

OPTIMAL FINANCING AND DIVIDEND DISTRIBUTION IN A GENERAL DIFFUSION MODEL WITH REGIME SWITCHING

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Abstract

We study the optimal financing and dividend distribution problem with restricted dividend rates in a diffusion type surplus model where the drift and volatility coefficients are general functions of the level of surplus and the external environment regime. The environment regime is modeled by a Markov process. Both capital injections and dividend payments incur expenses. The objective is to maximize the expectation of the total discounted dividends minus the total cost of capital injections. We prove that it is optimal to inject capitals only when the surplus tends to fall below zero and to pay out dividends at the maximal rate when the surplus is at or above the threshold dependent on the environment regime.

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1. Introduction

The optimal dividend strategy problem has gained extensive attention. In the diffusion setting, many works concerning dividend optimization use the Brownian motion model for the underlying cashflow process. [1] extends the basic model by assuming that the drift coefficient is a linear function of the level of cashflow and [2] uses the mean-reverting model and solves the optimization problem. [6] considers the optimization problem under the model where the drift coefficient is proportional to the level of cashflow and the diffusion coefficient is proportional to the square root of the cashflow level. [13], [11], [18] and

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some references therein address the optimization problems for the general diffusion model where the drift and diffusion coefficients are general functions of the cashflow level.

An interesting and different direction of extension is to include the impact of the changing external environments/conditions (for example, macroeconomic conditions and weather conditions) into modeling of the cashflows. A continuous time Markov chain can be used to model the state of the external environment condition, of which the use is supported by observation in financial markets. The optimal dividend problem with regular control for Markov-modulated risk processes has been investigated under a variety of assumptions. [14] solves the dividend optimization problem for a Markov-modulated Brownian motion model with both the drift and diffusion coefficients modulated by a two-state Markov Chain. [17] solves the problem for the Brownian motion model modulated by a multiple state Markov chain.

The optimality results in all the above works imply that distributing dividends according to the optimal strategy leads almost surely to ruin. [3] proposes to include capital injections (financing) to prevent the surplus becomes negative and therefore prevent ruin. Under the Brownian motion, [10] investigates the optimal dividend and financing problem, and [5] studies the problem with risk exposure control through control of reinsurance rate. The optimality problem with control in both capital injections and dividend distribution in a Cramér-Lundberg model is addressed in [12]. [16] solves the problem for dual model with transaction costs.

The purpose of this paper is to investigate optimal financing and dividend distribution problem with restricted dividend rates in a general diffusion model with regime switching. Under the model, the drift and volatility coefficients are general functions of the level of surplus and the external environment regime, which is modeled by a Markov process. Similar to the “reflection problem”, the company can control the financing /capital injections process (a deposit process) and the dividend distribution process (a “withdrawal” process). Both capital injections and dividend payments will incur transaction costs. Sufficient capital injections must be made to keep the controlled surplus process nonnegative and the dividend payment rate is capped. This paper can be considered as an extension of the existing works on the dividend optimization problem with restricted dividend rates for the diffusion models with or without regime switching. The model considered is more general as it assumes that 1. the drift and volatility are general functions of the cashflows; and 2. the model risk parameters (including drift, volatility and discount rates) are dependent on the external environment regime.

The rest of the paper is organized as follows. We formulate the optimization problem in Section 2. An auxiliary problem is introduced and solved in Section 3. Section 4 presents the optimality results. A conclusion is provided in Section 5. Proofs are relegated to Appendix.

2. Problem formulation

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{W_t; t \geq 0\}$ and $\{\xi_t; t \geq 0\}$ be respectively a standard Brownian motion and a Markov chain with the finite state space \mathcal{S} and the transition intensity matrix $Q = (q_{ij})_{i,j \in \mathcal{S}}$. The two stochastic processes $\{W_t; t \geq 0\}$ and $\{\xi_t; t \geq 0\}$ are independent. We use $\{\mathcal{F}_t; t \geq 0\}$ to denote the minimal complete σ -field generated by the stochastic process $\{(W_t, \xi_t); t \geq 0\}$. Let X_t denote the surplus at time t of a firm in absence of financing and dividend distribution. Assume that X_0 is \mathcal{F}_0 measurable and that X_t follows the dynamics, $dX_t = \mu(X_{t-}, \xi_{t-})dt + \sigma(X_{t-}, \xi_{t-})dW_t$ for $t \geq 0$, where the functions $\mu(\cdot, j)$ and $\sigma(\cdot, j)$ are Lipschitz continuous, differentiable and grow at most linearly on $[0, \infty)$ with $\mu(0, u) \geq 0$. Furthermore, the function $\mu(\cdot, j)$ is concave and the function $\sigma(\cdot, j)$ is positive and non-vanishing.

The firm must have nonnegative assets in order to continue its business. If necessary, the firm needs to raise money from the market. For each dollar of money raised, it includes c dollars of transaction cost and hence leads to an increase of $1 - c$ dollars in the surplus through capital injection. Let C_t denote the cumulative amount of capital injections up to time t . Then the total cost for capital injections up to time t is $\frac{C_t}{1-c}$. The company can distribute part of its assets to the shareholders as dividends. For each dollar of dividends received by the shareholders, there will be d dollars of cost incurred to them. Let D_t denote the cumulative amount of dividends paid out by the company up to time t . Then the total amount of dividends received by the shareholders up to time t is $\frac{D_t}{1+d}$. We consider the case where the dividend distribution rate is restricted. Let the random variable l_s denote the dividend payment rate at time s with the restriction $0 \leq l_s \leq \bar{l}$ where $\bar{l}(> 0)$ is constant. Then $D_t = \int_0^t l_s ds$. Both C_t and D_t are controlled by the company's decision makers. Define $\pi = \{(C_t, D_t); t \geq 0\}$. We call π a control strategy.

Taking financing and dividend distribution into consideration, the dynamics of the (controlled) surplus process with the strategy π becomes

$$dX_t^\pi = (\mu(X_{t-}^\pi, \xi_{t-}) - l_t)dt + \sigma(X_{t-}^\pi, \xi_{t-})dW_t + dC_t, \quad t \geq 0. \quad (2.1)$$

Define $P_{(x,i)}(\cdot) = P(\cdot | X_0 = x, \xi_0 = i)$, $E_{(x,i)}[\cdot] = E[\cdot | X_0 = x, \xi_0 = i]$, $P_i(\cdot) = P(\cdot | \xi_0 = i)$, and $E_i[\cdot] = E[\cdot | \xi_0 = i]$. The performance of a control strategy π is measured by its return function defined as follows:

$$R_\pi(x, i) = E_{(x,i)} \left[\int_0^\infty e^{-\Lambda_t} \frac{l_t}{1+d} dt - \int_0^\infty e^{-\Lambda_t} \frac{1}{1-c} dC_t \right], \quad x \geq 0, i \in \mathcal{S}, \quad (2.2)$$

where $\Lambda_t = \int_0^t \delta_{\xi_s} ds$ with δ_{ξ_s} representing the force of discount at time s . Assume $\delta_i > 0$, $i \in \mathcal{S}$.

A strategy $\pi = \{(C_t, D_t); t \geq 0\}$ is said to be *admissible* if (i) both $\{C_t; t \geq 0\}$ and $\{D_t; t \geq 0\}$ are nonnegative, increasing, càdlàg, and $\{\mathcal{F}_t; t \geq 0\}$ -adapted processes, (ii) there exists an $\{\mathcal{F}_t; t \geq 0\}$ -adapted

process $\{l_t; t \geq 0\}$ with $l_t \in [0, \bar{l}]$ such that $D_t = \int_0^t l_s ds$ and (iii) $X_t^\pi \geq 0$ for all $t > 0$. We use Π to denote the class of admissible strategies.

Since $\{C_t; t \geq 0\}$ is right continuous and increasing, we have the following decomposition: $C_t = \tilde{C}_t + C_t - C_{t-}$, where $\{\tilde{C}_t; t \geq 0\}$ represents the continuous part of $\{C_t; t \geq 0\}$.

For convenience, we use X , X^π , ξ and (X^π, ξ) to denote the stochastic processes $\{X_t; t \geq 0\}$, $\{X_t^\pi; t \geq 0\}$, $\{\xi_t; t \geq 0\}$ and $\{(X_t^\pi, \xi_t); t \geq 0\}$, respectively. Note that for any admissible strategy π , the stochastic process X^π is right-continuous and adapted to the filtration $\{\mathcal{F}_t; t \geq 0\}$.

The objective of this paper is to study the maximal return function (value function):

$$V(x, i) = \sup_{\pi \in \Pi} R_\pi(x, i), \quad (2.3)$$

and to identify the associated optimal admissible strategy, if any. Following the standard argument in stochastic control theory [4], we can show that the value function fulfils the following dynamic programming principle: $V(x, i) = \sup_{\pi \in \Pi} E_{(x, i)} \left[\int_0^\tau \frac{l_t e^{-\Lambda t}}{1+d} dt - \int_0^\tau \frac{e^{-\Lambda t}}{1-c} dC_t + e^{-\Lambda \tau} V(X_\tau^\pi, \xi_\tau^\pi) \right]$ for any stopping time τ .

