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An Algorithm for Higher Order Hopf Normal Forms

Normal form theory is important for studying the qualitative behavior of nonlinear oscillators. In some cases, higher order normal forms are required to understand the dynamic behavior near an equilibrium or a periodic orbit. However, the computation of high-order normal forms is usually quite complicated. This article provides an explicit formula for the normalization of nonlinear differential equations. The higher order normal form is given explicitly. Illustrative examples include a cubic system, a quadratic system and a Duffing–Van der Pol system. We use exact arithmetic and find that the undamped Duffing equation can be represented by an exact polynomial differential amplitude equation in a finite number of terms. © *1995 John Wily & Sons, Inc.*

INTRODUCTION

The invariant manifold of a nonlinear oscillator near an equilibrium or a periodic orbit is determined by the structure of its vector field. Two often-used mathematical tools to simplify the original system are center manifold and normal forms. The normal form theory is a technique of transforming the original nonlinear differential equation to a simpler standard form by appropriate changes of coordinates, so that the essential features of the manifold become more evident. Basic references on normal forms and their applications may be found in Poincaré (1889), Birkhoff (1927), Arnold (1983), Chow and Hole (1982), Guckenheimer and Holmes (1983), Iooss and Joseph (1980), Sethna and Sell (1978), Van der Beek (1989), Vakakis and Rand (1992), and Leung and Zhang (1994). In this article a computational approach based on the classical normal form theory of Poincare and Birkhoff is introduced. The relationship of the coefficients between the original equations and the normal form equations is explicitly constructed. The technique presented in our approach follows the idea of Takens (1973). A linear operator and its adjacent operator are defined with an inner product on the space of homogeneous polynomials. The resultant normal form keeps only the resonant terms, which cannot be eliminated by a nonlinear polynomial changes of variables, in the kernel of the adjacent linear operator. The normal form simplifies the original systems so that the dynamic stability and bifurcation can be studied in a standard manner and the classification of manifold in the neighborhood of an equilibrium or a periodic orbit can be achieved with relatively little efforts. This article is a further development of previous work (Leung and Zhang, 1994). The higher order normal forms of several typical nonlinear oscillators: *cubic system, quadratic system, and Duffing–Van der Pol system* are provided and the steady-state solutions of the method are compared with existing results.

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TRANSFORMATION TO NORMAL FORM

Consider the nonlinear ordinary differential equations

$$\dot{u} = f(u), \quad f \in \mathbf{C}^r(\mathbf{R}^n) \tag{1}$$

where f is an n-vector function of u, differentiable up to order r. Suppose Eq. (1) has a fixed point at $u = u_0$. We first perform a few linear transformations to simplify eq. (1). By the variable change $\nu = u - u_0$, we eliminate the constant terms and shift the fixed point to the origin under which Eq. (1) becomes

$$\dot{\nu} = f(\nu + u_0) \equiv H(\upsilon), \tag{2}$$

where $H(\nu)$ is at least linear in ν. We next split the linear part of the ordinary differential equation and write (2) as follows

$$\dot{\nu} = D_\nu H(0)\nu + \overline{H}(\nu), \tag{3}$$

where $\overline{H}(\nu) \equiv H(\nu) - D_\nu H(0)\nu$ and, $\overline{H}(\nu) = O(|\nu|^2)$ is at least quadratic in ν and D_ν denotes differentiation with respect to ν. We further transform $D_\nu H(0)$ into Jordan canonical form by the canonical matrix T, i.e., $\nu = Tx$, and obtain

$$\dot{x} = T^{-1}D_\nu H(0)Tx + T^{-1}\overline{H}(Tx). \tag{4}$$

which can be written alternatively as

$$\dot{x} = Jx + F(x). \tag{5}$$

where J is the Jordan Canonical form of $D_\nu H(0)$ and $F(x)$ is the nonlinear part of the equation. In Poincaré's normal form theory, the nonlinear function $F(x)$ is expanded by a series of homogeneous polynomials,

$$\begin{aligned}\dot{x} = Jx + F_2(x) + F_3(x) + \cdots + F_r(x) \\ + O(|x|^{r+1})\end{aligned} \tag{6}$$

where $F_k \in H_k$, which is the set of homogeneous polynomials or order k. To transform Eq. (6) into its normal form, Poincaré introduces a nearly identity nonlinear coordinate transformation of the form

$$x = y + h_k(y), \tag{7}$$

where $h_k(y)$ is kth order in y, $h_k(y) \in H_k$, $2 \leq k \leq r$.

Substituting Eq. (7) into Eq. (6) gives

$$\begin{aligned}\dot{x} = (I + D_y h_k(y))\dot{y} = J(y + h_k(y)) \\ + \cdots + F_k(y + h_k(y)) + O(|y|^{k+1}),\end{aligned} \tag{8}$$

where I denotes the $n \times n$ identity matrix and the term $(I + D_y h_k(y))$ is invertible for sufficient small y so that

$$(I + D_y h_k(y))^{-1} = I - D_y h_k(y) + O(|y|^2). \tag{9}$$

Substituting Eq. (9) into Eq. (8) and applying similar nearly identity nonlinear transformations up to the kth order, we obtain

$$\begin{aligned}\dot{y} &= Jy + F_2(y) + \cdots + F_{k-1}(y) + \overline{F}_k(y) \\ &\quad + (Jh_k(y) - D_y h_k(y)Jy) + O(|y|^{k+1}) \\ &= Jy + F_2(y) + \cdots + F_{k-1}(y) + \overline{F}_k(y) \\ &\quad + \mathrm{ad}_J h_k(y) + O(|y|^{k+1})\end{aligned} \tag{10}$$

where the overbar is used to represent the original polynomial matrices and $\mathrm{ad}_J h_k(y)$ denotes an adjacent operator equivalent to the function of the *Lie Bracket*,

$$\mathrm{ad}_J h_k(y) \equiv L_J(H_k) = Jh_k(y) - D_y h_k(y)Jy. \tag{11}$$

To simplify the terms of order k as much as possible, we choose a specific form for $h_k(y)$ such that

$$\overline{F}_k(y) = L_J(H_k). \tag{12}$$

If $\overline{F}_k(y)$ is the range of $L_J(H_k)$, then all terms of order k can be eliminated completely from *Eq.* (10). Otherwise we must find a complementary space G_k to $L_J(H_k)$ and let $H_k = L_J(H_k) \oplus G_k$ so that only terms of order k which are in G_k remain in the resultant expression. It is interesting to note that simplifying terms at order k would not affect the coefficients of any lower terms. However, terms of order higher than k will be changed. Therefore, it is only necessary to keep track of the way that the higher order terms are modified by the successive coordinate transformations. This will be discussed in the next section.

