THE PRODUCT OF A NONSYMMETRIC JACK POLYNOMIAL WITH A LINEAR FUNCTION

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(Communicated by John R. Stembridge)

ABSTRACT. In this paper a decomposition in terms of the nonsymmetric Jack polynomials is given for the product of any nonsymmetric Jack polynomial $E_\eta(z)$ with z_i. This decomposition generalises a recurrence formula satisfied by single variable orthogonal polynomials on the unit circle. The decomposition also allows the evaluation of the generalised binomial coefficients $\binom{\eta}{\nu}$ associated with the nonsymmetric Jack polynomials for $|\eta| = |\nu| + 1$.

1. INTRODUCTION

Let $\kappa := (\kappa_1, \ldots, \kappa_n)$ be a partition. The modulus of κ is defined by $|\kappa| := \sum_{i=1}^n \kappa_i$. The symmetric Jack polynomial $P_\kappa := P_\kappa(z; \alpha)$ is a function of n variables $z = (z_1, \ldots, z_n)$ and has coefficients in the field $\mathbb{Q}(\alpha)$ of rational functions of the indeterminant α. It can be defined as the unique symmetric eigenfunction of the differential operator

$$(1.1) \qquad D_2(\alpha) := \sum_{j=1}^n z_j^2 \frac{\partial^2}{\partial z_j^2} + \frac{2}{\alpha} \sum_{\substack{1 \le j,k \le n \\ j \ne k}} \frac{z_j^2}{z_j - z_k} \frac{\partial}{\partial z_j}$$

that is of the form

$$(1.2) \qquad P_\kappa(z; \alpha) = m_\kappa(z) + \sum_{\lambda < \kappa} u_{\kappa\lambda} m_\lambda(z).$$

In (1.2), $m_\kappa(z)$ is the monomial symmetric function in the variables $z_1, \ldots, z_n$. The ordering $<$ is the dominance ordering on partitions having the same modulus. It is defined by $\lambda < \kappa$ iff $\lambda \ne \kappa$ and $\sum_{i=1}^p (\kappa_i - \lambda_i) \ge 0$ for all $1 \le p \le n$.

The symmetric Jack polynomials are a class of multi-variable orthogonal polynomials on the unit circle. They possess a number of important properties including analogues of properties held by classical single variable orthogonal polynomials. In particular, they satisfy the Pieri formula (1.3) [13].
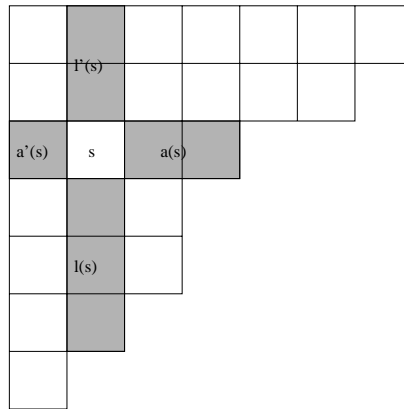
FIGURE 1. The Young diagram of $(7,6,4,3,3,2,1)$

Presenting the Pieri formula requires some notation. The Young diagram of κ is the set $\mathrm{diag}(\kappa) := \{(i,j) | 1 \le j \le \kappa_i, 1 \le i \le n\}$ which is drawn with i increasing from top to bottom and j from left to right. Each element of $\mathrm{diag}(\kappa)$ is called a node of κ. If $\mathrm{diag}(\kappa) \subset \mathrm{diag}(\lambda)$, then the skew diagram λ/κ is the set $\mathrm{diag}(\lambda)/\mathrm{diag}(\kappa)$. The skew diagram λ/κ is said to be a vertical r-strip if it consists of r nodes, all of which are in distinct rows. The Pieri formula is

$$(1.3) \qquad e_r(z)P_\kappa(z;\alpha) = \sum_{\lambda:\lambda/\kappa \text{ is a vertical } r\text{-strip}} \psi_{\lambda/\kappa}(\alpha)P_\lambda(z;\alpha),$$

where

$$(1.4) \qquad e_r(z) = \sum_{1 \le i_1 < \cdots < i_r \le n} z_{i_1} \cdots z_{i_r}$$