3. An auxiliary optimization problem

Motivated by [8], which introduces an auxiliary problem where the objective functional is modified in a way such that only the “returns” over the time period from the beginning up to the first regime switching are included plus a terminal value at the moment of the first regime switching, we start with a similar auxiliary problem first. The optimality results of this problem will play an essential role in solving the original optimization problem.

Throughout the paper, we define $\underline{\delta} = \min_{j \in \mathcal{S}} \delta_j$, $q_i = -q_{ii}$, and $\sigma_1 = \inf\{t > 0 : \xi_t \neq \xi_0\}$. Here, σ_1 is the first transition time of the Markov process ξ . For any function $g : \mathbb{R}^+ \times \mathcal{S} \rightarrow \mathbb{R}^+$, we use $g'(\cdot)$ and $g''(\cdot)$ to denote the first order and second order derivatives, respectively, with respect to the first argument. We start with introducing two special classes of functions.

Definition 3.1. (i) Let \mathcal{C} denote the class of functions $g : \mathbb{R}^+ \times \mathcal{S} \rightarrow \mathbb{R}$ such that for each $j \in \mathcal{S}$, $g(\cdot, j)$ is nondecreasing and $g(\cdot, j) \leq \frac{\bar{l}}{\underline{\delta}(1+d)}$. (ii) Let \mathcal{D} denote the class of functions $g \in \mathcal{C}$ such that for each $j \in \mathcal{S}$, $g(\cdot, j)$ is concave and $\frac{g(x, j) - g(y, j)}{x - y} \leq \frac{1}{1-c}$ for $0 \leq x < y$. (iii) Define the distance $\|\cdot\|$ by $\|f - g\| = \max_{x \geq 0, i \in \mathcal{S}} |f(x, i) - g(x, i)|$ for $f, g \in \mathcal{D}$.

Lemma 3.1. *The metric space $(\mathcal{D}, \|\cdot\|)$ is complete.*

Define a modified return function and the associated optimal return function by

$$R_{f,\pi}(x, i) = E_{(x,i)} \left[\int_0^{\sigma_1} \frac{l_t e^{-\Lambda t}}{1+d} dt - \int_0^{\sigma_1} \frac{e^{-\Lambda t}}{1-c} dC_t + e^{-\Lambda \sigma_1} f(X_{\sigma_1}^\pi, \xi_{\sigma_1}) \right], \quad x \geq 0, i \in \mathcal{S}, \quad (3.4)$$

$$V_f(x, i) = \sup_{\pi \in \Pi} R_{f,\pi}(x, i), \quad x \geq 0, i \in \mathcal{S}. \quad (3.5)$$

Lemma 3.2. For any $f \in \mathcal{C}$, $V, V_f \in \mathcal{C}$.

Notice that the un-controlled process (X, ξ) is a Markov process. For any $f \in \mathcal{C}$ and any $i \in \mathcal{S}$, the following Hamilton-Jacobi-Bellman (HJB) equation for the modified value function $V_f(\cdot, i)$ can be obtained by using standard arguments in stochastic control: for $x \geq 0$

$$\max \left\{ \max_{l \in [0, \bar{l}]} \left(\frac{\sigma^2(x, i)}{2} V_f''(x, i) + \mu(x, i) V_f'(x, i) - \delta_i V_f(x, i) + l \left(\frac{1}{1+d} - V_f'(x, i) \right) \right), V_f'(x, i) - \frac{1}{1-c} \right\} = 0.$$

Now we define a special class of admissible strategies, which has been shown in the literature to contain the optimal strategy for the original optimization problem if there is 1 regime only. Since the return function of the modified optimization includes the dividends and capital injections in the first regime only, this problem can be considered as a problem to maximize the returns up to an independent exponential time for a risk model with 1 regime. It is worth studying the special class of strategies mentioned above to see whether the optimal strategy of the modified problem falls into this class as well.

Definition 3.2. For any $b \geq 0$, define the strategy $\pi^{0,b} = \{(C_t^{0,b}, D_t^{0,b}); t \geq 0\}$ in the way such that the company pays dividends at the maximal rate \bar{l} when the surplus equals or exceeds b , pays no dividends when the surplus is below b and the company injects capital to maintain the surplus at level 0 whenever the surplus tends to go below 0 without capital injections.

We now investigate whether a strategy $\pi^{0,b}$ with an appropriate value for b is optimal or not for the modified optimization problem. We start with studying the associated return functions. For convenience, we write $X^{0,b} = X^{\pi^{0,b}}$ throughout the rest of the paper.

Remark 3.1. (i) It is not hard to see that $\pi^{0,b}$ is admissible and that both $\pi^{0,b}$ and $X^{0,b}$ are Markov processes. (ii) For any function $f \in \mathcal{C}$ and any $i \in \mathcal{S}$, by applying the comparison theorem used to prove the non-decreasing property of $V(\cdot, i)$ and $V_f(\cdot, i)$ in Lemma 3.2 we can show that the function $R_{f,\pi^{0,b}}(\cdot, i)$ is non-decreasing on $[0, \infty)$ as well.

For any $f \in \mathcal{C}$, $i \in \mathcal{S}$ and $b \geq 0$, define the operator $\mathcal{A}_{f,i,b}$ by

$$\mathcal{A}_{f,i,b} g(x) = \frac{\sigma^2(x, i)}{2} g''(x) + (\mu(x, i) - \bar{l}) g'(x) - (\delta_i + q_i) g(x) + \frac{\bar{l}}{1+d} + \sum_{j \neq i} q_{ij} f(x, j) = 0. \quad (3.6)$$

The following conditions will be required for the main theorems.

Condition 1: The functions $\mu(\cdot, i)$ and $\sigma(\cdot, i)$ are the ones such that for any given function $f \in \mathcal{D}$ and any given $i \in \mathcal{S}$, the ordinary differential equation $\mathcal{A}_{f,i,b} g(x) = 0$ with any finite initial value at $x = 0$ has a bounded solution over $(0, \infty)$.

A sufficient condition for Condition 1 to hold is that both the functions $\mu(\cdot, i)$ and $\sigma(\cdot, i)$ are bounded on $[0, \infty)$ (see Theorem 5.4.2 in [9]). However, this is far away from necessary. For example, when $\mu(\cdot, i)$ is a linear function with positive slope and $\sigma(\cdot, i)$ is a constant Condition 1 also holds (see section 4.4 of [18]).

Condition 2: $\mu'(x, i) \leq \delta_i$ for all $x \geq 0$ and $i \in \mathcal{S}$.

Define for any function $f \in \mathcal{C}$ and $i \in \mathcal{S}$,

$$A_{f,i} = \frac{\bar{l}/(1+d) + \sum_{j \neq i} q_{ij} f(\infty, j)}{q_i + \delta_i}. \quad (3.7)$$

Lemma 3.3. *Suppose Condition 1 holds. For any function $f \in \mathcal{D}$, any $i \in \mathcal{S}$, (i) the function $R_{f,\pi^{0,b}(\cdot, i)}$ for any $b \geq 0$, is a continuously differentiable solution on $[0, \infty)$ to the equations*

$$\frac{\sigma^2(x, i)}{2} g''(x) + \mu(x, i) g'(x) - (\delta_i + q_i) g(x) + \sum_{j \neq i} q_{ij} f(x, j) = 0, \quad 0 < x < b, \quad (3.8)$$

$$\frac{\sigma^2(x, i)}{2} g''(x) + (\mu(x, i) - \bar{l}) g'(x) - (\delta_i + q_i) g(x) + \sum_{j \neq i} q_{ij} f(x, j) = -\frac{\bar{l}}{1+d}, \quad x > b, \quad (3.9)$$

$$g'(0+) = \frac{1}{1-c}, \quad \lim_{x \rightarrow \infty} g(x) < \infty, \quad (3.10)$$

and is twice continuously differentiable on $(0, b) \cup (b, \infty)$; (ii) the function $h_{f,i}(b) := R'_{f,\pi^{0,b}}(b, i)$ is continuous with respect to b for $0 < b < \infty$.

Throughout the paper, we use $\frac{d^-}{dx} g(x, i)$ and $\frac{d^+}{dx} g(x, i)$ to represent the derivatives of g from the left- and right-hand side, respectively, with respect to x .

Corollary 3.1. *Suppose Condition 1 holds. For any $f \in \mathcal{D}$, $i \in \mathcal{S}$ and $b \geq 0$, (i) $R_{f,\pi^{0,b}(\cdot, i)}$ is increasing, bounded, continuously differentiable on $(0, \infty)$, and twice continuously differentiable on $(0, b) \cup (b, \infty)$ with $R'_{f,\pi^{0,b}}(0+, i) = \frac{1}{1-c}$, $\left[\frac{d^-}{dx} R'_{f,\pi^{0,b}}(x, i) \right]_{x=b} = \lim_{x \uparrow b} R''_{f,\pi^{0,b}}(x, i)$ and $\left[\frac{d^+}{dx} R'_{f,\pi^{0,b}}(x, i) \right]_{x=b} = \lim_{x \downarrow b} R''_{f,\pi^{0,b}}(x, i)$; and (ii) if $R'_{f,\pi^{0,b}}(b, i) = \frac{1}{1+d}$, then $R_{f,\pi^{0,b}}(x, i)$ is twice continuously differentiable with respect to x at $x = b$.*

We use $R'_{f,\pi^{0,b}}(0, i)$ and $R''_{f,\pi^{0,b}}(0, i)$ to denote $R'_{f,\pi^{0,b}}(0+, i)$ and $R''_{f,\pi^{0,b}}(0+, i)$, respectively.

