Optimal Two-Part Pricing under Demand Uncertainty *

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This paper examines the two-part pricing problem of a risk-neutral monopoly (the seller) for a good sold to buyers who face uncertainty about their demand for the good. If buyers are risk neutral, we show that marginal-cost pricing is not only profit-maximizing but also socially efficient. If buyers are risk averse, the demand uncertainty calls for the insurance need of buyers, which induces the seller to deviate from marginal-cost pricing. We show that the optimal unit price is higher or lower than the constant marginal cost, depending on the nature of the good (normal or inferior) and on the signs of cross-derivatives of buyers' multivariate utility function. Employing a quasi-linear specification that reduces the general multivariate utility function to a special univariate utility function, we show that the seller optimally raises (lowers) the unit price and lowers (raises) the fixed fee from their risk-neutral counterparts if buyers' total and marginal benefits are positively (negatively) correlated. We further show that these results are robust to the introduction of competition to the seller.

JEL classification: D11; D42; D81; L11

Keywords: Demand uncertainty; Insurance; Risk aversion; Two-part pricing

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Abstract

This paper examines the two-part pricing problem of a risk-neutral monopoly (the seller) for a good sold to buyers who face uncertainty about their demand for the good. If buyers are risk neutral, we show that marginal-cost pricing is not only profit-maximizing but also socially efficient. If buyers are risk averse, the demand uncertainty calls for the insurance need of buyers, which induces the seller to deviate from marginal-cost pricing. We show that the optimal unit price is higher or lower than the constant marginal cost, depending on the nature of the good (normal or inferior) and on the signs of cross-derivatives of buyers' multivariate utility function. Employing a quasi-linear specification that reduces the general multivariate utility function to a special univariate utility function, we show that the seller optimally raises (lowers) the unit price and lowers (raises) the fixed fee from their risk-neutral counterparts if buyers' total and marginal benefits are positively (negatively) correlated. We further show that these results are robust to the introduction of competition to the seller.

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1. Introduction

Two-part pricing requires consumers to pay a fixed fee upfront to secure the right to purchase a good or service at a predetermined unit price. Examples of firms using two-part pricing abound. The classical example is Disneyland that charges each visitor of its amusement park an admission fee and a price per ride (see Oi, 1971). Other examples include beach resorts, health clubs, mobile telephone companies, and commercial banks, to name just a few.

Since the seminal work of Oi (1971), there has been a large literature that examines two-part pricing as a means of price discrimination.[1] In the standard analysis wherein consumers are homogeneous and uncertainty is absent, the extant literature shows that a monopolistic firm using two-part pricing optimally sets the unit price equal to the constant marginal cost, and charges the fixed fee that seizes the entire consumer's surplus arising from marginal-cost pricing. A notable exception is Hayes (1987). She introduces uncertainty about future demand for a good to risk-averse buyers when they decide to subscribe to a two-part pricing

[1]For an excellent survey on price discrimination, see Armstrong (2006).

contract offered by a risk-neutral seller who supplies the good. Ex-ante demand uncertainty is driven by an unknown state variable (e.g., income, weather, health, etc.). Buyers make their consumption plans ex post after the state variable has been revealed. Hayes (1987) shows that the seller has incentives to deviate from marginal-cost pricing so as to cater the insurance need of buyers in the presence of demand uncertainty.[2]

To fix the idea, take beach resorts as an example. We can treat the uncertain weather as the state variable. Beach resorts charge risk-averse tourists for the hotel rooms, which can be regarded as the fixed fee. Suppose that tourists' marginal utility of income increases as the weather improves. If the weather is good (bad), tourists' demand for beach activities such as sailing and diving is likely to be high (low). In this case, the optimal two-part pricing is to price the beach activities below their marginal costs and charge a higher fixed fee. While risk-averse tourists suffers utility losses in bad weather because of the high fixed fee, they benefit from utility gains in good weather because of low usage charges. Since marginal utility of income increases as the weather improves, utility gains must out-weight utility losses so that tourists' expected utility increases. This is the insurance benefit offered by the optimal two-part pricing to risk-averse tourists. On the other hand, tourists' demand for indoor activities such as spa treatments and gym facilities is likely to be high (low) if the weather is bad (good). The optimal two-part pricing is then to price the indoor activities above their marginal costs and charges a lower fixed fee. Risk-averse tourists encounter utility losses in bad weather because of high usage charges, but they enjoy utility gains in good weather because of the low fixed fee. Since marginal utility of income increases as the weather improves, utility gains must out-weight utility losses, which offers insurance to risk-averse tourists. When tourists' marginal utility of income decreases as the weather improves, the opposite pricing rules would apply. In any case, the optimal two-part pricing plays an insurance role in protecting risk-averse buyers against demand uncertainty.

The purposes of this paper are to re-examine the model of Hayes (1987) in general, and derive sufficient conditions based on exogenous model restrictions under which the optimal pricing rule can be unambiguously characterized in particular.[3] When buyers are risk neutral, we show that marginal-cost pricing is not only optimal for the profit-maximizing seller, but also efficient in the social welfare perspective, where social welfare is defined as the expected joint surplus of buyers and the seller. These are the celebrated results of Oi (1971) extended to the case of demand uncertainty. When buyers are risk averse, we

[2]If buyers are risk neutral, marginal-cost pricing remains optimal in the presence of demand uncertainty (see Proposition 1). Hence, it is indeed the insurance need of buyers that induces the seller to deviate from marginal-cost pricing.

[3]Hayes (1987) shows that the optimal unit price is larger (smaller) than the constant marginal cost if the marginal utility of all other goods and the consumption of the good are negatively (positively) correlated at the optimum, which by itself is endogenously determined and thus cannot constitute a sufficient condition.

derive sufficient conditions under which the optimal unit price is higher or lower than the constant marginal cost. We show that the sufficient conditions include, among others, the differential Spence-Mirrlees single-crossing property that is commonly found in monotone comparative statics analysis (see Milgrom and Shannon, 1994; Edlin and Shannon, 1998). We then follow Png and Wang (2010) to consider a special case of Hayes (1987) wherein a quasi-linear specification is imposed to reduce buyers' general multivariate utility function to an univariate utility function. We show that the seller optimally raises (lowers) the unit price and lowers (raises) the fixed fee from their risk-neutral counterparts if buyers' total and marginal benefits are positively (negatively) correlated, which are consistent with the results of the general model of Hayes (1987).

As in Png and Wang (2010), we conduct comparative static analysis with respect to an increase in buyers' risk aversion. Confined to buyers' preferences that exhibit constant absolute risk aversion (CARA), we show that the marginal effect of increased risk aversion of buyers on the optimal two-part pricing contract inherits the global effect of risk aversion.[4] When buyers become more risk averse, there is greater insurance need that is catered for by moving the optimal unit price further away from the constant marginal cost. Our results complement to those of Png and Wang (2010) in that we analyze not only the impact on the optimal unit price but also that on the optimal fixed fee. Finally, we introduce competition to the seller in a reduced form by using the Nash bargaining solution to determine the optimal two-part pricing contract. This formulation subsumes monopoly and perfect competition as two extreme cases wherein the seller has all or none of the bargaining power, respectively, both of which have already been studied by Png and Wang (2010). We show that imposing competition onto the seller by means of the Nash bargaining solution does not qualitatively alter the optimal two-part pricing contract. Indeed, when buyers' preferences exhibit CARA, the optimal unit price is completely neutral to the extent of competition (as measured by the degree of bargaining power possessed by the seller), even though the optimal fixed fee increases as competition becomes lax.

The rest of this paper is organized as follows. Section 2 delineates the model of Hayes (1987). Section 3 derives the optimal two-part pricing contract when buyers are risk neutral, and that when buyers are risk averse. Section 4 considers a special case in which a quasi-linear specification is imposed to reduce buyers' multivariate utility function to an univariate utility function as in Png and Wang (2010). Within this simplified framework, we conduct comparative static analysis with respect to an increase in buyers' risk aversion and to the

[4]Due to the binding participation constraint of buyers at the optimum, the usual characterization of increased risk aversion (Arrow, 1965; Pratt, 1964; Diamond and Stiglitz, 1974) does not give us clear guidance to study the marginal effect of increased risk aversion of buyers on the optimal two-part pricing contract, except in the special case of CARA (see Section 4.1).

introduction of competition to the seller. The final section concludes.

2. The Model

Consider a monopoly (henceforth the seller) who produces a good at a known constant marginal cost, $c > 0$.[5] The seller sells the good to buyers who face uncertainty about their demand for the good. To facilitate sales, the seller uses a two-part pricing contract, (T, p), where T is the fixed fee paid (received if negative) by a buyer ex ante (i.e., before resolving the underlying uncertainty) to secure the right to purchase the good at the predetermined unit price, p, ex post (i.e., after resolving the underlying uncertainty). The seller is risk neutral and devises the two-part pricing contract, (T, p), ex ante to maximize his expected profit.[6]

All buyers have the same initial wealth, $I > 0$, to be allocated for the consumption of the good and all other goods. Following Hayes (1987), we assume that buyers are risk averse and possess the same ex-post utility function, $U(q, m, s)$, where q is the number of units of the good, m is the consumption of all other goods, and s is the realization of a state variable, $\tilde{s}$, that is unknown ex ante.[7] Let $F(s)$ be the known cumulative distribution function of the state variable, $\tilde{s}$, over support $[\underline{s}, \overline{s}]$ with $\underline{s} < \overline{s}$, which may either be independent or be perfectly correlated among buyers. Risk aversion requires that $U_q(q, m, s) > 0$, $U_m(q, m, s) > 0$, $U_{qq}(q, m, s) < 0$, and $U_{mm}(q, m, s) < 0$, where subscripts always signify partial derivatives throughout the paper.[8]

The sequence of moves is as follows. Before the state variable, $\tilde{s}$, is revealed, the seller offers the two-part pricing contract, (T, p), to buyers. If a buyer pays the fixed fee, T, to the seller, she obtains the right to purchase as many units of the good as she wants at the unit price, p, after $\tilde{s}$ has been revealed.[9] Specifically, the buyer chooses q and m to maximize her ex-post utility, $U(q, m, s)$, subject to the budget constraint, $T + pq + m \leq I$. The optimal consumption of the good, $q^\circ \equiv q(I, T, p, s)$, is characterized by the following

[5] In Section 4.2, we incorporate competition into the model by means of the Nash bargaining solution.

[6] Introducing risk aversion into the seller gives rise to a hedging motive. Specifically, the risk-averse seller would like to adopt marginal-cost pricing so as to eliminate the risk arising from his volatile operating profits. The wedge between the optimal unit price and the constant marginal cost is as such jointly determined by the efficient risk-sharing arrangement between buyers and the seller. Nevertheless, none of our qualitative results are affected when the seller is de facto risk averse.

