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Optimal Time-Weighted H_2 Model Reduction for Discrete Markovian Jump Systems

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Abstract: This paper considers the optimal time-weighted H_2 model reduction problem for discrete Markovian jump linear systems (MJLSs). The purpose is to find a mean square stable MJLS of lower order such that the time-weighted H_2 norm of the corresponding error system is minimized for a given mean square stable discrete MJLS. A new notation named time-weighted H_2 norm of discrete MJLS is defined for the model reduction purpose for the first time. Then a computational formula of the time-weighted H_2 norm is given. Based on this formula, a gradient flow method is proposed to solve the optimal time-weighted H_2 model reduction problem. Finally, a numerical example is used to illustrate the effectiveness of the proposed approach.

Key Words: H_2 norm, discrete Markovian jump system, Model reduction, Time-weighted error

1 Introduction

Mathematical modelling of physical systems often results in high-order models and it is desirable to replace these high-order models with reduced ones with respect to some given criteria. This has motivated the study of the model reduction problem with various approaches [2], [10], [13]–[15] For the model reduction problems of MJLS, the efforts are mainly concentrated on three kinds of reduction problems, namely, classical balanced truncation model reduction, H_∞ model reduction, and H_2 model reduction. A suboptimal balanced truncation algorithm is developed in [8] for discrete MJLSs. An H_∞ model reduction approach for MJLS is proposed both in continuous and discrete time case [21]. Lee and Huang [9] give an effective computational algorithm involving LMIs to solve the H_∞ structured model reduction problem for discrete MJLS. The H_∞ model reduction problems with the transition probabilities are partially unknown, uncertain and piecewise stationary are investigated in [18, 20, 23]. The study on H_∞ model reduction problem is also extended to the singular Markovian jump systems [11, 19]. It is proposed in [16] that an optimal H_2 model reduction method for continuous MJLS, in which the problem of time-weighted H_2 model reduction is introduced and solved via a gradient flow method. However, to the authors' best knowledge, there is no result on H_2 model reduction of discrete time MJLS nor the formulation of time-weighted H_2 model reduction problem even for the deterministic case (with no jump) in the literature to date.

The square of the H_2 norm of a linear discrete system can be regarded as the total energy of pulse response of the system in the time domain. This concept is concerned with the overall error on the infinite time horizon in the H_2 model reduction problem, hence the steady state of the error output in the model reduction problem is not emphasized in many cases. Based on this consideration, the time-weighted H_2 norm will be defined with a time-weighting factor k embedded into usual H_2 norm. Similar performance indices have been considered on optimal or suboptimal regulator design [3, 7], guaranteed cost control problems [12, 17], optimal output tracking [1]. Since the time-weighted H_2 norm lays a penalty on the steady state as time increases, hence the approximation error is expected to converge more rapidly in the time-weighed H_2 model reduction problem.

In this paper, the optimal time-weighted H_2 model reduction problem of discrete MJLS will be investigated. First, the time-weighted H_2 norm of discrete MJLS is defined, and then a computational method of this new performance index is constructed. Secondly, we propose a gradient flow method for the solution of the time-weighted H_2 model reduction problem. Finally, a numerical example is given to demonstrate the effectiveness of the proposed approach.

Notation. Throughout this paper, for a real symmetric matrix X, the notation $X \geq 0$ (respectively, $X > 0$) means that X is positive semi-definite (respectively, positive definite). The superscript "T" represents the transpose, and $\mathbf{P}\{\cdot\}$ denotes the probability. The notation $\mathrm{tr}(\cdot)$ represents the trace of a square matrix, I is the identity matrix with appropriate dimension. We denote $\mathcal{E}(\cdot)$ as the expectation.

2 Problem Formulation

Considering the following discrete Markovian jump linear system (MJLS)

$$(\Sigma): \begin{cases} x(k+1) = A(\theta(k))x(k) + B(\theta(k))u(k), \\ z(k) = C(\theta(k))x(k), \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ represents the state variable of the system, $u(k) \in \mathbb{R}^p$ is the control input vector and $z(k) \in \mathbb{R}^m$ is the output. The parameter $\theta(k)$ stands for the state of a Markov chain taking values in a finite set $S = \{1, 2, \ldots, N\}$ with transition probability matrix given by $\Pi = [p_{i,j}]$, where

$$p_{i,j} = \mathbf{P}\{\theta(k+1) = j|\theta(k) = i\} \geq 0,\ i,\ j = 1, \ldots, N,$$

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and $\sum_{j=1}^{N} p_{i,j} = 1,\ i = 1,\dots,N$. The Markov chain $\{\theta(k),\ k = 0,1,2,\dots\}$ is assumed to have an initial distribution $\mu = (\mu_1,\dots,\mu_N)$. For convenience, we denote $A(\theta(k)) = A_i,\ B(\theta(k)) = B_i,\ C(\theta(k)) = C_i$ when $\theta(k) = i \in S$.