Lemma 3.4. *Suppose Conditions 1 and 2 hold. For any fixed $f \in \mathcal{D}$, $i \in \mathcal{S}$ and $b \geq 0$, we have $R''_{f,\pi^{0,0}}(0+, i) \leq 0$, and in the case $b > 0$, $R''_{f,\pi^{0,b}}(0+, i) \leq 0$ if $R'_{f,\pi^{0,b}}(b, i) \leq \frac{1}{1-c}$.*

Lemma 3.5. *Suppose Conditions 1 and 2 hold. For any $f \in \mathcal{D}$ and $i \in \mathcal{S}$, (i) $R''_{f,\pi^{0,0}}(x, i) \leq 0$ for $x \geq 0$, and in the case $b > 0$, $R''_{f,\pi^{0,b}}(x, i) \leq 0$ for $x \geq 0$ if $R'_{f,\pi^{0,b}}(b, i) = \frac{1}{1+d}$; and (ii) for $b > 0$, if $R'_{f,\pi^{0,b}}(b, i) > \frac{1}{1+d}$, $R''_{f,\pi^{0,b}}(x, i) \leq 0$ for $x \in [0, b)$ and $R''_{f,\pi^{0,b}}(b-, 0) \leq 0$.*

Let $I\{\cdot\}$ be the indicator function. Define for any fixed $b \geq 0$ and any fixed $\pi \in \Pi$,

$$\tau_b^\pi = \inf\{t \geq 0 : X_t^\pi \geq b\}, \quad (3.11)$$

$$W_{f,b}(x, i) = \sup_{\pi \in \Pi} E_{(x,i)} \left[\int_0^{\tau_b^\pi \wedge \sigma_1} e^{-\Lambda_s} \frac{l_s}{1+d} ds - \int_0^{\tau_b^\pi \wedge \sigma_1} e^{-\Lambda_s} \frac{1}{1-c} dC_s + e^{-\Lambda_{\tau_b^\pi}} R_{f,\pi^{0,b}}(X_{\tau_b^\pi}^\pi, \xi_0) I\{\tau_b^\pi < \sigma_1\} + e^{-\Lambda_{\sigma_1}} f(X_{\sigma_1}^\pi, \xi_{\sigma_1}) I\{\sigma_1 \leq \tau_b^\pi\} \right]. \quad (3.12)$$

Theorem 3.1. *Suppose Conditions 1 and 2 hold. For any $f \in \mathcal{D}$, any $i \in \mathcal{S}$ and any $b > 0$, if $R'_{f,\pi^{0,b}}(b, i) > \frac{1}{1+d}$, then $R'_{f,\pi^{0,b}}(x, i) > \frac{1}{1+d}$ for $0 < x \leq b$ and $R_{f,\pi^{0,b}}(x, i) = W_{f,b}(x, i)$ for $x \geq 0$.*

We show in the following theorems that if b is chosen appropriately, the return function for the strategy $\pi^{0,b}$ coincides with the optimal return function of the modified problem.

Theorem 3.2. *Suppose that Conditions 1 and 2 hold. For any $f \in \mathcal{D}$ and any $i \in \mathcal{S}$, (i) if $R'_{f,\pi^{0,0}}(0+, i) \leq \frac{1}{1+d}$, then $V_f(x, i) = R_{f,\pi^{0,0}}(x, i)$ for $x \geq 0$; and (ii) if for a fixed $b > 0$, $R'_{f,\pi^{0,b}}(b, i) = \frac{1}{1+d}$, then $V_f(x, i) = R_{f,\pi^{0,b}}(x, i)$ for $x \geq 0$.*

Lemma 3.6. *Suppose Conditions 1 and 2 hold, $f \in \mathcal{D}$ and $i \in \mathcal{S}$. Let $R'_{f,\pi^{0,0}}(0, i)$ denote $R'_{f,\pi^{0,0}}(0+, i)$. If $R'_{f,\pi^{0,b}}(b, i) > \frac{1}{1+d}$ for all $b \geq 0$, then $V_f(x, i) = \lim_{b \rightarrow \infty} R_{f,\pi^{0,b}}(x, i)$ for $x \geq 0$.*

Again we use $R'_{f,\pi^{0,0}}(0, i)$ to denote $R'_{f,\pi^{0,0}}(0+, i)$. Define for any $f \in \mathcal{D}$ and $i \in \mathcal{S}$,

$$b_i^f = \infty \text{ if } R'_{f,\pi^{0,b}}(b, i) > \frac{1}{1+d} \text{ for all } b \geq 0, \text{ and } b_i^f = \inf\{b \geq 0 : R'_{f,\pi^{0,b}}(b, i) \leq \frac{1}{1+d}\} \text{ otherwise.} \quad (3.13)$$

We show in the following that the strategy π^{0,b_i^f} is optimal for the modified problem.

Theorem 3.3. *Suppose Conditions 1 and 2 hold. For any $f \in \mathcal{D}$ and any $i \in \mathcal{S}$, (i) $0 \leq b_i^f < \infty$; and (ii) $V_f(x, i) = R_{f,\pi^{0,b_i^f}}(x, i)$ for $x \geq 0$.*

4. The optimality results

We use the obtained optimality results for the modified optimization problem to address the original optimization problem. The starting point is to notice that the optimal return function of the original optimization V_f , when the fixed function f is chosen to be the value function of the original optimization, coincides with the value function V .

Theorem 4.1. *If Conditions 1 and 2 hold, (i) $V \in \mathcal{D}$; (ii) $b_i^V < \infty$ and $V(x, i) = R_{V, \pi^0, b_i^V}(x, i)$.*

Theorem 4.2. *Define π^* to be the strategy under which, the dividend pay-out rate at any time t is $\bar{l}I\{X_t^{\pi^*}\}$, and the company injects capital to maintain the surplus at level 0 whenever the surplus tends to go below 0 without capital injections. If Conditions 1 and 2 hold, then $V(x, i) = V^{\pi^*}(x, i)$ $i \in E$ and the strategy π^* is an optimal strategy.*

5. Conclusion

We have addressed the optimal dividend and financing problem for a regime-switching general diffusion model with restricted dividend rates. Our conclusion is that it is optimal to inject capitals only when necessary and at a minimal amount sufficient for the business to continue, and to pay out dividends at the maximal rate, \bar{l} , when the surplus exceeds the threshold dependent on the environmental state. This result is consistent with the findings for similar problems under simpler model configuration in the literature. For example, the optimal strategy with restricted dividend rates is of threshold type for the Brownian motion (see [15]), the general diffusion (see [18]), and the regime-switching Brownian motion (see [17]).

Appendix A. Proofs for Sections 3 and 4

For any $i \in \mathcal{S}$ and $b \geq 0$, define the operator \mathcal{B} by

$$\mathcal{B} g(x, i) = \frac{\sigma^2(x, i)}{2} g''(x, i) + \mu(x, i) g'(x, i) - \delta_i g(x, i). \quad (\text{A-1})$$

Proof of Lemma 3.1. Consider any convergent sequence $\{g_n; n = 1, 2, \dots\}$ in \mathcal{D} with limit g . It is sufficient to show $g \in \mathcal{D}$. As for any fixed i and n , $g_n(\cdot, i)$ is nondecreasing and concave, so is the function $g(\cdot, i)$. The inequality $g(\cdot, i) \leq \frac{\bar{l}}{\delta(1+d)}$ follows immediately by noticing $g_n(\cdot, i) \leq \frac{\bar{l}}{\delta(1+d)}$. It remains to show that $\frac{g(x, i) - g(y, i)}{x - y} \leq \frac{1}{1-c}$ for $0 \leq x < y$. We use proof by contradiction. Suppose that there exist x_0, y_0 with $0 \leq x_0 < y_0$ and j such that $\frac{g(x_0, j) - g(y_0, j)}{x_0 - y_0} > \frac{1}{1-c}$. Define $\epsilon_0 := \frac{1}{2} \left(\frac{g(x_0, j) - g(y_0, j)}{x_0 - y_0} - \frac{1}{1-c} \right)$. Clearly, $\epsilon_0 > 0$. As g_n converges to g , we can find an $N > 0$ such that for all $n \geq N$, $\|g_n - g\| \leq \epsilon_0(y_0 - x_0)$. Therefore, $|g_n(y_0, j) - g(y_0, j)| \leq \epsilon_0(y_0 - x_0)$ and $|g_n(x_0, j) - g(x_0, j)| \leq \epsilon_0(y_0 - x_0)$. As a result, $g_n(y_0, j) - g_n(x_0, j) \geq g(y_0, j) - \epsilon_0(y_0 - x_0) - (g(x_0, j) + \epsilon_0(y_0 - x_0)) = g(y_0, j) - g(x_0, j) - 2\epsilon_0(y_0 - x_0) = \frac{y_0 - x_0}{1-c}$. On the other hand, we have $\frac{g_n(y_0, j) - g_n(x_0, j)}{y_0 - x_0} < \frac{1}{1-c}$ (due to $g_n \in \mathcal{D}$), which is a contradiction.

