Optimal Dividends with Debts and Nonlinear Insurance Risk Processes

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Abstract

The optimal dividend problem is a classic problem in corporate finance though an early contribution to this problem can be tracked back to the classic work by an actuary, Bruno De Finetti, in the late 1950s. Nowadays, there is a leap of literature on optimal dividend problem. However, most of the literature focuses on linear insurance risk processes which fail to take into account some realistic features such as the nonlinear effect on the insurance risk processes. In this paper, we articulate this problem and consider an optimal dividend problem with nonlinear insurance risk processes attributed to internal competition factors. We also incorporate other realistic features such as the presence of debts, constraints in regular control variables, fixed and proportional transaction costs. This poses some theoretical challenge as the problem becomes a nonlinear regular-impulse control problem. Under some smooth hypothesis for the value function, we obtain the the structure of the value function by using its properties, not but guess its structure, which is widely used in most papers . By solving the corresponding HJB equation, closed-form solutions to the problem are obtained in various cases.

Keywords: Optimal dividend; Internal competition factors; Nonlinear risk processes; Transaction costs; Regular-impulse control; HJB equation; closed-form solution.

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1 Introduction

The optimal dividend problem is one of the major issues of corporate finance since its inception though an early contribution to this problem can be tracked back to the classic work by an actuary, Bruno De Finetti, in the late 1950s. The story goes back to the International Congress of Actuaries in New York in 1957, where Bruno presented his inaugural work De Finetti [7] on a mathematical approach to the optimal dividend problem of an insurance company. The key motivation of De Finetti's study is attributed to the observation that in the classical risk theory where evaluating the ruin probability of an insurance company is the main concern, the surplus of the company can increase indefinitely without bounds. This is, of course, not realistic. To articulate this problem, De Finetti [7] first introduced dividend payments of the company to the picture and considered the situation where the company wishes to maximize the expectation of the present value of all dividends before possible ruin. He modeled the surplus of the company as a simple discrete process with steps of size plus or minus one only. Under this assumption, he obtained an elegant result that the optimal dividend-payment strategy is a barrier strategy which devises that any surplus exceeding a certain barrier level should be paid as dividends to shareholders of the company. The classic work of De Finetti [7] has stimulated a leap of works on the optimal dividend problem. Some representative works include Bühlmann [3], Gerber [9], Asmussen and Taksar [2], Gerber and Shiu [10], [11], Højgaard and Taksar [15], Taksar [22], Alvarez [1] and Paulsen [19]. Nowadays, the optimal dividend problem becomes one of the central topics in actuarial risk theory.

Another key topics in actuarial risk theory is the optimal reinsurance problem. Reinsurance is a major tool for insurance companies to transfer their exposures to risk to another party, namely, a reinsurer. An effective use of reinsurance can protect an insurance company against unexpected large losses due to insurance claims and reduce the company's earning's volatility. There is a large amount of literature on optimal reinsurance. Schmidli [20], [21] considered the proportional reinsurance and determined an optimal proportional reinsurance strategy by minimizing the probability of ruin of an insurance company. Taksar and Markussen [23] extended the analysis using a diffusion model with investment and proportional reinsurance. Some of the other studies on reinsurance include Taksar [22], Højgaard and Taksar [15], Asmussen and Taksar [2], Choulli *et al.* [5], Choulli and Taksar [6], Irgens and Paulsen [16], Meng and Siu [18] and references therein. From a technical perspective, the optimal reinsurance problem can be viewed as a regular control problem, which is an important mathematical method for controlling and managing risks to which companies are exposed.

It appears that the vast literature on the optimal dividend problem and the optimal reinsurance problem mainly focus on linear insurance risk processes which may be motivated from controlling risks and dividend distributions, as well as from managing personnel (hiring/firing) policies for cooperations. However, linear insurance risk processes fail to take into account some realistic features such as the nonlinear effect on the insurance risk processes attributed to internal competition factors. The incorporation of nonlinear insurance risk processes is far more than a trivial issue. Indeed, it poses some theoretical challenge as the problem becomes a nonlinear regular-impulse control problem. Recently, Guo [12] and Guo, Liu and Zhou [13] introduced non-linear controlled dynamics in an optimal stochastic control problem which was motivated by a workforce control problem. They formulated the problem as a nonlinear regular-singular optimal control problem. The level of difficulty of their problem is similar to the nonlinear regular-impulse control problem.

In this paper, we consider an optimal dividend problem with nonlinear insurance risk processes, where the nonlinearity is attributed to internal competition factors of an insurance company. We also incorporate other realistic features such as constraints in regular control variables, fixed and proportional transaction costs. The nonlinear regular-impulse control problem is discussed using the dynamic programming approach. By solving the Hamiltonian-Jacobi-Bellman (HJB) equation, closed-form solutions to the problem are obtained in various cases.

This paper is structured as follows. The next section presents the modeling framework and formulates the optimization problem. Section 3 discusses some properties and structures of the value function. In Section 4, we derive closed-form expressions for the value function and the optimal dividend strategy in each of the cases. The final section summarizes the paper.

2 The Model

As usual, we consider a complete, filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\mathbb{P}$ is a real-world probability and the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions, (i.e., right-continuity and $\mathbb{P}$-completeness). Let $\{W_t\}_{t\geq 0}$ be an $(\{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$-standard Brownian motion. Consider the following controlled insurance risk process:

$$X_t = x + \int_0^t (\mu U(s) - aU^2(s) - \delta)ds + \int_0^t \sigma U(s)dW_s - \sum_{n=1}^{\infty} I_{\{\tau_n < t\}}\xi_n, \tag{2.1}$$

where $\mu, a, \sigma > 0$ and $\delta \geq 0$ are given parameters, and $x \geq 0$ is the initial state; $U(s) \in [l, u]$, where $0 < l < u < +\infty$; $\{\tau_i; i = 1, 2, \cdots\}$ is an increasing sequence of stopping times and $\{\xi_i; i = 1, 2, \cdots\}$ is a sequence of non-negative random variables, associated with amounts of the dividends paid to shareholders of an insurance company. The parameters μ and a describe the sensitivities of the expected surplus of an insurance company without dividend payments with respect to the linear risk factor and the nonlinear one, respectively. The parameter σ describes the volatility of the surplus of the company in the absence of dividend payments.

Definition 2.1: A pair

$$\pi \equiv \{U; \mathcal{S}\} \equiv \{U; \tau_1, \tau_2, \cdots, \tau_n, \cdots; \xi_1, \xi_2, \cdots, \xi_n, \cdots\}$$

is an admissible policy of an insurance company with initial capital x if it satisfies the following conditions:

1. For each $i = 1, 2, \cdots$ and each $t \geq 0$, $\{\tau_i \leq t\} \in \mathcal{F}_t$ and $\xi_i \in \mathcal{F}_{\tau_i}$;
2. $0 < \xi_i \leq X_{\tau_i -}$;
3. $U(t) \in [l, u]$;

We write $\Pi(x)$ for the space of these admissible policies.

Let $K (K > 0)$ be the fixed transaction cost attributed to the advisory and consulting fees and $k (0 < k < 1)$ the proportional transaction cost due to taxes on dividends. Then the optimization problem of the insurance company is to select $\pi \in \Pi(x)$ so as to maximize the following performance function:

$$J(x, \pi) := E\left[\sum_{n=1}^{\infty} e^{-r\tau_n}(-K + k\xi_n) I_{\{\tau_n \leq \tau^\pi\}}\right] , \tag{2.2}$$

where $r > 0$ is the impatient factor and the ruin time τ^π corresponding π is

$$\tau^\pi = \inf\{t : X_t^\pi < 0\}.$$

The goal of the insurance company is to select an optimal strategy $\pi \in \Pi(x)$ so as to maximize the expected present value of dividends before bankruptcy. That is, to determine the value function

$$V(x) := \sup\{J(x, \pi); \pi \in \Pi(x)\} , \tag{2.3}$$

and an optimal strategy π^* such that $V(x) = J(x, \pi^*)$.

The value function $V(x)$ is also called an optimal return function.

3 Properties and structure of the value function

In this section, we first derive some properties satisfied by the value function. Under certain smooth hypotheses for the value function, we obtain the the structure of the value function using its properties rather than guessing its structure.

Similar to the arguments in Proposition 3.1 of [4], we have:

Proposition 1. *The value function V satisfies, for all $x \in [0, \infty)$,*

$$V(x) \leq k(x + \nu/r), \tag{3.1}$$

where $\nu = \max_{U \in [l,u]} |\mu U - aU^2 - \delta|$.

To prove our main results, we first recall a result by Choulli, Taksar and Zhou [5].

Lemma 1. *Let Y_t be an Itô's process on a positive half line defined by:*

$$Y_t = x + \int_0^t m(u)du + \int_0^t s(u)dW_u, \tag{3.2}$$

where

$$0 < d \leq s(u) \leq g, \;\; b \leq m(u) \leq c, \tag{3.3}$$

for some constants b, c, d and g. Let $h > 0$ and $\zeta_h := \inf\{t \geq 0 : Y_t = h\}$. Then for any fixed $t > 0$,

$$P(\zeta_0 < \zeta_h \wedge t) \;\; \rightarrow \;\; 1, \tag{3.4}$$

$$E\left(\max_{0 \leq s \leq \zeta_0 \wedge t} Y_s\right) \;\; \rightarrow \;\; 0 \tag{3.5}$$

as $x \downarrow 0$ uniformly over all the processes Y_t with the drift and diffusion terms satisfying (3.3).

Proposition 2. *The value function $V(\cdot)$ is a continuous, nondecreasing function, and it satisfies:*

$$V(0+) := \lim_{x \downarrow 0} V(x) = 0. \tag{3.6}$$

Proof. Obviously, we have $V(x) > V(y)$ for $x > y$.