is the rth elementary symmetric function. For each node $s = (i,j)$ of κ, the number of points to the right, left, below and above s are its arm length $a_\kappa(s) := \kappa_i - j$, arm colength $a'_\kappa(s) := a'_\kappa(j) := j - 1$, leg length $l_\kappa(s) := \#\{k | k > i, j \le \kappa_k\}$, and leg colength $l'_\kappa(s) := l'_\kappa(i) := i - 1$. Let $\chi(\lambda/\kappa)$ denote the set of all nodes $(i,j) \in \mathrm{diag}(\lambda)$ such that $\kappa_i = \lambda_i$ and $l_\kappa(1,j) < l_\lambda(1,j)$. Then

$$(1.5) \qquad \psi_{\lambda/\kappa}(\alpha) := \prod_{s \in \chi(\lambda/\kappa)} \frac{(\alpha a_\lambda(s) + l_\lambda(s) + 1)(\alpha(a_\kappa(s) + 1) + l_\kappa(s))}{(\alpha a_\kappa(s) + l_\kappa(s) + 1)(\alpha(a_\lambda(s) + 1) + l_\lambda(s))}.$$

When $n = 1$, the Pieri formula reduces to

$$(1.6) \qquad zP_k(z;\alpha) = P_{k+1}(z;\alpha).$$

This is a special case of a recurrence formula which applies to any class of single variable orthogonal polynomials on the unit circle [14, p. 291].

The symmetric Jack polynomials are closely related to a more fundamental class of orthogonal polynomials called the nonsymmetric Jack polynomials [8]. These polynomials can be usefully employed to illuminate and simplify the theory of their symmetric counterparts [2]. They also possess many interesting properties in their own right. In this paper a decomposition in terms of the nonsymmetric Jack polynomials is given for the product of any nonsymmetric Jack polynomial with

z_i. The existence of this decomposition was conjectured by Cherednik [3]. Like the Pieri formula (1.3) the decomposition reduces to (1.6) when $n = 1$.

The derivation of this formula is similar to Knop and Sahi's derivation of (1.3) [6]. Knop and Sahi's derivation exploits the theory of the recently introduced interpolated symmetric Jack polynomials [9]. We shall proceed along similar lines using the interpolated nonsymmetric Jack polynomials which are nonsymmetric analogues of these polynomials [10, 5].

The interpolated nonsymmetric Jack polynomials are closely related to a class of generalised binomial coefficients $\binom{\eta}{\nu}$ indexed by pairs of compositions [1]. As a corollary we give an evaluation of these coefficients for $|\eta| = |\nu| + 1$.

2. PRELIMINARIES

Let $\eta := (\eta_1, \ldots, \eta_n)$ be a composition, that is, an n-tuple of non-negative integers. The modulus and Young diagram of η receive the same definition as that given in Section 1 for partitions. The partition η^+ is defined as the unique partition obtained by permuting the components of η. The nonsymmetric Jack polynomial E_η can be defined as the unique polynomial of the form

$$(2.1) \qquad E_\eta(z; \alpha) = z^\eta + \sum_{\mu \prec \eta} a_{\eta\mu} z^\mu,$$

that is, an eigenfunction of each of the Cherednik operators

$$(2.2)$$
$$\xi_i := \alpha z_i \frac{\partial}{\partial z_i} + \sum_{p < i} \frac{z_i}{z_i - z_p}(1 - s_{ip}) + \sum_{p > i} \frac{z_p}{z_i - z_p}(1 - s_{ip}) + 1 - i, \qquad i = 1, \ldots, n.$$

In (2.1), $z^\eta := z_1^{\eta_1} \cdots z_n^{\eta_n}$ and $\prec$ is a partial ordering on compositions having the same modulus. The ordering $\prec$ is defined by $\nu \prec \eta$ iff

$$(2.3) \qquad \text{either } \nu^+ < \eta^+, \text{ or else } \nu^+ = \eta^+ \text{ and } \nu < \eta.$$

The ordering $<$ in (2.3) is the dominance ordering on compositions having the same modulus and receiving the same definition as the dominance ordering on partitions. The eigenvalue of E_η with respect to ξ_i is

$$(2.4) \qquad \bar{\eta}_i := \alpha\eta_i - l'_\eta(i),$$

where $l'_\eta(i)$ is the leg colength of $s = (i, j)$ with respect to the composition η. The definition of leg colength given in Section 1 has to be extended to apply to compositions. The extension is [11]

$$(2.5) \qquad l'_\eta(i) := \#\{k | k < i, \eta_k \geq \eta_i\} + \#\{k | k > i, \eta_k > \eta_i\}.$$

