyapunov matrix $\hat{P}$ any more, but can be parametrized by a positive definite matrix X, which is fully independent of $\hat{P}$. Second, it follows from (13) that, if the error system in (6) is asymptotically stable and satisfies $\|\mathcal{G}_{ue}\|_\infty < \gamma$, the existence of X will be naturally guaranteed. Finally, one can see that the structure of X is rather flexible. To be specific, from the proof of ((ii)⇒(i)), we have that X takes the form $X = \alpha S$, where S can be *any positive definite matrix* with α being sufficiently large. The freedom on the structure of X will greatly facilitate the synthesis considered in this paper when additional constraints on G_r are imposed, which will be shown subsequently.

B. Synthesis of Positive Reduced-Order System

This subsection is devoted to the synthesis of the reduced-order system in (2). Based on the analysis in Subsection III-A, a necessary and sufficient condition for the existence of a solution to *Problem PP-H_∞-MR* is obtained. Then, an iterative LMI approach is developed to compute the reduced-order system matrices accordingly.

Theorem 2: Problem PP-H_∞-MR is solvable, if and only if there exists a matrix $\hat{P} > 0$, a diagonal matrix $X > 0$, matrices U, V, L_1, L_2, L_3 and L_4 such that

$$L = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} \in \mathbb{S}, \tag{14}$$

$$\Xi(U,V) \triangleq \begin{bmatrix} \Xi_{11} & \hat{P}\bar{F} + \bar{M}^T L^T & \Xi_{13} & \bar{C}^T \\ \# & -X & L\bar{N} & \bar{H}^T \\ \# & \# & \Xi_{33} & \bar{D}^T \\ \# & \# & \# & -\gamma I \end{bmatrix} < 0, \tag{15}$$

where

$$\begin{aligned} \Xi_{11} &= \text{Her}\left(\bar{A}^T\hat{P}\right) - \text{Her}\left(U^T L\bar{M}\right) + U^T X U, \\ \Xi_{13} &= \hat{P}\bar{B} - \bar{M}^T L^T V - U^T L\bar{N} + U^T X V, \\ \Xi_{33} &= -\text{Her}\left(V^T L\bar{N}\right) + V^T X V - \gamma I. \end{aligned}$$

In this case, the system matrices of (2) can be given as

$$G_r = X^{-1}L. \tag{16}$$

Proof: By expanding (11), we have

$$\begin{bmatrix} \text{Her}\left(\bar{A}^T\hat{P}\right) - \bar{M}^T G_r^T X G_r \bar{M} & \hat{P}\bar{F} + \bar{M}^T G_r^T X & \hat{P}\bar{B} - \bar{M}^T G_r^T X G_r \bar{N} & \bar{C}^T \\ \# & -X & X G_r \bar{N} & \bar{H}^T \\ \# & \# & -\bar{N}^T G_r^T X G_r \bar{N} - \gamma I & \bar{D}^T \\ \# & \# & \# & -\gamma I \end{bmatrix} < 0. \tag{17}$$

Sufficiency: It follows from (14) and $X > 0$ diagonal, we have that A_r Metzler, B_r, C_r and D_r positive. From (16), we have $L = XG_r$. Substituting this into (15), and observing that, for any U and V,

$$\begin{aligned} & -\Phi^T G_r^T X G_r \Phi \\ \leq & -\Phi^T G_r^T X G_r \Phi + (\Psi - G_r\Phi)^T X (\Psi - G_r\Phi) \\ = & -\text{Her}\left(\Psi^T X G_r \Phi\right) + \Psi^T X \Psi, \end{aligned}$$

where

$$\Phi = \begin{bmatrix} \bar{M} & 0 & \bar{N} \end{bmatrix}, \; \Psi = \begin{bmatrix} U & 0 & V \end{bmatrix}, \tag{18}$$

we obtain that (17) holds, which further indicates that (11) holds. According to Theorem 1, this completes the sufficiency proof.

Necessity: If *Problem PP-H_∞-MR* is solvable, then for the given error system in (6), it follows from Theorem 1 that there exists a matrix $\hat{P} > 0$, and a diagonal matrix $X > 0$

such that (11) holds, or equivalently, (17) holds. Then, by choosing $U = G_r \bar{M}$ and $V = G_r \bar{N}$, we have that

$$-\Phi^T G_r^T X G_r \Phi = -\mathrm{Her}\left(\Psi^T X G_r \Phi\right) + \Psi^T X \Psi,$$

where Φ and Ψ are defined in (18). Substituting this into (17), and letting $L = X G_r$, one has that (15) holds. This completes the whole proof. □

Remark 3: From the proof in Theorem 2, one can see that the construction matrix G_r is not coupled with $\hat{P}$, but can be parametrized by X, which makes the construction specification for $G_r \in \mathbb{S}$ possible. More specifically, due to the fact that the structure of X is rather flexible, we can designate X to be a positive diagonal matrix. As a matter of fact, X can be chosen as a positive diagonal matrix, or even a positive scalar matrix, whereas no conservatism will be introduced consequently.