(a) First we state (3.6). Let

$$Y_t := x + \int_0^t (\mu U(s) - aU^2(s) - \delta)ds + \int_0^t \sigma U(s)dW_s. \tag{3.7}$$

Then, for any $\varepsilon > 0$, there exists a $\rho > 0$ such that

$$\mathbb{P}_x\{\zeta_0 < \zeta_h \wedge t\} > 1 - \varepsilon \ for\ any\ 0 \le x < \rho, \tag{3.8}$$

$$\mathbb{E}_x\left[\sup_{0\le s\le \zeta_0\wedge t} Y_s\right] < \varepsilon \ for\ any\ 0 \le x < \rho. \tag{3.9}$$

Here $\mathbb{P}_x$ and $\mathbb{E}_x$ are the conditional probability measure and the conditional expectation under $\mathbb{P}$ given that $Y_0 = x$, respectively.

By observation, $\tau^\pi \le \zeta_0$, which, together with (3.8), implies that $\mathbb{P}_x\{\tau^\pi < \zeta_h \wedge t\} > 1 - \varepsilon$. Define

$$\varrho := \tau^\pi \wedge \zeta_h \wedge t. \tag{3.10}$$

Consequently,

$$\begin{aligned}
J(x,\pi) &= \mathbb{E}_x\left[\sum_{n=1}^\infty e^{-r\tau_n}(-K + k\xi_n)I_{\{0\le\tau_n\le\tau^\pi\}}\right] \\
&= \mathbb{E}_x\left[\sum_{n=1}^\infty e^{-r\tau_n}(-K + k\xi_n)I_{\{0\le\tau_n^\pi\le\varrho\}}\right] + \mathbb{E}_x\left[\sum_{n=1}^\infty e^{-r\tau_n}(-K + k\xi_n)I_{\{\varrho<\tau_n^\pi\le\tau^\pi\}}\right] \\
&:= J_1 + J_2.
\end{aligned}$$

However,

$$\begin{aligned}
J_1 &\le \mathbb{E}_x\left[\sum_{n=1}^\infty e^{-r\tau_n}(-K + k\xi_n)I_{\{0\le\tau_n^\pi\le\varrho\}}\right] \le \mathbb{E}_x\left[\sum_{n=1}^\infty e^{-r\tau_n}\xi_n I_{\{0\le\tau_n^\pi\le\varrho\}}\right] \\
&\le \mathbb{E}_x\left[\int_0^\varrho e^{-rs}dY_s\right] \le \mathbb{E}_x[Y_\varrho] \le \mathbb{E}_x\left[\max_{0\le s\le\varrho} Y_s\right] \le \mathbb{E}_x\left[\max_{0\le s\le\zeta_0\wedge t} Y_s\right] < \varepsilon.
\end{aligned} \tag{3.11}$$

On the other hand,

$$\begin{aligned}
J_2 &= \mathbb{E}_x\left[\sum_{n=1}^\infty e^{-r\tau_n}(-K + k\xi_n)I_{\{\varrho<\tau_n^\pi\le\tau^\pi\}}\right] \\
&= \mathbb{E}_x\left\{I_{\{\tau^\pi>\varrho\}}\mathbb{E}_x\left[\left(\sum_{n=1}^\infty e^{-r\tau_n}(-K + k\xi_n)I_{\{\varrho<\tau_n^\pi\le\tau^\pi\}}\right)\middle|\mathcal{F}_\varrho\right]\right\} \\
&\le \mathbb{E}_x[I_{\{\tau^\pi>\varrho\}}e^{-r\varrho}V(Y_\varrho)] \le \mathbb{E}_x[I_{\{\tau^\pi>\varrho\}}e^{-r\varrho}V(h)] \\
&\le V(h)\mathbb{P}_x(\tau^\pi > \varrho) \le V(h)\varepsilon.
\end{aligned}$$

Therefore $J(x,\pi) = J_1 + J_2 \leq (1+V(h))\varepsilon$. Taking the supremum over π, we obtain $V(x) \leq (1+V(h))\varepsilon$ for all $0 \leq x < \rho$. This proves that $V(0+) = \lim_{x\downarrow 0} V(x) = 0$.

(b) We next prove the continuity of V at any $x > 0$. Define, for each $0 < y < x$, $\chi_y = \inf\{t \geq 0 : X_t^x \leq y\}$. For any $\varepsilon > 0$, following the same line of the proof as for (3.11), we have, as $0 \leq y < \rho \wedge \varepsilon$,

$$\begin{aligned}
\mathbb{E}_x\left[\sum_{n=1}^{\infty} e^{-r\tau_n}(-K + k\xi_n) I_{\{\chi_y < \tau_n \leq \tau^\pi\}}\right] &< \mathbb{E}_y\left[\sum_{n=1}^{\infty} e^{-r\tau_n}(-K + k\xi_n) I_{\{0 < \tau_n \leq \tau^\pi\}}\right] \\
&\leq (1+V(h))\varepsilon.
\end{aligned}$$

Thus,

$$\begin{aligned}
J(x,\pi) &= \mathbb{E}_x\left[\sum_{n=1}^{\infty} e^{-r\tau_n}(-K + k\xi_n) I_{\{0 \leq \tau_n \leq \chi_y\}} + \sum_{n=1}^{\infty} e^{-r\tau_n}(-K + k\xi_n) I_{\{\chi_y < \tau_n \leq \tau^\pi\}}\right] \\
&< \mathbb{E}_{x-y}\left[\sum_{n=1}^{\infty} e^{-r\tau_n}(-K + k\xi_n) I_{\{0 \leq \tau_n \leq \tau^\pi\}}\right] + y + (1+V(h))\varepsilon \\
&\leq V(x-y) + (2+V(h))\varepsilon.
\end{aligned}$$

Consequently, taking the supremum over π gives:

$$0 \leq V(x) - V(x-y) \leq (2+V(h))\varepsilon, \tag{3.12}$$

which shows the left-continuity of $V(x)$ at x, for each $x > 0$. By proceeding exactly in the same manner, we can show the right-continuity of $V(x)$. $\square$

Remark 1: From the proof of the above Proposition, $V(x)$ is also a uniformly continuous function.

Consider, for each $U \in [l,u]$, the following operators:

$$\begin{aligned}
\mathcal{M}v(x) &= \sup_{0<\xi\leq x}\{v(x-\xi) + k\xi - K\}\ , \\
\mathcal{L}^U v(x) &= \frac{1}{2}\sigma^2 U^2 v''(x) + (\mu U - aU^2 - \delta)v'(x) - rv(x)\ .
\end{aligned}$$

Proposition 3. *$\mathcal{M}V(\cdot)$ is a continuous function.*

Proof. For any $\varepsilon > 0$, there exists a $0 < \rho < \frac{\varepsilon}{k}$ such that when $|y| < \rho$

$$-\varepsilon < V(x-\xi+y) - V(x-\xi) < \varepsilon,\ \textit{for any}\ x > 0, 0 < \xi \leq x$$

since V is uniformly continuous on $[0,\infty)$. Hence for $0<\xi\leq x$,

$$-\varepsilon+V(x-\xi)+k\xi-K<V(x-\xi+y)+k\xi-K<k\xi-K+V(x-\xi)+\varepsilon.$$

Taking the maximum for ξ gives:

$$-\varepsilon+\mathcal{M}V(x)<\sup_{0<\xi\leq x}\{V(x-\xi+y)+k\xi-K\}<\mathcal{M}V(x)+\varepsilon. \tag{3.13}$$

For $x<\xi\leq x+y$, we have

$$V(x-\xi+y)+k\xi-K\leq\varepsilon+kx+ky-K\leq kx-K+2\varepsilon\leq\mathcal{M}V(x)+2\varepsilon. \tag{3.14}$$

From (3.13),(3.14) and definition of $\mathcal{M}V(\cdot)$, we have

$$-2\varepsilon+\mathcal{M}V(x)<\mathcal{M}V(x+y)=\sup_{0<\xi\leq x+y}\{V(x-\xi+y)+k\xi-K\}<\mathcal{M}V(x)+2\varepsilon.$$

This completes the proof. □

Similar to P185 of [4], we can easily show:

Lemma 2. *$V(x)\geq\mathcal{M}V(x)$, for all $x\in\Re^+$.*

Define the continuation region $\mathcal{C}$ and the action region $\mathcal{A}$, respectively, by:

$$\begin{aligned}\mathcal{C} &:= \{x\in\Re^+:\mathcal{M}V(x)<V(x)\},\\ \mathcal{A} &:= \{x\in\Re^+:\mathcal{M}V(x)=V(x)\}.\end{aligned}$$

With the continuity of the value function in Proposition 1, the dynamic programming principle follows:

$$V(x)=\max_{\pi\in\Pi(x)}\left\{\mathbb{E}_x\left[\sum_{n=1}^{\infty}e^{-r\tau_n}(-K+k\xi_n)I_{\{0\leq\tau_n\leq\tau^\pi\wedge\varsigma\}}+e^{-r(\tau^\pi\wedge\varsigma)}V(X_{\tau^\pi\wedge\varsigma})\right]\right\}, \tag{3.15}$$

for any stopping time ς.

Using some standard results in stochastic optimal control, (see, for example, Fleming and Soner[8]), the Hamilton-Jacobi-Bellman (HJB) equation of the impulse control problem is given by:

$$\max\left\{\max_{U\in[l,u]}\mathcal{L}^U v(x),\mathcal{M}v(x)-v(x)\right\}=0\ ,\quad x>0\ , \tag{3.16}$$

with the boundary condition:

$$v(0) = 0 \ . \tag{3.17}$$

Since V and $\mathcal{M}V$ are continuous, we have the following lemma due to some topological properties.