Arm length, arm colength, and leg length are also extended. Arm length and colength receive the same definition as in Section 1. Leg length has definition [7]

$$(2.6) \qquad l_\eta(s) = \#\{k | k < i, j \leq \eta_k + 1 \leq \eta_i\} + \#\{k | k > i, j \leq \eta_k \leq \eta_i\}.$$

A diagrammatic computation of (2.5) and (2.6) is undertaken in Figure 2.

The interpolated Jack polynomials are generalisations of the classical shifted polynomials

$$(2.7) \qquad (-1)^p(-z)_p := z(z - 1) \cdots (z - p + 1).$$
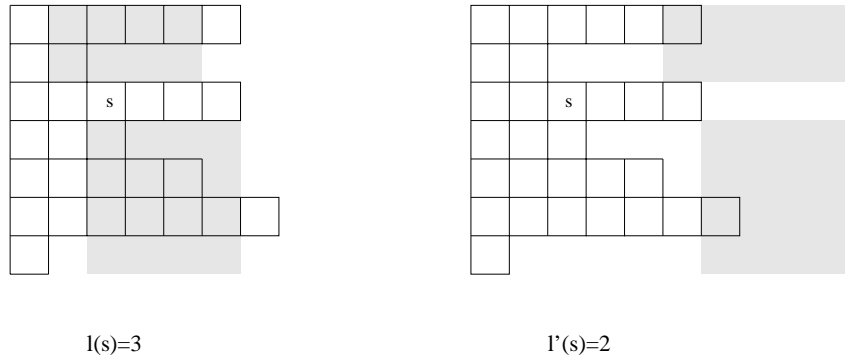
l(s)=3   l'(s)=2

FIGURE 2. In the left-hand side diagram the leg length is calculated as the number of rows which have shaded boxes and no unshaded boxes to the right of the shaded boxes. In the right-hand side diagram the leg colength is calculated as the number of rows which have any box shaded.

The interpolated nonsymmetric Jack polynomial E_η^* can be defined as the unique polynomial satisfying:

i): $E_\eta^*(\frac{\bar{\nu}}{\alpha}) = 0$, $|\nu| \le |\eta|$, $\nu \ne \eta$;
ii): $E_\eta^*(\frac{\bar{\eta}}{\alpha}) \ne 0$;
iii) : $E_\eta^*(z) = z^\eta + \sum_{\substack{|\nu| \le |\eta| \\ \nu \ne \eta}} b_{\eta\nu} z^\nu$.

Two properties of the polynomials E_η^* are given by the following two lemmas.

Lemma 2.1 ([5]). *If* $\eta_n = 0$, *then*

$$(2.8) \qquad E_\eta^*(z_1, \ldots, z_{n-1}, -\frac{n-1}{\alpha}) = E_{\eta'}^*(z_1, \ldots, z_{n-1}),$$

where $\eta' := (\eta_1, \ldots, \eta_{n-1})$.

Lemma 2.2 ([5]). *Define an operator* Δ *so that for any polynomial* $f(z)$,

$$(2.9) \qquad \Delta f(z_1, \ldots, z_n) := f(z_n - 1, z_1, \ldots, z_{n-1}).$$

We have

$$(2.10) \qquad E_{\Phi\eta}^*(z) = (z_n + \frac{n-1}{\alpha})\Delta E_\eta^*(z),$$

where $\Phi\eta := (\eta_2, \ldots, \eta_n, \eta_1 + 1)$.

The interpolated Jack polynomial E_η^* is defined to vanish for all $z = \frac{\bar{\nu}}{\alpha}$ where $|\nu| \le |\eta|$, $\nu \ne \eta$. It turns out that E_η^* vanishes on a larger domain. To state this domain requires introducing a further partial ordering on compositions. For compositions ν, η we say that $\nu \preceq' \eta$ iff there exists a permutation π such that

$$(2.11) \qquad \nu_i < \eta_{\pi(i)} \text{ if } i < \pi(i), \text{ and } \nu_i \le \eta_{\pi(i)} \text{ if } i \ge \pi(i).$$

This partial order reduces to the usual inclusion relation among diagrams when restricted to partitions.