Let us explain the conditions in Theorem 2 from a computational perspective. Obviously, the condition in (14) can be viewed as a set of LMIs, which can be readily verified by standard software. Now, we turn to inequality (15), which is generally not a linear matrix inequality with respect to the parameters $\hat{P}$, X, U, V and L. However, it can be easily observed that if U and V are held fixed, then it becomes an LMI problem with respect to the other remaining parameters. Note that the LMI problem is convex and can be efficiently solved if a feasible solution exists [22]. This leaves a natural problem about how to choose U and V properly. Define a scalar α satisfying

$$\Xi(U, V) < \alpha \Pi, \tag{19}$$

where

$$\Pi = \mathrm{diag}\begin{pmatrix} I & 0 & I & 0 \end{pmatrix} \tag{20}$$

and $\Xi(U, V)$ is defined in (15). Inspired by [23], it follows from the proof of Theorem 2 that α will achieve its minimum when $U = X^{-1} L \bar{M}$ and $V = X^{-1} L \bar{N}$, which leads to an iterative approach to solve inequality (15).

Now, we are in a position to develop the following iterative LMI algorithm:

Algorithm 1 (ILMI Approach):

1) **START**: Set $j = 1$. For a given H_∞ performance level γ, compute the initial matrices U_1 and V_1 such that the following auxiliary system,

$$\left\{ \begin{array}{rcl} \dot{\bar{x}}(t) & = & \bar{A}\bar{x}(t) + \bar{F}\bar{\vartheta}(t) + \bar{B}u(t), \\ e(t) & = & \bar{C}\bar{x}(t) + \bar{H}\bar{\vartheta}(t) + \bar{D}u(t), \end{array} \right. \tag{21}$$

with $\bar{\vartheta}(t) = U_1 \bar{x}(t) + V_1 u(t)$ is asymptotically stable and the transfer function $\mathcal{T}_{ue}(s)$ from $u(t)$ to $e(t)$ satisfies $\|\mathcal{T}_{ue}\|_\infty < \gamma$.

2) For fixed U_j and V_j, solve the following convex optimization problem for the parameters in $\Omega \triangleq \left\{\hat{P} > 0,\ X > 0 \text{ is diagonal},\ L_1,\ L_2,\ L_3 \text{ and } L_4\right\}$:

$$\alpha_j^* := \min_{\Omega} \alpha_j \quad \text{s.t.} \left\{ \begin{array}{l} L = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} \in \mathbb{S} \\ \Xi(U_j, V_j) < \alpha_j \Pi \end{array} \right. .$$

Denote the corresponding value of X and L as X_j and L_j, respectively.

3) If $\alpha_j^* \le 0$, then a desired parametric matrix G_r is obtained as $G_r := X_j^{-1} L_j$. **STOP**. If not, then go to next step.

4) If $\left|\left(\alpha_j^* - \alpha_{j-1}^*\right)/\alpha_j^*\right| < \delta_1$, where δ_1 is a prescribed tolerance, then go to next step. If not, update U_{j+1} and V_{j+1} as

$$U_{j+1} := X_j^{-1} L_j \bar{M},\ V_{j+1} := X_j^{-1} L_j \bar{N}.$$

Set $j := j + 1$, then go to Step 2.

5) A solution to *Problem PP-H_∞-MR* may not exist. **STOP**.

We give some remarks on *Algorithm 1* before ending this section.

Remark 4: The problem in Step 1 is convex, which can be regarded as a state-feedback H_∞ control problem. Furthermore, if there are no matrices U_1 and V_1 such that system (21) is stable and satisfies $\|\mathcal{T}_{ue}\|_\infty < \gamma$, then we can conclude immediately that there does not exist a solution to *Problem PP-H_∞-MR*. In addition, it follows from Lemma 2 that finding U_1 and V_1 is equivalent to finding $\bar{Q} > 0$, W_1 and V_1 such that

$$\begin{bmatrix} \mathrm{Her}\left(\bar{A}\bar{Q} + \bar{F}W_1\right) & \bar{B} + \bar{F}V_1 & \bar{Q}\bar{C}^T + W_1^T \bar{H}^T \\ \# & -\gamma I & \bar{D}^T + V_1^T \bar{H}^T \\ \# & \# & -\gamma I \end{bmatrix} < 0 \tag{22}$$

holds, then U_1 can be obtained as $U_1 = W_1 \bar{Q}^{-1}$, and V_1 can be given directly from (22).