Lemma 3. *The continuation region $\mathcal{C}$ is an open set.*

Lemma 4. *Suppose $x \in \mathcal{A}$, (i.e., x is in the action region). Then*
(1) the set

$$\Xi(x) := \{\xi \in \Re^+ : \mathcal{M}V(x) = V(x-\xi) + k\xi - K\} \tag{3.18}$$

is nonempty.

(2)For any $\xi_x \in \Xi(x)$, $x - \xi_x > 0$ and

$$V(x-\xi_x) > \mathcal{M}V(x-\xi_x) \ ,$$

so $x - \xi_x \in \mathcal{C}$.

Proof. (1) Fixed $x \in \mathcal{A}$, take sequence $\{\xi_n (0 < \xi_n \leq x)\}$ such that

$$\mathcal{M}V(x) \geq V(x-\xi_n) + k\xi_n - K \geq \mathcal{M}V(x) - \frac{1}{n}. \tag{3.19}$$

Consequently there exists a subsequence $\{\xi_{n_k}(0 < \xi_{n_k} \leq x)\}$ converging to $\xi^*(0 \leq \xi^* \leq x)$. This results in $\mathcal{M}V(x) = V(x-\xi^*) + k\xi^* - K$ by (3.19) as $n_k \to \infty$. If $\xi^* = 0$, we get $V(x) - K = \mathcal{M}V(x) = V(x)$, which is a contradiction, so $\xi^* \neq 0$.

(2)

$$\begin{aligned}
\mathcal{M}V(x) &= \sup_{0<\xi\leq x} \{V(x-\xi) + k\xi - K\} \\
&= \sup_{0<\eta\leq x-\xi_x} \{V(x-\xi_x-\eta) + k(\xi_x+\eta) - K\} \\
&\geq \sup_{0<\eta\leq x-\xi_x} \{V(x-\xi_x-\eta) + k\eta - K\} + k\xi_x \\
&= \mathcal{M}V(x-\xi_x) + k\xi_x.
\end{aligned}$$

On the other hand, $\mathcal{M}V(x) = V(x-\xi_x) + k\xi_x - K$, so $V(x-\xi_x) \geq \mathcal{M}V(x-\xi_x) + K$. This completes the proof. □

Lemma 5. *There exists a constant b such that $\mathcal{C}$ contains the interval $(0, b)$ with $b > 0$ and $\mathcal{C}$ does not contain any of interval $(a, +\infty)$ with $a \geq 0$.*

Proof. The first claim resembles to those in Section 6.1 of Meng and Siu [17]. We state the result without giving the proof. In the following, we shall prove the second claim.

Suppose $\mathcal{C} \supset (a, +\infty)$. By the following Lemma 6, for sufficiently large $a_1 > a$, there exists an $U_1 \in [l, u]$ such that

$$\frac{1}{2}\sigma^2 U_1^2 V''(x) + (\mu U_1 - aU_1^2 - \delta)V'(x) - rV(x) = 0, x \in [a_1, +\infty) \tag{3.20}$$

Consequently the ODE has a general solution

$$V(x) = A_1 e^{\lambda_1 x} + A_2 e^{\lambda_2 x}, \tag{3.21}$$

where $\lambda_1 < 0$ and $\lambda_2 > 0$.

From (3.1), $A_2 = 0$. This implies that $V(x) = A_1 e^{\lambda_1 x} \to 0$ as $x \to \infty$, which contradicts with $V(x) \to +\infty$. □

Under some smooth hypothesis for the value function, we obtain the the structure of the value function in the following theorem. In turn, we can easily verify that, in Section 4, the solution with this structure of the HJB equation really satisfies these smooth properties.

Theorem 1. *Assume that the value function V is C^1 on $(0, \infty)$, C^2 on the open set $\mathcal{C}$, we have:*

(i)For $x \in \mathcal{A}$, $V'(x) = k$.

(ii) $\mathcal{C}$ is connected, i.e., there exists a constant x^* such that

$$\begin{aligned} \mathcal{C} &:= \{x \in \Re^+ : \mathcal{M}V(x) < V(x)\} = (0, x^*) \ , \\ \mathcal{A} &:= \{x \in \Re^+ : \mathcal{M}V(x) = V(x)\} = [x^*, \infty) \ . \end{aligned}$$

(iii) There is $\tilde{x} \in (0, x^*)$ such that

$$V'(x^*) = V'(\tilde{x}) = k, \ V(x^*) = V(\tilde{x}) + k(x^* - \tilde{x}) - K,$$

or

$$V'(x^*) = k, \ V(x^*) = kx^* - K.$$

Proof. The idea of the proof of this theorem follows that in Section 5 of [14].

(i) For $x \in \mathcal{A}$, by Lemma 4, there exists $\xi_x (0 \le x - \xi_x \in \mathcal{C})$ such that

$$V(x) = \mathcal{M}V(x) = V(x - \xi_x) + k\xi_x - K \ .$$

If $x - \xi_x > 0$, we know that ξ_x is a global maximum of the function $V(x-y) + ky - K$ of y. For $x - \xi_x \in \mathcal{C}$, $\mathcal{C}$ is open and V is C^1 on $\mathcal{C}$,

$$V'(x - \xi_x) = k \ . \tag{3.22}$$

Now, for any $z \neq 0$, and $x + z \ge 0$,

$$V(x+z) \ge \mathcal{M}V(x+z) \ge V(x+z-\xi_x) + k\xi_x - K \ .$$

Then

$$V(x+z) - V(x+z-\xi_x) \ge k\xi_x - K = V(x) - V(x-\xi_x) \ .$$

Consequently,

$$\begin{aligned} \frac{V(x+z) - V(x)}{z} &\ge \frac{V(x+z-\xi_x) - V(x-\xi_x)}{z}, \ z > 0, \\ \frac{V(x+z) - V(x)}{z} &\le \frac{V(x+z-\xi_x) - V(x-\xi_x)}{z}, \ z < 0 \ . \end{aligned}$$

Letting $z \to 0^+$ $(z \to 0^-)$ give

$$V'^+(x) \ge k \ge V'^-(x). \tag{3.23}$$

Thus we obtain $V'^+(x) = V'^-(x) = k$.

If $x - \xi_x = 0$, similar to equation (3.22), we have $V'(x - \xi_x) < k$. ξ_x can be chosen as a continuous function of x, then there exists a neighborhood $\mathcal{H}$ of x such that $V'(y - \xi_y) < k$ for any $y \in \mathcal{H}$, and therefore $\xi_y = y$. Inserting $\xi_y = y$ into $\mathcal{M}V(y) = V(y)$ for $y \in \mathcal{H}$, we have $V(y) = ky - K$, which leads to $V'(x) = k$.

(ii) We prove this result by contradiction. Suppose that there exist some points $y_2 < y_1 < y_3$ such that $y_2, y_3 \in \mathcal{C}$ while $y_1 \in \mathcal{A}$. Define $x_1 := \sup\{x \ge y_1, [y_1, x] \subset \mathcal{A}\}$, where $y_1 = x_1$ is possible. From (i), we have $V'(x) = k$ for all $[y_1, x_1]$.

Step 1. We prove that

$$V(x) \ge k(x - x_1) + V(x_1), \forall x \ge x_1, \tag{3.24}$$

and the inequality is strict when $x > x_1$ and $x \in \mathcal{C}$.

Let $\xi_1 \in \Xi(x_1)$, we have

$$V(x) \geq \mathcal{M}V(x) \geq V(x_1 - \xi_1) + k\xi_1 - K + k(x - x_1) = V(x_1) + k(x - x_1),$$

where the first inequality is strict if $x \in \mathcal{C}$.

Step 2. We show that

$$\sup_{U \in [l,u]} (\mu U - aU^2 - \delta)k - rV(x_1) \leq 0. \tag{3.25}$$

Let $\varphi(x) = V(x_1) + k(x - x_1)$. For any admissible strategy $\pi \in \Pi$ and $h \in (0, \infty)$, let $\vartheta_\pi^h = h \wedge \inf\{t : R^\pi(t) \notin (x_1, x_1 + h)\}$. Then $\vartheta_\pi^h < \infty$ and $\vartheta_\pi^h \to 0$ as h does, almost surely. Define the strategy π by $U_\pi(t) = U$, for $t < \vartheta_\pi^h$ and $\xi_n = 0$ for $n = 1, 2, \cdots$, where $U \in [l, u]$ is a constant. Choose $h < x$, then $\vartheta_\pi^h < \tau^\pi$. From (3.15), with $\varsigma = \vartheta_\pi^h$, we get:

$$V(x_1) \geq E[e^{-r\vartheta_\pi^h} V(X^\pi(\vartheta_\pi^h))] \ . \tag{3.26}$$

Applying Itô's differentiation rule to φ and taking expectation then give:

$$E[e^{-r\vartheta_\pi^h} \varphi(X^\pi(\vartheta_\pi^h))] - \varphi(x_1) = E\left[\int_0^{\vartheta_\pi^h} e^{-rt} \mathcal{L}^U \varphi(X^\pi(t)) dt \right] \ . \tag{3.27}$$

Given that $V \geq \varphi$ in a right-neighborhood of x_1 and that $V(x_1) = \varphi(x_1)$. Then combining (3.26) and (3.27) as well as sending $h \to 0$ give:

$$\mathcal{L}^U \varphi(x_1) \leq 0 \ , \quad \forall U \in [l, u] \ ,$$

which is (3.25).