In the following, we denote $X = (X_1,\dots,X_N)$ for an N-tuple of matrices with the same dimensions, and denote

$$\begin{aligned}\mathcal{H}^{m,n} &= \{X = (X_1,\dots,X_N) \mid X_i \in \mathbb{R}^{m\times n},\ i \in S\},\\ \mathcal{H}^{n} &= \mathcal{H}^{n,n},\\ \mathcal{H}^{n}_{+} &= \{X = (X_1,\dots,X_N) \mid X \in \mathcal{H}^{n},\ X_i \geq 0,\ i \in S\}.\end{aligned}$$

For $A = (A,\dots,A_N) \in \mathcal{H}^n$, $X = (X_1,\dots,X_N) \in \mathcal{H}^n$, we define

$$\mathcal{L}_i(A,X) = X_i - \sum_{j=1}^{N} p_{i,j} A_i^T X_j A_i, \tag{2}$$

$$\mathcal{R}_i(A,X) = X_i - \sum_{j=1}^{N} p_{j,i} A_j X_j A_j^T. \tag{3}$$

Lemma 1 [5] *The following assertions are equivalent:*

(1) System (Σ) is mean-square stable (MSS).

(2) System (Σ) is exponentially mean-square stable (EMSS).

(3) There exists $X = (X_1,\dots,X_N) \in \mathcal{H}^n$, $X_i > 0$ for all $i \in S$ such that $\mathcal{L}_i(A,X) > 0$, $i \in S$.

(4) For any given $X = (X_1,\dots,X_N) \in \mathcal{H}^n$, $X_i > 0$, there exists a unique $R = (R_1,\dots,R_N) \in \mathcal{H}^n$ with $R_i > 0$, such that $\mathcal{L}_i(A,R) = X_i$, $i \in S$.

(5) For any given $X = (X_1,\dots,X_N) \in \mathcal{H}^n$, $X_i > 0$, there exists a unique $T = (T_1,\dots,T_N) \in \mathcal{H}^n$ with $T_i > 0$, such that $\mathcal{R}_i(A,T) = X_i$, $i \in S$.

In Proposition 1, if we set $X = (X_1,\dots,X_N) \in \mathcal{H}^n_+$, the uniqueness stated in (4) and (5) also hold for $R = (R_1,\dots,R_N) \in \mathcal{H}^n_+$, and $T = (T_1,\dots,T_N) \in \mathcal{H}^n_+$.

From Proposition 1, we can easily obtain the following result.

Lemma 2 *System (Σ) is MSS if and only if one of the following statements holds.*

(1) For any given $R_0 = (R_{0,1},\dots,R_{0,N}) \in \mathcal{H}^n_+$, there exist unique $R_l = (R_{l,1},\dots,R_{l,N}) \in \mathcal{H}^n_+$, $l = 1,2$, such that

$$\mathcal{L}_i(A,R_l) = R_{l-1,i}, \quad l = 1,2. \tag{4}$$

(2) For any given $T_0 = (T_{0,1},\dots,T_{0,N}) \in \mathcal{H}^n_+$, there exist unique $T_l = (T_{l,1},\dots,T_{l,N}) \in \mathcal{H}^n_+$, $l = 1,2$, such that

$$\mathcal{R}_i(A,T_l) = T_{l-1,i}, \quad l = 1,2. \tag{5}$$

Definition 1 [5] *The H_2-norm of system (Σ) is given by*

$$J_0(\Sigma) = \left(\sum_{s=1}^{p} \|z_s\|_2^2\right)^{1/2},$$

where $\|z_s\|_2^2 = \sum_{k=0}^{\infty} \mathcal{E}\left[z_s^T(k) z_s(k)\right]$ and z_s represents the output $\{z_s(k) \mid k = 0,1,2,\dots\}$ when

(1) the input is given by $u(0) = e_s$, $u(k) = 0$ when $k > 0$, where e_s is the p-dimensional unit vector formed by having 1 at the sth position and zero elsewhere;

(2) $x(0) = 0$ and $\theta(0) = i$ with probability $\mu_i > 0$ for $i \in S$.

Definition 2 *For discrete MSS system (Σ), the time-weighted H_2-norm is defined as*

$$J_1(\Sigma) = \left(\sum_{s=1}^{p}\sum_{k=0}^{\infty} \mathcal{E}\left[k z_s^T(k) z_s(k)\right]\right)^{1/2}, \tag{6}$$

where the input and initial conditions satisfy the same assumptions in Definition 1.

This paper is concerned with the model reduction problem for MJLS with a time-weighted H_2 error measure. That is, for a given $\bar{n}$th-order MSS Markovian jump linear system,

$$(\bar{\Sigma}):\ \begin{cases}\bar{x}(k+1) = & \bar{A}(\theta(k))\bar{x}(k) + \bar{B}(\theta(k))u(k),\\ \quad\ \bar{z}(k) = & \bar{C}(\theta(k))\bar{x}(k),\end{cases}$$