Proposition 2.3 ([5]). *If* $\eta \npreceq' \mu$, *then* $E_\eta^*(\frac{\bar{\mu}}{\alpha}) = 0$.

The minimal elements lying above η according to $\prec'$ play an important role in the next section. Let $I = \{i_1, \ldots, i_r\}$ where $i_1 < \ldots < i_r$. We define the composition $c_I(\eta) := \mu$ by:

 i): $\mu_{i_j} = \eta_{i_{j+1}}, \qquad j = 1, \ldots, r-1$;
 ii): $\mu_{i_r} = \eta_{i_1} + 1$;
 iii): $\mu_i = \eta_i, \qquad i \notin I$.

For example, if

$$\eta := (\ldots, \eta_{i_1}, \ldots, \eta_{i_2}, \ldots, \eta_{i_3}, \ldots, \eta_{i_{r-1}}, \ldots, \eta_{i_r}, \ldots),$$

then

$$c_I(\eta) := (\ldots, \eta_{i_2}, \ldots, \eta_{i_3}, \ldots, \eta_{i_4}, \ldots, \eta_{i_r}, \ldots, \eta_{i_1} + 1, \ldots).$$

Clearly $\eta \prec' c_I(\eta)$. The following gives the converse.

Lemma 2.4 ([5]). *If $\eta \prec' \mu$, then there exists an I such that $c_I(\eta) \preceq' \mu$.*

Define the operator σ_i by

$$(2.12) \qquad \sigma_i := s_i + \frac{1}{\alpha} \frac{1 - s_i}{z_i - z_{i+1}}, \qquad i = 1, \ldots, n-1,$$

where s_i is the transposition operator that acts on functions of $z := (z_1, \ldots, z_n)$ by interchanging the variables z_i and z_{i+1}. The interpolated Jack polynomial E_η^* is a simultaneous eigenfunction of the family of operators

$$(2.13) \qquad \tilde{\Xi}_i := z_i - \sigma_i \cdots \sigma_{n-1}(z_n + \frac{n-1}{\alpha})\Delta \, \sigma_1 \cdots \sigma_{i-1}, \qquad i = 1, \ldots, n-1.$$

These operators can be used to show that the interpolated Jack polynomials are triangular with respect to the nonsymmetric Jack polynomials. For each polynomial E_η^* we can write

$$(2.14) \qquad E_\eta^*(z) = E_\eta(z) + \sum_{|\nu| < |\eta|} \tilde{a}_{\eta\nu} E_\nu(z)$$

for coefficients $\tilde{a}_{\eta\nu}$. This decomposition shows that the top homogeneous component of any interpolated Jack polynomial $E_\eta^*(z)$ is the Jack polynomial $E_\eta(z)$. Using (2.14) we can define an isomorphism Ψ mapping each Jack polynomial E_η to its corresponding interpolated polynomial E_η^*. This isomorphism allows us to state the following important inversion formula.

Proposition 2.5 ([5]). *Define*

$$(2.15) \qquad \tilde{Z}_i := \sigma_i \cdots \sigma_{n-1}(z_n + \frac{n-1}{\alpha})\Delta\sigma_1 \cdots \sigma_{i-1}, \qquad i = 1, \ldots, n-1.$$

We have

$$(2.16) \qquad \tilde{Z}_i \Psi = \Psi z_i.$$

3. A JACK POLYNOMIAL DECOMPOSITION

We first give an evaluation formula for the interpolated Jack polynomials.

Proposition 3.1. *We have*

$$(3.1) \qquad E_\eta^*(\frac{\bar{\eta}}{\alpha}) = \frac{d'_\eta}{\alpha^{|\eta|}}$$

where

$$(3.2) \qquad d'_\eta := \prod_{s \in \operatorname{diag}(\eta)} \left(\alpha(a_\eta(s) + 1) + l_\eta(s) \right).$$

Proof. We proceed by induction on $n + |\eta|$. First suppose that $\eta_n = 0$. Then $(\frac{\bar{\eta}}{\alpha})_n = -\frac{n-1}{\alpha}$. Applying Lemma 2.1 and the inductive hypothesis we obtain

$$(3.3) \qquad E^*_\eta(\frac{\bar{\eta}}{\alpha}) = E^*_\mu(\frac{\bar{\mu}}{\alpha}) = \frac{1}{\alpha^{|\eta|}} \prod_{s \in \operatorname{diag}(\mu)} \left(\alpha(a_\mu(s) + 1) + l_\mu(s) \right)$$

where $\mu := (\eta_1, \ldots, \eta_{n-1})$. We obtain (3.1) since $a_\eta(s) = a_\mu(s)$ and $l_\eta(s) = l_\mu(s)$ for each $s \in \operatorname{diag}(\mu) = \operatorname{diag}(\eta)$.