[7] Throughout the paper, random variables have a tilde ($\tilde{}$), while their realizations do not.

[8] It is noteworthy mentioning that Hayes' (1987) model is along the line of Kihlstrom and Mirman (1974, 1981) for a general two-commodity choice problem with a single exogenous random variable.

[9] We exclude any possibilities of reselling the good to other buyers.

first-order condition:

$$U_q(q^\circ, m^\circ, s) - pU_m(q^\circ, m^\circ, s) = 0, \tag{1}$$

while the optimal consumption of all other goods, $m^\circ \equiv m(I, T, p, s)$, is determined by the binding budget constraint, i.e., $m^\circ = I - T - pq^\circ$. The second-order condition requires that

$$\Delta \equiv U_{qq}(q^\circ, m^\circ, s) - 2pU_{qm}(q^\circ, m^\circ, s) + p^2 U_{mm}(q^\circ, m^\circ, s) < 0, \tag{2}$$

which we assume to hold. Should the buyer elect not to pay the fixed fee, T, to the seller, her ex-post utility is equal to $U(0, I, s)$, for all $s \in [\underline{s}, \overline{s}]$, and the seller's profit from this buyer is zero.

Anticipating buyers' ex-post optimal consumption bundles, the seller devises the two-part pricing contract, (T, p), so as to maximize his expected profit ex ante:

$$\max_{T,p} \; T + (p - c)\mathrm{E}(\tilde{q}^\circ), \tag{3}$$

subject to the following participation constraint of buyers:

$$\mathrm{E}[U(\tilde{q}^\circ, \tilde{m}^\circ, \tilde{s})] \geq \mathrm{E}[U(0, I, \tilde{s})], \tag{4}$$

where $\tilde{q}^\circ \equiv q(I, T, p, \tilde{s})$, $\tilde{m}^\circ \equiv m(I, T, p, \tilde{s})$, and $\mathrm{E}(\cdot)$ is the expectation operator with respect to the cumulative distribution function, $F(s)$, of the state variable, $\tilde{s}$. It is clear from program (3) that buyers' participation constraint (4) must be binding at the optimum.[10] For a given unit price, p, we define the fixed fee, $T(p)$, as the unique solution to the following binding participation constraint:[11]

$$\mathrm{E}\Big\{U\{q[I, T(p), p, \tilde{s}], m[I, T(p), p, \tilde{s}], \tilde{s}\}\Big\} = \mathrm{E}[U(0, I, \tilde{s})]. \tag{5}$$

Differentiating Eq. (5) with respect to p and rearranging terms yields

$$T'(p) = -\frac{\mathrm{E}\Big\{U_m\{q[I, T(p), p, \tilde{s}], m[I, T(p), p, \tilde{s}], \tilde{s}\}q[I, T(p), p, \tilde{s}]\Big\}}{\mathrm{E}\Big\{U_m\{q[I, T(p), p, \tilde{s}], m[I, T(p), p, \tilde{s}], \tilde{s}\}\Big\}} < 0, \tag{6}$$

[10] Using Eq. (1), we have $\partial\mathrm{E}[U(\tilde{q}^\circ, \tilde{m}^\circ, \tilde{s})]/\partial T = -\mathrm{E}[U_m(\tilde{q}^\circ, \tilde{m}^\circ, \tilde{s})] < 0$. If buyers' participation constraint (4) is slack at the optimal two-part pricing contract, (T^*, p^*), the seller can increase the optimal fixed fee, T^*, by an amount, $\epsilon > 0$, such that $\mathrm{E}[U(\tilde{q}^*, \tilde{m}^*, \tilde{s})] > \mathrm{E}\{U[q(I, T^*+\epsilon, p^*, \tilde{s}), m(I, T^*+\epsilon, p^*, \tilde{s}), \tilde{s}]\} \geq \mathrm{E}[U(0, I, \tilde{s})]$. This contradicts the optimality of (T^*, p^*).

[11] Since $\partial\mathrm{E}[U(\tilde{q}^\circ, \tilde{m}^\circ, \tilde{s})]/\partial T = -\mathrm{E}[U_m(\tilde{q}^\circ, \tilde{m}^\circ, \tilde{s})] < 0$, where the equality follows from Eq. (1), and there is no restriction on T, $T(p)$ is indeed the unique solution to Eq. (5) for a given unit price, p.

which implies that a higher unit price is associated with a lower fixed fee so as to keep buyers' participation constraint (4) binding. The seller's ex-ante decision problem (3), therefore, reduces to

$$\max_p \; T(p) + (p - c)\mathrm{E}\{q[I, T(p), p, \tilde{s}]\}, \tag{7}$$

where $T(p)$ is implicitly defined by Eq. (5).

3. Solution to the Model

The first-order condition for program (7) is given by

$$(p^* - c)\Big\{\mathrm{E}\{q_p[I, T(p^*), p^*, \tilde{s}] + \mathrm{E}\{q_T[I, T(p^*), p^*, \tilde{s}]\}T'(p^*)\Big\}$$

$$+\mathrm{E}\{q[I, T(p^*), p^*, \tilde{s}]\} + T'(p^*) = 0, \tag{8}$$

where p^* is the optimal unit price. The second-order condition for program (7) is given by

$$T''(p^*) + 2\mathrm{E}\{q_T[I, T(p^*), p^*, \tilde{s}]\}T'(p^*) + 2\mathrm{E}\{q_p[I, T(p^*), p^*, \tilde{s}]\}$$

$$+(p^* - c)\Big\{\mathrm{E}\{q_T[I, T(p^*), p^*, \tilde{s}]\}T''(p^*) + \mathrm{E}\{q_{TT}[I, T(p^*), p^*, \tilde{s}]\}T'(p^*)^2$$

$$+2\mathrm{E}\{q_{Tp}[I, T(p^*), p^*, \tilde{s}]\}T'(p^*) + \mathrm{E}\{q_{pp}[I, T(p^*), p^*, \tilde{s}]\}\Big\} < 0, \tag{9}$$

which we assume to hold.

Let $\mathrm{Cov}(\cdot, \cdot)$ be the covariance operator with respect to the cumulative distribution function, $F(s)$, of the state variable, $\tilde{s}$. Substituting Eq. (6) with $p = p^*$ into Eq. (8) yields[12]

$$(p^* - c)\Big\{\mathrm{E}[q_p(I, T^*, p^*, \tilde{s})] - \mathrm{E}[q_T(I, T^*, p^*, \tilde{s})] \times \frac{\mathrm{E}[U_m(\tilde{q}^*, \tilde{m}^*, \tilde{s})\tilde{q}^*]}{\mathrm{E}[U_m(\tilde{q}^*, \tilde{m}^*, \tilde{s})]}\Big\}$$

$$-\frac{\mathrm{Cov}[U_m(\tilde{q}^*, \tilde{m}^*, \tilde{s}), \tilde{q}^*]}{\mathrm{E}[U_m(\tilde{q}^*, \tilde{m}^*, \tilde{s})]} = 0, \tag{10}$$

[12]For any two random variables, $\tilde{x}$ and $\tilde{y}$, we have $\mathrm{Cov}(\tilde{x}, \tilde{y}) = \mathrm{E}(\tilde{x}\tilde{y}) - \mathrm{E}(\tilde{x})\mathrm{E}(\tilde{y})$.

where $\tilde{q}^* \equiv q(I, T^*, p^*, \tilde{s})$, $\tilde{m}^* \equiv m(I, T^*, p^*, \tilde{s})$, and the optimal fixed fee, $T^* = T(p^*)$, is implicitly determined by the following binding participation constraint:

$$\mathrm{E}[U(\tilde{q}^*, \tilde{m}^*, \tilde{s})] = \mathrm{E}[U(0, I, \tilde{s})]. \tag{11}$$

Solving Eqs. (10) and (11) simultaneously gives us the optimal two-part pricing contract, (T^*, p^*).

3.1. Risk-Neutral Buyers

As a benchmark, suppose that buyers are risk neutral. According to Stiglitz (1969), risk neutrality is equivalent to linearity in wealth of the von Neumann-Morgenstern utility function for all fixed prices and states. Define $V(I, T, p, s) \equiv U(q^\circ, m^\circ, s)$ as the ex-post indirect utility function, where $q^\circ \equiv q(I, T, p, s)$ and $m^\circ \equiv m(I, T, p, s)$. It then follows from Eq. (1) that $V_I(I, T, p, s) = U_m(q^\circ, m^\circ, s)$. Hence, linearity of $V(I, T, p, s)$ in I for all fixed two-part pricing contracts and states implies that $U_m(q^\circ, m^\circ, s)$ is a constant, which, without any loss of generality, is normalized to unity.[13]

Using the fact that $U_m(q^\circ, m^\circ, s) = 1$ for all fixed two-part pricing contracts and states, we solve Eqs. (10) and (11) simultaneously to establish our first proposition. All proofs of propositions are relegated to Appendix A.

Proposition 1. *In the benchmark case of risk-neutral buyers, the profit-maximizing seller sets the optimal unit price equal to the constant marginal cost, c, and the optimal fixed fee,* T^0*, that solves* $\mathrm{E}\{U[q(I, T^0, c, \tilde{s}), m(I, T^0, c, \tilde{s}), \tilde{s}]\} = \mathrm{E}[U(0, I, \tilde{s})]$.

Proposition 1 extends the celebrated results of Oi (1971) to the case of two commodities and a single source of uncertainty. To see the intuition for why marginal-cost pricing is optimal when buyers are risk neutral, we first compute the expected joint surplus, $S(T, p)$, of buyers and the seller for a given two-part pricing contract, (T, p):

$$S(T, p) = \mathrm{E}[U(\tilde{q}^\circ, \tilde{m}^\circ, \tilde{s})] - \mathrm{E}[U(0, I, \tilde{s})] + T + (p - c)\mathrm{E}(\tilde{q}^\circ), \tag{12}$$

where $\tilde{q}^\circ \equiv q(I, T, p, \tilde{s})$ and $\tilde{m}^\circ \equiv m(I, T, p, \tilde{s})$. The first-order conditions for maximizing

[13] Under Png and Wang's (2010) quasi-linear specification (see Section 4), i.e., $U(q, m, s) = u[b(q, s) + m]$, the restriction due to risk neutrality on $U(q, m, s)$ implies that $U_m(q^\circ, m^\circ, s) = u'[b(q^\circ, s) + m^\circ] = 1$ for all fixed two-part pricing contracts and states. Hence, in this case, it must be true that $u'(x) = 1$, i.e., $u(x) = x$, which is indeed the definition of risk neutrality in the Arrow-Pratt sense in which utility is a function of one argument.

the expected joint surplus in Eq. (12) are given by

$$S_T(T^B, p^B) = -\mathrm{E}[U_m(\tilde{q}^B, \tilde{m}^B, \tilde{s})] + 1 = 0, \tag{13}$$

and

$$S_p(T^B, p^B) = -\mathrm{E}[U_m(\tilde{q}^B, \tilde{m}^B, \tilde{s})\tilde{q}^B] + \mathrm{E}(\tilde{q}^B) + (p^B - c)\mathrm{E}[q_p(I, T^B, p^B, \tilde{s})] = 0, \tag{14}$$

where (T^B, p^B) is the first-best two-part pricing contract, $\tilde{q}^B = q(I, T^B, p^B, \tilde{s})$, and $\tilde{m}^B = m(I, T^B, p^B, \tilde{s})$. Since $U_m(q^\circ, m^\circ, s) = 1$ for all fixed two-part pricing contracts and states, Eq. (13) holds for any value of T^B. In other words, we can regard the fixed fee as a pure transfer to the seller, thereby making the fixed fee irrelevant to the maximization of the expected joint surplus. Since $U_m(q^B, m^B, s) = 1$ for all $s \in [\underline{s}, \overline{s}]$, Eq. (14) implies that $p^B = c$. When buyers are risk neutral, marginal-cost pricing attains the maximum expected joint surplus, $S^B = S(T, c)$, for any value of T. The profit-maximizing seller as such optimally sets the unit price equal to the constant marginal cost. The optimal fixed fee, T^0, is then the one that extracts the entire amount of S^B, i.e., $T^0 = S(T^0, c) = S^B$.