$\bar{x}(k) \in \mathbb{R}^{\bar{n}}$, $u(k) \in \mathbb{R}^p$, $\bar{z}(k) \in \mathbb{R}^m$, our purpose is to find an $\hat{n}$th-order MSS Markovian jump system

$$(\hat{\Sigma}):\ \begin{cases}\hat{x}(k+1) = & \hat{A}(\theta(k))\hat{x}(k) + \hat{B}(\theta(k))u(k),\\ \quad\ \hat{z}(k) = & \hat{C}(\theta(k))\hat{x}(k),\end{cases}$$

where $\hat{x}(k) \in \mathbb{R}^{\hat{n}}$, $u(k) \in \mathbb{R}^p$, $\hat{z}(k) \in \mathbb{R}^m$, $\hat{n} < \bar{n}$, and $\theta(k)$ is the same Markov chain as in $(\bar{\Sigma})$, such that the time-weighted H_2 norm of the corresponding error system (Σ) as in (1) is minimized, where $x(k) \in \mathbb{R}^n$, $n = \hat{n} + \bar{n}$, $z(k) = \bar{z}(k) - \hat{z}(k)$, and

$$A_i = \begin{bmatrix}\bar{A}_i & 0\\ 0 & \hat{A}_i\end{bmatrix},\quad B_i = \begin{bmatrix}\bar{B}_i\\ \hat{B}_i\end{bmatrix}, C_i = \begin{bmatrix}\bar{C}_i & -\hat{C}_i\end{bmatrix}.$$

For convenience, we denote

$$\begin{aligned}\hat{\Sigma}_i &= \begin{bmatrix}\hat{A}_i & \hat{B}_i\\ \hat{C}_i & 0\end{bmatrix} \in \mathbb{R}^{(\hat{n}+m)\times(\hat{n}+p)},\ i = 1,\dots,N,\\ \hat{\Sigma} &= \begin{bmatrix}\hat{\Sigma}_1 & \cdots & \hat{\Sigma}_N\end{bmatrix} \in \mathcal{H}^{\hat{n}+m,\hat{n}+p}.\end{aligned} \tag{7}$$

Then, the time-weighted H_2 model reduction problem of discrete MJLS can be described as the following optimal problem:

$$\min J_1^2(\Sigma), \tag{8}$$

subject to

$$\hat{\Sigma} \in \Omega \triangleq \left\{\hat{\Sigma} \in \mathcal{H}^{\hat{n}+m,\hat{n}+p} \mid (\hat{\Sigma}) \text{ is MSS}\right\}. \tag{9}$$

3 Computation of Time-weighted H_2 Norm

In this section, the time-weighted H_2 norm of the error system will be computed. By using the properties of the trace we can easily obtain the following two results.

Lemma 3 *For any given $A = (A_1,\dots,A_N) \in \mathcal{H}^n$, $R = (R_1,\dots,R_N) \in \mathcal{H}^n$, $T = (T_1,\dots,T_N) \in \mathcal{H}^n$, we have*

$$\sum_{i=1}^{N} \operatorname{tr}\left[T_i \mathcal{L}_i(A,R)\right] = \sum_{i=1}^{N} \operatorname{tr}\left[\mathcal{R}_i(A,T) R_i\right].$$

Lemma 4 *If there exist $R_l \in \mathcal{H}^n_+$, $T_l \in \mathcal{H}^n_+$, $l = 0,1,2$, satisfying (4), (5), then*

$$\sum_{i=1}^{N} \operatorname{tr}\left[R_{l,i} T_{2-l,i}\right] = \sum_{i=1}^{N} \operatorname{tr}\left[R_{l+1,i} T_{1-l,i}\right], \quad l = 0,1.$$

Lemma 5 *[5] Denote* $X_i(k) = \mathcal{E}\left[x(k)x(k)^T 1_{\{\theta(k)=i\}}\right]$, $i \in S$, *for any* $k = 0, 1, 2, \ldots$, *where* $1_{\{\cdot\}}$ *stands for the Dirac measure, then the following equality holds:*

$$X_i(k+1) = X_i(k) - \mathcal{R}_i\left(A, X(k)\right).$$

Lemma 6 *Consider MSS MJLS* (Σ) *with* $u(k) = 0$, $k = 1, 2, \ldots$, *if* $R_l \in \mathcal{H}_+^n$, $l = 1, 2$, *is the solution of (4), then we have*

$$\begin{aligned}&\sum_{k=1}^{\infty}\mathcal{E}\left[kx^T(k)R_{0,\theta(k)}x(k)\right]\\ &= \mathcal{E}\left[x^T(1)R_{1,\theta(1)}x(1)\right] + \mathcal{E}\left[x^T(2)R_{2,\theta(2)}x(2)\right].\end{aligned}$$