NORMAL FORMS OF OSCILLATING SYSTEMS

We now specialize the normal form formulation in two-dimensions for the Hopf bifurcation below. Consider a system with a small perturbed parameter μ and an equilibrium point with eigenvalues $\pm i\omega_0$, $\omega_0 > 0$.

$$\dot{x} = f(x, \mu), \quad x \in \mathbf{R}^2, \mu \in \mathbf{R}^1 \tag{13}$$

where we suppose, after shifting of origin and canonical transformation,

$$f(0, 0) = 0, \quad D_x f(0, 0) = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix}.$$

Furthermore, the Taylor expansion of Eq. (13) gives

$$\dot{x} = Ax + F_2(x) + F_3(x) + \cdots + F_r(x) + O(|x|^{r+1}), \tag{14}$$

where

$$A \cong D_x f(0, 0) + D^2_{x\mu} f(0, 0)\mu = \begin{bmatrix} \beta\mu & -(\alpha\mu + \omega_0) \\ (\alpha\mu + \omega_0) & \beta\mu \end{bmatrix}, \tag{15}$$

$$F_2(y) = \begin{bmatrix} F_{21}(y) \\ F_{22}(y) \end{bmatrix} = \begin{bmatrix} a_{20} & a_{11} & a_{02} \\ b_{20} & b_{11} & b_{02} \end{bmatrix} \begin{Bmatrix} y_1^2 \\ y_1 y_2 \\ y_2^2 \end{Bmatrix}. \tag{16}$$

We would like to find a coordinate transformation (7) so that the nonlinear terms of order $k + 1$ in the new system of function y vanish rather than of order k in the original system of function x. Inserting $y = (y_1, y_2)^T \in R^2$ and $h_k(y) = (h_{k1}(y), h_{k2}(y))^T \in H_k$ into (13), we have the Lie bracket,

$$L_J(H_k) = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \begin{bmatrix} h_{k1} \\ h_{k2} \end{bmatrix} - \begin{bmatrix} \partial h_{k1}/\partial y_1 & \partial h_{k1}/\partial y_2 \\ \partial h_{k2}/\partial y_1 & \partial h_{k2}/\partial y_2 \end{bmatrix} \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{17}$$

Second-Order Normal Form

If the smallest order of nonlinear terms appearing in (13) is two, we try to find a transformation h_2 of the form

$$x = y + h_2(y). \tag{18}$$

where

$$h_2(y) = \begin{bmatrix} h_{21}(y) \\ h_{22}(y) \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{2} c_{ij} y_1^i y_2^j \\ \sum_{j=0}^{2} d_{ij} y_1^i y_2^j \end{bmatrix}; \quad i = 2 - j. \tag{19}$$

Perform the Lie bracket operation to each basis element on H_2,

$$H_2 = \operatorname{span}\left\{ \begin{pmatrix} y_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 y_2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ y_1 y_2 \end{pmatrix}, \begin{pmatrix} 0 \\ y_2^2 \end{pmatrix} \right\}$$

thus,

$$L_J \begin{pmatrix} y_1^2 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \begin{bmatrix} y_1^2 \\ 0 \end{bmatrix} - \begin{bmatrix} 2y_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \omega_0 \begin{bmatrix} 2y_1 y_2 \\ y_1^2 \end{bmatrix}, \text{ and}$$

$$L_J \begin{pmatrix} y_1 y_2 \\ 0 \end{pmatrix} = \omega_0 \begin{pmatrix} y_2^2 - y_1^2 \\ y_1 y_2 \end{pmatrix}, \quad L_J \begin{pmatrix} y_2^2 \\ 0 \end{pmatrix} = \omega_0 \begin{pmatrix} -y_1 y_2 \\ y_2^2 \end{pmatrix}, \quad L_J \begin{pmatrix} 0 \\ y_1^2 \end{pmatrix} = \omega_0 \begin{pmatrix} -y_1^2 \\ 2y_1 y_2 \end{pmatrix},$$

$$L_J \begin{pmatrix} 0 \\ y_1 y_2 \end{pmatrix} = \omega_0 \begin{pmatrix} -y_1 y_2 \\ y_2^2 - y_1^2 \end{pmatrix}, \quad L_J \begin{pmatrix} 0 \\ y_2^2 \end{pmatrix} = -\omega_0 \begin{pmatrix} y_2^2 \\ 2y_1 y_2 \end{pmatrix}.$$

The matrix representation of $L_J(H_2)$ can then be written as

$$L_J(H_2) = \operatorname{span} \left\{ \begin{bmatrix} y_1^2 & y_1 y_2 & y_2^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_1^2 & y_1 y_2 & y_2^2 \end{bmatrix} \cdot A_J^2 \right\} = H_2 \cdot A_J^2 \tag{20}$$

where

$$A_J = \omega_0 \begin{bmatrix} 0 & -1 & 0 & -1 & 0 & 0 \\ 2 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Because the $\det[A_J^2] = 8\omega_0 > 0$, which does not vanish for $\omega_0 > 0$, we obtain the following null complementary space for the homogeneous equations $A_J^2\,\xi = \{0\}$,

$$G_2 = \xi \equiv \{0\}, \quad \xi \in R^6. \tag{21}$$

Therefore, we can eliminate the quadratic term completely from Eq. (10) by means of the second-order coordinate transformation (19) and get the second-order normal form

$$\begin{cases} \dot{y}_1 = (\beta\mu\, y_1 - (\alpha\mu + \omega_0)y_2 + O(|y_1|^3, |y_2|^3) \\ \dot{y}_2 = (\alpha\mu + \omega_0)y_1 + \beta\mu\, y_2 + O(|y_1|^3, |y_2|^3) \end{cases}. \tag{22}$$

The coefficients in Eq. (19) are required for further development and are found by Eqs. (10)–(12). Substituting Eq. (19) into (10) and truncating to the Mth degree yields

$$\dot{y} = \sum_{N=0}^{M} (-1)^N [D_y h_2(y)]^N$$

$$\left\{Ay + Ah_2(y) + \sum_{n=2}^{M}\sum_{m=0}^{n} \frac{D_y^m F_n(y)}{m!}[h_2(y)]^m\right\}$$

$$= Ay + \sum_{n=2}^{M} F_n^{(1)}(y) \tag{23}$$

where

$$F_n^{(1)}(y) = \begin{bmatrix} F_{n1}^{(1)}(y) \\ F_{n2}^{(2)}(y) \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{n} a_{ij}^{(1)} y_1^i y_2^j \\ \sum_{j=0}^{n} b_{ij}^{(1)} y_1^i y_2^j \end{bmatrix}; \tag{24}$$

$$n = 2, \cdots, M, i = 2 - j.$$

in which, the number in the superscript parentheses refers to the index of coordinate transformation. From Eq. (22) we know that the complementary space of second-order normal form, G_2, is null so that we have the following equation

$$F_2^{(1)}(y) = F_2(y) + Jh_2(y) - D_y h_2(y) \cdot J \cdot y = 0. \tag{25}$$

Then, we find coefficients c_{ij}, d_{ij} in Eq. (19) by solving the linear algebraic equation, Eq. (25),

$$\begin{bmatrix} c_{20} \\ c_{11} \\ c_{02} \\ d_{20} \\ d_{11} \\ d_{02} \end{bmatrix} = \frac{-1}{3\omega 0} \begin{bmatrix} b_{20} + a_{11} + 2b_{02} \\ -2a_{20} - b_{11} + 2a_{02} \\ 2b_{20} - a_{11} + b_{02} \\ -a_{20} + b_{11} - 2a_{02} \\ -2b_{20} + a_{11} + 2b_{02} \\ -2a_{20} - b_{11} - a_{02} \end{bmatrix}. \tag{26}$$

Third-Order Normal Form

We further perform the nearly identity transformation (7) to third order to obtain the more accurate nonlinear description of Eq. (14) in the neighborhood of the original singular point.

$$x = y + h_3(y). \tag{27}$$

where y represents the new coordinate and x is its old one.

$$h_3(y) = \begin{bmatrix} h_{31}(y) \\ h_{32}(y) \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{3} c_{ij} y_1^i y_2^j \\ \sum_{j=0}^{3} d_{ij} y_1^i y_2^j \end{bmatrix}; \quad i = 3 - j. \tag{28}$$