Now suppose that $\eta_n > 0$. Applying Lemma 2.2 gives

$$(3.4) \qquad E^*_\eta(z) = (z_n + \frac{n-1}{\alpha}) E^*_\mu(z_n - 1, z_1, \ldots, z_{n-1})$$

where $\mu := \Phi^{-1}\eta$. Evaluating (3.4) at $z = \frac{\bar{\eta}}{\alpha}$ we obtain

$$\begin{aligned} E^*_\eta(\frac{\bar{\eta}}{\alpha}) &= (\eta_n + \frac{n - 1 - l'_\eta(n)}{\alpha}) E^*_\mu(\frac{\bar{\mu}}{\alpha}) \\ (3.5) \qquad &= \frac{1}{\alpha^{|\eta|}} \left(\alpha \left(a_\eta(n, 1) + 1 \right) + l_\eta(n, 1) \right) \prod_{s \in \operatorname{diag}(\mu)} \left(\alpha \left(a_\mu(s) + 1 \right) + l_\mu(s) \right). \end{aligned}$$

In the second line we have used the inductive hypothesis together with the identities $\eta_n = a_\eta(n, 1) + 1$ and $l_\eta(n, 1) = n - 1 - l'_\eta(n)$. Now consider the bijection $\psi : \operatorname{diag}(\mu) \mapsto \operatorname{diag}(\Phi\mu)/\{(n, 1)\}$ where

$$(3.6) \qquad \psi(i, j) := \begin{cases} (n, j + 1) & \text{if } i = 1, \\ (i - 1, j) & \text{if } i > 1. \end{cases}$$

We obtain (3.1) by noting that $a_\mu(s) = a_{\Phi\mu}(\psi s)$ and $l_\mu(s) = l_{\Phi\mu}(\psi s)$ for each $s \in \operatorname{diag}(\mu)$. $\qquad\qquad\square$

Proposition 3.2. *The action of $\tilde{Z}_i$ on $f(z)$ is given by*

$$(3.7) \qquad \tilde{Z}_i f(z) = \sum_{\substack{I \subseteq \{1, \ldots, n\} \\ i \in I}} a_I^{(i)}(z) f(c_I^{-1}(z)).$$

The rational function $a_I^{(i)}(z)$ can be expressed as

$$(3.8) \qquad a_I^{(i)}(z) = \chi_I^{(i)}(z) A_I(z) B_I(z)$$

where

$$(3.9) \qquad A_I(z) := \frac{1}{\alpha^{s-1}(z_{t_s} - z_{t_1} - 1)} \prod_{u=1}^{s-1} \frac{1}{(z_{t_u} - z_{t_{u+1}})},$$

$$(3.10) \qquad B_I(z) := \left(\prod_{j=t_s+1}^{n} \frac{z_{t_s} - z_j - \frac{1}{\alpha}}{z_{t_s} - z_j} \right) \left(z_{t_s} + \frac{n-1}{\alpha} \right)$$

$$\times \left(\prod_{j=1}^{t_1-1} \frac{z_{t_s} - 1 - z_j - \frac{1}{\alpha}}{z_{t_s} - 1 - z_j} \right) \left(\prod_{u=1}^{s-1} \prod_{j=t_u+1}^{t_{u+1}-1} \frac{z_{t_u} - z_j - \frac{1}{\alpha}}{z_{t_u} - z_j} \right),$$

$$(3.11) \qquad \chi_I^{(i)}(z) := \begin{cases} z_{t_{k-1}} - z_i & \text{if } i = t_k, \ k \in \{2,3,\ldots,s\}, \\ z_{t_s} - z_i - 1 & \text{if } i = t_1, \end{cases}$$

and $I = \{t_1, \ldots, t_s\}$, $1 \le t_1 < \cdots < t_s \le n$.