Proof. Denote $X(k) = \left[\, X_1(k) \ \cdots \ X_N(k) \,\right] \in \mathcal{H}^n$, then it is easy to show from (4), Lemma 3 and Lemma 5 that

$$\begin{aligned}&\mathcal{E}\left[x^T(k)R_{d,\theta(k)}x(k)\right]\\ &= \operatorname{tr}\sum_{i=1}^{N}\mathcal{E}\left[x(k)x^T(k)1_{\{\theta(k)=i\}}\right]R_{d,i}\\ &= \operatorname{tr}\sum_{i=1}^{N}X_i(k)\mathcal{L}_i\left(A, R_{d+1}\right)\\ &= \operatorname{tr}\sum_{i=1}^{N}\mathcal{R}_i\left(A, X(k)\right)R_{d+1,i}\\ &= \operatorname{tr}\sum_{i=1}^{N}\left[X_i(k) - X_i(k+1)\right]R_{d+1,i}\\ &= \operatorname{tr}\sum_{i=1}^{N}\left\{\mathcal{E}\left[x(k)x(k)^T R_{d+1,\theta(k)}1_{\{\theta(k)=i\}}\right]\right.\\ &\quad \left. -\mathcal{E}\left[x(k+1)x(k+1)^T R_{d+1,\theta(k+1)}\ 1_{\{\theta(k+1)=i\}}\right]\right\}\\ &= \mathcal{E}\left[x(k)^T R_{d+1,\theta(k)}x(k)\right]\\ &\quad -\mathcal{E}[x(k+1)^T R_{d+1,\theta(k+1)}x(k+1)], \quad d = 0, 1,\end{aligned}$$

hence

$$\begin{aligned}&\sum_{k=1}^{\infty}\mathcal{E}\left[kx^T(k)R_{0,\theta(k)}x(k)\right]\\ &= \sum_{k=1}^{\infty}\mathcal{E}\left[kx^T(k)R_{1,\theta(k)}x(k)\right.\\ &\quad \left. - kx^T(k+1)R_{1,\theta(k+1)}x(k+1)\right]\\ &= \sum_{k=1}^{\infty}\mathcal{E}\left[kx^T(k)R_{1,\theta(k)}x(k)\right.\\ &\quad \left. - (k+1)x^T(k+1)R_{1,\theta(k+1)}x(k+1)\right]\\ &\quad + \sum_{k=1}^{\infty}\mathcal{E}[x^T(k+1)R_{1,\theta(k+1)}x(k+1)],\end{aligned} \tag{10}$$

and

$$\begin{aligned}&\sum_{k=1}^{\infty}\mathcal{E}\left[x^T(k+1)R_{1,\theta(k+1)}x(k+1)\right]\\ &= \sum_{k=2}^{\infty}\mathcal{E}\left[x^T(k)R_{1,\theta(k)}x(k)\right]\\ &= \sum_{k=2}^{\infty}\mathcal{E}\left[x^T(k)R_{2,\theta(k)}x(k)\right.\\ &\quad \left. - x^T(k+1)R_{2,\theta(k+1)}x(k+1)\right].\end{aligned} \tag{11}$$

From Lemma 1, system (Σ) is MSS and at the same time it is EMSS. Therefore

$$\lim_{k\to\infty}\mathcal{E}\left[x^T(k)R_{2,\theta(k)}x(k)\right] = 0,$$

$$\lim_{k\to\infty}\mathcal{E}\left[kx^T(k)R_{1,\theta(k)}x(k)\right] = 0,$$

which together with (10)–(11) imply

$$\begin{aligned}&\sum_{k=1}^{\infty}\mathcal{E}\left[kx^T(k)R_{1,\theta(k)}x(k)\right.\\ &\quad \left. -(k+1)\ x^T(k+1)R_{1,\theta(k+1)}x(k+1)\right]\\ &= \mathcal{E}[x^T(1)R_{1,\theta(1)}x(1)].\end{aligned} \tag{12}$$

and

$$\begin{aligned}&\sum_{k=1}^{\infty}\mathcal{E}\left[x^T(k+1)R_{1,\theta(k+1)}x(k+1)\right]\\ &= \mathcal{E}[x^T(2)R_{2,\theta(2)}x(2)].\end{aligned} \tag{13}$$

Combining (10)–(13) can complete the proof. □

Theorem 1 *Consider MSS system* (Σ), *set* $A = (A_1, \ldots, A_N)$, *and*

$$P_{0,i} = \sum_{j=1}^{N}p_{j,i}\mu_j B_j B_j^T, \quad Q_{0,i} = C_i^T C_i, \quad i \in S. \tag{14}$$

Suppose $P_l = (P_{l,1}, \ldots, P_{l,N}) \in \mathcal{H}_+^n$, $Q_l = (Q_{l,1}, \ldots, Q_{l,N}) \in \mathcal{H}_+^n$, $l = 1, 2$, *are the unique solutions of the following equations:*

$$\mathcal{L}_i(A, Q_1) = Q_{0,i},\ \mathcal{L}_i(A, Q_2) = Q_{1,i}, \tag{15}$$

$$\mathcal{R}_i(A, P_1) = P_{0,i},\ \mathcal{R}_i(A, P_2) = P_{1,i}, \tag{16}$$

then the square of the time-weighted H_2 *norm of* (Σ) *can be computed as*

$$\begin{aligned}J_1^2\left(\Sigma\right) &= \operatorname{tr}\left[\sum_{i=1}^{N}\mu_i B_i^T\left(\sum_{j=1}^{N}p_{i,j}Q_{2,j}\right)B_i\right]\\ &= \operatorname{tr}\left(\sum_{i=1}^{N}C_i P_{2,i}C_i^T\right).\end{aligned} \tag{17}$$