The third-order monomial base is given by

$$H_3 = \operatorname{span}\left\{\begin{pmatrix} y_1^3 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1^2 y_2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 y_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_2^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y_1^3 \end{pmatrix}, \begin{pmatrix} 0 \\ y_1^2 y_2 \end{pmatrix}, \begin{pmatrix} 0 \\ y_1 y_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ y_2^3 \end{pmatrix}\right\}. \tag{29}$$

Similar to Eq. (20), the matrix representation of $L_J(H_3)$ is given by

$$L_J(H_3 = H_3 \cdot A_J^3 \tag{30}$$

where

$$A_J^3 = \omega_0 \begin{bmatrix} 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 3 & 0 & -2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & -3 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We observe that the homogenous equations $A^3{}_J\xi = \{0\}$, $\xi \in R^8$ have two zero eigenvectors for the matrix A_J^3 corresponding to its two zero eigenvalues,

$$e^T = \{(1, 0, 1, 0, 0, 1, 0, 1)^T \\ (0, -1, 0, -1, 1, 0, 1, 0)^T\}. \tag{31}$$

Thus, A_J^3 has a complementary space spanned by the monomial basis (29),

$$G_3 = H_3 \cdot e^T \\ = \text{span}\left\{\begin{pmatrix} y_1(y_1^2 + y_2^2) \\ y_2(y_1^2 + y_2^2)\end{pmatrix} \begin{pmatrix} -y_2(y_1^2 + y_2^2) \\ y_1(y_1^2 + y_2^2)\end{pmatrix}\right\}. \tag{32}$$

Finally, through a proper coordinate transformation, we determine the third-order normal form of the original system,

$$\begin{cases} \dot{y}_1 = \beta\mu\, y_1 - (\alpha\mu + \omega_0)y_2 + a_1 y_1(y_1^2 + y_2^2) \\ \qquad - b_1 y_2(y_1^2 + y_2^2) + O(|y_1|^5, |y_2|^5) \\ \dot{y}_2 = (\alpha\mu \quad \omega_0)y_1 + \beta\mu y_2 + a_1 y_2(y_1^2 + y_2^2) \\ \qquad + b_1 y_1(y_1^2 + y_2^2) + O(|y_1|^5, |y_2|^5) \end{cases}, \tag{33}$$

where a_i and b_i are to be determined.

We now want to develop a systematic procedure to evaluate the coefficients of normal forms and the normal transformations of third order. We see from Eq. (11)

$$\dot{x} = \sum_{N=0}^{M} (-1)^N [Dh_3(y)]^N \\ \left\{Ay + Ah_3(y) + \sum_{n=2}^{M}\sum_{m=0}^{n} \frac{D^m F_n(y)}{m!}[h_3(y)]^m\right\} \\ = Ay + F_2^{(1)}(y) + \sum_{n=3}^{M} F_n^{(2)}(y) \tag{34}$$

where

$$F_n^{(2)}(y) = \begin{bmatrix} F_{n1}^{(2)}(y) \\ F_{n2}^{(2)}(y)\end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{n} a_{ij}^{(2)} y_1^i y_2^j \\ \sum_{j=0}^{n} b_{ij}^{(2)} y_1^i y_2^j \end{bmatrix}; \\ n = 3, \cdots, M, i = 2 - j. \tag{35}$$

We know from Eq. (24) that the third-order normal form is equal to its complement space so that

$$F_3^{(2)}(y) = F_3^{(1)}(y) + J \cdot h_3(y) - Dh_3(y) \cdot J \cdot y \\ = G_3(y). \tag{36}$$

The resultant linear equation is

$$A_J^3 \cdot \xi = \eta \tag{37}$$

where

$$\xi = \{c_{30}, c_{21}, c_{12}, c_{03}, d_{30}, d_{21}, d_{12}, d_{03}\}^T$$

$$\eta = \{a_{30}^{(1)} - a_1, a_{21}^{(1)} + b_1, a_{12}^{(1)} - a_1, a_{03}^{(1)} + b_1, b_{30}^{(1)} \\ - b_1, b_{21}^{(1)} - a_1, b_{12}^{(1)} - b_1, a_{03}^{(1)} - a_1\}^T$$

or

$$\omega_0 \begin{bmatrix} B & -I \\ I & B \end{bmatrix}\begin{bmatrix} \bar{c} \\ \bar{d}\end{bmatrix} = \begin{bmatrix} \bar{a} \\ \bar{b}\end{bmatrix} \tag{38}$$

in which

$$B = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 3 & 0 & -2 & 0 \\ 0 & 2 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \quad \bar{c} = \begin{bmatrix} c_{30} \\ c_{21} \\ c_{12} \\ c_{03}\end{bmatrix};$$

$$\bar{d} = \begin{bmatrix} d_{30} \\ d_{21} \\ d_{12} \\ d_{03}\end{bmatrix}; \quad \bar{a} = \begin{bmatrix} a_{30}^{(1)} - a_1 \\ a_{21}^{(1)} + b_1 \\ a_{12}^{(1)} - a_1 \\ a_{03}^{(1)} + b_1 \end{bmatrix};$$

$$\bar{b} = \begin{bmatrix} b_{30}^{(1)} - b_1 \\ b_{21}^{(1)} - a_1 \\ b_{12}^{(1)} - b_1 \\ b_{03}^{(1)} - a_1 \end{bmatrix}.$$

Dividing Eq. (38) into two parts, we obtain

$$\begin{cases} \omega_0(B^2 + I) \cdot \bar{c} = B \cdot \bar{a} - \bar{b} \\ \omega_0(B^2 + I) \cdot \bar{d} = B \cdot \bar{b} + \bar{a} \end{cases}. \tag{39}$$

Note that there are two more unknowns existing in Eq. (39) than the number of equations. However, the coefficients of the normal form a_1, b_1 can be evaluated independently by performing an orthogonal linear transformation on one of the equations in Eq. (39),

$$T = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -3 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \tag{40}$$

where T is the canonical matrix consisting of the eigenvectors of $B^2 + I$. The second equation in (39) becomes

$$\text{LHS} = T^{-1}(B^2 + I)T \cdot \tilde{d} = \omega_0 \begin{bmatrix} 0 & \frac{-1}{4} & 0 & \frac{1}{4} \\ \frac{-1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 & 2 & 0 \\ 0 & -6 & 0 & 6 \\ 6 & 0 & -6 & 0 \\ 0 & 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 1 \\ -3 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} . \tilde{d}$$

$$= \omega_0 \begin{bmatrix} 0 & 2 & 0 & -2 \\ 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tilde{d} \tag{41}$$

$$\text{RHS} = T^{-1}(B \cdot b + a) = \begin{bmatrix} 0 & \frac{-1}{4} & 0 & \frac{1}{4} \\ \frac{-1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} -2a_1 + a_{30}^{(1)} + b_{21}^{(1)} \\ a_{21}^{(1)} + 2b_1 + 2b_{12}^{(1)} - 3b_{30}^{(1)} \\ -2a_1 + a_{12}^{(1)} + 3b_{03}^{(1)} - 2b_{21}^{(1)} \\ a_{03}^{(1)} + 2b_1 - b_{12}^{(1)} \end{bmatrix}$$

$$= \begin{bmatrix} a_{03}^{(1)} - a_{21}^{(1)} - 3b_{12}^{(1)} + 3b_{30}^{(1)} \\ a_{12}^{(1)} - a_{30}^{(1)} + 3b_{03}^{(1)} - 3b_{21}^{(1)} \\ 3a_{03}^{(1)} + a_{21}^{(1)} + 8b_1 - b_{12}^{(1)} - 3b_{30}^{(1)} \\ -8a_1 + a_{12}^{(1)} + 3a_{30}^{(1)} + 3b_{03}^{(1)} + b_{21}^{(1)} \end{bmatrix}.$$