Proof. The action of σ_i on any polynomial $f(z)$ is given by

$$(3.12) \qquad \sigma_i f(z) = \mathcal{A}_i f(z) + \mathcal{B}_i f(z),$$

where

$$\mathcal{A}_i := \frac{1}{\alpha(z_i - z_{i+1})}, \qquad \mathcal{B}_i := \frac{z_i - z_{i+1} - \frac{1}{\alpha}}{z_i - z_{i+1}} s_i.$$

Given definition (2.15) of $\tilde{Z}_i$ we can write

$$(3.13) \qquad \tilde{Z}_i = \sum_{\substack{1 \le t_1 < \cdots < t_s \le n \\ i = t_k}} U_{t_k} \cdots U_{t_s} U_{t_1} \cdots U_{t_{k-1}},$$

where

$$(3.14) \qquad U_{t_j} := \mathcal{A}_{t_j} \mathcal{B}_{t_j+1} \cdots \mathcal{B}_{t_{j+1}-1}, \qquad 1 \le j < k,$$

$$(3.15) \qquad U_{t_s} := \mathcal{A}_{t_s-1} \mathcal{B}_{t_s} \cdots \mathcal{B}_{n-1}(z_n + \frac{n-1}{\alpha})\Delta \mathcal{B}_1 \cdots \mathcal{B}_{t_1-1},$$

$$(3.16) \qquad U_{t_j} := \mathcal{A}_{t_{j-1}} \mathcal{B}_{t_j} \cdots \mathcal{B}_{t_{j+1}-2}, \qquad k < j < s,$$

and

$$(3.17) \qquad U_i := \begin{cases} \mathcal{B}_i \cdots \mathcal{B}_{t_{k+1}-2} & \text{if } i \ne s, \\ \mathcal{B}_i \cdots \mathcal{B}_{n-1}(z_n + \frac{n-1}{\alpha})\Delta \mathcal{B}_1 \cdots \mathcal{B}_{t_1-1} & \text{if } i = s. \end{cases}$$

By applying (3.13) to $f(z)$ we can directly compute (3.7). $\qquad \square$

We are now in a position to derive an expansion of $z_i E_\eta$ in terms of the Jack polynomials.

Proposition 3.3. *We have*

$$(3.18) \qquad z_i E_\eta(z) = \alpha d'_\eta \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ i \in I}} \frac{a_I^{(i)}(\overline{\frac{c_I(\eta)}{\alpha}})}{d'_{c_I(\eta)}} E_{c_I(\eta)}(z).$$

Proof. The nonsymmetric Jack polynomials E_η with $|\eta| = d$ form a basis for the homogeneous polynomials of degree d. Hence we can write

$$(3.19) \qquad z_i E_\eta(z) = \sum_{|\mu|=|\eta|+1} c_\mu E_\mu(z)$$

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for coefficients c_μ. Applying Ψ to both sides of (3.19) and using the inversion formula (2.16) gives

$$(3.20) \qquad \tilde{Z}_i E_\eta^*(z) = \sum_{|\mu|=|\eta|+1} c_\mu E_\mu^*(z).$$

We now evaluate (3.20) at $z = \frac{\bar{\lambda}}{\alpha}$ where $|\lambda| = |\eta| + 1$. It follows from the vanishing properties of the interpolated Jack polynomials and Proposition 3.2 that

$$(3.21) \qquad c_\lambda = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ i \in I, c_I(\eta)=\lambda}} \frac{a_I^{(i)}(\overline{\frac{c_I(\eta)}{\alpha}})E_\eta^*(\frac{\bar{\eta}}{\alpha})}{E_{c_I(\eta)}^*(\overline{\frac{c_I(\eta)}{\alpha}})}.$$