Proof. Suppose $z_s(k)$ is the output corresponding to the input and initial conditions as in Definition 2. By using Lemma 6 we have

$$\begin{aligned}&\sum_{k=0}^{\infty}\mathcal{E}\left[kz_s^T(k)z_s(k)\right]\\ &= \sum_{k=1}^{\infty}\mathcal{E}\left[kx^T(k)Q_{0,\theta(k)}x(k)\right]\\ &= \mathcal{E}\left[x^T(1)Q_{1,\theta(1)}x(1)\right] + \mathcal{E}\left[x^T(2)Q_{2,\theta(2)}x(2)\right]\end{aligned} \tag{18}$$

It is not difficult to show that

$$\begin{aligned}&\mathcal{E}\left[x^T(2)Q_{2,\theta(2)}x(2)\right]\\ &= \sum_{i=1}^{N}\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\mu_i p_{i,i_1}p_{i_1,i_2}e_s^T B_i^T A_{i_1}^T Q_{2,i_2}A_{i_1}B_i e_s\end{aligned}$$

$$= \sum_{i=1}^{N} \left[\mu_i e_s^T B_i^T \left(\sum_{i_1=1}^{N} p_{i,i_1} A_{i_1}^T \left(\sum_{i_2=1}^{N} p_{i_1,i_2} Q_{2,i_2} \right) \cdot A_{i_1} \right) B_i e_s \right]$$
$$= \sum_{i=1}^{N} \left[\mu_i e_s^T B_i^T \left(\sum_{i_1=1}^{N} p_{i,i_1} \left(Q_{2,i_1} - Q_{1,i_1} \right) \right) B_i e_s \right]. \tag{19}$$

Similarly, it can be derived that

$$\mathcal{E}\left[x^T(1) Q_{1,\theta(1)} x(1)\right] = \sum_{i=1}^{N} \left[\mu_i e_s^T B_i^T \left(\sum_{i_1=1}^{N} p_{i,i_1} Q_{1,i_1} \right) B_i e_s \right]. \tag{20}$$

According to (18)–(20) we have that

$$J_1^2(\Sigma) = \sum_{s=1}^{p} \sum_{k=0}^{\infty} \mathcal{E}\left[k z_s^T(k) z_s(k)\right] = \sum_{i=1}^{N} \mathrm{tr} \left[\mu_i B_i^T \left(\sum_{j=1}^{N} p_{i,j} Q_{2,j} \right) B_i \right], \tag{21}$$

which implies

$$J_1^2(\Sigma) = \sum_{i=1}^{N} \mathrm{tr} \left[\left(\sum_{j=1}^{N} p_{j,i} \mu_j B_j B_j^T \right) Q_{2,i} \right] = \sum_{i=1}^{N} \mathrm{tr}\left(P_{0,i} Q_{2,i}\right).$$

It can be obtained from Lemma 4 and (15)–(16) that

$$\sum_{i=1}^{N} \mathrm{tr}\left(P_{0,i} Q_{2,i}\right) = \sum_{i=1}^{N} \mathrm{tr}\left(P_{1,i} Q_{1,i}\right) = \sum_{i=1}^{N} \mathrm{tr}\left(P_{2,i} Q_{0,i}\right), \tag{22}$$

Combining (21)–(22) and (14) yields (17).

This completes the proof. □

From (21)–(22), we can get equivalent expressions in the following theorem.

Theorem 2 *Consider MSS system (Σ), its time-weighted H_2 norm is given by*

$$J_1^2(\Sigma) = \sum_{i=1}^{N} \mathrm{tr}\left(P_{l,i} Q_{2-l,i}\right), \tag{23}$$

where $l \in \{0,1,2\}$ and $P_l = (P_{l,1}, \ldots, P_{l,N}) \in \mathcal{H}_+^n$, $Q_l = (Q_{l,1}, \ldots, Q_{l,N}) \in \mathcal{H}_+^n$, $l = 0,1,2$, are defined as that in Theorem 1.

4 Time-weighted H_2 Model Reduction

To solve the optimization problem (8), the gradient flow method will be adopted in this paper. This method has been successfully applied to solve many control problems [6, 16]. First, the partial derivatives of J_1^2 will be given.

Theorem 3 *The partial derivatives of J_1^2 with respect to $\hat{A}_d$, $\hat{B}_d$, $\hat{C}_d$ are given by*

$$\frac{\partial J_1^2}{\partial \hat{A}_d} = 2 \left[0_{\hat{n}\times\bar{n}} \ I_{\hat{n}} \right] \sum_{i=1}^{N} p_{d,i} \left(Q_{1,i} A_d P_{2,d} + Q_{2,i} A_d P_{1,d} \right) \begin{bmatrix} 0_{\bar{n}\times\hat{n}} \\ I_{\hat{n}} \end{bmatrix}, \tag{24}$$

$$\frac{\partial J_1^2}{\partial \hat{B}_d} = 2\mu_d \left[0_{\hat{n}\times\bar{n}} \ I_{\hat{n}} \right] \sum_{i=1}^{N} \left(p_{d,i} Q_{2,i} \right) B_d, \tag{25}$$

$$\frac{\partial J_1^2}{\partial \hat{C}_d} = 2 C_d P_{2,d} \begin{bmatrix} 0_{\bar{n}\times\hat{n}} \\ -I_{\hat{n}} \end{bmatrix}, \tag{26}$$

where $d = 1, \ldots, N$, and $P_l = \left[P_{l,1} \ \cdots \ P_{l,N} \right]$, $Q_l = \left[Q_{l,1} \ \cdots \ Q_{l,N} \right]$, $l = 1,2$, are the solutions of (15)–(16).