By comparing the terms in the last two rows of Eq. (41), we obtain

$$\begin{cases} a_1 = \dfrac{1}{8}(a_{12}^{(1)} + 3a_{03}^{(1)} + b_{21}^{(1)} + 3b_{03}^{(1)}) \\ b_1 = \dfrac{-1}{8}(a_{21}^{(1)} + 3a_{03}^{(1)} - b_{12}^{(1)} - 3b_{30}^{(1)}) \end{cases}. \tag{42}$$

Substituting Eq. (42) into (39) and solving Eq. (39), the third-order homogeneous polynomial (28) is then determined by

$$\begin{bmatrix} c_{30} \\ c_{21} \\ c_{12} \\ c_{03} \end{bmatrix} = \frac{1}{8\omega_0} \begin{bmatrix} 0 \\ -3a_{12}^{(1)} - b_{03}^{(1)} + 3a_{30}^{(1)} + b_{21}^{(1)} \\ -3a_{03}^{(1)} + b_{12}^{(1)} + 3a_{21}^{(1)} + b_{30}^{(1)} \\ 0 \end{bmatrix},$$
$$\begin{bmatrix} d_{30} \\ d_{21} \\ d_{12} \\ d_{03} \end{bmatrix} = \frac{1}{4\omega_0} \begin{bmatrix} a_{12}^{(1)} - b_{03}^{(1)} + a_{30}^{(1)} - b_{21}^{(1)} \\ 3a_{03}^{(1)} + 5b_{30}^{(1)} + a_{21}^{(1)} - b_{12}^{(1)} \\ a_{12}^{(1)} + 5b_{03}^{(1)} + 3a_{30}^{(1)} + b_{21}^{(1)} \\ a_{03}^{(1)} + b_{30}^{(1)} + a_{21}^{(1)} + b_{12}^{(1)} \end{bmatrix}. \tag{43}$$

Higher Order Normal Form

The higher order normal form in a rectangular coordinate system is derived accordingly and can be written as follows,

$$\dot{y} = A\,y + \sum_{i=1}^{L} W_{2i+1}(y) + O(|y|^{2L+3}), \tag{44}$$

where

$$A = \begin{bmatrix} \beta\mu & -(\alpha\mu + \omega_0) \\ (\alpha\mu + \omega_0) & \beta\mu \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

$$W_{2i+1}(y) = (y_1^2 + y_2^2)^i \begin{bmatrix} a_i & -b_i \\ b_i & a_i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$W_{2i+1} \in G_{2i+1}.$$

and L is a given degree of the normal form equation. The validity of such simplification is guaranteed by the implicit function theorem, as for each μ near μ_0, there will be an equilibrium $p(\mu)$ near $p(\mu_0)$ that varies smoothly with μ. The normal form of Eq. (44) in polar coordinates can then be written as the form

$$\begin{cases} y_1 = r\cos\theta, \quad y_2 = r\sin\theta \\ \dot{r} = r(\beta\mu + \sum_{i=1}^{L} a_i r^{2i}) + \text{hot.} \\ \dot{\theta} = \omega_0 + \alpha\mu + \sum_{i=1}^{L} b_i r^{2i} + \text{hot} \end{cases} \tag{45}$$

A general formula for each a_i, b_i is not presently available but we can derive their expressions up to any desired higher order using the algorithm mentioned above recursively, such that,

$$\begin{cases} a_3 = \dfrac{1}{128}\left(35a^{(5)}_{70} + 5a^{(5)}_{52} + 3a^{(5)}_{34} + 5a^{(5)}_{16} + 5b^{(5)}_{61} + 3b^{(5)}_{43} + 5b^{(5)}_{25} + 35b^{(5)}_{07}\right), \\ b_3 = \dfrac{-1}{128}\left(5a^{(5)}_{61} + 3a^{(5)}_{43} + 5a^{(5)}_{25} + 35a^{(5)}_{07} - 35b^{(5)}_{70} - 5b^{(5)}_{52} - 3b^{(5)}_{34} - 5b^{(5)}_{16}\right); \end{cases}$$

$$\begin{cases} a_4 = \dfrac{1}{256}\left(63a^{(7)}_{90} + 7a^{(7)}_{72} + 3a^{(7)}_{54} + 3a^{(7)}_{36} + 7a^{(7)}_{18} + 7b^{(7)}_{81} + 3b^{(7)}_{63} + 3b^{(7)}_{45} + 7b^{(7)}_{27} + 63b^{(7)}_{09}\right), \\ b_4 = \dfrac{-1}{256}\left(7a^{(7)}_{81} + 3a^{(7)}_{63} + 3a^{(7)}_{45} + 7a^{(7)}_{27} + 63a^{(7)}_{09} - 63b^{(7)}_{90} - 7b^{(7)}_{72} - 3b^{(7)}_{54} - 3b^{(7)}_{36} - 7b^{(7)}_{18}\right); \end{cases}$$

and for the eleventh order,

$$\begin{cases} a_5 = \dfrac{1}{1024}\left(231a^{(9)}_{B0} + 21a^{(9)}_{92} + 7a^{(9)}_{74} + 5a^{(9)}_{56} \right. \\ \quad + 7a^{(9)}_{38} + 21a^{(9)}_{1A} + 21b^{(9)}_{A1} + 7b^{(9)}_{83} + 5b^{(9)}_{65} \\ \quad \left. + 7b^{(9)}_{47} + 21b^{(9)}_{29} + 231b^{(9)}_{0B}\right), \\ b_5 = \dfrac{-1}{1024}\left(21a^{(9)}_{A1} + 7a^{(9)}_{83} + 5a^{(9)}_{65} + 7a^{(9)}_{47} \right. \\ \quad + 21a^{(9)}_{29} + 231a^{(9)}_{0B} - 231b^{(9)}_{B0} - 21b^{(9)}_{92} \\ \quad \left. - 7b^{(9)}_{74} - 5b^{(9)}_{56} - 7b^{(9)}_{38} - 21b^{(9)}_{1A}\right); \end{cases}$$

here the subscripts A, B refer to the indices 10 and 11, respectively.

The complexity of higher order normal forms rapidly becomes apparent as pointed out by Leung and Zhang (1994). Thus, the algorithm is required to be implemented with the symbolic manipulation in *Mathematica* (Wolfram, 1991). In the case of $m > 5$, however, it may also cause overflow in the computers equipped with conventional memory if we apply the nearly identity normal transformations directly in Eqs. (23) and (34). A general explicit formula representing the homogeneous polynomial terms is therefore very useful to reduce the size of the problem.