Step 3. There exists a point $x_2 > x_1$ such that

$$\sup_{U \in [l,u]} (\mu U - aU^2 - \delta)k - rV(x_2) \geq 0. \tag{3.28}$$

Let $(x_1, c_1) \subset \mathcal{C}$, where $c_1 = \sup\{c : (x_1, c) \subset \mathcal{C}\}$. Obviously, $c_1 < +\infty$ by Lemma 5 and $c_1 \in \mathcal{A}$. Thus $V'(c_1) = k$ by above (i), and there exists $d \in (x_1, c_1)$ such that $V'(d) > k$ since $V(x) > k(x - x_1) + V(x_1)$ for $x \in (x_1, c_1)$. Define

$$x_2 = \inf\{d \leq x \leq c_1 : V'(x) = k\}. \tag{3.29}$$

Obviously, $x_2 > d$, and $V'(x) > k = V'(x_2)$ for $d \leq x < x_2$. So by definition $V''(x_2) \leq 0$. Thus

$$
\begin{aligned}
0 &= \sup_{U\in[l,u]} \{\frac{1}{2}\sigma^2 U^2 V''(x_2) + (\mu U - aU^2 - \delta)V(x_2) - rV(x_2)\} \\
&\leq \sup_{U\in[l,u]} (\mu U - aU^2 - \delta)k - rV(x_2). \qquad (3.30)
\end{aligned}
$$

Step 4. From (3.25) and (3.30), we have $V(x_2) - V(x_1) \leq 0$, which contradicts with $V(x_2) > V(x_1) + k(x_2 - x_1)$.

Thus we prove that $\mathcal{C}$ is connected. Then combined with Lemma 5, the desired results follow.

(iii) Let $\xi_{x^*} \in \Xi(x^*)$ and $\tilde{x} := x^* - \xi_{x^*}$. Then $\tilde{x} \in [0, x^*)$ from Lemma 4. From the process of the proof of (i), we know that if $\tilde{x} \in (0, x^*)$, we have $V'(\tilde{x}) = V'(x^*) = k$, and

$$
V(x^*) = \mathcal{M}V(x^*) = V(\tilde{x}) + k(x^* - \tilde{x}) - K \ ; \qquad (3.31)
$$

if $\tilde{x} = 0$, obviously we have

$$
V'(x^*) = k, \ V(x^*) = kx^* - K.
$$

$\square$

In what follows, by theorem 1, we try to find some x^* and $\tilde{x}$ and a smooth function $\upsilon(x)$ on $(0, x^*)$ such that

$$
\sup_{U\in[l,u]} \{\frac{1}{2}\sigma^2 U^2 \upsilon''(x) + (\mu U - aU^2 - \delta)\upsilon'(x) - r\upsilon(x)\} = 0, x \in (0, x^*), \qquad (3.32)
$$

$$
\upsilon'(x) = k, x \in [x^*, \infty), \qquad (3.33)
$$

with boundary conditions

$$
\upsilon'(x^*) = \upsilon'(\tilde{x}) = k, \ \upsilon(x^*) = \upsilon(\tilde{x}) + k(x^* - \tilde{x}) - K, \upsilon(0) = 0,
$$

or

$$
\upsilon'(x^*) = k, \ \upsilon(x^*) = kx^* - K, \upsilon(0) = 0.
$$

4 Explicit solutions in different cases

For $\mu^2 \le 4a\delta$, we have $\mu U - aU^2 - \delta \le 0$ for $U \in [l, u]$. For this case, in Section 4.2, we can see that it is trivial. So we first discuss the value function with $\mu^2 > 4a\delta$.

4.1 $\mu^2 > 4a\delta$

In this case, we have $\frac{2\delta}{\mu} < \frac{\mu}{2a}$. Thus $[l, u] \cap [\frac{2\delta}{\mu}, \frac{\mu}{2a}]$ will have six different cases. In the following, we shall discuss the six cases in some details.

4.1.1 $2\delta/\mu < l < u < \mu/2a$

The HJB equation (3.16) reduces to

$$\max_{l \le U \le u} \left\{ \frac{1}{2}\sigma^2 U^2 v''(x) + (\mu U - aU^2 - \delta)v'(x) - rv(x) \right\} = 0, v(0) = 0, 0 < x < x^*. \quad (4.1)$$

Our objective is to find

$$U^*(x) := \arg \left\{ \max_{l \le U \le u} \{ \frac{1}{2}\sigma^2 U^2 v''(x) + (\mu U - aU^2 - \delta)v'(x) - rv(x) \} \right\} \quad (4.2)$$

for $0 \le x < x^*$.

To obtain the expression for $U^*(x)$, we define, without constraint,

$$\theta(x) := \arg \left\{ \max_{-\infty < U < \infty} \{ \frac{1}{2}\sigma^2 U^2 v''(x) + (\mu U - aU^2 - \delta)v'(x) - rv(x) \} \right\}, \quad (4.3)$$

for $0 < x < x^*$.

Applying the zero-derivative solution condition to (4.3) gives

$$[\mu - 2a\theta(x)]v'(x) + \sigma^2\theta(x)v''(x) = 0. \quad (4.4)$$

Thus we have

$$\theta(x) = \frac{\mu v'(x)}{2av'(x) - \sigma^2 v''(x)}, \ 0 < x < x^*. \quad (4.5)$$

Substituting the above to (4.1) without constraint, we obtain

$$\theta(x) = \frac{2r\upsilon(x)}{\mu\upsilon'(x)} + \frac{2\delta}{\mu}, \tag{4.6}$$

which becomes:

$$-r\upsilon(x) + \left(\frac{1}{2}\mu\theta(x) - \delta\right)\upsilon'(x) = 0. \tag{4.7}$$

From (4.6) we know $\theta(0) = \frac{2\delta}{\mu}$ and from (4.5) we have $\theta(x) \leq \frac{\mu}{2a}$.

Taking derivative on the above equation and combining it with (4.4) give:

$$[\mu\theta'(x) - 2r]\theta(x)\sigma^2 = [\mu - 2a\theta(x)][\mu\theta(x) - 2\delta]. \tag{4.8}$$

Similar to Lemma 5 of Guo, Liu and Zhou [13] we have the following lemma.

Lemma 6. *The unique strictly increasing solution $\theta(x)$ to the following first-order, nonlinear differential equation:*

$$\begin{cases} [\mu\theta'(x) - 2r]\theta(x)\sigma^2 = [\mu - 2a\theta(x)][\mu\theta(x) - 2\delta], \\ \theta(\bar{x}) = \eta \in [2\delta/\mu, \mu/(2a)], for\ some\ \bar{x} > 0\ , \end{cases} \tag{4.9}$$

satisfies the following nonlinear equation:

$$[G - \theta(x)]^{G/(G-H)}[\theta(x) - H]^{-H/(G-H)} = P(\eta)\exp\left[-\frac{2a}{\sigma^2}(x - \bar{x})\right], \tag{4.10}$$

where

$$G = \frac{2r\sigma^2 + \mu^2 + 4a\delta + \sqrt{(2r\sigma^2 + \mu^2 + 4a\delta)^2 - 16a\mu^2\delta}}{4a\mu} < \frac{2\delta}{\mu}, \tag{4.11}$$

$$H = \frac{2r\sigma^2 + \mu^2 + 4a\delta - \sqrt{(2r\sigma^2 + \mu^2 + 4a\delta)^2 - 16a\mu^2\delta}}{4a\mu} > \frac{\mu}{2a}\ , \tag{4.12}$$

and

$$P(\eta) := (G - \eta)^{G/(G-H)}(\eta - H)^{-H/(G-H)}\ . \tag{4.13}$$

This shows that there exist $0 \leq x_1 < x_2 \leq x^*$ such that

$$U^*(x) = \begin{cases} l, & 0 \leq x \leq x_1, \\ \theta(x), & x_1 < x < x_2, \\ u, & x_2 \leq x \leq x^*\ , \end{cases} \tag{4.14}$$

with $\theta(x_1) = l$ and $\theta(x_2) = u$.

In the following, we shall solve the HJB equation (3.16) in several intervals.

Step 1: $x \in [0, x_1]$. In this case, we have $U^*(x) = l$. Therefore, equation (4.1) becomes

$$\frac{1}{2}\sigma^2 l^2 v''(x) + (\mu l - al^2 - \delta)v'(x) - rv(x) = 0, \ v(0) = 0. \tag{4.15}$$

The solution of (4.15) is given by:

$$v(x) = A(e^{\alpha_1 x} - e^{\beta_1 x}), \ 0 \le x \le x_1, \tag{4.16}$$

where A is an undetermined constant, α_1 and β_1 are the solutions of the following quadratic equation:

$$\frac{1}{2}\sigma^2 l^2 y^2 + (\mu l - al^2 - \delta)y - r = 0, \tag{4.17}$$

that is,

$$\alpha_1 = \frac{-(\mu l - al^2 - \delta) + \sqrt{(\mu l - al^2 - \delta)^2 + 2r\sigma^2 l^2}}{\sigma^2 l^2}, \tag{4.18}$$

$$\beta_1 = \frac{-(\mu l - al^2 - \delta) - \sqrt{(\mu l - al^2 - \delta)^2 + 2r\sigma^2 l^2}}{\sigma^2 l^2}. \tag{4.19}$$

Step 2: $x \in (x_1, x_2)$. For $x \in (x_1, x_2)$, we have $U^*(x) = \theta(x)$ with $\theta(x_1) = l$ and $\theta(x_2) = u$, and

$$v(x) = D \exp\left(-\int_x^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y) - \delta} dy\right), \tag{4.20}$$

where D is an undetermined constant.