Proof of Lemma 3.2. Noting that $l_s \leq \bar{l}$ and that σ_1 is exponentially distributed with mean $\frac{1}{q_i}$ and $\Lambda_s = \delta_i s$ for $s \leq \sigma_1$, the upper-bounds follow easily from (2.2), (2.3) and (3.5).

Fix x and y with $y > x \geq 0$. Let $\{X_t^x; t \geq 0\}$ and $\{X_t^y; t \geq 0\}$ denote the surplus processes in absence of control with initial surplus x and y , respectively. We use $\pi^x = \{(C_t^x, D_t^x) : t \geq 0\}$ with $D_t^x = \int_0^t l_s^x ds$

to denote any admissible control strategy for the process $\{X_t^x; t \geq 0\}$. Noting that $\{C_t^x; t \geq 0\}$ is right-continuous and increasing, we have the following decomposition: $C_t^x = \int_0^t e^x ds + \sum_{0 < s \leq t} (C_s^x - C_{s-}^x)$. Define $\zeta_0 = 0$, $\zeta_1 = \inf\{s > 0 : C_s^x - C_{s-}^x > 0 \text{ or } \xi_s \neq \xi_{s-}\}$ and $\zeta_{n+1} = \{s > \zeta_n : C_s^x - C_{s-}^x > 0 \text{ or } \xi_s \neq \xi_{s-}\}$ for $n = 1, 2, \dots$. Note that $\xi_t = \xi_{\zeta_n}$ for $t \in [\zeta_n, \zeta_{n+1})$ and hence, $dX_t^{x, \pi^x} = (\mu(X_{t-}^{x, \pi^x}, \xi_{\zeta_n}) - l_t^x + e_t^x)dt + \sigma(X_{t-}^{x, \pi^x}, \xi_{\zeta_n})dW_t$ and $dX_t^{y, \pi^x} = (\mu(X_{t-}^{y, \pi^x}, \xi_{\zeta_n}) - l_t^y + e_t^y)dt + \sigma(X_{t-}^{y, \pi^x}, \xi_{\zeta_n})dW_t$ for $t \in (\zeta_n, \zeta_{n+1})$, $n = 0, 1, \dots$. By noting $X_0^{x, \pi^x} = X_0^x = x < y = X_0^y = X_0^{y, \pi^x}$ and applying the comparison theorem for solutions of stochastic differential equations (see [7]), we can show that with probability one, $X_t^{x, \pi^x} \leq X_t^{y, \pi^x}$ for $t \in [0, \zeta_1)$. Further notice that any discontinuity of a surplus process is caused by a jump in the associated process C^x at the same time and hence, $X_{\zeta_1}^{x, \pi^x} = X_{\zeta_1-}^{x, \pi^x} + (C_{\zeta_1}^x - C_{\zeta_1-}^x) \leq X_{\zeta_1-}^{y, \pi^x} + (C_{\zeta_1}^x - C_{\zeta_1-}^x) = X_{\zeta_1}^{y, \pi^x}$ with probability one. As a result, by applying the comparison theorem on (ζ_1, ζ_2) we can see $X_t^{x, \pi^x} \leq X_t^{y, \pi^x}$ for $t \in (\zeta_1, \zeta_2)$ with probability one. Repeating the same procedure, we can show that $X_t^{x, \pi^x} \leq X_t^{y, \pi^x}$ for $t \in (\zeta_n, \zeta_{n+1}]$ with probability one. In conclusion, $X_t^{x, \pi^x} \leq X_t^{y, \pi^x}$ for all $t \geq 0$ with probability one. Therefore, π^x satisfies all the requirements for being an admissible strategy for the risk process X^y and hence, $R_{f, \pi^x}(y, i) \leq V_f(y, i)$ and $R_{\pi^x}(y, i) \leq V(y, i)$. Using this and (3.4) we can show $R_{f, \pi^x}(x, i) \leq R_{f, \pi^x}(y, i) \leq V_f(y, i)$. Similarly we can obtain $R_{\pi^x}(x, i) \leq V(y, i)$. By the arbitrariness of π^x , we conclude that $V_f(x, i) \leq V_f(y, i)$ and $V(x, i) \leq V(y, i)$ for $0 \leq x < y$.

Lemma A.1. For any $f \in \mathcal{C}$ and $i \in \mathcal{S}$, suppose the function $w_{f,i} : \mathbb{R} \times \mathcal{S} \rightarrow \mathbb{R}$ with $w_{f,i}(\cdot, j) = f(\cdot, j)$ if $j \neq i$, is bounded, continuously differentiable and piecewise twice continuously differentiable with respect to the first argument on $[0, \infty)$, and the function $w_{f,i}(\cdot, i)$ satisfies the ordinary differential equations (3.8) and (3.9). Then, for any $\pi \in \Pi$, there exists a positive sequence of stopping times $\{\tau_n; n = 1, 2, \dots\}$ with $\lim_{n \rightarrow \infty} \tau_n = \infty$ such that

$$\begin{aligned} w_{f,i}(x, i) &= E_{(x,i)} \left[e^{-\Lambda_{\tau_n \wedge \sigma_1 \wedge t}} w_{f,i}(X_{\tau_n \wedge \sigma_1 \wedge t}^\pi, \xi_{\tau_n \wedge \sigma_1 \wedge t}) + \int_0^{\tau_n \wedge \sigma_1 \wedge t} l_s e^{-\Lambda_s} w'_{f,i}(X_{\tau_n \wedge \sigma_1 \wedge t}^\pi, \xi_{\tau_n \wedge \sigma_1 \wedge t}) ds \right] \\ &- E_{(x,i)} \left[\sum_{0 < s \leq \tau_n \wedge \sigma_1 \wedge t} e^{-\Lambda_s} (w_{f,i}(X_s^\pi, \xi_{s-}) - w_{f,i}(X_{s-}^\pi, \xi_{s-})) + \int_0^{\tau_n \wedge \sigma_1 \wedge t} e^{-\Lambda_s} w'_{f,i}(X_{s-}^\pi, \xi_{s-}) d\tilde{C}_s \right] \\ &- E_{(x,i)} \left[\int_0^{\tau_n \wedge \sigma_1 \wedge t} e^{-\Lambda_s} \bar{l}(w'_{f,i}(X_{s-}^\pi, \xi_{s-}) - \frac{1}{1+d}) I\{X_{s-}^\pi \geq b\} ds \right]. \end{aligned} \quad (\text{A-2})$$

Proof. The result follows by applying Itô's formula to $e^{-\Lambda_{\tau_n \wedge \sigma_1 \wedge t}} w_{f,i}(X_{\tau_n \wedge \sigma_1 \wedge t}^\pi, \xi_{\tau_n \wedge \sigma_1 \wedge t})$. The full detail of the proof can found on <http://arxiv.org/abs/1506.08360>.

Proof of Lemma 3.3. (i) The existence of a $[0, \infty)$ -continuously differentiable and $[0, b) \cup (b, \infty)$ -twice continuously differentiable bounded solution to (3.8), (3.9), and (3.10) can be proven by constructing a general form of the solution from the sets of independent linear solutions to (3.8) and (3.9), respectively, and then specifying the coefficients of the general form using the continuity and differentiability of the

solution at b and letting the solution satisfying (3.10). Denote the solution by $g_{b,i}(x)$. It suffices to show $R_{f,\pi^0,b}(x,i) = g_{b,i}(x)$ for $x \geq 0$. This can be done by defining $w_{f,i}$ by $w_{f,i}(x,j) = g_{b,i}(x)$ if $j = i$, and $w_{f,i}(x,j) = f(x,j)$ if $j \neq i$, and then applying Lemma A.1 with π there being set to $\pi^{0,b}$ and using the properties of the strategy $\pi^{0,b}$. The full detail of the proof can be found on <http://arxiv.org/abs/1506.08360>.

(ii) Note by (3.4) that $\lim_{x \rightarrow \infty} g_{b,i}(x) = \lim_{x \rightarrow \infty} R_{f,\pi^0,b}(x,i) = A_{f,i}$, where the second last equality follows by noticing that given $X_0 = x$, $X_s^{0,b} \rightarrow \infty$ as $x \rightarrow \infty$ and hence $C_s^{0,b} \rightarrow 0$ as $x \rightarrow \infty$, and the last equality follows by noting that, given $(X_0, \xi_0) = (x, i)$, σ_1 is exponentially distributed with mean $\frac{1}{q_i}$, and using the definition of $A_{f,i}$ in (3.7). So the constants K_1, K_2, K_3 and K_4 are solutions to the equations $K_1 v_1(b; i) + K_2 v_2(b; i) + B_1(b; i) = K_3 v_3(b; i) + K_4 v_4(b; i) + B_2(b; i)$, $K_1 v'_1(b; i) + K_2 v'_2(b; i) + B'_1(b; i) = K_3 v'_3(b; i) + K_4 v'_4(b; i) + B'_2(b; i)$, $K_1 v'_1(0; i) + K_2 v'_2(0; i) = \frac{1}{1-c}$ and $K_3 v_3(\infty) + K_4 v_4(\infty) + B_2(\infty) = A_{f,i}$. Note that the coefficients of the above system of equations are either constants or continuous functions of b . Hence, K_1, K_2, K_3 and K_4 are continuous functions of b , denoted by $K_1(b), K_2(b), K_3(b)$ and $K_4(b)$ here. As a result, the function $h_{f,i}(b) = g'_{b,i}(b) = K_1(b)v'_1(b) + K_2(b)v'_2(b) + B'_1(b; i)$ is continuous for $0 < b < \infty$.