If the nearly identical change of coordinate of order k

$$x = y + h_k(y) \tag{46}$$

is applied then the new homogeneous polynomials of order n can be obtained by

$$n<k, \quad F^*_n(y) = F_n(y); \tag{47}$$

$$n = k, \quad F^*_n(y) = F_n(y) + J \cdot h_k(y) - Dh_k(y) \cdot J \cdot y; \tag{48}$$

$$\begin{aligned} n > k, \quad F^*_n(y) = {} & F_n(y) + [DF_{n-k+1}(y) \cdot h_k(y) - Dh_k(y) \cdot F_{n-k+1}(y)]_{n \geq 2k-1} \\ & + \sum_{m=2}^{k_m} \left[\frac{D^m_y F_{n+m(1-k)}(y)}{m!}\,[h_k(y)]^m\right]_{n+m(1-k) \geq k}; \end{aligned} \tag{49}$$

where $F^*_n(y)$ denotes the resulting homogeneous polynomials, $k_m = \min[k_1, k_2]$, and

$$k_1 = \text{floor}\left[\frac{n-k}{k-1}\right], \quad k_2 = \text{floor}\left[\frac{n}{k}\right]$$

The operator floor[·] gives the greatest integer less than or eual to the varible in the bracket. The result confirms that only odd-order terms would appear in the normal form equation, corresponding to the *Poincaré resonance* (Guckenheimer and Holmes, 1983).

OSCILLATORS WITH ODD NONLINEARITY

Here we consider the nonlinear differential equations with $f(u)$ to be an odd function of u. To illustrate this, two examples are presented. The first one is the well-known Duffing oscillator, which arises from various physical and engineering problems. The solution of this oscillator has been thoroughly studied so that the accuracy of the results can be compared with existing methods. In the second example, we work with an oscillator having a term of the fifth power.

Duffing Oscillator

Consider the Duffing oscillator

$$\ddot{x} + \omega_0^2 x = -\varepsilon x^3 \tag{50}$$

with the initial conditions

$$x(0) = a, \quad \dot{x}(0) = 0, \tag{51}$$

where the parameter $\varepsilon > 0$, ω_0 is the linear frequency. In classical perturbation methods, it is usually assumed that ε is small. In the present case, however, ε need not be small. With the transformation $x = x_1$, $\dot{x} = -\omega_0 x_2$, we rewrite Eq. (50) as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{\varepsilon}{\omega_0} x_1^3 \end{bmatrix}. \tag{52}$$

Comparing Eq. (52) with the standard third-order normal form $\{\dot{x}\} = \{x\} + F_3(x)$, we have $b_{30} = (\varepsilon/\omega_0)$ and all the other coefficients are zero. If we substitute these coefficients into Eq. (42), we obtain $a_1 = 0$, $b_1 = (3\varepsilon/8\omega_0)$.

The third-order nonlinear transformation of coordinate $\{x\} = [J]\{y\} + h_3(y)$ can be determined by Eq. (43) and the coefficients of $h_3(y)$ are $c_{30} = c_{03} = c_{21} = d_{30} = d_{12} = 0$,

$$c_{12} = \frac{\varepsilon}{8\omega_0^2}, \quad d_{21} = \frac{5\varepsilon}{8\omega_0^2}, \quad d_{03} = \frac{\varepsilon}{4\omega_0^2}.$$

Here we use y to denote the new coordinates and x the old ones. While under the fourth order nonlinear transformation, all the coefficients of the homogeneous polynomial are identically zero,

$$c_{40} = c_{31} = c_{22} = c_{13} = c_{04} = d_{40} = d_{31} = d_{22} = d_{13} = d_{04} = 0.$$

According to the method mentioned in the last section, such computation can be repeated up the the desired orders. Finally, we found that all the coefficients of the even-order coordinate homogeneous polynomials $h_{2n}(y)$ are equal to zero and all the even-order terms are already in their simplest forms. Thus, only the odd-order homogeneous polynomials need to be handled in our computation.

The coefficients of normal form a_2 and b_2 can be obtained by taking the change of variable

$$\{y\} = \{z\} + h_5(z); \quad a_2 = 0, \quad b_2 = \frac{-21\varepsilon^2}{256\omega_0^3}.$$

Then, the fifth-order nonlinear coordinate change $h_5(z)$ is given by

$$c_{50} = c_{05} = c_{23} = c_{41} = d_{50} = d_{14} = d_{32} = 0,$$

$$c_{32} = \frac{21\varepsilon^2}{256\omega_0^4}, \quad c_{14} = \frac{25\varepsilon^2}{256\omega_0^4}, \quad d_{41} = \frac{-39\varepsilon^2}{256\omega_0^4},$$

$$d_{23} = \frac{-67\varepsilon^2}{256\omega_0^4}, \quad d_{05} = \frac{-\varepsilon^2}{32\omega_0^4}.$$

Similarly, after taking the seventh-order polynomial change of variables,

$$\{z\} = \{w\} + h_7(w),$$

we get

$$a_3 = 0, \quad b_3 = \frac{237\varepsilon^3}{4096\omega_0^5},$$

and $h_7(w)$ is given by

$$c_{70} = c_{07} = c_{61} = c_{43} = c_{25} = d_{70} = d_{52} = d_{34} = d_{16} = 0$$

$$c_{52} = \frac{-183\varepsilon^3}{4096\omega_0^6}, \quad c_{34} = \frac{-91\varepsilon^3}{1024\omega_0^6}, \quad c_{16} = \frac{-133\varepsilon^3}{4096\omega_0^6},$$

$$d_{61} = \frac{363\varepsilon^3}{4096\omega_0^6}, \quad d_{43} = \frac{217\varepsilon^3}{4096\omega_0^6}, \quad d_{25} = \frac{633\varepsilon^3}{4096\omega_0^6},$$

$$d_{07} = \frac{5\varepsilon^3}{256\omega_0^6}.$$

Performing the variable changes further to the ninth order, $\{w\} = \{u\} + h_9(u)$, yields

$$a_4 = 0, \quad b_4 = \frac{-11241\varepsilon^4}{262144\omega_0^7}$$

and $h_9(u)$ is given by

$$c_{90} = c_{09} = c_{81} = c_{63} = c_{45} = c_{27} = d_{90} = d_{72} = d_{54} = d_{36} = d_{18} = 0,$$

$$c_{72} = \frac{7089\varepsilon^4}{262144\omega_0^8}, \quad c_{54} = \frac{19215\varepsilon^4}{262144\omega_0^8}, \quad c_{36} = \frac{19003\varepsilon^4}{262144\omega_0^8}, \quad c_{18} = \frac{7901\varepsilon^4}{262144\omega_0^8},$$

$$d_{81} = \frac{-16035\varepsilon^4}{262144\omega_0^8}, \quad d_{63} = \frac{-46821\varepsilon^4}{262144\omega_0^8}, \quad d_{45} = \frac{-57969\varepsilon^4}{262144\omega_0^8}, \quad d_{27} = \frac{-38575\varepsilon^4}{262144\omega_0^8}, \quad d_{09} = \frac{-29\varepsilon^4}{2048\omega_0^8}$$

If further transformations beyond the ninth order were performed, it is interesting to note that all the following higher order coefficients of normal forms a_i, b_i $(i = 5, 6, 7 \ldots)$ as well as the coordinate transformation $h_k(y)$ $(k = 10, 11, 12, \ldots)$ vanish. Therefore, the normal form of our cubic nonlinear differential system has an ultimate degree of nine. The dynamic system can be represented by an exact polynomial differential amplitude equation in a finite number of terms.