3.2. Risk-Averse Buyers

We now resume the original case that buyers are risk averse. Evaluating the left-hand side of Eq. (10) at $p^* = c$ yields

$$-\frac{\mathrm{Cov}\Big\{U_m\{q[I,T(c),c,\tilde{s}], m[I,T(c),c,\tilde{s}],\tilde{s}\}, q[I,T(c),c,\tilde{s}]\Big\}}{\mathrm{E}\Big\{U_m\{q[I,T(c),c,\tilde{s}], m[I,T(c),c,\tilde{s}],\tilde{s}\}\Big\}}. \tag{15}$$

Expression (15) is positive (negative) if the covariance of the marginal utility of all other goods, $U_m\{q[I,T(c),c,\tilde{s}], m[I,T(c),c,\tilde{s}],\tilde{s}\}$, and the consumption of the good, $q[I,T(c),c,\tilde{s}]$, is negative (positive). In this case, it follows from the second-order condition (9) and Eq. (10) that $p^* > (<)\ c$, which is exactly the same condition stated in Hayes (1987). Of course, such a condition is not that informative since it depends on endogenously chosen variables. It is thus of great interest to look for sufficient conditions under which we can unambiguously sign expression (15).

It is well known that the covariance of two monotonically increasing functions of a random variable is positive (see Theorem 236 in Hardy et al., 1964). Hence, to sign expression (15), we must delve into the co-movement of the marginal utility of all other goods at the optimum, $U_m(q^\circ, m^\circ, s)$, and the optimal consumption of the good, q°, when the realized state, s, varies. As is shown in Proposition 2, the following single-crossing property:

$$\frac{\partial}{\partial s}\left[\frac{U_q(q,m,s)}{U_m(q,m,s)}\right] > (<)\ 0, \tag{16}$$

is proved to be crucial for determining the sign of expression (15).

Proposition 2. *Given that condition (16) holds and the good is a normal good, i.e., $q_I(I,T,p,s) > 0$, the profit-maximizing seller sets the optimal unit price, p^*, lower (higher) than the constant marginal cost, c, if $U_{ms}(q,m,s) \geq (\leq)\ 0$. Given that condition (16) holds and the good is an inferior good, i.e., $q_I(I,T,p,s) < 0$, then $p^* > (<)\ c$ if $U_{ms}(q,m,s) \leq (\geq)\ 0$. Given that condition (16) holds and the optimal consumption of the good has no wealth effect, i.e., $q_I(I,T,p,s) = 0$, then $p^* < (>)\ c$ if $U_{ms}(q,m,s) > 0$, and $p^* > (<)\ c$ if $U_{ms}(q,m,s) < 0$.*

Condition (16) exhibits the differential Spence-Mirrlees single-crossing property that is central for monotone comparative statics analysis (see Milgrom and Shannon, 1994; Edlin and Shannon, 1998). It reflects the notion that the tradeoff between the consumption of the good, q, and that of all other goods, m, results in an increase in q and a decrease in m when the realized state, s, is higher (lower), i.e., $q_s(I,T,p,s) > (<)\ 0$. If the good is a normal good, $q_s(I,T,p,s) > (<)\ 0$ implies that the marginal utility of all other goods at the optimum is enhanced (reduced) when the realized state increases, which reinforces the direct effect that $U_{ms}(q,m,s) \geq (\leq)\ 0$, so that $U_m(q^\circ, m^\circ, s)$ unambiguously increases (decreases) with an increase in s. Expression (15) as such is negative (positive), thereby rendering $p^* < (>)\ c$.[14] On the other hand, if the good is an inferior good, $q_s(I,T,p,s) > (<)\ 0$ implies that the marginal utility of all other goods at the optimum is reduced (enhanced) when the realized state increases, which reinforces the direct effect that $U_{ms}(q,m,s) \leq (\geq)\ 0$, so that $U_m(q^\circ, m^\circ, s)$ unambiguously decreases (increases) with an increase in s. It then follows that expression (15) is positive (negative) and thus $p^* > (<)\ c$. Finally, if the optimal consumption of the good has no wealth effect, the sate variable affects the marginal utility of all other goods at the optimum solely through the direct effect, $U_{ms}(q,m,s)$. Hence, given condition (16), we have $p^* < (>)\ c$ if $U_{ms}(q,m,s) > 0$, and $p^* > (<)\ c$ if $U_{ms}(q,m,s) < 0$.

To see the intuition underlying Proposition 2, consider first the case that the marginal

[14]As an example, consider the following ex-post utility function employed in Hayes (1987): $U(q,m,s) = s\sqrt{q} + \sqrt{m}$. In this example, we have $U_{ms}(q,m,s) = 0$. Furthermore, the single-crossing property is satisfied, i.e., $\partial[U_q(q,m,s)/U_m(q,m,s)]/\partial s = \sqrt{m/q} > 0$. The optimal consumption of the good is $q^\circ = (I-T)/(p+p^2/s^2)$ so that the good is a normal good. From Proposition 2, the optimal unit price, p^*, must be less than the constant marginal cost, c, which is indeed what Hayes (1987) shows numerically by using a two-state discrete distribution of $\tilde{s}$.

utility of all other goods at the optimum, $U_m(q^\circ, m^\circ, s)$, increases with an increase in the realized state, s.[15] Suppose that the seller offers the two-part pricing contract, $[T(c), c]$, to buyers. In this case, the seller earns the expected profit equal to $T(c)$. Given condition (16) that ensures $q_s(I, T, p, s) > (<)\ 0$, Figure 1 depicts the optimal consumption of the good at $s = \underline{s}$, i.e., $q^c(\underline{s}) \equiv q[I, T(c), c, \underline{s}]$, and that at $s = \overline{s}$, i.e., $q^c(\overline{s}) \equiv q[I, T(c), c, \overline{s}]$ in Panel A (B) so that $q^c(\underline{s}) < (>)\ q^c(\overline{s})$. Let $[\hat{T}(p), p]$ be the two-part pricing contract that generates the same expected profit, i.e., $\hat{T}(p) + (p - c)\mathrm{E}\{q[I, \hat{T}(p), p, \tilde{s}]\} = T(c)$. If the seller chooses to lower (raise) the unit price, p, below (above) the constant marginal cost, c, he has to adjust the fixed fee, $\hat{T}(p)$, upward (downward) from $T(c)$ so as to keep the expected profit constant at $T(c)$. Given condition (16) that ensures $q_s(I, T, p, s) > (<)\ 0$, the seller alters the two-part pricing contract from $[T(c), c]$ to $[\hat{T}(p^*), p^*]$ with $p^* < (>)\ c$ and $\hat{T}(p^*) > (<)\ T(c)$. In this case, the optimal consumption of the good at $s = \underline{s}$ becomes $\hat{q}(\underline{s}) \equiv q[I, \hat{T}(p^*), p^*, \underline{s}]$, and that at $s = \overline{s}$ becomes $\hat{q}(\overline{s}) \equiv q[I, \hat{T}(p^*), p^*, \overline{s}]$. As is shown in Panels A and B of Figure 1, buyers' utility falls when the realized state is low and rises when the realized state is high. Since $U_m(q^\circ, m^\circ, s)$ increases with an increase in s, the utility loss in low states must be out-weighted by the utility gain in high states so that buyers' expected utility increases when the seller alters the two-part pricing contract from $[T(c), c]$ to $[\hat{T}(p^*), p^*]$:

$$\mathrm{E}\Big\{U\{q[I, \hat{T}(p^*), p^*, \tilde{s}], m[I, \hat{T}(p^*), p^*, \tilde{s}], \tilde{s}\}\Big\}$$

$$> \mathrm{E}\Big\{U\{q[I, T(c), c, \tilde{s}], m[I, T(c), c, \tilde{s}], \tilde{s}\}\Big\} = \mathrm{E}[U(0, I, \tilde{s})], \tag{17}$$

which reflects the insurance benefit that the two-part pricing contract offers to buyers. Since buyers' participation constraint (17) is slack, the seller raises the fixed fee from $\hat{T}(p^*)$ to $T^* = T(p^*)$, making the resulting expected profit exceed $T(c)$. Given condition (16) and that $U_m(q^\circ, m^\circ, s)$ increases with an increase in s, the seller indeed finds it optimal to set the unit price below (above) the constant marginal cost, c.

(Insert Figure 1 here.)

Consider now the case that the marginal utility of all other goods at the optimum, $U_m(q^\circ, m^\circ, s)$, decreases with an increase in the realized state, s.[16] Given the two-part

[15]If the good is a normal (an inferior) good, this is the case when $q_s(I, T, p, s) > (<)\ 0$ and $U_{ms}(q, m, s) \geq 0$. If the optimal consumption of the good has no wealth effect, this is the case when $U_{ms}(q, m, s) > 0$.