For any $f \in \mathcal{C}$, $i \in \mathcal{S}$ and $b \geq 0$, define the functions h and \bar{h} by

$$\begin{aligned} h_{f,i,b}(x) &= (\delta_i + q_i)R_{f,\pi^0,b}(x,i) - \mu(x,i)R'_{f,\pi^0,b}(x,i) - \sum_{j \neq i} q_{ij}f(x,j) \\ &\quad - \bar{l} \left(\frac{1}{1+d} - R'_{f,\pi^0,b}(x,i) \right) I\{x \geq b\}, \end{aligned} \quad (\text{A-3})$$

$$\begin{aligned} \bar{h}_{f,i,b}(x) &= (\delta_i + q_i)R_{f,\pi^0,b}(x,i) - \mu(x,i)R'_{f,\pi^0,b}(x,i) - \sum_{j \neq i} q_{ij}f(x,j) \\ &\quad - \bar{l} \left(\frac{1}{1+d} - R'_{f,\pi^0,b}(x,i) \right) I\{x > b\}. \end{aligned} \quad (\text{A-4})$$

Proof of Corollary 3.1. (i) is a result of Remark 3.1 and Lemma 3.3 (i). (ii) By (i) and Lemma 3.3(i) we have $\left[\frac{d^-}{dx} R'_{f,\pi^0,b}(x,i) \right]_{x=b} = \lim_{x \downarrow b} \frac{2h_{f,i,b}(b,i)}{\sigma^2(b,i)}$ and $\left[\frac{d^+}{dx} R'_{f,\pi^0,b}(x,i) \right]_{x=b} = \lim_{x \downarrow b} \frac{2h_{f,i,b}(b,i)}{\sigma^2(b,i)}$. Noting $R'_{f,\pi^0,b}(b,i) = \frac{1}{1+d}$, we obtain $\left[\frac{d^-}{dx} R'_{f,\pi^0,b}(x,i) \right]_{x=b} = \left[\frac{d^+}{dx} R'_{f,\pi^0,b}(x,i) \right]_{x=b}$.

For any sequence $\{y_n\}$, define

$$k_{f,b}(x,i; \{y_n\}) = (\delta_i + q_i - \mu'(x,i))R'_{f,\pi^0,b}(x,i) - \sum_{j \neq i} q_{ij} \lim_{n \rightarrow \infty} \frac{f(y_n, j) - f(x, j)}{y_n - x}. \quad (\text{A-5})$$

Proof of Lemma 3.4. Throughout the proof, we assume $f \in \mathcal{D}$, $i \in \mathcal{S}$ and $b \geq 0$, unless stated otherwise. We use proof by contradiction. Suppose $R''_{f,\pi^0,b}(0+, i) > 0$.

Since $R_{f,\pi^{0,0}}(\cdot, i)$ is bounded, we can find a large enough x such that $R'_{f,\pi^{0,0}}(x, i) < \frac{1}{1-c} = R'_{f,\pi^{0,0}}(0+, i)$, where the last equality is by Lemma 3.3 (i). Hence there exists an $x > 0$ such that $R''_{f,\pi^{0,0}}(x, i) < 0$. In the case $b > 0$, notice that $R'_{f,\pi^{0,b}}(0+, i) = \frac{1}{1-c} \geq R'_{f,\pi^{0,b}}(b, i)$. So for $b > 0$ there exists an $x \in (0, b)$ such that $R''_{f,\pi^{0,b}}(x, i) \leq 0$. Define $x_1 = \inf\{x > 0 : R''_{f,\pi^{0,b}}(x, i) \leq 0\}$. Then $x_1 > 0$ in the case $b = 0$ and $x_1 \in (0, b)$ in the case $b > 0$, and for $b \geq 0$,

$$R''_{f,\pi^{0,b}}(x_1, i) = 0, \quad R''_{f,\pi^{0,b}}(x, i) > 0 \text{ for } x \in [0, x_1]. \quad (\text{A-6})$$

As a result, for $b \geq 0$,

$$R'_{f,\pi^{0,b}}(x, i) > R'_{f,\pi^{0,b}}(0+, i) = \frac{1}{1-c} \text{ for } x \in (0, x_1]. \quad (\text{A-7})$$

Write $R_{f,\pi^{0,b},i}(x) = R_{f,\pi^{0,b}}(x, i)$. It follows by Lemma 3.3 that for $b \geq 0$, $\mathcal{A}_{f,i,b}R_{f,\pi^{0,b},i}(x) = 0$ for $x > 0$. Therefore, it follows by (A-6) and (A-3) that for $b \geq 0$, $h_{f,i,b}(x) = \frac{\sigma^2(x,i)}{2}R''_{f,\pi^{0,b}}(x, i) > 0$ for $0 < x < x_1$ and $h_{f,i,b}(x_1) = \frac{\sigma^2(x_1,i)}{2}R''_{f,\pi^{0,b}}(x_1, i) = 0$. Hence, we obtain that for $b \geq 0$,

$$\frac{h_{f,i,b}(x, i) - h_{f,i,b}(x_1, i)}{x - x_1} < 0, \quad 0 < x < x_1. \quad (\text{A-8})$$

Note that $x_1 > b$ in the case $b = 0$, and that $x_1 < b$ in the case $b > 0$. Therefore, we can find a non-negative sequence $\{x_{1n}\}$ with $b < x_{1n} \leq x_1$ in the case $b = 0$, $x_{1n} \leq x_1 < b$ in the case $b > 0$, and $\lim_{n \rightarrow \infty} x_{1n} = x_1$ such that $\lim_{n \rightarrow \infty} \frac{f(x_{1n},j) - f(x_1,j)}{x_{1n} - x_1}$ exists. By replacing x in (A-8) by x_{1n} and then letting $n \rightarrow \infty$ on both sides of (A-8) gives $k_{f,b}(x_1, i; \{x_{1n}\}) - (\mu(x_1, i) - \bar{I}I\{b=0\})R''_{f,\pi^{0,b}}(x_1, i) \geq 0$, which combined with (A-6) implies $\left(\sum_{j \neq i} q_{ij} \lim_{n \rightarrow \infty} \frac{f(x_{1n},j) - f(x_1,j)}{x_{1n} - x_1} - q_i R'_{f,\pi^{0,b}}(x_1, i)\right) + (\mu'(x_1, i) - \delta_i) R'_{f,\pi^{0,b}}(x_1, i) \leq 0$. It follows by this inequality, $R'_{f,\pi^{0,b}}(x_1, i) > \frac{1}{1-c}$ (see (A-7)) and $\lim_{n \rightarrow \infty} \frac{f(x_{1n},j) - f(x_1,j)}{x_{1n} - x_1} \leq \frac{1}{1-c}$ (due to $f \in \mathcal{D}$) that $(\mu'(x_1, i) - \delta_i) R'_{f,\pi^{0,b}}(x_1, i) > 0$, which combined with (A-7) implies $\mu'(x_1, i) - \delta_i > 0$. This contradicts the assumption that $\mu'(x_1, i) \leq \delta_i$.

Proof of Lemma 3.5. We consider any fixed $f \in \mathcal{D}$ and $i \in \mathcal{S}$ throughout the proof. We first show that there exists a positive sequence $\{x_n\}$ with $\lim_{n \rightarrow \infty} x_n = \infty$ such that for $b \geq 0$,

$$R''_{f,\pi^{0,b}}(x_n, i) \leq 0. \quad (\text{A-9})$$

Suppose the contrary: for some $M > 0$, $R''_{f,\pi^{0,b}}(x, i) > 0$ for all $x \geq M$. This implies $R'_{f,\pi^{0,b}}(x, i) > R'_{f,\pi^{0,b}}(M+1, i) > R'_{f,\pi^{0,b}}(M, i) \geq 0$ for $x > M+1$, where the last inequality follows by the increasing property of $R_{f,\pi^{0,b}}(\cdot, i)$ (see Corollary 3.1(i)). As a result, $R_{f,\pi^{0,b}}(x, i) > R_{f,\pi^{0,b}}(M+1, i) + R'_{f,\pi^{0,b}}(M+1, i)(x - M - 1)$ for $x > M+1$, which implies $\lim_{x \rightarrow \infty} R_{f,\pi^{0,b}}(x, i) = \infty$. This contradicts the boundedness of $R_{f,\pi^{0,b}}(\cdot, i)$ (Corollary 3.1(i)).

