Generalized Gauge for Multi-scale Inhomogeneous Media

W. C. Chew[1], **Q. I. Dai**[1], **S. Sun**[2], **A. Y. Liu**[1], **C. J. Ryu**[1], **S. Chen**[1], **Y. L. Li**[2], **and W. E. I. Sha**[2]

[1]University of Illinois at Urbana-Champaign, USA
[2]The University of Hong Kong, Hong Kong SAR, China

Abstract— The vector potential $\mathbf{A}$ has no direct physical meaning in classical electromagnetics. However, it manifests itself in quantum physics in terms of the Aharonov-Bohm effect. The vector potential $\mathbf{A}$ is similar to momentum. By itself, it is hard to detect classically, but its time variation generates a force in terms of electric field. Hence, the $\mathbf{E}$ field is of the form

$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla\Phi \tag{1}$$

where the electric field, which exerts a force on a charge, is generated by a time varying $\mathbf{A}$ and the gradient of the scalar potential Φ. The magnetic flux is given by $\mathbf{B} = \nabla \times \mathbf{A}$

By using Lorentz gauge

$$\nabla \cdot \mathbf{A} = -\mu\varepsilon\partial_t\Phi \tag{2}$$

Maxwell's equations in vacuum reduce to

$$\nabla^2\Phi - \mu\varepsilon\partial_t^2\Phi = -\rho/\varepsilon, \tag{3}$$

$$\nabla^2\mathbf{A} - \mu\varepsilon\partial_t^2\mathbf{A} = -\mu\mathbf{J} \tag{4}$$

For inhomogeneous medium, we pick the generalized gauge

$$\varepsilon^{-1}\nabla \cdot \varepsilon\mathbf{A} = -\mu\varepsilon\partial_t\Phi. \tag{5}$$

Then it can be shown that Maxwell's equations reduce to

$$\varepsilon^{-1}\nabla \cdot \varepsilon\nabla\Phi - \mu\varepsilon\partial_t^2\Phi = -\rho/\varepsilon, \tag{6}$$

$$-\mu\nabla \times \mu^{-1}\nabla \times \mathbf{A} - \mu\varepsilon\partial_t^2\mathbf{A} + \mu\varepsilon\nabla\frac{1}{\mu\varepsilon}\varepsilon^{-1}\nabla \cdot \varepsilon\mathbf{A} = -\mu\mathbf{J}. \tag{7}$$

For homogeneous medium, (6) and (7) reduce to (3) and (4).

The above equations have no low-frequency breakdown when solved numerically irrespective of how small the meshes are. Moreover, since $\mathbf{A}$ and Φ are needed in writing the Hamiltonian of an atom-field system, it is particularly suited for solving Maxwell-Schrodinger system of equations.

The discretization of the above equations can be inspired by differential forms from differential geometry. The vector potential $\mathbf{A}$ can be regarded as a one form which is curl-conforming. But the permittivity function can be regarded as a Hodge operator that converts a one form to a two form. Hence, $\varepsilon\mathbf{A}$ becomes a two form which has to be divergence conforming. The Hodge operator can also be implemented numerically.