[16]If the good is a normal (an inferior) good, this is the case when $q_s(I, T, p, s) < (>)\ 0$ and $U_{ms}(q, m, s) \leq 0$. If the optimal consumption of the good has no wealth effect, this is the case when $U_{ms}(q, m, s) < 0$.

pricing contract, $[T(c), c]$, Figure 2 depicts the optimal consumption of the good, $q^c(\underline{s})$, at $s = \underline{s}$, and $q^c(\overline{s})$ at $s = \overline{s}$. Since condition (16) implies that $q_s(I, T, p, s) > (<)\ 0$, Panel A (B) of Figure 2 shows that $q^c(\underline{s}) < (>)\ q^c(\overline{s})$. If the seller alters the two-part pricing contract from $[T(c), c]$ to $[\hat{T}(p^*), p^*]$ with $p^* > (<)\ c$ and $\hat{T}(p^*) < (>)\ T(c)$, the optimal consumption of the good becomes $\hat{q}(\underline{s})$ at $s = \underline{s}$ and $\hat{q}(\overline{s})$ at $s = \overline{s}$. As is evident from Panels A and B of Figure 2, buyers' utility falls when the realized state is high and rises when the realized state is low. Since $U_m(q^\circ, m^\circ, s)$ decreases with an increase in s, the utility loss in high states must be out-weighted by the utility gain in low states so that Eq. (17) holds. Since buyers' participation constraint (17) is slack, the seller raises the fixed fee from $\hat{T}(p^*)$ to $T^* = T(p^*)$, thereby rendering the expected profit to exceed $T(c)$. Given condition (16) and that $U_m(q^\circ, m^\circ, s)$ decreases with an increase in s, we thus conclude that the optimal unit price, p^*, is indeed above (below) the constant marginal cost, c.

(Insert Figure 2 here.)

4. A Quasi-Linear Specification

Png and Wang (2010) have recently revisited the model of Hayes (1987) by using a quasi-linear specification of the ex-post utility function, $U(q, m, s)$. Succinctly, Png and Wang (2010) assume that $U(q, m, s) = u[b(q, s) + m]$, where $b(q, s) + m$ is the total benefit derived from the consumption of q units of the good and m of all other goods when the prevailing state is s, and $u(x)$ is a von Neumann-Morgenstern utility function defined over the ex-post total benefit, x, with $u'(x) > 0$ and $u''(x) < 0$, indicating the presence of risk aversion. Imposing such a quasi-linear specification not only greatly enhances the tractability of Hayes' (1987) model, but also clearly identifies the underlying uncertainty as the uncertainty about buyers' demand (i.e., marginal benefit) for the good.

Png and Wang (2010) assume that buyers' ex-post marginal benefit from consuming the good is positive but diminishing, i.e., $b_q(q, s) > 0$ and $b_{qq}(q, s) < 0$, which is consistent with $U_q(q, m, s) > 0$ and $U_{qq}(q, m, s) < 0$.[17] Png and Wang (2010) order the realizations of $\tilde{s}$ in such a way that $b_s(q, s) > 0$ for all $s \in [\underline{s}, \overline{s}]$. Under the quasi-linear specification, condition (16) reduces to $b_{qs}(q, s) > (<)\ 0$, which perfectly matches the two cases, (i) $b_{qs}(q, s) > 0$ and (ii) $b_{qs}(q, s) < 0$, that are the focuses of Png and Wang (2010). Since $b_s(q, s) > 0$, case (i) implies that buyers' total and marginal benefits are positively correlated ex ante, while case (ii) implies that they are negatively correlated ex ante.

[17]Since $U_q(q, m, s) = u'[b(q, s) + m]b_q(q, s)$ and $U_{qq}(q, m, s) = u''[b(q, s) + m]b_q(q, s)^2 + u'[b(q, s) + m]b_{qq}(q, s)$, it follows that $b_q(q, s) > 0$ and $b_{qq}(q, s) < 0$ imply that $U_q(q, m, s) > 0$ and $U_{qq}(q, m, s) < 0$, respectively. Furthermore, $U_m(q, m, s) = u'[b(q, s) + m] > 0$ and $U_{mm}(q, m, s) = u''[b(q, s) + m] < 0$.

Given that $U(q,m,s) = u[b(q,s)+m]$, Eq. (1) reduces to $b_q[q(p,s),s] = p$, so that the optimal consumption of the good, $q(p,s)$, depends neither on the initial wealth, I, nor on the fixed fee, T, i.e., there is no wealth effect.[18] Solving the binding budget constraint yields the optimal consumption of all other goods, $m(I,T,p,s) = I - T - pq(p,s)$. Hence, the ex-post total benefit of buyers at the optimum is given by

$$x(I,T,p,s) \equiv b[q(p,s),s] + m(I,T,p,s) = b[q(p,s),s] + I - T - pq(p,s). \tag{18}$$

Since $U_{ms}(q,m,s) = u''[b(q,s)+m]b_s(q,s) < 0$, it follows immediately from Proposition 2 that $p^* > (<)\ c$ if $b_{qs}(q,s) > (<)\ 0$, which is consistent with the findings in Png and Wang (2010). The optimal fixed fee, T^*, is implicitly determined by the following binding participation constraint:

$$\mathrm{E}\{u[x(I,T^*,p^*,\tilde{s})]\} = \mathrm{E}\{u[b(0,\tilde{s})+I]\}, \tag{19}$$

where $x(I,T,p,s)$ is given by Eq. (18).

In the benchmark case that buyers are risk neutral, we have $u(x) = x$ so that $U(q,m,s) = b(q,s)+m$.[19] It follows from Proposition 1 that the optimal unit price is equal to the constant marginal cost, c.[20] The optimal fixed fee, T^0, is then determined by the following binding participation constraint under risk neutrality:

$$\mathrm{E}[x(I,T^0,c,\tilde{s})] = \mathrm{E}[b(0,\tilde{s})+I]. \tag{20}$$

Substituting Eq. (18) into Eq. (20) and solving for T^0 yields[21]

$$T^0 = \mathrm{E}\{b[q(c,\tilde{s}),\tilde{s}]\} - \mathrm{E}[b(0,\tilde{s})] - c\mathrm{E}[q(c,\tilde{s})] > 0. \tag{21}$$

We show in the following proposition that the optimal fixed fee, T^*, is smaller or greater than T^0, depending on whether buyers' total and marginal benefits are positively or negatively correlated, respectively.

Proposition 3. *Given that buyers have the ex-post utility function, $u[b(q,s)+m]$, satisfying $b_{qs}(q,s) > (<)\ 0$, the profit-maximizing seller sets the optimal unit price, p^*, higher (lower)*

[18] The second-order condition (2) is satisfied since $b_{qq}(q,s) < 0$.

[19] Using the results in Hosoya (2011), we can interpret $b(q,s)+m$ as the least concave utility function of quasi-linear preferences.

[20] When buyers are risk neutral so that $u(x) = x$ and $p^* = c$, Eq. (9) reduces to $\mathrm{E}[q_p(c,\tilde{s})] < 0$, which holds since $q_p(p,s) = 1/b_{qq}[q(p,s),s] < 0$.

[21] Since $b[q(c,s),s] - cq(c,s) > b(0,s)$ for all $s \in [\underline{s},\overline{s}]$, we have $T^0 > 0$.

than the constant marginal cost, c, and the optimal fixed fee, $T^* > 0$, lower (higher) than the risk-neutral counterpart, $T^0 > 0$.

The intuition for Proposition 3 is the same as that for Proposition 2 when the marginal utility of all other goods at the optimum satisfies that $U_m(q^\circ, m^\circ, s)$ decreases with an increase in the realized state, s. To see why $T^* < (>)\ T^0$ if $b_{qs}(q, s) > (<)\ 0$, suppose that the seller offers the two-part pricing contract, (T^0, c), which is optimal under risk neutrality, to buyers who are risk averse. If buyers were risk neutral, they would have been indifferent between accepting and rejecting (T^0, c) given Eq. (20), which implies that buyers derive the same expected total benefit with and without the right to purchase the good from the seller. Since $\partial[x(I, T^0, c, s) - b(0, s)]/\partial s = b_s[q(c, s), s] - b_s(0, s) > (<)\ 0$ if $b_{qs}(q, s) > (<)\ 0$, it follows from Eq. (20) that there must exist a critical realization of $\tilde{s}$ below which $x(I, T^0, c, s)$ is less (greater) than $b(0, s) + I$, and above which $x(I, T^0, c, s)$ is greater (less) than $b(0, s) + I$, thereby rendering $x(I, T^0, c, \tilde{s})$ to be more (less) volatile than $b(0, \tilde{s}) + I$. Indeed, following the arguments in the proof of Proposition 3, we can easily verify that the distribution of $x(I, T^0, c, \tilde{s})$ is a mean-preserving-spread of the distribution of $b(0, \tilde{s}) + I$ in the sense of Rothschild and Stiglitz (1970) if $b_{qs}(q, s) > 0$, and that the converse is true if $b_{qs}(q, s) < 0$. Since buyers are risk averse, it follows from Rothschild and Stiglitz (1971) that $\mathrm{E}\{u[x(I, T^0, c, \tilde{s})]\} < \mathrm{E}\{u[b(0, \tilde{s}) + I]\}$ and thus buyers strictly prefer $b(0, \tilde{s}) + I$ to $x(I, T^0, c, \tilde{s})$ if $b_{qs}(q, s) > 0$. To induce buyers to accept the two-part pricing contract, the seller has to cut the fixed fee below T^0 to $T(c)$. Since $p^* > c$, we have $T^* = T(p^*) < T(c) < T^0$. On the other hand, if $b_{qs}(q, s) < 0$, risk-averse buyers strictly prefer $x(I, T^0, c, \tilde{s})$ to $b(0, \tilde{s}) + I$. The seller as such raises the fixed fee above T^0 to $T(c)$ to extract more rent from buyers. Since $p^* < c$, we have $T^* = T(p^*) > T(c) > T^0$.[22]

4.1. Risk Aversion

Proposition 3 shows the global effect of imposing risk aversion onto buyers on the optimal two-part pricing contract, (T^*, p^*), i.e., (T^*, p^*) is compared with the optimal one under risk neutrality, (T^0, c). A related issue of interest is to study the marginal effect of increased risk aversion of buyers on (T^*, p^*). To this end, we follow Diamond and Stiglitz (1974) to work with a differentiable family of utility functions, $u(x, \alpha)$, where α is a one sided index of risk aversion. Given this notation, Diamond and Stiglitz (1974) show that an increase in α represents an increase in risk aversion if, and only if, the coefficient of absolute risk

[22]Png and Wang (2010) show that $T(c) < (>)\ T(p^*) = T^*$ if $b_{qs}(q, s) > (<)\ 0$. However, they do not compare T^0 with either $T(c)$ or T^*. Proposition 3 as such fills this gap.

aversion (Arrow, 1965; Pratt, 1964), $-u_{xx}(x,\alpha)/u_x(x,\alpha)$, satisfies that

$$\frac{\partial}{\partial\alpha}\left[-\frac{u_{xx}(x,\alpha)}{u_x(x,\alpha)}\right]=\frac{u_{xx}(x,\alpha)u_{x\alpha}(x,\alpha)-u_x(x,\alpha)u_{xx\alpha}(x,\alpha)}{u_x(x,\alpha)^2}>0, \tag{22}$$

i.e., an increase in α increases the Arrow-Pratt measure of absolute risk aversion for all $x \geq 0$.