Write $R_{f,\pi^0,b,i}(x) = R_{f,\pi^0,b}(x,i)$. By Lemma 3.3 it follows that

$$\mathcal{A}_{f,i,b}R_{f,\pi^0,b,i}(x) = 0 \text{ for } x > 0. \quad (\text{A-10})$$

(i) By Lemma 3.3 and Corollary 3.1 we can see that $R_{f,\pi^0,b,i}(\cdot)$ is twice continuously differentiable on $[0, \infty)$ with the differentiability at 0 referring to the differentiability from the right-hand side. It follows by noting $R'_{f,\pi^0,b,i}(b) = R'_{f,\pi^0,b}(b,i) = \frac{1}{1+d} \leq \frac{1}{1-c}$ for $b > 0$, and Lemma 3.4 that

$$R''_{f,\pi^0,b,i}(0+) \leq 0 \text{ for } b \geq 0. \quad (\text{A-11})$$

We use proof by contradiction to prove the statement in (i). Suppose that the statement in (i) is not true. Then there exists a $b \geq 0$ and a $y_0 > 0$ such that $R''_{f,\pi^0,b,i}(y_0) = R''_{f,\pi^0,b}(y_0,i) > 0$. Let $\{x_n\}$ be the sequence defined as before. We can find a positive integer N such that $x_N > y_0$. By noting $R''_{f,\pi^0,b,i}(x_N) = R''_{f,\pi^0,b}(x_N,i) \leq 0$ (due to (A-9)), (A-11) and the continuity of $R''_{f,\pi^0,b,i}(\cdot)$, we can find y_1, y_2 with $0 \leq y_1 < y_0 < y_2 \leq x_N$ such that

$$R''_{f,\pi^0,b}(y_1,i) = 0, \quad R''_{f,\pi^0,b}(y_2,i) = 0, \quad \text{and} \quad R''_{f,\pi^0,b}(x,i) > 0 \text{ for } x \in (y_1, y_2). \quad (\text{A-12})$$

Hence,

$$R'_{f,\pi^0,b,i}(y_2) > R'_{f,\pi^0,b,i}(y_1). \quad (\text{A-13})$$

It follows by (A-10) and (A-3) that $-\frac{\sigma^2(x,i)}{2}R''_{f,\pi^0,b,i}(x) = h_{f,b,i}(x)$ for $x > 0$. Note that for $x > 0$, $I\{x \geq b\} = I\{x > b\}$ in the case $b = 0$, and that in the case $b > 0$, $\frac{1}{1+d} - R'_{f,\pi^0,b}(b,i) = 0$ and hence, $\bar{l}\left(\frac{1}{1+d} - R'_{f,\pi^0,b}(x,i)\right)I\{x \geq b\} = \bar{l}\left(\frac{1}{1+d} - R'_{f,\pi^0,b}(x,i)\right)I\{x > b\}$ for $x > 0$. Therefore, for $x > 0$, $\frac{\sigma^2(x,i)}{2}R''_{f,\pi^0,b}(x,i) = \bar{h}_{f,i,b}(x)$, which combined with (A-12) implies that for $x \in (y_1, y_2)$,

$$\bar{h}_{f,i,b}(y_1) = \frac{\sigma^2(y_1,i)}{2}R''_{f,\pi^0,b}(y_1,i) = 0 < \frac{\sigma^2(x,i)}{2}R''_{f,\pi^0,b}(x,i) = \bar{h}_{f,i,b}(x), \quad (\text{A-14})$$

$$\bar{h}_{f,i,b}(y_2) = \frac{\sigma^2(y_2,i)}{2}R''_{f,\pi^0,b}(y_2,i) = 0 < \frac{\sigma^2(x,i)}{2}R''_{f,\pi^0,b}(x,i) = \bar{h}_{f,i,b}(x). \quad (\text{A-15})$$

Let $\{y_{1n}\}$ and $\{y_{2n}\}$ be two sequences with $y_{1n} \downarrow y_1$ and $y_{2n} \uparrow y_2$ as $n \rightarrow \infty$ such that $\lim_{n \rightarrow \infty} \frac{f(y_{1n,j}) - f(y_1,j)}{y_{1n} - y_1}$ and $\lim_{n \rightarrow \infty} \frac{f(y_{2n,j}) - f(y_2,j)}{y_{2n} - y_2}$ exist for all $j \in \mathcal{S}$. It follows by (A-14) and (A-15) that $\frac{\bar{h}_{f,i,b}(y_{1n}) - \bar{h}_{f,i,b}(y_1)}{y_{1n} - y_1} > 0 > \frac{\bar{h}_{f,i,b}(y_{2n}) - \bar{h}_{f,i,b}(y_2)}{y_{2n} - y_2}$. By letting $n \rightarrow \infty$, we obtain

$$k_{f,b}(y_1, i; \{y_{1n}\}) - \mu(y_1, i)R''_{f,\pi^0,b}(y_1, i) + \bar{l}R''_{f,\pi^0,b}(y_1, i)I\{y_1 > b\} \geq 0$$

and $k_{f,b}(y_2, i; \{y_{2n}\}) - \mu(y_2, i)R''_{f,\pi^0,b}(y_2, i) + \bar{l}R''_{f,\pi^0,b}(y_2, i)I\{y_2 > b\} \leq 0$. Therefore, by noting $R''_{f,\pi^0,b}(y_1, i) = 0 = R''_{f,\pi^0,b}(y_2, i)$ (see (A-12)) we have

$$k_{f,b}(y_1, i; \{y_{1n}\}) \geq 0 \geq k_{f,b}(y_2, i; \{y_{2n}\}). \quad (\text{A-16})$$

On the other hand, note that $0 < \delta_i + q_i - \mu'(y_1, i) \leq \delta_i + q_i - \mu'(y_2, i)$ (due to the concavity of $\mu(\cdot, i)$), $R'_{f, \pi^0, b}(y_1, i) < R'_{f, \pi^0, b}(y_2, i)$ (see (A-13)), $\lim_{n \rightarrow \infty} \frac{f(y_{1n}, j) - f(y_1, j)}{y_{1n} - y_1} \geq \lim_{n \rightarrow \infty} \frac{f(y_{2n}, j) - f(y_2, j)}{y_{2n} - y_2}$ (due to the concavity of $f(\cdot, j)$). As a result, $k_{f, b}(y_1, i; \{y_{1n}\}) < k_{f, b}(y_2, i; \{y_{2n}\})$, which is a contradiction to (A-16).

(ii) We distinguish two cases: (a) $R''_{f, \pi^0, b}(b+, i) > 0$ and (b) $R''_{f, \pi^0, b}(b+, i) \leq 0$.

(a) Suppose $R''_{f, \pi^0, b}(b+, i) > 0$. By (A-9) we can find $N > 0$ such that $x_N > b$ and $R''_{f, \pi^0, b}(x_N, i) \leq 0$. Then by the continuity of the function $R''_{f, \pi^0, b}(\cdot, i)$ on (b, ∞) (see Corollary 3.1(i)) we know that there exists a $y_2 \in (b, x_N]$ such that $R''_{f, \pi^0, b}(y_2, i) = 0$ and $R''_{f, \pi^0, b}(x, i) > 0$ for $x \in (b, y_2)$. We now proceed to show that $R''_{f, \pi^0, b}(b-, i) \leq 0$. Suppose the contrary, i.e., $R''_{f, \pi^0, b}(b-, i) > 0$. By noting $R''_{f, \pi^0, b}(0+, i) \leq 0$ (see (A-11)), it follows that there exists a $y_1 \in (0, b)$ such that $R''_{f, \pi^0, b}(y_1, i) = 0$ and $R''_{f, \pi^0, b}(x, i) > 0$ for $x \in (y_1, b)$. In summary, (A-12) holds for $x \in (y_1, y_2) - \{b\}$. Repeating the argument right below (A-12) in (i), we obtain a contradiction.

(b) Suppose $R''_{f, \pi^0, b}(b+, i) \leq 0$. It follows by (A-10) and the assumption $R'_{f, \pi^0, b}(b, i) > \frac{1}{1+d}$ that

$$R''_{f, \pi^0, b}(b-, i) = \lim_{x \uparrow b} \frac{2h_{f, i, b}(x, i)}{\sigma^2(x, i)} < \lim_{x \downarrow b} \frac{2h_{f, i, b}(x, i)}{\sigma^2(x, i)} = R''_{f, \pi^0, b}(b+, i) \leq 0. \quad (\text{A-17})$$

We now show that $R''_{f, \pi^0, b}(x, i) \leq 0$ for all $x \in [0, b)$. Suppose the contrary. That is, there exists some $x \in [0, b)$ such that $R''_{f, \pi^0, b}(x, i) > 0$. Then by noting $R''_{f, \pi^0, b}(0+, i) \leq 0$ (see (A-11)) and $R''_{f, \pi^0, b}(b-, i) < 0$ (see (A-17)), we can find y_1 and y_2 with $0 \leq y_1 < y_2 < b$ such that $R''_{f, \pi^0, b}(y_1, i) = 0$, $R''_{f, \pi^0, b}(y_2, i) = 0$ and $R''_{f, \pi^0, b}(x, i) > 0$ for $x \in (y_1, y_2)$. Repeating again the argument right after (A-12) in (i), we can obtain a contradiction.