We obtain (3.18) by applying Proposition 3.1 to (3.21). $\square$

We can make two improvements to the formula (3.18). First, we can simplify the coefficient $a^{(i)}(\overline{\frac{c_I(\eta)}{\alpha}})$.[1] Let

$$(3.22) \qquad \tilde{B}_I(z) := \left(\prod_{j=t_s+1}^{n} \frac{z_{t_1} + 1 - z_j - \frac{1}{\alpha}}{z_{t_1} + 1 - z_j} \right) \left(z_{t_1} + 1 + \frac{n-1}{\alpha} \right)$$

$$\times \left(\prod_{j=1}^{t_1-1} \frac{z_{t_1} - z_j - \frac{1}{\alpha}}{z_{t_1} - z_j} \right) \left(\prod_{u=2}^{s} \prod_{j=t_{u-1}+1}^{t_u-1} \frac{z_{t_u} - z_j - \frac{1}{\alpha}}{z_{t_u} - z_j} \right),$$

$$(3.23) \qquad \tilde{\chi}_I^{(i)}(z) := \begin{cases} z_i - z_{t_{k+1}} & \text{if } i = t_k, \ k \in \{1,2,3,\ldots,s-1\}, \\ z_i - z_{t_1} - 1 & \text{if } i = t_s, \end{cases}$$

where $I = \{t_1,\ldots,t_s\}$, $1 \le t_1 < \cdots < t_s \le n$. We have $\tilde{B}_I(\frac{\bar{\eta}}{\alpha}) = B_I(\overline{\frac{c_I(\eta)}{\alpha}})$ and $\tilde{\chi}_I^{(i)}(\frac{\bar{\eta}}{\alpha}) = \chi_I^{(i)}(\overline{\frac{c_I(\eta)}{\alpha}})$. Since $A_I(\frac{\bar{\eta}}{\alpha}) = A_I(\overline{\frac{c_I(\eta)}{\alpha}})$, it follows that

$$(3.24) \qquad a_I^{(i)}(\overline{\frac{c_I(\eta)}{\alpha}}) = \tilde{\chi}_I^{(i)}(\frac{\bar{\eta}}{\alpha})A_I(\frac{\bar{\eta}}{\alpha})\tilde{B}_I(\frac{\bar{\eta}}{\alpha}).$$

Second, we can restrict the domain of summation in (3.18) by removing a number of vanishing terms. This allows us to give a unique decomposition of $z_i E_\eta$ in terms of the nonsymmetric Jack polynomials.

Proposition 3.4. *Let $I := \{t_1,\ldots,t_s\}$ with $1 \le t_1 < \cdots < t_s \le n$ and $I \ne \emptyset$. We call I maximal with respect to η iff:*

i): $\eta_j \ne \eta_{t_1}$, $\qquad j = 1,\ldots,t_1-1$,
ii): $\eta_j \ne \eta_{t_u}$, $\qquad j = t_{u-1}+1,\ldots,t_u-1$ *and* $u = 2,\ldots,s$,
iii): $\eta_j \ne \eta_{t_1} + 1$, $\qquad j = t_s+1,\ldots,n$.

Let $\mathbb{J}_\eta$ be the set of non-empty subsets of $\{1,\ldots,n\}$ which are maximal with respect to η. The decomposition of $z_i E_\eta$ in terms of the nonsymmetric Jack polynomials is given by

$$(3.25) \qquad z_i E_\eta(z) = \alpha d'_\eta \sum_{\substack{I \in \mathbb{J}_\eta \\ i \in I}} \frac{\tilde{\chi}_I^{(i)}(\frac{\bar{\eta}}{\alpha})A_I(\frac{\bar{\eta}}{\alpha})\tilde{B}_I(\frac{\bar{\eta}}{\alpha})}{d'_{c_I(\eta)}} E_{c_I(\eta)}(z).$$

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[1]This simplification was pointed out to me by David McAnally.

Proof. If I is a non-empty subset of $\{1, \ldots, n\}$ with $I \notin \mathbb{J}_\eta$, then $\tilde{B}_I(\frac{\bar{\eta}}{\alpha}) = 0$. Hence, we can restrict (3.18) to range over $\mathbb{J}_\eta$. Furthermore, if $I, I' \in \mathbb{J}_\eta$ and $c_I(\eta) = c_{I'}(\eta)$, then $I = I'$. It follows that each Jack polynomial in the restricted sum only appears once. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. GENERALISED BINOMIAL COEFFICIENTS