We state the binding participation constraint of buyers as

$$\mathrm{E}\Big\{u\{x[I,T(p,\alpha),p,\tilde{s}],\alpha\}\Big\}=\mathrm{E}\{u[b(0,\tilde{s})+I,\alpha]\}, \tag{23}$$

which defines the fixed fee, $T(p,\alpha)$, as a function of the unit price, p, and the index of risk aversion, α. The first-order condition, Eq. (8), becomes

$$(p^*-c)\mathrm{E}[q_p(p^*,\tilde{s})]+\mathrm{E}[q(p^*,\tilde{s})]+T'(p^*,\alpha)=0. \tag{24}$$

Totally differentiating $T^*=T(p^*,\alpha)$ with respect to α yields

$$\frac{\mathrm{d}T^*}{\mathrm{d}\alpha}=T_\alpha(p^*,\alpha)+T_p(p^*,\alpha)\times\frac{\mathrm{d}p^*}{\mathrm{d}\alpha}, \tag{25}$$

where we differentiate Eq. (23) with respect to p and α separately, and evaluate the resulting derivatives at $p=p^*$ to yield

$$T_p(p^*,\alpha)=-\frac{\mathrm{E}\{u_x[x(I,T^*,p^*,\tilde{s}),\alpha]q(p^*,\tilde{s})\}}{\mathrm{E}\{u_x[x(I,T^*,p^*,\tilde{s}),\alpha]\}}<0, \tag{26}$$

and

$$T_\alpha(p^*,\alpha)=\frac{\mathrm{E}\{u_\alpha[x(I,T^*,p^*,\tilde{s}),\alpha]\}-\mathrm{E}\{u_\alpha[b(0,\tilde{s})+I,\alpha]\}}{\mathrm{E}\{u_x[x(I,T^*,p^*,\tilde{s}),\alpha]\}}. \tag{27}$$

The first term on the right-hand side of Eq. (25) is the direct effect of increased risk aversion on the optimal fixed fee, T^*. The second term on the right-hand side of Eq. (25) captures the indirect effect of increased risk aversion on T^* via its effect on the optimal unit price, p^*. Eq. (26) implies that the indirect effect has the opposite sign to that of $\mathrm{d}p^*/\mathrm{d}\alpha$.

Totally differentiating Eq. (24) with respect to α yields

$$\frac{\mathrm{d}p^*}{\mathrm{d}\alpha}=-\frac{T_{p\alpha}(p^*,\alpha)}{T_{pp}(p^*,\alpha)+2\mathrm{E}[q_p(p^*,\tilde{s})]+(p^*-c)\mathrm{E}[q_{pp}(p^*,\tilde{s})]}, \tag{28}$$

where we differentiate Eq. (23) twice with respect to p and α, and evaluate the resulting derivative at $p = p^*$ to yield

$$T_{p\alpha}(p^*, \alpha) = -\frac{\mathrm{E}\{u_{x\alpha}[x(I, T^*, p^*, \tilde{s}), \alpha][T_p(p^*, \alpha) + q(p^*, \tilde{s})]\}}{\mathrm{E}\{u_x[x(I, T^*, p^*, \tilde{s}), \alpha]\}}$$

$$+T_\alpha(p^*, \alpha) \times \frac{\mathrm{E}\{u_{xx}[x(I, T^*, p^*, \tilde{s}), \alpha][T_p(p^*, \alpha) + q(p^*, \tilde{s})]\}}{\mathrm{E}\{u_x[x(I, T^*, p^*, \tilde{s}), \alpha]\}}. \tag{29}$$

It follows from the second-order condition (9) and Eq. (28) that the sign of $\mathrm{d}p^*/\mathrm{d}\alpha$ is the same as that of $T_{p\alpha}(p^*, \alpha)$. Inspection of Eqs. (28) and (29) reveals that there are two effects that jointly determine how increased risk aversion influences the optimal unit price, p^*. The first term on the right-hand side of Eq. (29) captures the pure effect of increased risk aversion on p^*. The second term on the right-hand side of Eq. (29) captures the wealth effect that arises from the direct effect of increased risk aversion on the optimal fixed fee, T^*. Specifically, if $T_\alpha(p^*, \alpha) > (<)\ 0$, buyers' wealth is reduced (increased) by this amount for all $s \in [\underline{s}, \overline{s}]$ as they become more risk averse, thereby giving rise to the wealth effect.[23]

The literature on risk aversion is completely silent about how increased risk aversion affects marginal utility, i.e., the signs of $u_\alpha(x, \alpha)$, $u_{x\alpha}(x, \alpha)$, and $u_{xx\alpha}(x, \alpha)$. Thus, there is no acceptable guidance that we can follow to determine the sign of $T_\alpha(p^*, \alpha)$, except in the special case that buyers' preferences exhibit constant absolute risk aversion (CARA), i.e., $-u_{xx}(x, \alpha)/u_x(x, \alpha) = \alpha$ for all $x \geq 0$, where $\alpha > 0$ is the constant coefficient of absolute risk aversion. Note that this is also the case considered by Png and Wang (2010) when they examine how increased risk aversion of buyers affects the optimal two-part-pricing contract, (T^*, p^*). Under CARA, we have $u_x(x, \alpha) = -\alpha u_{xx}(x, \alpha)$. It then follows from Eq. (26) that the second term on the right-hand side of Eq. (29) vanishes. This is the well-known result that CARA induces no wealth effect. Thus, $\mathrm{d}p^*/\mathrm{d}\alpha$ is solely determined by the pure effect of increased risk aversion. The following proposition shows how an increase in buyers' risk aversion affects the optimal two-part pricing contract, (T^*, p^*), under CARA.

Proposition 4. *Given that buyers have the ex-post utility function, $u[b(q, s) + m]$, satisfying $b_{qs}(q, s) > (<)\ 0$, and that $u(x)$ exhibits constant absolute risk aversion, the profit-maximizing seller raises (lowers) the optimal unit price, p^*, and lowers (raises) the optimal fixed fee, T^*, as the constant coefficient of absolute risk aversion, α, increases, i.e., $\mathrm{d}p^*/\mathrm{d}\alpha > (<)\ 0$ and $\mathrm{d}T^*/\mathrm{d}\alpha < (>)\ 0$.*

[23] In standard decision making problems, increased risk aversion seldom creates any wealth effect. In our context, the wealth effect is driven by the binding participation constraint, Eq. (23), which is evidently affected by the index of risk aversion, α. See also Wong (1997).

When buyers' preferences exhibit CARA, Png and Wang (2010) derive the same comparative static results regarding the optimal unit price, p^*. Ignoring the direct effect, $T_\alpha(p^*, \alpha)$, Png and Wang (2010) conclude that the effect of an increase in buyers' risk aversion on the optimal fixed fee, T^*, is governed solely by the indirect effect, $T_p(p^*, \alpha) \times \mathrm{d}p^*/\mathrm{d}\alpha$. Indeed, we show in Proposition 4 that the direct effect reinforces the indirect effect under CARA, thereby making the overall effect of increased risk aversion on the optimal fixed fee, T^*, unambiguous under CARA.

4.2. Competition

To study how competition affects the optimal two-part pricing contract, (T^*, p^*), we model the determination of (T^*, p^*) by the Nash bargaining solution. Let η be the bargaining power of the seller, and $1-\eta$ be that of buyers, where $\eta \in [0, 1]$. The Nash bargaining solution is given by

$$\max_{T,p} \left\{T + (p-c)\mathrm{E}[q(p, \tilde{s})]\right\}^{\eta}\left\{\mathrm{E}\{u[x(I, T, p, \tilde{s})]\} - \mathrm{E}\{u[b(0, \tilde{s}) + I]\}\right\}^{1-\eta}, \tag{30}$$

since buyers' reservation utility would be $\mathrm{E}\{u[b(0, \tilde{s}) + I]\}$ and the seller's profit would be zero should no agreement be reached. When $\eta = 1$, the seller has all the bargaining power and, thereby, is a monopoly. This extreme case has been thoroughly analyzed above (see also Png and Wang, 2010). When $\eta = 0$, the seller has no bargaining power and, thereby, is perfectly competitive. In this extreme case, Png and Wang (2010) show that the optimal unit price is above or below the constant marginal cost depending on whether buyers' total and marginal benefits are positively or negatively correlated, respectively, which are qualitatively the same as those when $\eta = 1$. The seller makes zero expected profit so that the optimal fixed fee is set to exactly offset the expected operating profit. In general when $\eta \in (0, 1)$, the seller possesses some, but not all, bargaining power and, thereby, is imperfectly competitive. We hereafter interpret a lower value of η as a greater extent of competition confronted by the seller.

The first-order conditions for program (30) are given by[24]

$$\eta\left\{\mathrm{E}\{u[x(I, T^*, p^*, \tilde{s})]\} - \mathrm{E}\{u[b(0, \tilde{s}) + I]\}\right\}$$

$$-(1-\eta)\{T^* + (p^* - c)\mathrm{E}[q(p^*, \tilde{s})]\}\mathrm{E}\{u'[x(I, T^*, p^*, \tilde{s})]\} = 0, \tag{31}$$

[24]The second-order conditions for program (30) are assumed to hold.

and

$$\eta\Big\{\mathrm{E}\{u[x(I,T^*,p^*,\tilde{s})]\} - \mathrm{E}\{u[b(0,\tilde{s})+I]\}\Big\}\{\mathrm{E}[q(p^*,\tilde{s})] + (p^*-c)\mathrm{E}[q_p(p^*,\tilde{s})]\}$$

$$-(1-\eta)\{T^* + (p^*-c)\mathrm{E}[q(p^*,\tilde{s})]\}\mathrm{E}\{u'[x(I,T^*,p^*,\tilde{s})]q(p^*,\tilde{s})\} = 0. \tag{32}$$

Multiplying $\mathrm{E}\{u'[x(I,T^*,p^*,\tilde{s})]q(p^*,\tilde{s})\}/\mathrm{E}\{u'[x(I,T^*,p^*,\tilde{s})]\}$ to Eq. (31) and subtracting the resulting equation from Eq. (32) yields

$$\eta\Big\{\mathrm{E}\{u[x(I,T^*,p^*,\tilde{s})]\} - \mathrm{E}\{u[b(0,\tilde{s})+I]\}\Big\}$$

$$\times\Big\{\mathrm{E}[q(p^*,\tilde{s})] + (p^*-c)\mathrm{E}[q_p(p^*,\tilde{s})] - \frac{\mathrm{E}\{u'[x(I,T^*,p^*,\tilde{s})]q(p^*,\tilde{s})\}}{\mathrm{E}\{u'[x(I,T^*,p^*,\tilde{s})]\}}\Big\} = 0. \tag{33}$$

If $\eta \in (0,1)$, Eq. (33) reduces to

$$\mathrm{E}[q(p^*,\tilde{s})] + (p^*-c)\mathrm{E}[q_p(p^*,\tilde{s})] = \frac{\mathrm{E}\{u'[x(I,T^*,p^*,\tilde{s})]q(p^*,\tilde{s})\}}{\mathrm{E}\{u'[x(I,T^*,p^*,\tilde{s})]\}}. \tag{34}$$

In the two extreme cases that $\eta = 0$ and $\eta = 1$, we know from Png and Wang (2010) that Eq. (34) remains to be the optimality condition. Hence, the results of Proposition 3 regarding the optimal unit price, p^*, apply to all $\eta \in [0,1]$.