From now on, define for any $f \in \mathcal{C}$ and $i \in \mathcal{S}$, define the function $w_{f, i} : \mathbb{R} \times \mathcal{S} \rightarrow \mathbb{R}$ by

$$w_{f, i}(\cdot, i) = R_{f, \pi^0, b}(\cdot, i) \text{ and } w_{f, i}(\cdot, j) = f(\cdot, j) \text{ if } j \neq i.$$

Proof of Theorem 3.1. Note that $\tau_b^\pi = 0$ given $X_0^\pi \geq b$. Hence, it follows from the definition (3.12) that

$$W_{f, b}(x, i) = \sup_{\pi \in \Pi} E_{(x, i)} [R_{f, \pi^0, b}(X_0^\pi, \xi_0)] = R_{f, \pi^0, b}(x, i) \text{ for } x \geq b \text{ and } b = 0. \quad (\text{A-18})$$

We consider the case $b > 0$. By Lemma 3.5 (ii) we know that $R''_{f, \pi^0, b}(x, i) \leq 0$ for $x \in [0, b)$, and $R''_{f, \pi^0, b}(b-, i) \leq 0$. Therefore, it follows by Corollary 3.1(i) that

$$\frac{1}{1-c} = R'_{f, \pi^0, b}(0+, i) \geq R'_{f, \pi^0, b}(x, i) \geq R'_{f, \pi^0, b}(b, i) > \frac{1}{1+d} \text{ for } 0 < x \leq b. \quad (\text{A-19})$$

It follows by Corollary 3.1(i) and Lemma 3.3 we know that $w_i(\cdot, j)$ satisfies the conditions in Lemma A.1. Then by applying Lemma A.1 we know that for some positive sequence of stopping times $\{\tau_n; n = 1, 2, \dots\}$ with $\lim_{n \rightarrow \infty} \tau_n = \infty$, the equation (A-2) holds. By letting t in (A-2) be $\tau_b^\pi \wedge t$, noting that $X_s^\pi - X_{s-}^\pi = C_s - C_{s-} \geq 0$, and that given $(X_0, \xi_0) = (x, i)$, $X_{s-}^\pi \in [0, b)$ and $w_i(X_{s-}^\pi, \xi_{s-}) = R_{f, \pi^0, b}(X_{s-}^\pi, i)$ for

$s \leq \sigma_1 \wedge \tau_b^\pi$, that $\sum_{0 < s \leq \tau_n \wedge \sigma_1 \wedge \tau_b^\pi \wedge t} e^{-\Lambda s} \frac{X_s^\pi - X_{s-}^\pi}{1-c} + \int_0^{\tau_n \wedge \sigma_1 \wedge \tau_b^\pi \wedge t} \frac{e^{-\Lambda s}}{1-c} d\tilde{C}_s = \int_0^{\tau_n \wedge \sigma_1 \wedge \tau_b^\pi \wedge t} \frac{e^{-\Lambda s}}{1-c} dC_s$, and using (A-19), we derive that for any $\pi \in \Pi$, $t > 0$ and $0 \leq x \leq b$,

$$\begin{aligned} & E_{(x,i)} \left[\int_0^{\tau_n \wedge \sigma_1 \wedge \tau_b^\pi \wedge t} \frac{l_s e^{-\Lambda s}}{1+d} ds - \int_0^{\tau_n \wedge \sigma_1 \wedge \tau_b^\pi \wedge t} \frac{e^{-\Lambda s}}{1-c} dC_s \right. \\ & \left. + e^{-\Lambda \tau_n \wedge \sigma_1 \wedge \tau_b^\pi \wedge t} w_i(X_{\tau_n \wedge \sigma_1 \wedge \tau_b^\pi \wedge t}^\pi, \xi_{\tau_n \wedge \sigma_1 \wedge \tau_b^\pi \wedge t}) \right] \leq R_{f,\pi^0,b}(x,i). \end{aligned} \quad (\text{A-20})$$

Note that the functions $R_{f,\pi^0,b}(\cdot, j)$ and $f(\cdot, j)$ $j \in \mathcal{S}$ are all bounded. Hence, the functions $w_i(\cdot, j)$ $j \in \mathcal{S}$ are also bounded. By letting $\tau_n \rightarrow \infty$ and $t \rightarrow \infty$ on both sides of (A-20), using the monotone convergence theorem and the dominated convergence theorem and noticing that due to $\xi_s = \xi_0$ for $0 \leq s < \sigma_1$ we have $E_{(x,i)} \left[e^{-\Lambda \tau_b^\pi \wedge \sigma_1} w_{f,i}(X_{\tau_b^\pi \wedge \sigma_1}^\pi, \xi_{\tau_b^\pi \wedge \sigma_1}) \right] = E_{(x,i)} \left[e^{-\Lambda \tau_b^\pi} R_{f,\pi^0,b}(b, \xi_0) I\{\tau_b^\pi < \sigma_1\} + e^{-\Lambda \sigma_1} f(X_{\sigma_1}, \xi_{\sigma_1}) I\{\sigma_1 \leq \tau_b^\pi\} \right]$ and that π is an arbitrary admissible strategy and (3.12), we can conclude

$$W_{f,b}(x,i) \leq R_{f,\pi^0,b}(x,i) \quad \text{for } 0 \leq x \leq b. \quad (\text{A-21})$$

Note that $\{(X_t^{0,b}, \xi_t); t \geq 0\}$ is a strong Markov process and that by the Markov property it follows that

$$\begin{aligned} R_{f,\pi^0,b}(x,i) &= E_{(x,i)} \left[\int_0^{\tau_b^{\pi^0,b} \wedge \sigma_1} \frac{\bar{l} e^{-\Lambda s}}{1+d} I\{X_s^{0,b} \geq b\} ds - \int_0^{\tau_b^{\pi^0,b} \wedge \sigma_1} \frac{e^{-\Lambda s}}{1-c} dC_s \right. \\ & \left. + e^{-\delta(\tau_b^{\pi^0,b} \wedge \sigma_1)} R_{f,\pi^0,b}(X_{\tau_b^{\pi^0,b} \wedge \sigma_1}^{0,b}, \xi_{\tau_b^{\pi^0,b} \wedge \sigma_1}) \right] \leq W_{f,b}(x,i) \quad \text{for } x \geq 0, \end{aligned} \quad (\text{A-22})$$

where the last inequality follows by noting $\pi^{0,b} \in \Pi$ and the definition (3.12). Combining (A-18), (A-21) and (A-22) completes the proof.

Proof of Theorem 3.2. We first show that

$$R'_{f,\pi^0,b}(x,i) \leq R'_{f,\pi^0,b}(b,i) = \frac{1}{1+d} \quad \text{for } x > b, b \geq 0. \quad (\text{A-23})$$

By Lemma 3.5(i) it follows that $R'_{f,\pi^0,0}(x,i) \leq 0$ for $x \geq 0$. As a result, (A-23) holds for $b = 0$. Now suppose $b > 0$. By Lemma 3.3 (i) we know that $R'_{f,\pi^0,b}(0+, i) = \frac{1}{1-c}$. Since $R'_{f,\pi^0,b}(b,i) = \frac{1}{1+d}$, it follows by Corollary 3.1 (ii) that $R_{f,\pi^0,b}(\cdot, i)$ is twice continuously differentiable on $[0, \infty)$ and by Lemma 3.5 (i) that $R''_{f,0,b}(x,i) \leq 0$ for $x \geq 0$. Hence, (A-23) holds for $b > 0$ as well, and

$$\frac{1}{1-c} = R'_{f,\pi^0,b}(0+, i) \geq R'_{f,\pi^0,b}(x,i) \geq R'_{f,\pi^0,b}(b,i) = \frac{1}{1+d} \quad \text{for } x \in [0, b]. \quad (\text{A-24})$$

It follows by using (A-23) and (A-24), and noting $\bar{l} \geq l_s$ for $s \geq 0$ we obtain that for $b \geq 0$,

$$\begin{aligned} & \bar{l} I\{X_s^\pi \geq b\} \left(R'_{f,\pi^0,b}(X_{s-}^\pi, i) - \frac{1}{1+d} \right) - l_s R'_{f,\pi^0,b}(X_{s-}^\pi, i) \\ &= (\bar{l} - l_s) I\{X_s^\pi \geq b\} R'_{f,\pi^0,b}(X_{s-}^\pi, i) - \frac{\bar{l}}{1+d} I\{X_s^\pi \geq b\} - l_s I\{X_s^\pi < b\} R'_{f,\pi^0,b}(X_{s-}^\pi, i) \\ &\leq \frac{\bar{l} - l_s}{1+d} I\{X_s^\pi \geq b\} - \frac{\bar{l}}{1+d} I\{X_s^\pi \geq b\} - \frac{l_s}{1+d} I\{X_s^\pi < b\} = -\frac{l_s}{1+d}, \end{aligned} \quad (\text{A-25})$$

By (A-23) again we can obtain

$$R'_{f,\pi^0,b}(x,i) \leq \frac{1}{1-c} \quad \text{for } b \geq 0 \text{ and } x > b. \quad (\text{A-26})$$

Further, note that for $b \geq 0$ and any $t \geq 0$,

$$\begin{aligned} & E_{(x,i)} \left[\int_{0 < s \leq \sigma_1 \wedge t} e^{-\Lambda s} R'_{f,\pi^0,b}(X_s^\pi, \xi_{s-}) d\tilde{C}_s + \sum_{0 < s \leq \sigma_1 \wedge t} e^{-\Lambda s} (R_{f,\pi^0,b}(X_s^\pi, \xi_{s-}) - R_{f,\pi^0,b}(X_{s-}^\pi, \xi_{s-})) \right] \\ & \leq E_{(x,i)} \left[\int_0^{\sigma_1 \wedge t} \frac{e^{-\Lambda s}}{1-c} d\tilde{C}_s + \sum_{0 < s \leq \sigma_1 \wedge t} \frac{e^{-\Lambda s}}{1-c} (X_s^\pi - X_{s-}^\pi) \right] = E_{(x,i)} \left[\sum_{0 < s \leq \sigma_1 \wedge t} \frac{e^{-\Lambda s}}{1-c} dC_s \right], \end{aligned} \quad (\text{A-27})$$

where the last inequality follows by (A-24), (A-26), $d\tilde{C}_s \geq 0$, $X_s^\pi - X_{s-}^\pi = C_s - C_{s-} \geq 0$ and $dC_s = d\tilde{C}_s + C_s - C_{s-}$.