Proof. The proof of (25)–(26) is very easy and we only give the proof of (24) here. Denote $a_d^{u,v}$, $b_d^{u,v}$ be the uth row, vth column element of $\hat{A}_d$ respectively, and set

$$E_{u,v} = \begin{bmatrix} 0_{\bar{n}\times\bar{n}} & 0_{\bar{n}\times\hat{n}} \\ 0_{\hat{n}\times\bar{n}} & \theta_u \theta_v^T \end{bmatrix}$$

with θ_i be the ith standard basis of $\mathbb{R}^{\hat{n}}$, respectively.

Set $\frac{\partial Q_l}{\partial a_d^{u,v}} = \left[\frac{\partial Q_{l,1}}{\partial a_d^{u,v}}, \ldots, \frac{\partial Q_{l,N}}{\partial a_d^{u,v}} \right]$ and take partial derivative with respect to $a_d^{u,v}$ on both sides of (14)–(15) for $d = 1, 2, \ldots, N$, yields

$$\frac{\partial Q_{0,i}}{\partial a_d^{u,v}} = 0, \tag{27}$$

$$\mathcal{L}_i\left(A, \frac{\partial Q_l}{\partial a_d^{u,v}}\right) = \frac{\partial Q_{l-1,i}}{\partial a_d^{u,v}}, \quad \text{if } i \neq d, \tag{28}$$

$$\mathcal{L}_d\left(A, \frac{\partial Q_l}{\partial a_d^{u,v}}\right) = \frac{\partial Q_{l-1,d}}{\partial a_d^{u,v}} + \sum_{j=1}^{N} p_{d,j} \left[E_{u,v}^T Q_{l,j} A_d + A_d^T Q_{l,j} E_{u,v} \right]. \tag{29}$$

It follows from Lemma 4 and (15), (27)–(29) that

$$\sum_{i=1}^{N} \mathrm{tr}\left(P_{0,i} \frac{\partial Q_{2,i}}{\partial a_d^{u,v}} \right) = 2\mathrm{tr} \sum_{j=1}^{N} E_{u,v}^T p_{d,j} \left(Q_{1,j} A_d P_{2,d} + Q_{2,j} A_d P_{1,d} \right). \tag{30}$$

Combining (23) and (30) yields (24).

This completes the proof. □

Similarly we can get the following result based on the computational formula of H_2 norm of discrete MJLS [4, 5].

Theorem 4 *The partial derivatives of J_0^2 with respect to $\hat{A}_d$, $\hat{B}_d$, $\hat{C}_d$ for $d = 1, \ldots, N$, are given by*

$$\frac{\partial J_0^2}{\partial \hat{A}_d} = 2 \left[0_{\hat{n}\times\bar{n}} \ I_{\hat{n}} \right] \sum_{i=1}^{N} p_{d,i} Q_{1,i} A_d P_{1,d} \begin{bmatrix} 0_{\bar{n}\times\hat{n}} \\ I_{\hat{n}} \end{bmatrix},$$

$$\frac{\partial J_0^2}{\partial \hat{B}_d} = 2\mu_d \left[0_{\hat{n}\times\bar{n}} \ I_{\hat{n}} \right] \sum_{i=1}^{N} (p_{d,i} Q_{1,i}) B_d,$$

$$\frac{\partial J_0^2}{\partial \hat{C}_d} = 2 C_d P_{1,d} \begin{bmatrix} 0_{\bar{n}\times\hat{n}} \\ -I_{\hat{n}} \end{bmatrix},$$

where $P_1 = \left[P_{1,1} \ \cdots \ P_{1,N} \right]$, $Q_l = \left[Q_{1,1} \ \cdots \ Q_{1,N} \right]$, are the solutions of (15)–(16).

When we apply the gradient flow method to the optimal time-weighted H_2 model reduction problem (8), we have to take two issues into account in the process of seeking an optimum. One is how to ensure the error system is MSS. The other is how to keep the parameters of the reduced-order model within a bounded domain. In order to solve the optimal time-weighted H_2 model reduction problem under the above conditions, an alternative optimization problem will be considered with an auxiliary objective function:

$$\min \bar{J}_1^2(\Sigma), \tag{31}$$

where $\bar{J}_1^2(\Sigma) \triangleq J_1^2(\Sigma) + \tilde{J}^2(\hat{\Sigma})$, and

$$\begin{aligned} \tilde{J}^2(\hat{\Sigma}) &\triangleq \varepsilon_1 \left\|\hat{\Sigma}\right\|_F^2 + \varepsilon_2 \sum_{i=1}^{N} \text{tr}\left(X_i^{-2}\right) \\ &\equiv \varepsilon_1 \sum_{i=1}^{N} \text{tr}\left(\hat{A}_i\hat{A}_i^T + \hat{B}_i\hat{B}_i^T + \hat{C}_i^T\hat{C}_i\right) \\ &\quad + \varepsilon_2 \sum_{i=1}^{N} \text{tr}\left(X_i^{-2}\right), \end{aligned} \tag{32}$$

and $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ are sufficiently small scalars, $X = (X_1, \ldots, X_N) \in \mathcal{H}^{\hat{n}}$ with $X_i > 0$ for $i \in S$, is the solution of

$$\mathcal{L}_i(A, X) = Z_i, \tag{33}$$

for some given constant matrices $Z_i > 0$ for each $i \in S$.