The classical binomial coefficients

$$(4.1) \qquad \binom{l}{p} := \frac{l!}{(l-p)!\,p!}$$

occur in the identities

$$(4.2) \qquad e^z z^p = \sum_{l=p}^{\infty} \binom{l}{p} \frac{p!}{l!} z^l$$

and

$$(4.3) \qquad (1+z)^l = \sum_{p=0}^{l} \binom{l}{p} z^p.$$

Rather than using (4.1) to define the binomial coefficients we could use either (4.2) or (4.3). Taking this latter perspective, we can define generalised binomial coefficients by an appropriate Jack polynomial generalisation of (4.2) or (4.3) [1, 12]. For $|\eta| \geq |\nu|$, we can define the generalised nonsymmetric binomial coefficients $\binom{\eta}{\nu}$ by

$$(4.4) \qquad \frac{E_\eta(1+z)}{E_\eta(1^n)} = \sum_{\nu | |\nu| \leq |\eta|} \binom{\eta}{\nu} \frac{E_\nu(z)}{E_\nu(1^n)}.$$

This identity reduces to (4.3) when $n = 1$. Using (4.4) to define $\binom{\eta}{\nu}$ we can derive the natural Jack polynomial generalisation of (4.2) (see e.g. [4])

$$(4.5) \qquad \frac{\alpha^{|\eta|}}{d'_\eta} e^{e_1(z)} E_\eta(z) = \sum_{\nu | |\nu| \geq |\eta|} \frac{\alpha^{|\nu|}}{d'_\nu} \binom{\nu}{\eta} E_\nu(z)$$

where $e_1(z)$ is the first elementary symmetric function defined by (1.4). Alternatively, we may use (4.5) to define the generalised binomial coefficients and derive (4.4) as a consequence.

The generalised binomial coefficients are closely related to the interpolated Jack polynomials via the formula [12]

$$(4.6) \qquad \binom{\eta}{\nu} = \frac{E_\nu^*(\eta)}{E_\nu^*(\nu)}.$$

This formula generalises the classical identity

$$(4.7) \qquad \binom{l}{p} = \frac{(-1)^p(-l)_p}{(-1)^p(-p)_p},$$

which relates the binomial coefficients $\binom{l}{p}$ to the classical shifted polynomials.

Using the decomposition of the product of $E_\eta(\eta)$ with z_i we can give an evaluation of the generalised binomial coefficients for a restricted domain. We first require an analogue of (1.3) when $r = 1$.

1826 DAN MARSHALL

Proposition 4.1.

$$(4.8) \qquad e_1(z)E_\eta(z) = -\alpha d'_\eta \sum_{I \in \mathbb{J}_\eta} \frac{A_I(\frac{\bar{\eta}}{\alpha})\tilde{B}_I(\frac{\bar{\eta}}{\alpha})}{d'_{c_I(\eta)}} E_{c_I(\eta)}(z).$$

Proof. It follows from (3.25) that

$$(4.9) \qquad (\sum_{i=1}^{n} z_i)E_\eta(z) = \alpha d'_\eta \sum_{I \in \mathbb{J}_\eta} (\sum_{i \in I} \tilde{\chi}_I^{(i)}(\frac{\bar{\eta}}{\alpha})) \frac{A_I(\frac{\bar{\eta}}{\alpha})\tilde{B}_I(\frac{\bar{\eta}}{\alpha})}{d'_{c_I(\eta)}} E_{c_I(\eta)}(z).$$

The identity (4.8) follows since $e_1(z) = \sum_{i=1}^{n} z_i$ and $\sum_{i \in I} \tilde{\chi}_I^{(i)}(\frac{\bar{\eta}}{\alpha}) = -1$. □

Corollary 4.2. *Suppose* $|\mu| = |\eta| + 1$. *Then*

$$(4.10) \qquad \binom{\mu}{\eta} = -A_I(\frac{\bar{\eta}}{\alpha})\tilde{B}_I(\frac{\bar{\eta}}{\alpha}),$$

where $\mu = c_I(\eta)$. *If there is no* I *such that* $\mu = c_I(\eta)$, *then* $\binom{\mu}{\eta} = 0$.

Proof. It follows from (4.5) that

$$(4.11) \qquad e_1(z)E_\eta(z) = \alpha d'_\eta \sum_{|\nu| = |\eta| + 1} \frac{1}{d'_\nu} \binom{\nu}{\eta} E_\nu(z).$$

Comparison with (4.8) gives (4.10). □

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