As a benchmark, suppose that buyers are risk neutral so that $u(x) = x$. It follows from Eq. (34) with $u'(x) = 1$ that $p^* = c$. The optimal fixed fee, denoted by T^0, is then determined by Eq. (31) with $u(x) = x$ and $p^* = c$:

$$\eta\{\mathrm{E}[x(I,T^0,c,\tilde{s})] - \mathrm{E}[b(0,\tilde{s})+I]\} - (1-\eta)T^0 = 0. \tag{35}$$

Substituting Eq. (18) into Eq. (35) and solving for T^0 yields

$$T^0 = \eta\Big\{\mathrm{E}\{b[q(c,\tilde{s}),\tilde{s}]\} - \mathrm{E}[b(0,\tilde{s})] - c\mathrm{E}[q(c,\tilde{s})]\Big\} \geq 0, \tag{36}$$

where the equality holds when $\eta = 0$. Inspection of the optimal two-part pricing contract, (T^0, c), under risk neutrality reveals that introducing competition to the seller, i.e., setting $\eta < 1$, renders buyers to enjoy a lower fixed fee, but has no effect on the optimal unit price. To see why the optimal unit price is neutral to competition, we use Eqs. (12) and (18) to write the expected joint surplus of buyers and the seller as

$$S(T,p) = \mathrm{E}[x(I,T,p,\tilde{s})] - \mathrm{E}[b(0,\tilde{s})+I] + T + (p-c)\mathrm{E}[q(p,\tilde{s})]$$

$$= \mathrm{E}\{b[q(p,\tilde{s}),\tilde{s}]\} - \mathrm{E}[b(0,\tilde{s})] - c\mathrm{E}[q(p,\tilde{s})], \tag{37}$$

which is maximized at $p = c$. Hence, Eq. (37) implies that the maximum expected joint surplus is given by $S^B = S(T,c) = \mathrm{E}\{b[q(c,\tilde{s}),\tilde{s}]\} - \mathrm{E}[b(0,\tilde{s})] - c\mathrm{E}[q(c,\tilde{s})]$. The fixed fee, T^0, is a pure transfer of this surplus, S^B, between buyers and the seller. According to their relative bargaining power, the seller receives ηS^B and buyers receive $(1-\eta)S^B$, as is evident from Eqs. (35) and (36).

We characterize the optimal two-part pricing contract, (T^*, p^*), of the Nash bargaining solution, which solves Eqs. (31) and (34) simultaneously, in the following proposition.[25]

Proposition 5. *Given that buyers have the ex-post utility function, $u[b(q,s)+m]$, satisfying $b_{qs}(q,s) > (<)\ 0$, the Nash bargaining solution is the one at which the optimal unit price, p^*, is set higher (lower) than the constant marginal cost, c, and the optimal fixed fee, T^*, is set lower (higher) than the risk-neutral counterpart, $T^0 \geq 0$.*

The intuition for Proposition 5 is similar to that for Proposition 3 and thus is omitted. An immediate implication of Proposition 5 is that imposing competition onto the seller does not seem to change the optimal two-part pricing contract in a qualitative manner. Indeed, consider the case that buyers' preferences exhibit constant absolute risk aversion (CARA) so that the utility function, $u(x)$, must take on the exponential form. Without any loss of generality, let $u(x) = -e^{-\alpha x}$, where $\alpha > 0$ is the constant coefficient of absolute risk aversion. In this case, Eq. (34) reduces to

$$\mathrm{E}[q(p^*,\tilde{s})] + (p^*-c)\mathrm{E}[q_p(p^*,\tilde{s})] = \frac{\mathrm{E}\{e^{-\alpha\{b[q(p^*,\tilde{s}),\tilde{s}]-p^*q(p^*,\tilde{s})\}}q(p^*,\tilde{s})\}}{\mathrm{E}\{e^{-\alpha\{b[q(p^*,\tilde{s}),\tilde{s}]-p^*q(p^*,\tilde{s})\}}\}}. \tag{38}$$

Inspection of Eq. (38) reveals that the optimal unit price, p^*, does not depend on the extent of competition, η, confronted by the seller in the case of CARA. When $u(x) = -e^{-\alpha x}$, using the fact that $\mathrm{d}p^*/\mathrm{d}\eta = 0$, we differentiate Eq. (31) with respect to η to yield

$$\frac{\mathrm{d}T^*}{\mathrm{d}\eta} = \frac{T^* + (p^*-c)\mathrm{E}[q(p^*,\tilde{s})]}{\eta + \alpha\eta(1-\eta)\{T^* + (p^*-c)\mathrm{E}[q(p^*,\tilde{s})]\}} > 0, \tag{39}$$

for all $\eta \in (0,1)$. Hence, Eq. (39) implies that as competition becomes lax, the seller with greater bargaining power is able to charge a higher fixed fee under CARA. These results are qualitatively the same as those under risk neutrality.

[25]If $b_{qs}(q,s) > 0$, prudence, i.e., $u'''(x) > 0$, is called for to ensure that $T^* < T^0$. As convincingly argued by Kimball (1990, 1993), prudence is a reasonable behavioral assumption for decision making under uncertainty. See also Eeckhoudt et al. (2007) for extending the concepts of prudence to multivariate utility functions.

5. Conclusion

In this paper, we re-examine the model of Hayes (1987) wherein a risk-neutral, profit-maximizing monopoly (the seller) sells a good or service to buyers via a two-part pricing contact. Demand uncertainty is modeled by a state variable that affects buyers' ex-post demand for the good. Prior to knowing the realization of the state variable, buyers have to decide whether to subscribe to the two-part pricing contract offered by the seller or not. When buyers are risk neutral, we show that marginal-cost pricing is not only profit-maximizing but also socially efficient. When buyers are risk averse, demand uncertainty calls for the insurance need of buyers, which induces the seller to deviate from marginal-cost pricing. We show that the optimal unit price is higher or lower than the constant marginal cost, depending on the nature of the good (normal or inferior) and on the signs of cross-derivatives of buyers' multivariate utility function.

Following Png and Wang (2010) to employ a quasi-linear specification that reduces the general multivariate utility function to a special univariate utility function, we show that the seller optimally raises (lowers) the unit price and lowers (raises) the fixed fee from their risk-neutral counterparts if buyers' total and marginal benefits are positively (negatively) correlated, which are consistent with the results of the general model of Hayes (1987). Confined to buyers' preferences that exhibit constant absolute risk aversion (CARA), we show that the marginal effect of increased risk aversion of buyers on the optimal two-part pricing contract inherits the global effect of risk aversion. Finally, we introduce competition to the seller in a reduced form by using the Nash bargaining solution to determine the optimal two-part pricing contract. We show that imposing competition onto the seller by means of the Nash bargaining solution does not qualitatively alter the optimal two-part pricing contract.

Appendix A

PROOF OF PROPOSITION 1. Since $U_m(q^*, m^*, s) = 1$ for all $s \in [\underline{s}, \overline{s}]$, Eq. (10) reduces to

$$(p^* - c)\{\mathrm{E}[q_p(I, T^*, p^*, \tilde{s})] - \mathrm{E}[q_T(I, T^*, p^*, \tilde{s})]\mathrm{E}(\tilde{q}^*)\} = 0. \tag{A.1}$$

Differentiating Eq. (1) with respect to T and rearranging terms yields

$$q_T(I, T, p, s) = -\frac{1}{\Delta}[pU_{mm}(q^\circ, m^\circ, s) - U_{qm}(q^\circ, m^\circ, s)]. \tag{A.2}$$

Differentiating Eq. (1) with respect to p and rearranging terms yields

$$q_p(I,T,p,s) = -\frac{1}{\Delta}\{[pU_{mm}(q^\circ,m^\circ,s) - U_{qm}(q^\circ,m^\circ,s)]q^\circ - U_m(q^\circ,m^\circ,s)\}$$

$$= \frac{1}{\Delta} + q_T(I,T,p,s)q^\circ, \tag{A.3}$$

where the second equality follows from Eq. (A.2) and the fact that $U_m(q^\circ,m^\circ,s) = 1$. Substituting Eqs. (A.2) and (A.3) with $T = T^*$ and $p = p^*$ into Eq. (A.1) yields

$$(p^* - c)\left\{\mathrm{E}\left(\frac{1}{\tilde{\Delta}}\right) + \mathrm{Cov}[q_T(I,T^*,p^*,\tilde{s}),\tilde{q}^*]\right\} = 0. \tag{A.4}$$

The expression inside the curly brackets on the left-hand side of Eq. (A.4) vanishes only in very special cases with measure zero. Hence, Eq. (A.4) implies that $p^* = c$. From Eq. (14), $p^* = c$ is also the optimal unit price that maximizes the expected joint surplus, $S(T,p)$, in Eq. (12). The optimal fixed fee, T^0, is then determined by solving Eq. (5) with $p = c$. Hence, the seller's maximum expected profit is equal to $T^0 = S(T^0,c) = S^B$, thereby rendering the optimality of the two-part pricing contract, (T^0,c).

Proof of Proposition 2. Differentiating Eq. (1) with respect to s and rearranging terms yields

$$q_s(I,T,p,s) = -\frac{1}{\Delta}[U_{qs}(q^\circ,m^\circ,s) - pU_{ms}(q^\circ,m^\circ,s)]$$

$$= -\frac{1}{\Delta}\left[\frac{U_m(q^\circ,m^\circ,s)U_{qs}(q^\circ,m^\circ,s) - U_q(q^\circ,m^\circ,s)U_{ms}(q^\circ,m^\circ,s)}{U_m(q^\circ,m^\circ,s)}\right], \tag{A.5}$$

where the second equality follows from Eq. (1), and $\Delta < 0$ from the second-order condition (2). It follows from Eq. (A.5) that $q_s(I,T,p,s) > (<)\ 0$ given condition (16). Differentiating $U_m(q^\circ,m^\circ,s)$ with respect to s yields

$$\frac{\partial}{\partial s}U_m(q^\circ,m^\circ,s)$$

$$= [U_{qm}(q^\circ,m^\circ,s) - pU_{mm}(q^\circ,m^\circ,s)]q_s(I,T,p,s) + U_{ms}(q^\circ,m^\circ,s). \tag{A.6}$$

Differentiating Eq. (1) with respect to I and rearranging terms yields

$$q_I(I,T,p,s) = -\frac{1}{\Delta}[U_{qm}(q^\circ,m^\circ,s) - pU_{mm}(q^\circ,m^\circ,s)]. \tag{A.7}$$

According to Eq. (A.7), the good is a normal (an inferior) good, i.e., $q_I(I, T, p, s) > (<)\ 0$, if $U_{qm}(q^\circ, m^\circ, s) - pU_{mm}(q^\circ, m^\circ, s) > (<)\ 0$.