Remark 5: It can be easily seen that α_j^* is monotonically decreasing with respect to j, that is, $\alpha_{j+1}^* \le \alpha_j^*$. If α_j^* does not converge to a positive scalar, then α_j^* will eventually be negative after running *Algorithm 1* with sufficient iterations, which corresponds to the stopping criterion in Step 3, and further indicates that there exists a feasible solution to *Problem PP-H_∞-MR*. Thus, the nonconvergent case is trivial, and we only need to consider the convergent situation.

IV. Illustrative Example

In this section, we present an illustrative example to demonstrate the applicability of the proposed results.

Consider a positive system in (1) with parameters as follows:

$$A = \begin{bmatrix} -2.0 & 0.8 & 1.5 \\ 0.6 & -1.6 & 0 \\ 0.4 & 0 & -1.5 \end{bmatrix},\ B = \begin{bmatrix} 0.4 \\ 0 \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},\ D = 0.5.$$

It can be easily verified that this positive system is asymptotically stable, and we assume that the H_∞ performance level is prescribed as $\gamma = 0.155$. The aim of this example is to construct a positive first-order system in the form of (2) to approximate the original system.

By implementing *Algorithm 1* via Yalmip [24], an initial value of U_1 and V_1 in Step 1 can be obtained from (22) as

$$U_1 = \begin{bmatrix} 0 & 0 & 0 & -0.5 \\ 1.0 & 0 & 0 & 0 \end{bmatrix},\ V_1 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}.$$

Subsequently, it can be found that the conditions in Theorem 2 are feasible with the following solution:

$$X = 10^3 \times \begin{bmatrix} 0.7357 & 0 \\ 0 & 2.2793 \end{bmatrix},$$

$$L = 10^3 \times \begin{bmatrix} -0.6934 & 0.0002 \\ 0.4167 & 1.4903 \end{bmatrix}.$$

Then, according to (16), a desired positive first-order model in (2) can be readily obtained with the system matrices given as

$$\left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right] = \left[\begin{array}{c|c} -0.9425 & 0.0003 \\ \hline 0.1828 & 0.6538 \end{array}\right],$$

that is,

$$\begin{cases} \dot{x}_r(t) &= -0.9425x_r(t) + 0.0003u(t), \\ y_r(t) &= 0.1828x_r(t) + 0.6538u(t). \end{cases}$$

It can be easily verified that the H_∞ performance of the associated error system is 0.1538, which is less than the prescribed H_∞ norm bound $\gamma = 0.155$. This is also demonstrated in Figure 1, which gives the singular value plot of the associated error system.

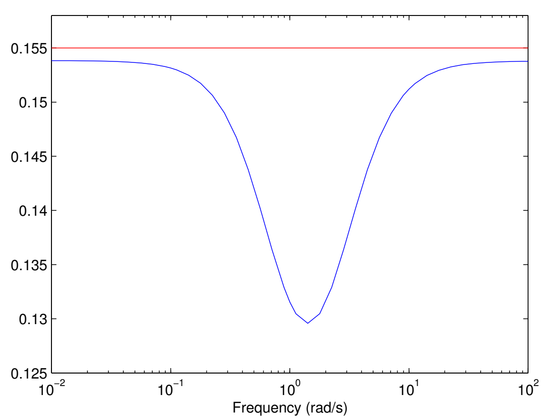

Fig. 1: Singular value plot of associated error system.

V. Conclusion

In this paper, we have presented a model reduction approach that preserves positivity and stability with H_∞ performance of positive systems. In particular, we have proposed a novel characterization on the stability and H_∞ performance of the associated error system by means of a system augmentation method, which ensures the separation of the reduced-order system matrices to be constructed from the Lyapunov matrix. Based on this new characterization, a necessary and sufficient condition for the existence of a desired reduced-order system has been established in terms of matrix equalities, and an iterative LMI approach has been developed to solve the condition. Finally, the effectiveness of the proposed method has been illustrated by a numerical example. The approach adopted in this paper can be applied to tackle problems involving some constraints on elements of the required system matrices, such as positivity and boundedness.

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