Step 3: $x \in [x_2, x^*)$. In this case, we have $U^*(x) = u$. Therefore, equation (4.1) can be rewritten as:

$$\frac{1}{2}\sigma^2 u^2 v''(x) + (\mu u - au^2 - \delta)v'(x) - rv(x) = 0, \ v(0) = 0. \tag{4.21}$$

The solution of (4.21) is given by:

$$v(x) = Be^{\alpha_2 x} + Ce^{\beta_2 x}, \ x_2 \le x < x^*, \tag{4.22}$$

where B and C are two undetermined constants, α_2 and β_2 are the solutions of the following quadratic equation:

$$\frac{1}{2}\sigma^2 u^2 y^2 + (\mu u - au^2 - \delta)y - r = 0, \tag{4.23}$$

that is,

$$\alpha_2 = \frac{-(\mu u - au^2 - \delta) + \sqrt{(\mu u - au^2 - \delta)^2 + 2r\sigma^2 u^2}}{\sigma^2 u^2}, \tag{4.24}$$

$$\beta_2 = \frac{-(\mu u - au^2 - \delta) - \sqrt{(\mu u - al^2 - \delta)^2 + 2r\sigma^2 u^2}}{\sigma^2 u^2}. \tag{4.25}$$

Step 4: $x \in [x^*, \infty)$. In this case, since $\upsilon'(x) = k$, we have

$$\upsilon(x) = \upsilon(x^*) + k(x - x^*). \tag{4.26}$$

Step 5: Solve x_1, x_2, x^*, A, B, C and D.

Firstly, applying the principle of smooth fit to $\upsilon(x)$ at $x = x_1$ gives:

$$\upsilon(x_1+) = \upsilon(x_1-), \upsilon'(x_1+) = \upsilon'(x_1-),$$

which results in

$$A(\alpha_1 e^{\alpha_1 x_1} - \beta_1 e^{\beta_1 x_1}) = \frac{rA(e^{\alpha_1 x_1} - e^{\beta_1 x_1})}{\frac{1}{2}\mu l - \delta}. \tag{4.27}$$

From (4.27), we conclude

$$x_1 = \frac{1}{\alpha_1 - \beta_1} \log\left[\frac{r - \beta_1(\frac{1}{2}\mu l - \delta)}{r - \alpha_1(\frac{1}{2}\mu l - \delta)}\right], \tag{4.28}$$

and

$$D = A(e^{\alpha_1 x_1} - e^{\beta_1 x_1}) \int_{x_1}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y) - \delta} dy. \tag{4.29}$$

Define

$$x_2 := \inf\{x > x_1 | \theta(x) = u\}. \tag{4.30}$$

By setting $\bar{x} = x_1$, $\eta = l$ and $x = x_2$ in (4.10) and then solving for x_2, we have

$$x_2 = x_1 + \frac{\sigma^2}{2a}\left[\frac{G}{G-H}\log\left(\frac{G-l}{G-u}\right) - \frac{H}{G-H}\log\left(\frac{l-H}{u-H}\right)\right]. \tag{4.31}$$

Second, smooth fit at $x = x_2$ yields:

$$\begin{aligned} Be^{\alpha_2 x_2} + Ce^{\beta_2 x_2} &= D, \\ B\alpha_2 e^{\alpha_2 x_2} + C\beta_2 e^{\beta_2 x_2} &= \frac{Dr}{\frac{1}{2}\mu u - \delta}. \end{aligned}$$

Solving the above equations leads to:

$$B = \frac{D\left(\beta_2 - \frac{r}{\frac{1}{2}\mu u - \delta}\right)}{(\beta_2 - \alpha_2)e^{\alpha_2 x_2}}, \tag{4.32}$$

$$C = \frac{D\left(\alpha_2 - \frac{r}{\frac{1}{2}\mu u - \delta}\right)}{(\alpha_2 - \beta_2)e^{\beta_2 x_2}}. \tag{4.33}$$

Consequently, the solution has the following form:

$$\upsilon(x) = \begin{cases} A(e^{\alpha_1 x} - e^{\beta_1 x}), & 0 \le x < x_1, \\ A(e^{\alpha_1 x_1} - e^{\beta_1 x_1})\exp\left(\int_{x_1}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y) - \delta}dy\right)\exp\left(-\int_{x}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y) - \delta}dy\right), & x_1 < x < x_2, \\ A(e^{\alpha_1 x_1} - e^{\beta_1 x_1})\exp\left(\int_{x_1}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y) - \delta}dy\right)\left\{\left[\frac{\left(\beta_2 - \frac{r}{\frac{1}{2}\mu u - \delta}\right)}{(\beta_2 - \alpha_2)e^{\alpha_2 x_2}}\right]e^{\alpha_2 x}\right. & \\ \left. + \left[\frac{\left(\alpha_2 - \frac{r}{\frac{1}{2}\mu u - \delta}\right)}{(\alpha_2 - \beta_2)e^{\beta_2 x_2}}\right]e^{\beta_2 x}\right\}, & x_2 < x < x^*, \\ \upsilon(x^*) + k(x - x^*), & x \ge x^*. \end{cases}$$

In what follows, we shall determine A, $\tilde{x}$ and x^* by the following conditions:

$$\upsilon'(x^*) = \upsilon'(\tilde{x}) = k, \upsilon(x^*) = \upsilon(\tilde{x}) + k(x^* - \tilde{x}) - K,$$

or

$$\upsilon'(x^*) = k,\ \upsilon(x^*) = kx^* - K.$$

We start by constructing the following function $H(x)$, for $x \geq 0$:

$$H(x) = \begin{cases} \alpha_1 e^{\alpha_1 x} - \beta_1 e^{\beta_1 x}, & 0 \leq x < x_1, \\ (e^{\alpha_1 x_1} - e^{\beta_1 x_1}) \exp\left(\int_{x_1}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta} dy\right) \exp\left(-\int_{x}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta} dy\right) & \\ \times \frac{r}{\frac{1}{2}\mu\theta(x)-\delta}, & x_1 \leq x < x_2, \\ (e^{\alpha_1 x_1} - e^{\beta_1 x_1}) \exp\left(\int_{x_1}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta} dy\right) \left\{ \alpha_2 \left[\frac{\left(\beta_2 - \frac{r}{\frac{1}{2}\mu u - \delta}\right)}{(\beta_2-\alpha_2)e^{\alpha_2 x_2}} \right] e^{\alpha_2 x} \right. & \\ \left. +\beta_2 \left[\frac{\left(\alpha_2 - \frac{r}{\frac{1}{2}\mu u - \delta}\right)}{(\alpha_2-\beta_2)e^{\beta_2 x_2}} \right] e^{\beta_2 x} \right\}, & x \geq x_2 . \end{cases} \tag{4.34}$$

Lemma 7. *The function $H(x)$ defined by (4.34) is a convex and decreasing function on $(0, x_1)$, decreasing function on (x_1, x_2) and convex function on (x_2, ∞).*

Proof. For $0 \leq x \leq x_1$, we can easily see $H''(x) = \alpha_1^3 e^{\alpha_1 x} - \beta_1^3 e^{\beta_1 x} > 0$ with $\alpha_1 > 0$ and $\beta_1 < 0$ and

$$H'(x) = \alpha_1^2 e^{\alpha_1 x} - \beta_1^2 e^{\beta_1 x} < 0, \ x \in \left(0, \frac{1}{\alpha_1 - \beta_1} \log\left(\frac{\beta_1}{\alpha_1}\right)^2\right). \tag{4.35}$$

where

$$\frac{1}{\alpha_1 - \beta_1} \log\left(\frac{\beta_1}{\alpha_1}\right)^2 > x_1, \tag{4.36}$$

which shows that on $(0, x_1)$ the function $H(x)$ is a convex and decreasing function. For $x \in [x_1, x_2]$, we have

$$\begin{aligned} H'(x) &= -(e^{\alpha_1 x_1} - e^{\beta_1 x_1}) \exp\left(\int_{x_1}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y) - \delta} dy\right) \exp\left(-\int_{x}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y) - \delta} dy\right) \\ &\quad \times \frac{2r(\mu - 2a\theta(x))}{\sigma^2\theta(x)(\mu\theta(x) - 2\delta)} < 0, \end{aligned}$$

which shows that $H(x)$ is a decreasing function on $x \in (x_1, x_2)$.

Since α_2 satisfies

$$\frac{1}{2}\sigma^2 u^2 \alpha_2^2 + (\mu u - a u^2 - \delta)\alpha_2 - r = 0, \tag{4.37}$$

$$(\frac{1}{2}\mu u - \delta)\alpha_2 - r = -\frac{1}{2}\sigma^2 u^2 \alpha_2^2 - \left(\frac{1}{2}\mu - a u\right) u < 0. \tag{4.38}$$

Thus,

$$\alpha_2 - \frac{r}{\frac{1}{2}\mu u - \delta} = \frac{(\frac{1}{2}\mu u - \delta)\alpha_2 - r}{\frac{1}{2}\mu u - \delta} < 0. \tag{4.39}$$

Therefore, on (x_2, ∞), we have

$$\begin{aligned} H''(x) &= \alpha_2^3(e^{\alpha_1 x_1} - e^{\beta_1 x_1}) \exp\left(\int_{x_1}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y) - \delta} dy\right) \left\{ \left[\frac{\left(\beta_2 - \frac{r}{\frac{1}{2}\mu u - \delta}\right)}{(\beta_2 - \alpha_2)e^{\alpha_2 x_2}} \right] e^{\alpha_2 x} \right. \\ &\quad \left. + \beta_2^3 \left[\frac{\left(\alpha_2 - \frac{r}{\frac{1}{2}\mu u - \delta}\right)}{(\alpha_2 - \beta_2)e^{\beta_2 x_2}} \right] e^{\beta_2 x} \right\} \\ &\geq 0, \end{aligned}$$

which shows that $H(x)$ is convex function on (x_2, ∞). $\square$

From the above analysis, H is a continuously differentiable function. It is easy to see that

$$\lim_{x \to 0} H(x) = \alpha_1 - \beta_1 > 0, \lim_{x \to \infty} H(x) = \infty \ .$$

Consequently there exists a point $\dot{x} > x_2$ such that

$$H'(\dot{x}) = 0 \ .$$

Let $\alpha = H(\dot{x}) > 0$. If $0 < A < k/\alpha$, then there exist two points $\check{x}^A < \dot{x} < \hat{x}^A$ such that $AH(\check{x}^A) = AH(\hat{x}^A) = k$. Obviously, if $A = k/\alpha$, then $\check{x}^A = \dot{x} = \hat{x}^A$. It is easy to see that $\check{x}^A$ is an increasing function of A, while $\hat{x}^A$ is a decreasing function of A, for $A \in (0, k/\alpha]$.