Condition (16) implies that $q_s(I, T, p, s) > (<)\ 0$. If the good is a normal good and $U_{ms}(q, m, s) \geq (\leq)\ 0$, it then follows from Eq. (A.6) that $U_m(q^\circ, m^\circ, s)$ is an increasing (a decreasing) function of s. Expression (15) is, therefore, negative (positive) so that $p^* < (>)\ c$. On the other hand, if the good is an inferior good and $U_{ms}(q, m, s) \leq (\geq)\ 0$, it then follows from Eq. (A.6) that $U_m(q^\circ, m^\circ, s)$ is a decreasing (an increasing) function of s. Expression (15) is, therefore, positive (negative) so that $p^* > (<)\ c$. Finally, if the optimal consumption of the good has no wealth effect, i.e., $q_I(I, T, p, s) = 0$, it then follows from Eq. (A.6) that $U_m(q^\circ, m^\circ, s)$ is an increasing (a decreasing) function of s if $U_{ms}(q, m, s) < (>)\ 0$. Given condition (16), expression (15) is negative (positive) so that $p^* < (>)\ c$ if $U_{ms}(q, m, s) > 0$, and expression (15) is positive (negative) so that $p^* > (<)\ c$ if $U_{ms}(q, m, s) < 0$

Proof of Proposition 3. From Proposition 2, we have $p^* > (<)\ c$ if $b_{qs}(q, s) > (<)\ 0$. Suppose that $T^* \leq 0$. Then, $x(I, T^*, p^*, s) \geq b[q(p^*, s), s] + I - p^*q(p^*, s) > b(0, s) + I$ for all $s \in [\underline{s}, \overline{s}]$, and thus $\mathrm{E}\{u[x(I, T^*, p^*, \tilde{s})]\} > \mathrm{E}\{u[b(0, \tilde{s}) + I]\}$, a contradiction to Eq. (19). Hence, we have $T^* > 0$. It remains to show that $T^* < (>)\ T^0$ if $b_{qs}(q, s) > (<)\ 0$.

Let T^n be the fixed fee that solves the following equation:

$$\mathrm{E}[x(I, T^n, p^*, \tilde{s})] = \mathrm{E}[b(0, \tilde{s}) + I], \tag{A.8}$$

where p^* is the optimal unit price. Substituting Eq. (18) into Eq. (A.8) and solving for T^n yields

$$T^n = \mathrm{E}\{b[q(p^*, \tilde{s}), \tilde{s}]\} - \mathrm{E}[b(0, \tilde{s})] - p^*\mathrm{E}[q(p^*, \tilde{s})] > 0. \tag{A.9}$$

We want to show that $\mathrm{E}\{u[x(I, T^n, p^*, \tilde{s})]\} < (>)\ \mathrm{E}\{u[b(0, \tilde{s}) + I]\}$ if $b_{qs}(q, s) > (<)\ 0$, which is done by showing that the distribution of $x(I, T^n, p^*, \tilde{s})$ is a mean-preserving-spread of the distribution of $b(0, \tilde{s}) + I$ in the sense of Rothschild and Stiglitz (1970) if $b_{qs}(q, s) > 0$, and that the converse holds if $b_{qs}(q, s) < 0$.

Let $\Phi(x)$ be the cumulative distribution function of $\tilde{x} = x(I, T^n, p^*, \tilde{s})$. Since $\partial x/\partial s = b_s[q(p^*, s), s] > 0$, the inverse of $x = x(I, T^n, p^*, s)$ exists and is denoted by $s = f(x)$. Using the change-of-variable technique (see, e.g., Hogg and Craig, 1989), $\tilde{x}$ has support $[x(I, T^n, p^*, \underline{s}), x(I, T^n, p^*, \overline{s})]$ and $\Phi(x) = G[f(x)]$. We can express $\mathrm{E}\{u[x(I, T^n, p^*, \tilde{s})]\}$ in terms of $\Phi(x)$:

$$\mathrm{E}\{u[x(I, T^n, p^*, \tilde{s})]\} = \int_{x(I,T^n,p^*,\underline{s})}^{x(I,T^n,p^*,\overline{s})} u(x)\ \mathrm{d}\Phi(x). \tag{A.10}$$

Likewise, let $\Psi(z)$ be the cumulative distribution function of $\tilde{z} = b(0, \tilde{s}) + I$. Since $\partial z/\partial s = b_s(0, s) > 0$, the inverse of $z = b(0, s)+I$ exists and is denoted by $s = g(z)$. Using the change-of-variable technique (see, e.g., Hogg and Craig, 1989), $\tilde{z}$ has support $[b(0, \underline{s}) + I, b(0, \overline{s}) + I]$ and $\Psi(z) = G[g(z)]$. We can express $\mathrm{E}\{u[b(0, \tilde{s}) + I]\}$ in terms of $\Psi(z)$:

$$\mathrm{E}\{u[b(0, \tilde{s}) + I]\} = \int_{b(0,\underline{s})+I}^{b(0,\overline{s})+I} u(z)\ \mathrm{d}\Psi(z). \tag{A.11}$$

Note that $\partial(x - z)/\partial s = b_s[q(p^*, s), s] - b_s(0, s)$. If $b_{qs}(q, s) > 0$, we have $\partial(x - z)/\partial s > 0$. It then follows from Eq. (A.8) that $b(0, \underline{s}) + I > x(I, T^n, p^*, \underline{s})$, $b(0, \overline{s}) + I < x(I, T^n, p^*, \overline{s})$, and that there exists a unique point, $s^{\mathrm{o}} \in (\underline{s}, \overline{s})$, such that $x > (<)\ z$ for all $s > (<)\ s^{\mathrm{o}}$ and $x = z = x^{\mathrm{o}} \in (b(0, \underline{s}) + I, b(0, \overline{s}) + I)$ at $s = s^{\mathrm{o}}$. These observations imply that $f(x) > g(x)$ for all $x \in [b(0, \underline{s}) + I, x^{\mathrm{o}})$ and $f(x) < g(x)$ for all $x \in (x^{\mathrm{o}}, b(0, \overline{s}) + I]$. On the other hand, if $b_{qs}(q, s) < 0$, we have $\partial(x - z)/\partial s < 0$. It then follows from Eq. (A.8) that $b(0, \underline{s}) + I < x(I, T^n, p^*, \underline{s})$, $b(0, \overline{s}) + I > x(I, T^n, p^*, \overline{s})$, and that there exists a unique point, $s^{\diamond} \in (\underline{s}, \overline{s})$, such that $x > (<)\ z$ for all $s < (>)\ s^{\diamond}$ and $x = z = x^{\diamond} \in (x(I, T^n, p^*, \underline{s}), x(I, T^n, p^*, \overline{s}))$ at $s = s^{\diamond}$. These observations imply that $f(x) < g(x)$ for all $x \in [x(I, T^n, p^*, \underline{s}), x^{\diamond})$ and $f(x) > g(x)$ for all $x \in (x^{\diamond}, x(I, T^n, p^*, \overline{s})]$.

Consider first the case that $b_{qs}(q, s) > 0$. Subtracting Eq. (A.10) from Eq. (A.11) yields

$$\mathrm{E}\{u[x(I, T^n, p^*, \tilde{s})]\} - \mathrm{E}\{u[b(0, \tilde{s}) + I]\} = \int_{x(I,T^n,p^*,\underline{s})}^{x(I,T^n,p^*,\overline{s})} u(x)\ \mathrm{d}[\Phi(x) - \Psi(x)], \tag{A.12}$$

since $\mathrm{d}\Psi(x) = 0$ for all $x \in [x(I, T^n, p^*, \underline{s}), b(0, \underline{s}) + I) \cup (b(0, \overline{s}) + I, x(I, T^n, p^*, \overline{s})]$. Consider the following function:

$$D(x) = \int_{x(I,T^n,p^*,\underline{s})}^{x} [\Phi(y) - \Psi(y)]\ \mathrm{d}y, \tag{A.13}$$

for all $x \in [x(I, T^n, p^*, \underline{s}), x(I, T^n, p^*, \overline{s})]$. Using Leibniz's rule, we differentiate Eq. (A.13) with respect to x to yield

$$D'(x) = \begin{cases} \Phi(x) & \text{if } x \in [x(I, T^n, p^*, \underline{s}), b(0, \underline{s}) + I), \\ \Phi(x) - \Psi(x) & \text{if } x \in [b(0, \underline{s}) + I, b(0, \overline{s}) + I], \\ \Phi(x) - 1 & \text{if } x \in (b(0, \overline{s}) + I, x(I, T^n, p^*, \overline{s})]. \end{cases} \tag{A.14}$$

For all $x \in [b(0, \underline{s})+I, x^{\mathrm{o}})$, we have $f(x) > g(x)$ so that $\Phi(x)-\Psi(x) = G[f(x)]-G[g(x)] > 0$. On the other hand, for all $x \in (x^{\mathrm{o}}, b(0, \overline{s}) + I]$, we have $f(x) < g(x)$ so that $\Phi(x) - \Psi(x) = G[f(x)]-G[g(x)] < 0$. It then follows from Eq. (A.14) that $D(x)$ is strictly increasing for all

$x \in [x(I, T^n, p^*, \underline{s}), x^{\rm o})$ and strictly decreasing for all $x \in (x^{\rm o}, x(I, T^n, p^*, \overline{s})]$. Furthermore, we have

$$D[x(I, T^n, p^*, \overline{s})] = \int_{x(I,T^n,p^*,\underline{s})}^{x(I,T^n,p^*,\overline{s})} [\Phi(x) - \Psi(x)] \, \mathrm{d}x$$

$$= \int_{b(0,\underline{s})+I}^{b(0,\overline{s})+I} x \, \mathrm{d}\Psi(x) - \int_{x(I,T^n,p^*,\underline{s})}^{x(I,T^n,p^*,\overline{s})} x \, \mathrm{d}\Phi(x), \tag{A.15}$$

which vanishes from Eq. (A.8). Hence, it follows from Eqs. (A.14) and (A.15) that $D(x) > 0$ for all $x \in (x(I, T^n, p^*, \underline{s}), x(I, T^n, p^*, \overline{s}))$. In other words, $\Phi(x)$ is a mean-preserving-spread of $\Psi(x)$ in the sense of Rothschild and Stiglitz (1970). Integrating the right-hand side of Eq. (A.12) by parts twice yields

$$\mathrm{E}\{u[x(I, T^n, p^*, \tilde{s})]\} - \mathrm{E}\{u[b(0, \tilde{s}) + I]\} = \int_{x(I,T^n,p^*,\underline{s})}^{x(I,T^n,p^*,\overline{s})} u''(x)H(x) \, \mathrm{d}x < 0, \tag{A.16}$$

since $u''(x) < 0$ (see also Rothschild and Stiglitz, 1971).