The asymptotic solution of this normal-form equation is

$$\begin{cases} u_1 = r\cos\theta \\ \dot{r} = 0 \\ \dot{\theta} = \omega_0 + \dfrac{3\varepsilon r^2}{8\omega_0} - \dfrac{21\varepsilon^2 r^4}{256\omega_0^3} + \dfrac{237\varepsilon^3 r^6}{4096\omega_0^5} - \dfrac{11241\varepsilon^4 r^8}{262144\omega_0^7} \end{cases} \tag{53}$$

The steady-state periodic solution of Eq. (53) is

$$\begin{cases} u_1 = a\cos\theta, \quad u_2 = a\sin\theta \\ \theta = \left(\omega_0 + \dfrac{3\varepsilon a^2}{8\omega_0} - \dfrac{21\varepsilon^2 a^4}{256\omega_0^3} + \dfrac{237\varepsilon^3 a^6}{4096\omega_0^5} - \dfrac{11241\varepsilon^4 a^8}{262144\omega_0^7}\right)t. \end{cases} \tag{54}$$

To get the steady-state periodic solution in the original coordinate, we could trace back all the transformations, for example,

$$\begin{aligned} \{x\} &= \{y\} + h_3(y) \\ &= \{z\} + h_5(z) + h_3(z + h_5(z)) \\ &= \{w\} + h_7(w) + h_5(w + h_7(w)) + h_3(\{w\} \\ &\qquad + h_7(w) + h_5(u + h_7(w))) \\ &= \{u\} + h_9(u) + h_7(\{u\} + h_9(u)) + h_5(\{u\} \\ &\qquad + h_9(u) + h_7(\{u\} + h_9(u))) \\ &\quad + h_3(\{u\} + h_9(u) + h_7(\{u\} + h_9(u)) + h_5(\{u\} \\ &\qquad + h_9(u) + h_7(\{u\} + h_9(u)))). \end{aligned} \tag{55}$$

The steady-state periodic solution of the Duffing equation is

$$\begin{cases} x = \\ \left(a - \dfrac{\varepsilon a^3}{32\omega_0^2} + \dfrac{23\varepsilon^2 a^5}{1024\omega_0^4} - \dfrac{167\varepsilon^3 a^7}{16384\omega_0^6} + \dfrac{3431\varepsilon^3 a^9}{524288\omega_0^4}\right)\cos\theta \\ + \left(\dfrac{\varepsilon a^3}{32\omega_0^2} - \dfrac{3\varepsilon^2 a^5}{128\omega_0^4} + \dfrac{309\varepsilon^3 a^7}{32768\omega_0^6} - \dfrac{7033\varepsilon^4 a^9}{1048576\omega_0^4}\right)\cos 3\theta \\ + \left(\dfrac{\varepsilon^2 a^5}{1024\omega_0^4} + \dfrac{31\varepsilon^3 a^7}{32768\omega_0^6} + \dfrac{191\varepsilon^4 a^9}{524288\omega_0^4}\right)\cos 5\theta. \\ - \left(\dfrac{3\varepsilon^3 a^7}{16384\omega_0^6} + \dfrac{227\varepsilon^4 a^9}{1048576\omega_0^4}\right)\cos 7\theta \\ \qquad + \left(\dfrac{\varepsilon^4 a^9}{65536\omega_0^4}\right)\cos 9\theta \\ \theta = \left(\omega_0 + \dfrac{3\varepsilon a^2}{8\omega_0} - \dfrac{21\varepsilon^2 a^4}{256\omega_0^3} + \dfrac{237\varepsilon^3 a^6}{4096\omega_0^5} \right. \\ \qquad \left. - \dfrac{11241\varepsilon^4 a^8}{262144\omega_0^7}\right)t. \end{cases} \tag{56}$$

We compare the solution by the Lindstedt–Poincaré (L–P) method with the same initial conditions below,

$$\begin{cases} x = \left(a - \dfrac{\varepsilon a^3}{32\omega_0^2} + \dfrac{23\varepsilon^2 a^5}{1024\omega_0^4} - \dfrac{547\varepsilon^3 a^7}{32768\omega_0^6} \right. \\ \qquad \left. + \dfrac{6713\varepsilon^3 a^9}{524288\omega_0^4}\right)\cos\theta \\ + \left(\dfrac{\varepsilon a^3}{32\omega_0^2} - \dfrac{3\varepsilon^2 a^5}{128\omega_0^4} + \dfrac{297\varepsilon^3 a^7}{16384\omega_0^6} \right. \\ \qquad \left. - \dfrac{15121\varepsilon^4 a^9}{1048576\omega_0^4}\right)\cos 3\theta \\ + \left(\dfrac{\varepsilon^2 a^5}{1024\omega_0^4} - \dfrac{3\varepsilon^3 a^7}{2048\omega_0^6} + \dfrac{883\varepsilon^4 a^9}{524288\omega_0^4}\right)\cos 5\theta \\ + \left(\dfrac{\varepsilon^3 a^7}{32768\omega_0^6} - \dfrac{9\varepsilon^4 a^9}{131072\omega_0^4}\right)\cos 7\theta \\ \qquad + \left(\dfrac{\varepsilon^4 a^9}{1048576\omega_0^4}\right)\cos 9\theta \\ \theta = \left(\omega_0 + \dfrac{3\varepsilon a^2}{8\omega_0} - \dfrac{21\varepsilon^2 a^4}{256\omega_0^3} + \dfrac{81\varepsilon^3 a^6}{2048\omega_0^5} - \dfrac{6549\varepsilon^4 a^8}{262144\omega_0^7} \right. \\ \qquad \left. + O(|a|^{10})\right)t \end{cases} \tag{57}$$

It is clear to see that the coefficients of the first three orders of ε in Eq. (57) are exactly the same as those given in Eq. (55). Nevertheless, Eq. (55) is different from Eq. (57) in the higher order terms. The result derived by normal form theory gives a finite, asymptotic power series, but using the L–P method the solution was represented by an infinite one.

Oscillator with Fifth-Power Nonlinearity

Consider an oscillator with a fifth-power nonlinearity as the second example,

$$\ddot{x} + \omega_0^2 x = -\varepsilon x^5 \tag{58}$$

with initial conditions (51). Using the normal form method described previously, one obtains the following results:

$$a_1 = 0, \quad b_1 = \frac{5\varepsilon}{16\omega_0}$$

and the fifth-order nonlinear coordinate change is given by

$$c_{50} = c_{05} = c_{23} = c_{41} = d_{50} = d_{14} = d_{32} = 0,$$

$$c_{32} = \frac{-3\varepsilon}{16\omega_0^2}, \quad c_{14} = \frac{-7\varepsilon}{48\omega_0^2}, \quad d_{41} = \frac{11\varepsilon}{16\omega_0^2},$$

$$d_{23} = \frac{31\varepsilon}{48\omega_0^2}, \quad d_{05} = \frac{\varepsilon}{6\omega_0^2}.$$

Then, taking seventh-order transformation of coordinates, we obtain $a_2 = 0$, $b_2 = 0$, with $h_7(w) = \{0\}$. After that, we directly take the ninth-order change of variable which yields $a_3 = 0$, $b_3 = (-215\varepsilon^2/3072\omega_0^3)$, and

$$c_{90} = c_{09} = c_{81} = c_{63} = c_{45} = c_{27} = d_{90} = d_{72} = d_{54} = d_{36} = d_{18} = 0,$$

$$c_{72} = \frac{295\varepsilon^2}{3072\omega_0^4}, \quad c_{54} = \frac{2915\varepsilon^2}{9216\omega_0^4}, \quad c_{36} = \frac{327\varepsilon^2}{1024\omega_0^4}, \quad c_{18} = \frac{937\varepsilon^{-}}{9216\omega_0^4},$$

$$d_{81} = \frac{-445\varepsilon^2}{3072\omega_0^4}, \quad d_{63} = \frac{-5825\varepsilon^2}{9216\omega_0^4}, \quad d_{63} = \frac{-6173\varepsilon^2}{9216\omega_0^4},$$

$$d_{63} = \frac{-2027\varepsilon^2}{9216\omega_0^4}, \quad d_{09} = \frac{-\varepsilon^2}{72\omega_0^4}.$$