Define

$$I(A) := \int_{\check{x}^A \vee 0}^{\hat{x}^A} (k - AH(y))dy \ .$$

We can easily show that $I(A) \to \infty$ as $A \to 0$ and $I(A)$ is a decreasing function of A. Let $\bar{A}$ be a constant satisfying:

$$\bar{A}H(0) = k, \tag{4.40}$$

that is,

$$\bar{A} = \frac{k}{\alpha_1 - \beta_1}. \tag{4.41}$$

Since $I(k/\alpha) = 0$,

$$0 \leq I(A) < \infty. \tag{4.42}$$

From the above analysis, we know that if $K \leq I(\bar{A})$, there exists an A^* such that

$$I(A^*) := \int_{\check{x}^{A^*}}^{\hat{x}^{A^*}} (k - A^* H(y)) dy = K, \tag{4.43}$$

with $\hat{x}^{A^*} > \check{x}^{A^*} \geq 0$.

If $K > I(\bar{A})$, there exists an A^* such that

$$I(A^*) := \int_{0}^{\hat{x}^{A^*}} (k - A^* H(y)) dy = K, \tag{4.44}$$

with $\hat{x}^{A^*} > 0$.

Define the function $\hat{T}(x)$ by

$$\hat{T}(x) = \begin{cases} A^* H(x), & 0 \leq x \leq \hat{x}^{A^*}, \\ k, & x \geq \hat{x}^{A^*}, \end{cases} \tag{4.45}$$

we have

$$\upsilon(x) = \int_0^x \hat{T}(y) dy.$$

Thus, if $K \leq I(\bar{A})$, we have

$$\upsilon(x) = \begin{cases} A^*(e^{\alpha_1 x} - e^{\beta_1 x}), & 0 \leq x < x_1, \\ A^*(e^{\alpha_1 x_1} - e^{\beta_1 x_1}) \exp\left(\int_{x_1}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y) - \delta} dy\right) \exp\left(-\int_{x}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y) - \delta} dy\right), & x_1 < x < x_2, \\ A^*(e^{\alpha_1 x_1} - e^{\beta_1 x_1}) \exp\left(\int_{x_1}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y) - \delta} dy\right) \left\{ \left[\frac{\left(\beta_2 - \frac{r}{\frac{1}{2}\mu u - \delta}\right)}{(\beta_2 - \alpha_2) e^{\alpha_2 x_2}} \right] e^{\alpha_2 x} \right. & \\ \left. + \left[\frac{\left(\alpha_2 - \frac{r}{\frac{1}{2}\mu u - \delta}\right)}{(\alpha_2 - \beta_2) e^{\beta_2 x_2}} \right] e^{\beta_2 x} \right\}, & x_2 < x < \hat{x}^{A^*}, \\ \upsilon(\check{x}^{A^*}) + k(x - \check{x}^{A^*}) - K, & x \geq \hat{x}^{A^*}, \end{cases} \tag{4.46}$$

and if $K > I(\bar{A})$, we have

$$\upsilon(x) = \begin{cases} A^*(e^{\alpha_1 x} - e^{\beta_1 x}), & 0 \le x < x_1, \\ A^*(e^{\alpha_1 x_1} - e^{\beta_1 x_1}) \exp\left(\int_{x_1}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta} dy\right) \exp\left(-\int_{x}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta} dy\right), & x_1 < x < x_2, \\ A^*(e^{\alpha_1 x_1} - e^{\beta_1 x_1}) \exp\left(\int_{x_1}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta} dy\right) \left\{ \left[\frac{\left(\beta_2 - \frac{r}{\frac{1}{2}\mu u-\delta}\right)}{(\beta_2-\alpha_2)e^{\alpha_2 x_2}}\right] e^{\alpha_2 x} \right. & \\ \left. + \left[\frac{\left(\alpha_2 - \frac{r}{\frac{1}{2}\mu u-\delta}\right)}{(\alpha_2-\beta_2)e^{\beta_2 x_2}}\right] e^{\beta_2 x} \right\}, & x_2 < x < \hat{x}^{A^*}, \\ kx - K, & x \ge \hat{x}^{A^*}. \end{cases} \tag{4.47}$$

Setting $x^* = \hat{x}^{A^*}$ and $\tilde{x} = \check{x}^{A^*}$, we have the following theorem which gives the value function and the optimal policy of the impulse control problem.

Theorem 2. *Let*

$$U^*(x) = \begin{cases} l, & 0 \le x \le x_1, \\ \theta(x), & x_1 < x < x_2, \\ u, & x_2 \le x \le \hat{x}^{A^*}. \end{cases}$$

If $K \le I(\bar{A})$, then the value function $V(x)$ is given by $V(x) = \upsilon(x)$, defined by (4.46). The optimal policy $\pi^* := (U^*, Q^*)$ satisfies

$$\begin{cases} X_t^{\pi^*} = x + \int_0^t (\mu U^*(X_s^{\pi^*}) - aU^{*2}(X_s^{\pi^*}) - \delta) ds + \int_0^t \sigma U^*(X_s^{\pi^*}) dW_s - \sum_{n=1}^{\infty} I_{\{\tau_n^* < t\}} \xi_n^*, \\ 0 \le X_t^{\pi^*} \le \hat{x}^{A^*}. \end{cases}$$

where $\tau_0^{\pi^*} := 0$, $\tau_i^{\pi^*} := \inf\{t > \tau_{i-1}^{\pi^*} : X_{t-}^{\pi^*} = \hat{x}^{A^*}\}$, and $\xi_i^{\pi^*} := \hat{x}^{A^*} - \check{x}^{A^*}$, $i = 1, 2, \cdots$.

If $K > I(\bar{A})$, then the value function $V(x)$ is given by $V(x) = \upsilon(x)$, defined by (4.47). The optimal policy $\pi^* := (U^*, Q^*)$ satisfies

$$\begin{cases} X_t^{\pi^*} = x + \int_0^t (\mu U^*(X_s^{\pi^*}) - aU^{*2}(X_s^{\pi^*}) - \delta) ds + \int_0^t \sigma U^*(X_s^{\pi^*}) dW_s - \sum_{n=1}^{\infty} I_{\{\tau_n^* < t\}} \xi_n^*, \\ 0 \le X_t^{\pi^*} \le \hat{x}^{A^*}. \end{cases}$$

where $\tau_0^{\pi^*} := 0$, $\tau_i^{\pi^*} := \inf\{t > \tau_{i-1}^{\pi^*} : X_{t-}^{\pi^*} = \hat{x}^{A^*}\}$, and $\xi_i^{\pi^*} := \hat{x}^{A^*}$, $i = 1, 2, \cdots$.

Proof: Under the condition $K \le I(\bar{A})$ or $K > I(\bar{A})$ respectivly, we can easily show $\upsilon(x)$ is the solution of HJB equation, satisfying twice continuously differentiable on $(0, \hat{x}^{A^*})$ and linear on $[\hat{x}^{A^*}, \infty)$. Similar to Theorem 3.4 of Cadenillas et al [4], we have $V(x) \le \upsilon(x)$. From the above discussion, we know that the control π^* is a admissible control strategy with $\upsilon(x)$. Thus $V(x) = \upsilon(x)$ with optimal control strategy π^*. $\square$

Remark 2:

(I) In case when $K \leq I(\bar{A})$: if $X_{t-} < \hat{x}^{A^*}$, no dividends happen, and when $X_{t-} = \hat{x}^{A^*}$, there is an impulse dividend $X_{t-} - X_t = \hat{x}^{A^*} - \check{x}^{A^*}$. This dividend strategy is called a continuation strategy;

(II) In case when $K > I(\bar{A})$: if $X_{t-} < \hat{x}^{A^*}$, no dividends happen, and when $X_{t-} = \hat{x}^{A^*}$, there is an impulse dividend $X_{t-} - X_t = \hat{x}^{A^*}$, which leads to bankruptcy. This dividend strategy is called a ruin strategy.

4.1.2 $l \leq 2\delta/\mu \leq u < \mu/2a$

In this case, by (4.6), we get

$$l \leq \frac{2\delta}{\mu} = \theta(0) \leq u. \tag{4.48}$$

Thus we have

$$U^*(x) = \begin{cases} \theta(x), & 0 \leq x \leq x_2, \\ u, & x_2 \leq x \leq \hat{x}^{A^*} \end{cases}. \tag{4.49}$$

By Lemma 6, we know that $\theta(x)$ can be determined by the following equation:

$$[G - \theta(x)]^{G/(G-H)}[\theta(x) - H]^{-H/(G-H)} = P\left(\frac{2\delta}{\mu}\right)\exp\left[-\frac{2a}{\sigma^2}x\right], \ 0 \leq x < x_2 \tag{4.50}$$

with $\theta(0) = \frac{2\delta}{\mu}$.