When both ε_1 and ε_2 tend to zero, the added term $\tilde{J}^2(\hat{\Sigma})$ will converge to zero if the matrices $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$, $i \in S$ are bounded and X_i, $i \in S$ are nonsingular. At the same time, the term $\tilde{J}^2(\hat{\Sigma})$ is a penalty function which ensure the two technical requirements are satisfied. If $\tilde{J}^2$ is bounded, then the parameters of $(\hat{\Sigma})$ are bounded and at the same time, X_i, $i \in S$, are nonsingular. In the model reduction computation, we treat $\hat{\Sigma}(t)$ as time-varying parameters. When the initial reduced-order model $\hat{\Sigma}(0)$ is MSS, the unique solution $X_i(0), i \in S$ of (33) is nonsingular. Hence we can keep the boundness of $\sum_{i=1}^{N} \text{tr}\left(X_i^2(t)\right)^{-1}$ for all values of t such that the reduced-order model $\hat{\Sigma}(t)$ keeps its MSS property. Now we can translate the model reduction problem with constraints into (31) which is unconstrained.

It is worth pointing out that a similar penalty term has shown to be effective when applying to optimization problems [6, 16, 22].

The partial derivatives of $\bar{J}_1^2(\Sigma)$ with respect to $\hat{A}_d$, $\hat{B}_d$, $\hat{C}_d$, $d = 1, \ldots, N$, can be computed in the same way as in [16] and is omitted here.

From above, the gradient flow of $\bar{J}_1^2$ can be written in the form of the following ordinary differential equation

$$\dot{\hat{\Sigma}}(t) = -\frac{1}{2}\text{grad}\bar{J}_1^2(\hat{\Sigma}), \tag{34}$$

where $\dot{\hat{\Sigma}}(t) = \left[\dot{\hat{\Sigma}}_1(t) \cdots \dot{\hat{\Sigma}}_N(t)\right]$, and for $d = 1, \ldots, N$,

$$\dot{\hat{\Sigma}}_d(t) = \begin{bmatrix} \frac{d}{dt}\hat{A}_d(t) & \frac{d}{dt}\hat{B}_d(t) \\ \frac{d}{dt}\hat{C}_d(t) & 0 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} \frac{\partial \bar{J}_1^2}{\partial \hat{A}_d} & \frac{\partial \bar{J}_r^2}{\partial \hat{B}_d} \\ \frac{\partial \bar{J}_1^2}{\partial \hat{C}_d} & 0 \end{bmatrix}.$$

In unweighted case the gradient flow of $\bar{J}_0^2$ can be formed in a similar way. There are some properties concerning ordinary differential equation in the form of (34) which is similar to that in [6, 16, 22]. These properties show that if we choose an initial reduced-order model which is MSS, then we can definitely get a reduced-order model with minimum time-weighted H_2 error using above gradient flow method.

5 Numerical Example

In this section, we present a numerical example to illustrate above results.

Example 1 *Consider discrete MJLS $(\bar{\Sigma})$ with two modes having the following parameters*

$$\bar{A}_1 = \begin{bmatrix} 0.3 & 0 & 0 & 1 \\ 0 & 0.2 & 0 & 0 \\ 0.5 & -0.1 & -0.5 & 0.2 \\ 0.1 & 0 & 0 & -0.3 \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} -0.1 \\ -0.5 \\ 0 \\ 0.2 \end{bmatrix},$$

$$\bar{C}_1 = \begin{bmatrix} 0.5 & 0 & -0.1 & 0.2 \end{bmatrix},$$

$$\bar{A}_2 = \begin{bmatrix} -0.5 & 0 & 0 & 1 \\ 0 & 0.1 & 0 & -0.2 \\ 1 & 0.5 & -0.3 & 0 \\ -0.2 & 0.8 & 0 & 0.4 \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} -0.1 \\ 0.3 \\ 0.5 \\ 0 \end{bmatrix},$$

$$\bar{C}_2 = \begin{bmatrix} 0 & 0.3 & 0 & 0.2 \end{bmatrix}.$$

For the Markov chain, we have the transition probability matrix given by

$$\Pi = \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix},$$

and the initial distribution is $\mu_1 = 0.2$, $\mu_2 = 0.8$.