The steady-state periodic solution of the oscillator with fifth power non-linearity is

$$\begin{cases} x = \left(a - \dfrac{\varepsilon a^5}{24\omega_0^2} + \dfrac{3791\varepsilon^2 a^9}{147456\omega_0^4}\right)\cos\theta + \left(\dfrac{5\varepsilon a^5}{128\omega_0^2} - \dfrac{5^2\varepsilon a^9}{192\omega_0^4}\right)\cos 3\theta \\ \quad + \left(\dfrac{\varepsilon a^5}{384\omega_0^2}\right)\cos 5\theta + \left(\dfrac{95\varepsilon^2 a^9}{294912\omega_0^4}\right)\cos 7\theta + \left(\dfrac{\varepsilon^2 a^9}{98304\omega_0^4}\right)\cos 9\theta. \\ \theta = \left(\omega_0 + \dfrac{5\varepsilon a^4}{16\omega_0} - \dfrac{215\varepsilon^2 a^8}{3072\omega_0^3}\right)t \end{cases} \tag{59}$$

The terms of asymptotic normal form series expansion of Eq. (59) is also finite with all higher order normal forms equal to zero.

OSCILLATORS WITH EVEN NONLINEARITY

The normal form method can also be easily extended to analyze an oscillator with even nonlinearity. A quadratic nonlinearity equation is considered in the following calculation.

Quadratic Nonlinear Oscillator

Consider a system having quadratic nonlinearity,

$$\ddot{x} + \omega_0^2 x = -\varepsilon x^2 \tag{60}$$

with initial conditions (51). According to the normal form method, one needs to do a second-order homogeneous polynomial transformation of coordinates to simplify the second-order terms in the original equations. The coefficients of this change can be evaluated by Eq. (26), the results are,

$$c_{20} = \frac{-\varepsilon}{3\omega_0^2}, \quad c_{11} = 0, \quad c_{02} = \frac{-2\varepsilon}{3\omega_0^2}, \quad d_{20} = 0,$$

$$d_{11} = \frac{2\varepsilon}{3\omega_0^2}, \quad d_{02} = 0.$$

The subsequent third-order variable change gives

$$a_1 = 0, \quad b_1 = \frac{-5\varepsilon^2}{12\omega_0^3}.$$

and

$$c_{30} = c_{03} = c_{21} = d_{30} = d_{12} = 0,$$

$$c_{12} = -\frac{\varepsilon^2}{12\omega_0^4}, \quad d_{21} = -\frac{\varepsilon^2}{4\omega_0^4}, \quad d_{03} = -\frac{\varepsilon^2}{2\omega_0^4}.$$

The fourth-order homogeneous polynomial of transformation cannot be overlooked in this case and finally its coefficients are found to be

$$c_{31} = c_{13} = d_{40} = d_{22} = 0,$$

$$c_{40} = -\frac{\varepsilon^3}{9\omega_0^6}, \quad c_{22} = -\frac{8\varepsilon^3}{9\omega_0^6}, \quad c_{04} = -\frac{28\varepsilon^3}{27\omega_0^6}$$

$$d_{31} = \frac{4\varepsilon^3}{9\omega_0^6}, \quad d_{13} = \frac{16\varepsilon^3}{27\omega_0^6}.$$

Then normal-form coefficients a_2, b_2 and the fifth-order homogeneous polynomial of coordinate transformation can be evaluated as

$$a_2 = 0, \quad b_2 = \frac{-455\varepsilon^4}{1728\omega_0^7}$$

with

$$c_{50} = c_{05} = c_{23} = c_{41} = d_{50} = d_{14} = d_{32} = 0,$$

$$c_{32} = \frac{23\varepsilon^4}{1728\omega_0^8}, \quad c_{14} = \frac{217\varepsilon^4}{5184\omega_0^8}, \quad d_{41} = \frac{-365\varepsilon^4}{1728\omega_0^8}$$

$$d_{23} = \frac{-499\varepsilon^4}{5184\omega_0^8}, \quad d_{05} = \frac{263\varepsilon^4}{648\omega_0^8}$$

The sixth-order homogeneous polynomial of the transformation is determined by its coefficients,

$$c_{51} = c_{33} = c_{15} = d_{60} = d_{42} = d_{24} = d_{06} = 0,$$

$$c_{60} = -\frac{25\varepsilon^5}{324\omega_0^{10}}, \quad c_{42} = -\frac{23\varepsilon^5}{54\omega_0^{10}}, \quad c_{24} = \frac{5\varepsilon^5}{243\omega_0^{10}}$$

$$c_{06} = \frac{182\varepsilon^5}{243\omega_0^{10}},$$

$$d_{51} = \frac{25\varepsilon^5}{81\omega_0^{10}}, \quad d_{33} = \frac{88\varepsilon^5}{243\omega_0^{10}}, \quad d_{15} = -\frac{32\varepsilon^5}{243\omega_0^{10}}.$$

The coefficients of the normal form a_3, b_3 are

$$a_3 = 0, \quad b_3 = \frac{-28675\varepsilon^6}{124416\omega_0^{11}}.$$

The seventh-order coordinate transformation is determined by

$$c_{70} = c_{07} = c_{61} = c_{43} = c_{25} = d_{70} = d_{52} = d_{34} = d_{16} = 0,$$

$$c_{52} = -\frac{4967\varepsilon^6}{124416\omega_0^{12}}, \quad c_{34} = \frac{19105\varepsilon^6}{93312\omega_0^{12}}, \quad c_{16} = -\frac{51263\varepsilon^6}{373248\omega_0^{12}},$$

$$d_{61} = -\frac{37285\varepsilon^6}{124416\omega_0^{12}}, \quad d_{43} = \frac{-77285\varepsilon^6}{93312\omega_0^{12}},$$

$$d_{25} = -\frac{114743\varepsilon^6}{124416\omega_0^{12}}, \quad d_{07} = -\frac{11201\varepsilon^6}{23328\omega_0^{12}}.$$

Substituting the transformation into the original equation, we have the steady-state periodic solution of the oscillator with quadratic nonlinearity

$$\begin{cases} x = -\dfrac{\varepsilon a^2}{2\omega^2} - \dfrac{13\varepsilon^3 a^4}{24\omega_0^6} + \dfrac{319\varepsilon^5 a^6}{1728\omega_0^{10}} + \left(a - \dfrac{\varepsilon^2 a^3}{48\omega_0^4} + \dfrac{143\varepsilon^4 a^5}{20736\omega_0^8} - \dfrac{35005\varepsilon^6 a^7}{20736\omega_0^{12}}\right)\cos\theta \\ \quad + \left(\dfrac{\varepsilon a^2}{6\omega_0^2} + \dfrac{25\varepsilon^3 a^4}{54\omega_0^6} - \dfrac{12479\varepsilon^5 a^6}{31104\omega_0^{10}}\right)\cos 2\theta + \left(\dfrac{\varepsilon^2 a^3}{48\omega_0^4} - \dfrac{5\varepsilon^4 a^5}{576\omega_0^8} + \dfrac{9799\varepsilon^6 a^7}{331776\omega_0^{12}}\right)\cos 3\theta \\ \quad + \left(-\dfrac{7\varepsilon^3 a^4}{216\omega_0^6} + \dfrac{2353\varepsilon^5 a^6}{15552\omega_0^{10}}\right)\cos 4\theta + \left(\dfrac{37\varepsilon^4 a^5}{20736\omega_0^8} - \dfrac{5633\varepsilon^6 a^7}{995328\omega_0^{12}}\right)\cos 5\theta \\ \quad - \left(\dfrac{41\varepsilon^5 a^6}{3456\omega_0^{10}}\right)\cos 6\theta - \left(\dfrac{641\varepsilon^6 a^7}{1492992\omega_0^{12}}\right)\cos 7\theta \\ \theta = \left(\omega_0 - \dfrac{5\varepsilon^2 a^2}{12\omega_0^3} - \dfrac{455\varepsilon^4 a^4}{1278\omega_0^7} - \dfrac{28675\varepsilon^6 a^6}{124416\omega_0^{11}} + O(|a|^8)\right)t \end{cases} \quad (61)$$