Accordingly, x_2 becomes:

$$x_2 = \frac{\sigma^2}{2a}\left[\frac{G}{G-H}\log\left(\frac{G - 2\delta/\mu}{G-u}\right) - \frac{H}{G-H}\log\left(\frac{2\delta/\mu - H}{u-H}\right)\right]. \tag{4.51}$$

Thus, if $K \leq I(\bar{A})$, we have

$$\upsilon(x) = \begin{cases} A^* \exp\left(-\int_x^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y) - \delta} dy\right), & 0 < x < x_2, \\ A^* \left[\frac{\left(\beta_2 - \frac{r}{\frac{1}{2}\mu u - \delta}\right)}{(\beta_2 - \alpha_2)e^{\alpha_2 x_2}}\right] e^{\alpha_2 x} + A^* \left[\frac{\left(\alpha_2 - \frac{r}{\frac{1}{2}\mu u - \delta}\right)}{(\alpha_2 - \beta_2)e^{\beta_2 x_2}}\right] e^{\beta_2 x}, & x_2 < x < \hat{x}^{A^*}, \\ \upsilon(\check{x}^{A^*}) + k(x - \check{x}^{A^*}) - K, & x \geq \hat{x}^{A^*}, \end{cases} \tag{4.52}$$

and if $K > I(\bar{A})$, we have

$$\upsilon(x) = \begin{cases} A^* \exp\left(-\int_x^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta} dy\right), & 0 < x < x_2, \\ A^* \left[\frac{\left(\beta_2 - \frac{r}{\frac{1}{2}\mu u - \delta}\right)}{(\beta_2-\alpha_2)e^{\alpha_2 x_2}}\right] e^{\alpha_2 x} + A^* \left[\frac{\left(\alpha_2 - \frac{r}{\frac{1}{2}\mu u - \delta}\right)}{(\alpha_2-\beta_2)e^{\beta_2 x_2}}\right] e^{\beta_2 x}, & x_2 < x < \hat{x}^{A^*}, \\ kx - K, & x \geq \hat{x}^{A^*}, \end{cases} \tag{4.53}$$

In this case, the optimal regular control is given by (4.49) and the optimal impulse control Q^* is the corresponding regulatory process at $(\check{x}^{A^*}, \hat{x}^{A^*})$ or $(0, \hat{x}^{A^*})$ and $U^*(x) \in [2\delta/\mu, u]$.

4.1.3 $2\delta/\mu < l \leq \mu/2a \leq u$

In this case, it follows from (4.5) that

$$l \leq \mu/2a = \theta(x_2) \leq u. \tag{4.54}$$

Therefore, we have

$$U^*(x) = \begin{cases} l, & 0 \leq x \leq x_1, \\ \theta(x), & x_1 < x < x_2, \\ \mu/2a, & x_2 \leq x \leq \hat{x}^{A^*}. \end{cases} \tag{4.55}$$

The point x_1 is determined by exactly the same formula (4.28), and

$$x_2 = x_1 + \frac{\sigma^2}{2a}\left[\frac{G}{G-H}\log\left(\frac{G-l}{G-\mu/2a}\right) - \frac{H}{G-H}\log\left(\frac{l-H}{\mu/2a-H}\right)\right]. \tag{4.56}$$

Thus if $K \leq I(\bar{A})$,we have

$$\upsilon(x) = \begin{cases} A^*(e^{\alpha_1 x} - e^{\beta_1 x}), & 0 \leq x < x_1, \\ A^*(e^{\alpha_1 x_1} - e^{\beta_1 x_1}) \exp\left(\int_{x_1}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta} dy\right) \exp\left(-\int_x^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta} dy\right), & x_1 < x < x_2, \\ A^*(e^{\alpha_1 x_1} - e^{\beta_1 x_1}) \exp\left(\int_{x_1}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta} dy\right) \left\{\left[\frac{\left(\beta_3 - \frac{r}{\frac{1}{4a}\mu^2-\delta}\right)}{(\beta_3-\alpha_3)e^{\alpha_3 x_2}}\right] e^{\alpha_3 x} \right. & \\ \left. + \left[\frac{\left(\alpha_3 - \frac{r}{\frac{1}{4a}\mu^2-\delta}\right)}{(\alpha_3-\beta_3)e^{\beta_3 x_2}}\right] e^{\beta_3 x}\right\}, & x_2 < x < \hat{x}^{A^*}, \\ \upsilon(\check{x}^{A^*}) + k(x - \check{x}^{A^*}) - K, & x \geq \hat{x}^{A^*}, \end{cases} \tag{4.57}$$

and if $K > I(\bar{A})$, we have

$$\upsilon(x) = \begin{cases} A^*(e^{\alpha_1 x} - e^{\beta_1 x}), & 0 \le x < x_1, \\ A^*(e^{\alpha_1 x_1} - e^{\beta_1 x_1}) \exp\left(\int_{x_1}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta} dy\right) \exp\left(-\int_{x}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta} dy\right), & x_1 < x < x_2, \\ A^*(e^{\alpha_1 x_1} - e^{\beta_1 x_1}) \exp\left(\int_{x_1}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta} dy\right) \left\{ \left[\frac{\left(\beta_3 - \frac{r}{\frac{1}{4a}\mu^2-\delta}\right)}{(\beta_3-\alpha_3)e^{\alpha_3 x_2}}\right] e^{\alpha_3 x} \right. & \\ \left. + \left[\frac{\left(\alpha_3 - \frac{r}{\frac{1}{4a}\mu^2-\delta}\right)}{(\alpha_3-\beta_3)e^{\beta_3 x_2}}\right] e^{\beta_3 x} \right\}, & x_2 < x < \hat{x}^{A^*}, \\ kx - K, & x \ge \hat{x}^{A^*}. \end{cases} \tag{4.58}$$

Here α_3 and β_3 are the positive root and the negative root of the equation:

$$\frac{1}{2}\sigma^2(\mu/2a)^2 y^2 + (\mu^2/2a - a(\mu/2a)^2 - \delta)y - r = 0, \tag{4.59}$$

In this case, the optimal regular control is given by (4.55) and the optimal impulse control Q^* is the corresponding regulatory process at $(\check{x}^{A^*}, \hat{x}^{A^*})$ or $(0, \hat{x}^{A^*})$ and $U^*(x) \in [l, \mu/(2a)]$.

4.1.4 $l < 2\delta/\mu < \mu/2a < u$

In this case,

$$U^*(x) = \begin{cases} \theta(x), & 0 \le x \le x_2, \\ \mu/2a, & x_2 \le x \le \hat{x}^{A^*}. \end{cases} \tag{4.60}$$

Therefore, similar arguments lead to:

if $K \le I(\bar{A})$, we have

$$\upsilon(x) = \begin{cases} A^* \exp\left(-\int_{x}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta} dy\right), & 0 < x < x_2, \\ A^* \left[\frac{\left(\beta_3 - \frac{r}{\frac{1}{4a}\mu^2-\delta}\right)}{(\beta_3-\alpha_3)e^{\alpha_3 x_2}}\right] e^{\alpha_3 x} + A^* \left[\frac{\left(\alpha_3 - \frac{r}{\frac{1}{4a}\mu^2-\delta}\right)}{(\alpha_3-\beta_3)e^{\beta_3 x_2}}\right] e^{\beta_3 x}, & x_2 < x < \hat{x}^{A^*}, \\ \upsilon(\check{x}^{A^*}) + k(x - \check{x}^{A^*}) - K, & x \ge \hat{x}^{A^*}, \end{cases} \tag{4.61}$$

and if $K > I(\bar{A})$, we have

$$\upsilon(x) = \begin{cases} A^* \exp\left(-\int_{x}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta} dy\right), & 0 < x < x_2, \\ A^* \left[\frac{\left(\beta_3 - \frac{r}{\frac{1}{4a}\mu^2-\delta}\right)}{(\beta_3-\alpha_3)e^{\alpha_3 x_2}}\right] e^{\alpha_3 x} + A^* \left[\frac{\left(\alpha_3 - \frac{r}{\frac{1}{4a}\mu^2-\delta}\right)}{(\alpha_3-\beta_3)e^{\beta_3 x_2}}\right] e^{\beta_3 x}, & x_2 < x < \hat{x}^{A^*}, \\ kx - K, & x \ge \hat{x}^{A^*}, \end{cases} \tag{4.62}$$

where

$$x_2 = \frac{\sigma^2}{2a}\left[\frac{G}{G-H}\log\left(\frac{G-2\delta/\mu}{G-\mu/(2a)}\right) - \frac{H}{G-H}\log\left(\frac{2\delta/\mu - H}{\mu/(2a)-H}\right)\right]. \tag{4.63}$$

In this case, the optimal regular control is given by (4.60) and the optimal impulse control Q^* is the corresponding regulatory process at $(\check{x}^{A^*}, \hat{x}^{A^*})$ or $(0, \hat{x}^{A^*})$ and $U^*(x) \in [2\delta/\mu, \mu/(2a)]$.

4.1.5 $l > \mu/2a, \mu l - al^2 - \delta > 0$

In this case, since $\theta(x^*) = \mu/2a < l$, we have

$$U^*(x) = l,\ 0 \le x \le \hat{x}^{A^*}. \tag{4.64}$$

Thus if $K \le I(\bar{A})$,we have

$$v(x) = \begin{cases} A^*(e^{\alpha_1 x} - e^{\beta_1 x}), & 0 \le x < \hat{x}^{A^*}, \\ v(\check{x}^{A^*}) + k(x - \check{x}^{A^*}) - K, & x \ge \hat{x}^{A^*}, \end{cases} \tag{4.65}$$

and if $K > I(\bar{A})$, we have

$$v(x) = \begin{cases} A^*(e^{\alpha_1 x} - e^{\beta_1 x}), & 0 \le x < \hat{x}^{A^*}, \\ kx - K, & x \ge \hat{x}^{A^*}. \end{cases} \tag{4.66}$$

In this case, the optimal regular control is given by (4.64) and the optimal impulse control Q^* is the corresponding regulatory process at $(\check{x}^{A^*}, \hat{x}^{A^*})$ or $(0, \hat{x}^{A^*})$ and $U^*(x) = l$.