It is easy to verify by Proposition 1 that this jump system is MSS. By using the presented method, we obtain two optimal second-order reduced models for time-weighted case $(\hat{\Sigma}^1)$ and unweighted case $(\hat{\Sigma}^0)$, respectively. The parameters of the reduced-order model $(\hat{\Sigma}^0)$ can be obtained as

$$\hat{A}_1^{(0)} = \begin{bmatrix} 0.5306 & -0.0519 \\ -0.1310 & 0.2472 \end{bmatrix}, \quad \hat{B}_1^{(0)} = \begin{bmatrix} 0.0000 \\ -0.3334 \end{bmatrix},$$

$$\hat{C}_1^{(0)} = \begin{bmatrix} 0.7939 & -0.0369 \end{bmatrix},$$

$$\hat{A}_2^{(0)} = \begin{bmatrix} 0.0235 & 0.6472 \\ -0.4938 & 0.5589 \end{bmatrix}, \quad \hat{B}_2^{(0)} = \begin{bmatrix} -0.0771 \\ 0.2542 \end{bmatrix},$$

$$\hat{C}_2^{(0)} = \begin{bmatrix} 0.0006 & 0.3485 \end{bmatrix}.$$

The parameters of the reduced-order model $(\hat{\Sigma}^1)$ are computed as

$$\hat{A}_1^{(1)} = \begin{bmatrix} 0.5311 & -0.0154 \\ -0.1861 & 0.2374 \end{bmatrix}, \quad \hat{B}_1^{(1)} = \begin{bmatrix} 0.0297 \\ -0.3281 \end{bmatrix},$$

$$\hat{C}_1^{(1)} = \begin{bmatrix} 0.7988 & -0.0273 \end{bmatrix},$$

$$\hat{A}_2^{(1)} = \begin{bmatrix} -0.0138 & 0.6680 \\ -0.4927 & 0.5543 \end{bmatrix}, \quad \hat{B}_2^{(1)} = \begin{bmatrix} -0.0685 \\ 0.2601 \end{bmatrix},$$

$$\hat{C}_2^{(1)} = \begin{bmatrix} 0.0162 & 0.3364 \end{bmatrix}.$$

Both above reduced-order models are MSS, and their optimal time-weighted H_2 norm are computed as $J_0(\Sigma^0) = 0.0448$, $J_1(\Sigma^1) = 0.0718$, respectively. Randomly creates a sample path of a Markov chain from $k = 0$ to $k = 15$ as the ordered sequence $\{\theta_0(k)\} = \{2, 2, 2, 2, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2\}$. Figure 1 shows the pulse responses of the original system, the reduced-order models both with H_2 performance and time-weighted H_2 performance corresponding to this path. Figure 2 shows the approximate errors of the pulse response. It can be seen from Figure 1 that both reduced-order models are good approximations of the original system. When we use the time-

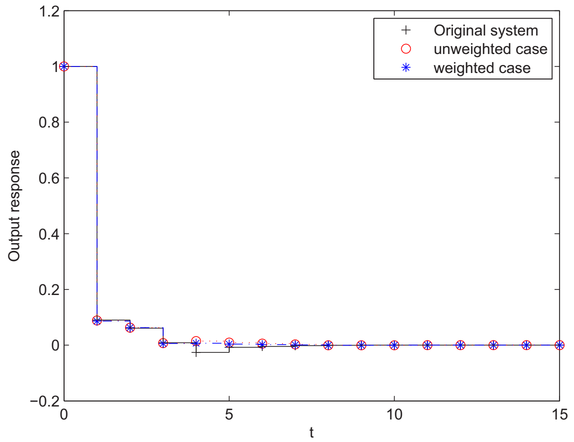

Fig. 1. Pulse responses.

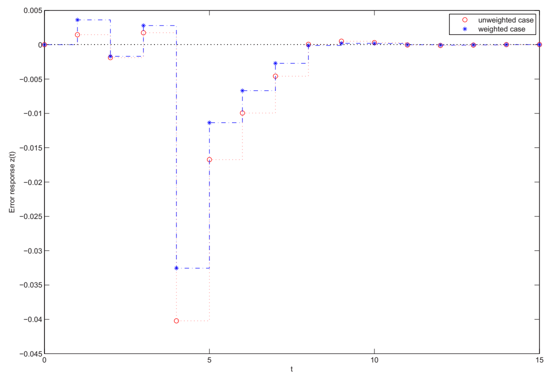

Fig. 2. Approximate errors of the pulse response.

weighted H_2 performance, the approximate error falls to zero more quickly than unweighted case which can be observed from Figure 2.

6 Conclusion

In this paper, time-weighted H_2 model reduction problem of discrete Markovian jump linear systems has been considered. The time-weighted H_2 norm of discrete MJLSs for model reduction purpose is defined for the first time. The computational formula of the time-weighted H_2 norm of discrete MJLS is given in terms of the solutions of two sets of discrete Markovian jump Lyapunov equations. The optimal time-weighted H_2 model reduction problem of discrete MJLS is introduced and the gradient flow method is adopted to solve the optimal reduction problem. The gradient flow is formed by a set of ordinary differential equations to seek an optimal reduced-order model. A numerical example has shown the effectiveness of the proposed reduction approach and the superiority of the time-weighted model reduction in terms of convergence rate over the unweighted case.

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