DUFFING–VAN DER POL OSCILLATOR

In the problem of flow-induced oscillations, the governing differential equation can be described by a well-known Duffing–van der Pol equation (Blevins, 1977)

$$\ddot{x} - 2\varepsilon\,\lambda\dot{x} + x + \varepsilon\,x^3 + \varepsilon\,x^2\,\dot{x} = 0 \quad (62)$$

with initial conditions (50). Here we assume that λ is a control variable of the system and ε is a bookkeeping device for small perturbations. Equation (59) can further be approximately expressed by a Jordan canonical matrix form,

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \varepsilon\lambda & -1 \\ 1 & \varepsilon\lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \varepsilon \begin{bmatrix} y_1 y_2^2 + y_2^3 \\ 0 \end{bmatrix}, \quad (63)$$

in which, the first-order near identity transformation is given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\varepsilon\lambda & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad (64)$$

where we choose $x_1 = x$, $x_2 = -\dot{x}$. Note that the higher order terms in ε are neglected during the computation. We take transformations up to order five. The resultant normal form of Eq. (63) in polar coordinates is expressed as follows,

$$\begin{cases} y_1 = r\cos\theta, \quad y_2 = r\sin\theta \\ \dot{r} = \varepsilon r\left(\lambda - \dfrac{1}{8}r^2\right) \\ \theta = 1 + \dfrac{3\varepsilon}{8}r^2 \end{cases} \tag{65}$$

The coefficients of the intermeidate coordinate transformations are

$$c_{30} = c_{03} = 0, \quad c_{21} = c_{12} = \frac{3\varepsilon}{8},$$

$$d_{30} = d_{03} = \frac{-\varepsilon}{4}, \quad d_{21} = \frac{-3\varepsilon}{8}, \quad d_{12} = \frac{-\varepsilon}{8},$$

respectively. The asymptotic solution of Eq. (65) could also be obtained if we have traced back all the nearly identity coordinate transformations

$$\begin{aligned} x = {} & \varepsilon a\left(-\lambda + \frac{7}{32}a^2\right)\cos\theta + \frac{\varepsilon}{32}a^3\cos 3\theta \\ & + \left(-a + \frac{9\varepsilon}{32}a^3\right)\sin\theta + \frac{\varepsilon}{32}a^3\sin 3\theta \end{aligned} \tag{66}$$

where the constant a denotes the amplitude defined by the initial condition. Further discussion of the normal-form amplitude equation of (65) reveals the Hopf bifurcation of the Duffing–van der Pol equation on a parametric plane $r - \lambda$. As shown in Figure 1, there is a spiral sink at (0, 0) for $\lambda < 0$ and a source at (0, 0) surrounded by a limit cycle for $\lambda > 0$. The limit cycle evolves continuously from the center at (0, 0) for $\lambda = 0$. The Hopf bifurcation is of importance in situations where a flow-induced oscillator is subjected to flutters or self-exciting movements. At such circumstances, the orbits of the steady-state periodic solutions stay on the surface of the paraboloid rotated by $\lambda = (1/8)\,r^2$.

CONCLUSION

We have presented an arithmetic algorithm to compute the higher order normal forms. By applying the explicit formula proposed in this work, we can achieve, in a standard manner, the desired higher order normal forms for nonlinear differential polynomial equations. We found that the steady-state solution of the undamped Duffing equation can be represented by a finite cosine

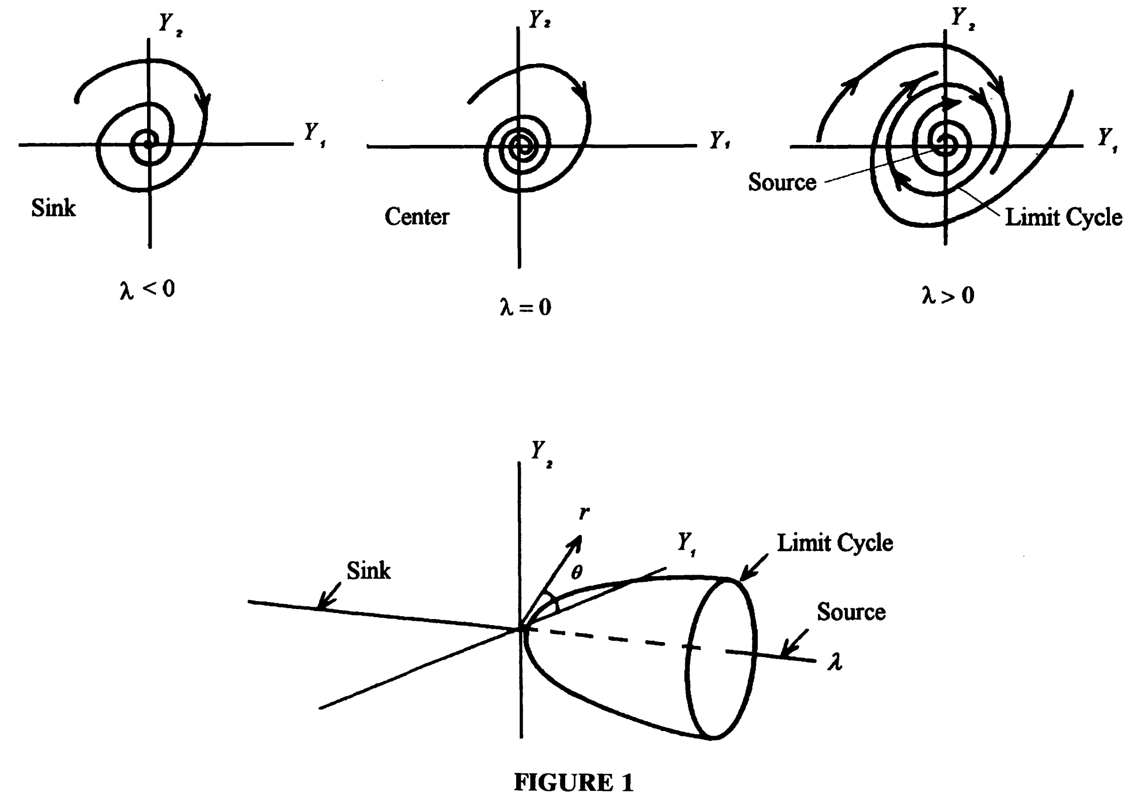

FIGURE 1

series with varying phase in finite polynomial terms. To show the versatility of the algorithm, we illustrated an example in which the order of nonlinearity is not restricted to odd numbers. The application to limit cycle bifurcation is also demonstrated by the Duffing–van der Pol oscillator.

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