It follows by Corollary 3.1(i) and Lemma 3.3 we know that the conditions in Lemma 3.3 are satisfied by $w_i(\cdot, j)$. By applying Lemma A.1 we know that for some positive sequence of stopping times $\{\tau_n; n = 1, 2, \dots\}$ with $\lim_{n \rightarrow \infty} \tau_n = \infty$, the equation (A-2) holds for any $\pi \in \Pi$, any $b, t > 0$ and any $n \in \mathbb{N}$. By using (A-2), (A-25) and (A-27) (setting $t = t \wedge \tau_n$) we arrive at $R_{f,\pi^0,b}(x,i) \geq E_{(x,i)} \left[\int_0^{\sigma_1 \wedge t \wedge \tau_n} \frac{l_s e^{-\Lambda s}}{1+d} ds - \sum_0^{\sigma_1 \wedge t \wedge \tau_n} \frac{e^{-\Lambda s}}{1-c} dC_s + e^{-\Lambda \sigma_1 \wedge t \wedge \tau_n} w_{f,i}(X_{\sigma_1 \wedge t \wedge \tau_n}^\pi, \xi_{\sigma_1 \wedge t \wedge \tau_n}) \right]$ for $b \geq 0$. By noting that the functions $R_{f,\pi^0,b}(\cdot, i)$ and $f(\cdot, j)$, $j \in \mathcal{S}$ are bounded and letting $t \rightarrow \infty$ and then $n \rightarrow \infty$ and then using the monotone convergence theorem for the first two terms inside the expectation and the dominated convergence theorem for the last term, we obtain that for $b \geq 0$, $R_{f,\pi^0,b}(x,i) \geq E_{(x,i)} \left[\int_0^{\sigma_1} \frac{l_s e^{-\Lambda s}}{1+d} ds - \int_0^{\sigma_1} \frac{e^{-\Lambda s}}{1-c} dC_s + e^{-\Lambda \sigma_1} w_{f,i}(X_{\sigma_1}^\pi, \xi_{\sigma_1}) \right]$. By noting $w_{f,i}(X_{\sigma_1}^\pi, \xi_{\sigma_1}) = f(X_{\sigma_1}^\pi, \xi_{\sigma_1})$ given $\xi_0 = i$, the arbitrariness of π and the definition of V_f in (3.5) we conclude $R_{f,\pi^0,b}(x,i) \geq V_f(x,i)$ for $x \geq 0$. On the other hand, $R_{f,\pi^0,b}(x,i) \leq V_f(x,i)$ for $x \geq 0$ according to the definition (3.5). Consequently, $R_{f,\pi^0,b}(x,i) = V_f(x,i)$ for $x \geq 0$.

Proof of Lemma 3.6. Recall that τ_b^π is defined in (3.11). By Theorem 3.1 it follows that for any large enough b and any $x \geq 0$,

$$\begin{aligned} R_{f,\pi^0,b}(x,i) &= W_{f,b}(x,i) = \sup_{\pi \in \Pi} E_{(x,i)} \left[\int_0^{\sigma_1 \wedge \tau_b^\pi} \frac{l_s e^{-\Lambda s}}{1+d} ds - \int_0^{\sigma_1 \wedge \tau_b^\pi} \frac{e^{-\Lambda s}}{1-c} dC_s \right. \\ & \quad \left. + e^{-\Lambda \tau_b^\pi} R_{f,\pi^0,b}(b, \xi_0) I\{\tau_b^\pi < \sigma_1\} + e^{-\Lambda \sigma_1} f(X_{\sigma_1}^\pi, \xi_{\sigma_1}) I\{\sigma_1 \leq \tau_b^\pi\} \right] \\ & \geq \sup_{\pi \in \Pi} E_x \left[\int_0^{\sigma_1 \wedge \tau_b^\pi} \frac{l_s e^{-\Lambda s}}{1+d} ds - \int_0^{\sigma_1 \wedge \tau_b^\pi} \frac{e^{-\Lambda s}}{1-c} dC_s + e^{-\Lambda \sigma_1} f(X_{\sigma_1}^\pi, \xi_{\sigma_1}) I\{\sigma_1 \leq \tau_b^\pi\} \right]. \end{aligned}$$

Note $\lim_{b \rightarrow \infty} \tau_b^\pi = \infty$ and f is bounded. Then it follows by letting $b \rightarrow \infty$ on both sides, and then using the monotone convergence theorem twice and the dominated convergence that $\liminf_{b \rightarrow \infty} R_{f,\pi^0,b}(x,i) \geq \sup_{\pi \in \Pi} E_{(x,i)} \left[\int_0^{\sigma_1} \frac{l_s e^{-\Lambda s}}{1+d} ds - \int_0^{\sigma_1} \frac{e^{-\Lambda s}}{1-c} dC_s + e^{-\Lambda \sigma_1} f(X_{\sigma_1}^\pi, \xi_{\sigma_1}) \right] = V_f(x,i)$ for $x \geq 0$. This combined with the fact $R_{f,\pi^0,b}(x,i) \leq V_f(x,i)$ for $x \geq 0$ completes the proof.

Proof of Theorem 3.3. (i) $b_i^f \geq 0$ is obvious by the definition. We just need to prove $b_i^f < \infty$. Suppose the contrary. Then by (3.13) we have $R'_{f,\pi^{0,b}}(b, i) > \frac{1}{1+d}$ for all $b \geq 0$. Hence, it follows by Lemma 3.6 that $V_f(x, i) = \lim_{b \rightarrow \infty} R_{f,\pi^{0,b}}(x, i)$ for $x \geq 0$. For any $b \geq 0$, by Theorem 3.1 we know $R'_{f,\pi^{0,b}}(x, i) > \frac{1}{1+d}$ for $x \in (0, b]$, which implies $R_{f,\pi^{0,b}}(x, i) > R_{f,\pi^{0,b}}(0, i) + \frac{x}{1+d}$ for $x \in (0, b]$. Hence, for any $x \geq 0$, we can find a $b > x$ such that $V_f(x, i) \geq R_{f,\pi^{0,b}}(x, i) > R_{f,\pi^{0,b}}(0, i) + \frac{x}{1+d}$. Hence, $\lim_{x \rightarrow \infty} V_f(x, i) = +\infty$, which contradicts $V_f(x, i) \leq \frac{\bar{l}}{\delta(1+d)}$ for $x \geq 0$ (see Lemma 3.2). (ii) is a result of (i) and Theorem 3.2.

Proof of Theorem 4.1. (i) Define an operator \mathcal{P} by $\mathcal{P}(f)(x, i) = R_{f,\pi^{0,b_i^f}}(x, i)$. By applying the results about $R_{f,\pi^{0,b_i^f}}$ obtained in Section 3, we can show that \mathcal{P} is non-decreasing and a contraction on the complete space $(\mathcal{D}, \|\cdot\|)$. Using the monotonicity of \mathcal{P} and the fixed point theory, we can show that $\lim_{n \rightarrow \infty} \mathcal{P}^n(g_2) \geq V \geq \lim_{n \rightarrow \infty} \mathcal{P}^n(g_2) = \lim_{n \rightarrow \infty} \mathcal{P}^n(g_2)$, where $g_1(x, i) = 0$ and $g_2(x, i) = \frac{\bar{l}}{\delta(1+d)}$. As a result, $V \in \mathcal{D}$. The full detail of the proof can be found on <http://arxiv.org/abs/1506.08360>. (ii) The results follow by (i) and Theorem 3.3.

Proof of Theorem 4.2. Note $b_i^V < \infty$ for all $i \in \mathcal{S}$. Define an operator \mathcal{Q} by $\mathcal{Q}(f)(x, i) = R_{f,\pi^{0,b_i^V}}(x, i)$. We can show that \mathcal{Q} is a contraction on $(\mathcal{C}, \|\cdot\|)$, and both V and \mathcal{R}_{π^*} are fixed points in $(\mathcal{C}, \|\cdot\|)$. By the fixed point theory, we conclude $V = \mathcal{R}_{\pi^*}$. The full detail of the proof can be found on <http://arxiv.org/abs/1506.08360>.

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