4.1.6 $u < 2\delta/\mu, \mu u - au^2 - \delta > 0$

In this case, since $\theta(x^*) = 2\delta/\mu > u$, we have

$$U^*(x) = u,\ 0 \le x \le \hat{x}^{A^*}. \tag{4.67}$$

Thus if $K \le I(\bar{A})$,we have

$$v(x) = \begin{cases} A^*(e^{\alpha_2 x} - e^{\beta_2 x}), & 0 \le x < \hat{x}^{A^*}, \\ v(\check{x}^{A^*}) + k(x - \check{x}^{A^*}) - K, & x \ge \hat{x}^{A^*}, \end{cases} \tag{4.68}$$

and if $K > I(\bar{A})$, we have

$$\upsilon(x) = \begin{cases} A^*(e^{\alpha_2 x} - e^{\beta_2 x}), & 0 \le x < \hat{x}^{A^*}, \\ kx - K, & x \ge \hat{x}^{A^*}. \end{cases} \tag{4.69}$$

In this case, the optimal regular control is given by (4.67) and the optimal impulse control Q^* is the corresponding regulatory process at $(\check{x}^{A^*}, \hat{x}^{A^*})$ or $(0, \hat{x}^{A^*})$ and $U^*(x) = u$.

4.2 $\mu^2 \le 4a\delta$

In this case, we have

$$\max_{U\in[l,u]} \left\{ \frac{1}{2}\sigma^2 U^2 \upsilon''(x) + (\mu U - aU^2 - \delta)\upsilon'(x) - r\upsilon(x) \right\} = 0 \ , \quad x < \hat{x}^{A^*} \ .(4.70)$$

Noting that $\upsilon'(x) \ge 0$ and $\mu U - aU^2 - \delta \le 0$ for $U \in [l, u]$, we obtain $\upsilon''(x) \ge 0$ by equation (4.70). So,

$$U^*(x) = \arg \max_{U\in[l,u]} \left\{ \frac{1}{2}\sigma^2 U^2 \upsilon''(x) + (\mu U - aU^2 - \delta)\upsilon'(x) - r\upsilon(x) \right\} \equiv u.$$

In this case we note $K > I(\bar{A})$ forever, so

$$\upsilon(x) = \begin{cases} A^*(e^{\alpha_2 x} - e^{\beta_2 x}), & 0 \le x < \hat{x}^{A^*}, \\ kx - K, & x \ge \hat{x}^{A^*}. \end{cases} \tag{4.71}$$

In this case, the optimal regular control is given by (4.71) and the optimal impulse control Q^* is the corresponding regulatory process at $(\check{x}^{A^*}, \hat{x}^{A^*})$ or $(0, \hat{x}^{A^*})$ and $U^*(x) = u$.

4.3 A special case: $\delta = 0$

In the interesting case that $\delta = 0$, (i.e., no-debt), the value function is completely explicit. That is, we obtain an explicit expression for $\theta(x)$. In this subsection, we only present the result for $0 < l < u < \mu/2a$. For the other cases, we can give the corresponding results similarly.

In the current case, equation (4.9) becomes:

$$\begin{cases} \theta'(x) + \frac{2a}{\sigma^2}\theta(x) - \frac{\mu^2+2r\sigma^2}{\mu\sigma^2} = 0, \\ \theta(x_1) = l, \end{cases}$$

which has the solution

$$\theta(x) = Re^{-(2a/\sigma^2)x} + \frac{\mu^2 + 2r\sigma^2}{2a\mu}, \tag{4.72}$$

where

$$R = -e^{(2a/\sigma^2)x_1}\left(\frac{\mu^2 + 2r\sigma^2}{2a\mu} - l\right) < \frac{2a}{\mu}\ , \tag{4.73}$$

and

$$x_1 = \frac{1}{\alpha_1 - \beta_1}\log\left[\frac{r - \frac{1}{2}\beta_1\mu l}{r - \frac{1}{2}\alpha_1\mu l}\right]. \tag{4.74}$$

Setting $\theta(x_2) = u$, we obtain

$$x_2 = x_1 + \frac{\sigma^2}{2a}\log\left(\frac{\frac{\mu^2+2r\sigma^2}{2a\mu} - l}{\frac{\mu^2+2r\sigma^2}{2a\mu} - u}\right). \tag{4.75}$$

After some simple calculations, we get

$$\upsilon(x) = \upsilon(x_1)\left(\frac{e^{2a(x-x_1)/\sigma^2} - 1 + \frac{2a\mu l}{\mu^2+2r\sigma^2}}{\frac{2a\mu l}{\mu^2+2r\sigma^2}}\right)^{2r\sigma^2/(\mu^2+2r\sigma^2)}, \ x_1 \le x \le x_2. \tag{4.76}$$

Thus we have

Theorem 3. *If $\delta = 0$, and $0 < l < u \le \mu/2a$, then*

(1) The optimal regular control $U^*(x)$ is given as a feedback control

$$U^*(x) = \begin{cases} l, & 0 \le x \le x_1, \\ \frac{\mu^2+2r\sigma^2}{2a\mu} - \left(\frac{\mu^2+2r\sigma^2}{2a\mu} - l\right)e^{-(2a/\sigma^2)(x_1-x)}, & x_1 \le x \le x_2, \\ u, & x_2 \le x \le \hat{x}^{A^*}\ . \end{cases} \tag{4.77}$$

(2) If $K \le I(\bar{A})$, we have

$$
\upsilon(x) = \begin{cases} A^*(e^{\alpha_1 x} - e^{\beta_1 x}), & 0 \le x < x_1, \\ A^*(e^{\alpha_1 x_1} - e^{\beta_1 x_1}) \left(\frac{e^{2a(x-x_1)/\sigma^2} - 1 + \frac{2a\mu l}{\mu^2+2r\sigma^2}}{\frac{2a\mu l}{\mu^2+2r\sigma^2}} \right)^{2r\sigma^2/(\mu^2+2r\sigma^2)}, & x_1 < x < x_2, \\ A^*(e^{\alpha_1 x_1} - e^{\beta_1 x_1}) \left[L_1 e^{\alpha_2 x} + L_2 e^{\beta_2 x} \right], & x_2 < x < \hat{x}^{A^*}, \\ \upsilon(\check{x}^{A^*}) + k(x - \check{x}^{A^*}) - K, & x \ge \hat{x}^{A^*}, \end{cases} \tag{4.78}
$$

and if $K > I(\bar{A})$, we have

$$
\upsilon(x) = \begin{cases} A^*(e^{\alpha_1 x} - e^{\beta_1 x}), & 0 \le x < x_1, \\ A^*(e^{\alpha_1 x_1} - e^{\beta_1 x_1}) \left(\frac{e^{2a(x-x_1)/\sigma^2} - 1 + \frac{2a\mu l}{\mu^2+2r\sigma^2}}{\frac{2a\mu l}{\mu^2+2r\sigma^2}} \right)^{2r\sigma^2/(\mu^2+2r\sigma^2)}, & x_1 < x < x_2, \\ A^*(e^{\alpha_1 x_1} - e^{\beta_1 x_1}) \left[L_1 e^{\alpha_2 x} + L_2 e^{\beta_2 x} \right], & x_2 < x < \hat{x}^{A^*}, \\ kx - K, & x \ge \hat{x}^{A^*}, \end{cases} \tag{4.79}
$$

where

$$
\begin{aligned}
L_1 &= \frac{e^{-\alpha_2 x_2}}{2a\mu l(\beta_2 - \alpha_2)} \left(\frac{e^{2a(x_2-x_1)/\sigma^2} - 1 + \frac{2a\mu l}{\mu^2+2r\sigma^2}}{\frac{2a\mu l}{\mu^2+2r\sigma^2}} \right)^{-\mu^2/(\mu^2+2r\sigma^2)} \\
&\quad \times \left[\alpha_2(\mu^2 + 2r\sigma^2 - 4ar) e^{2a(x_2-x_1)/\sigma^2} + \alpha_2(2a\mu l - \mu^2 - 2r\sigma^2) \right], \\
L_2 &= \frac{e^{-\beta_2 x_2}}{2a\mu l(\alpha_2 - \beta_2)} \left(\frac{e^{2a(x_2-x_1)/\sigma^2} - 1 + \frac{2a\mu l}{\mu^2+2r\sigma^2}}{\frac{2a\mu l}{\mu^2+2r\sigma^2}} \right)^{-\mu^2/(\mu^2+2r\sigma^2)} \\
&\quad \times \left[\beta_2(\mu^2 + 2r\sigma^2 - 4ar) e^{2a(x_2-x_1)/\sigma^2} + \beta_2(2a\mu l - \mu^2 - 2r\sigma^2) \right].
\end{aligned}
$$

5 Conclusion

An optimal dividend problem, where the insurance risk dynamics are nonlinear due to internal competition effect, was discussed. We considered the situation where both fixed and proportional transactions costs were present and discussed the problem of maximizing the expected present value of dividend payments by selecting a sequence of optimal dividend payment times and a sequence of optimal dividend payment amounts. The problem was formulated as a nonlinear, regular-impulse control and the HJB dynamic programming approach was used to discuss the problem. Explicit characterizations of the solutions of the problem were given by considering different cases. In the case when the debt is absent, we obtain an explicit solution to the optimal dividend problem.

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