Consider now the case that $b_{qs}(q, s) < 0$. Subtracting Eq. (A.11) from Eq. (A.10) yields

$$\mathrm{E}\{u[b(0, \tilde{s}) + I]\} - \mathrm{E}\{u[x(I, T^n, p^*, \tilde{s})]\} = \int_{b(0,\underline{s})+I}^{b(0,\overline{s})+I} u(x) \, \mathrm{d}[\Psi(x) - \Phi(x)], \tag{A.17}$$

since $\mathrm{d}\Phi(x) = 0$ for all $x \in [b(0, \underline{s}) + I, x(I, T^n, p^*, \underline{s})) \cup (x(I, T^n, p^*, \overline{s}), b(0, \overline{s}) + I]$. Consider the following function:

$$\hat{D}(x) = \int_{b(0,\underline{s})+I}^{x} [\Psi(y) - \Phi(y)] \, \mathrm{d}y, \tag{A.18}$$

for all $x \in [b(0, \underline{s}) + I, b(0, \overline{s}) + I]$. Using Leibniz's rule, we differentiate Eq. (A.18) with respect to x to yield

$$\hat{D}'(x) = \begin{cases} \Psi(x) & \text{if } x \in [b(0, \underline{s}) + I, x(I, T^n, p^*, \underline{s})), \\ \Psi(x) - \Phi(x) & \text{if } x \in [x(I, T^n, p^*, \underline{s}), x(I, T^n, p^*, \overline{s})], \\ \Psi(x) - 1 & \text{if } x \in (x(I, T^n, p^*, \overline{s}), b(0, \overline{s}) + I]. \end{cases} \tag{A.19}$$

For all $x \in [x(I, T^n, p^*, \underline{s}), x^{\rm o})$, we have $g(x) > f(x)$ so that $\Psi(x) - \Phi(x) = G[g(x)] - G[f(x)] > 0$. For all $x \in (x^{\rm o}, x(I, T^n, p^*, \overline{s})]$, we have $g(x) < f(x)$ so that $\Psi(x) - \Phi(x) = G[g(x)] - G[f(x)] < 0$. It then follows from Eq. (A.19) that $\hat{D}(x)$ is strictly increasing for all $x \in [b(0,\underline{s})+I, x^{\diamond})$ and strictly decreasing for all $x \in (x^{\diamond}, b(0,\overline{s})+I]$. Furthermore, we have

$$\hat{D}[b(0,\overline{s})+I] = \int_{b(0,\underline{s})+I}^{b(0,\overline{s})+I} [\Psi(x)-\Phi(x)]\ \mathrm{d}x$$

$$= \int_{x(I,T^n,p^*,\underline{s})}^{x(I,T^n,p^*,\overline{s})} x\ \mathrm{d}\Phi(x) - \int_{b(0,\underline{s})+I}^{b(0,\overline{s})+I} x\ \mathrm{d}\Psi(x), \tag{A.20}$$

which vanishes from Eq. (A.8). Hence, it follows from Eqs. (A.19) and (A.20) that $\hat{D}(x) > 0$ for all $x \in (b(0,\underline{s})+I, b(0,\overline{s})+I)$. In other words, $\Psi(x)$ is a mean-preserving-spread of $\Phi(x)$ in the sense of Rothschild and Stiglitz (1970). Integrating the right-hand side of Eq. (A.17) by parts twice yields

$$\mathrm{E}\{u[b(0,\tilde{s})+I]\} - \mathrm{E}\{u[x(I,T^n,p^*,\tilde{s})]\} = \int_{b(0,\underline{s})+I}^{b(0,\overline{s})+I} u''(x)\hat{H}(x)\ \mathrm{d}x < 0, \tag{A.21}$$

since $u''(x) < 0$ (see also Rothschild and Stiglitz, 1971).

If $b_{qs}(q,s) > (<)\ 0$, we have

$$\mathrm{E}\{u[x(I,T^n,p^*,\tilde{s})]\} < (>)\ \mathrm{E}\{u[b(0,\tilde{s})+I]\} = \mathrm{E}\{u[x(I,T^*,p^*,\tilde{s})]\}. \tag{A.22}$$

Since $\partial\mathrm{E}\{u[x(I,T,p^*,\tilde{s})]\}/\partial T = -\mathrm{E}\{u'[x(I,T,p^*,\tilde{s})]\} < 0$, it follows from Eq. (A.22) that $T^* < (>)\ T^n$ if $b_{qs}(q,s) > (<)\ 0$. Since

$$\frac{\partial}{\partial p}\left\{\mathrm{E}\{b[q(p,\tilde{s}),\tilde{s}]\} - p\mathrm{E}[q(p,\tilde{s})]\right\} = -\mathrm{E}[q(p,\tilde{s})] < 0, \tag{A.23}$$

and $p^* > (<)\ c$ if $b_{qs}(q,s) > (<)\ 0$, Eqs. (21), (A.9), and (A.23) imply that $T^0 > (<)\ T^n$ if $b_{qs}(q,s) > (<)\ 0$. Hence, we conclude that $T^* < (>)\ T^0$ if $b_{qs}(q,s) > (<)\ 0$.

Proof of Proposition 4. Define $\hat{s}$ as the one at which $q(p^*,\hat{s}) = -T_p(p^*,\alpha)$. Using $h(x) = -u_{x\alpha}(x,\alpha)/u_x(x,\alpha)$ and Eq. (26), we can write

$$-\mathrm{E}\{u_{x\alpha}[x(I,T^*,p^*,\tilde{s}),\alpha][T_p(p^*,\alpha)+q(p^*,\tilde{s})]\}$$

$$= \mathrm{E}\bigg\{\{h[x(I,T^*,p^*,\tilde{s})] - h[x(I,T^*,p^*,\hat{s})]\}$$

$$\times u_x[x(I,T^*,p^*,\tilde{s}),\alpha][T_p(p^*,\alpha)+q(p^*,\tilde{s})]\bigg\}. \tag{A.24}$$

Note that

$$h'(x) = \frac{u_{xx}(x,\alpha)u_{x\alpha}(x,\alpha) - u_x(x,\alpha)u_{xx\alpha}(x,\alpha)}{u_x(x,\alpha)^2} > 0, \tag{A.25}$$

where the inequality follows from Eq. (22). Since $b_s(q,s) > 0$, it follows from Eq. (A.25) that $\partial h[x(I,T,p,s)]/\partial s = h'[x(I,T,p,s)]b_s[q(p,s),s] > 0$. Since $q_s(p,s) > (<)\ 0$ if $b_{qs}(q,s) > (<)\ 0$, the sign of $h[x(I,T^*,p^*,\tilde{s})] - h[x(I,T^*,p^*,\hat{s})]$ must be the same as (opposite to) that of $T_p(p^*,I) + q(p^*,\tilde{s})$ if $b_{qs}(q,s) > (<)\ 0$. Hence, the right-hand side of Eq. (A.24) is positive (negative) if $b_{qs}(q,s) > (<)\ 0$. Under CARA, we have $u_x(x,\alpha) = -\alpha u_{xx}(x,\alpha)$. It then follows from Eq. (26) that the second term of Eq. (29) vanishes, thereby implying that $\mathrm{d}p^*/\mathrm{d}\alpha > (<)\ 0$ if $b_{qs}(q,s) > (<)\ 0$.

Under CARA, we can write buyers' utility function as $u(x,\alpha) = -\alpha^n e^{-\alpha x}$, where $n \geq 0$ is a constant. Define the two-part pricing contract, (T^a, p^a), that solves the following system of equations:

$$\mathrm{E}[x(I,T^a,p^a,\tilde{s})] = \mathrm{E}[b(0,\tilde{s}) + I], \tag{A.26}$$

and

$$\mathrm{E}\{u_\alpha[x(I,T^a,p^a,\tilde{s}),\alpha]\} = \mathrm{E}\{u_\alpha[x(I,T^*,p^*,\tilde{s}),\alpha]\}. \tag{A.27}$$

Using similar arguments as in the proof of Proposition 3, it follows from Eq. (A.26) that the distribution of $x(I,T^a,p^a,\tilde{s})$ is a mean-preserving-spread of the distribution of $b(0,\tilde{s}) + I$ in the sense of Rothschild and Stiglitz (1970) if $b_{qs}(q,s) > 0$, and that the converse holds if $b_{qs}(q,s) < 0$. Since $u(x,\alpha) = -\alpha^n e^{-\alpha x}$, we have $u_{xx\alpha}(x,\alpha) = \alpha^{n+1}e^{-\alpha x}(\alpha x - n - 2) < 0$ for all $x \geq 0$ if n is sufficiently large. In this case, we know from Rothschild and Stiglitz (1971) that

$$\mathrm{E}\{u_\alpha[b(0,\tilde{s}) + I,\alpha]\} > (<)\ \mathrm{E}\{u_\alpha[x(I,T^a,p^a,\tilde{s}),\alpha]\}, \tag{A.28}$$

if $b_{qs}(q,s) > (<)\ 0$. It then follows from Eqs. (27), (A.27), and (A.28) that $T_\alpha(p^*,\alpha) < (>)\ 0$ if $b_{qs}(q,s) > (<)\ 0$. Since $\mathrm{d}p^*/\mathrm{d}\alpha > (<)\ 0$ if $b_{qs}(q,s) > (<)\ 0$, it follows from Eq. (25) that $\mathrm{d}T^*/\mathrm{d}\alpha < (>)\ 0$ if $b_{qs}(q,s) > (<)\ 0$. Since von Neumann-Morgenstern utility functions are unique up to affine transformations (see Varian, 1992), the optimal two-part pricing contract, (T^*,p^*), must be invariant to changes in n. Hence, the results on $\mathrm{d}T^*/\mathrm{d}\alpha$ when n is sufficiently large must remain valid for all $n \geq 0$.

Proof of Proposition 5. Since buyers are risk averse, it follows from Eq. (34) that $p^* > (<)\ c$ if $b_{qs}(q,s) > (<)\ 0$ for all $\eta \in [0,1]$. When $\eta = 1$, we know from Proposition 3 that $T^* < (>)\ T^0$ if $b_{qs}(q,s) > (<)\ 0$. When $\eta = 0$, we know from Png and Wang (2010) that $T^* = -(p^* - c)\mathrm{E}[q(p^*, \tilde{s})] < (>)\ T^0 = 0$ if $b_{qs}(q,s) > (<)\ 0$. It remains to show that $T^* < (>)\ T^0$ if $b_{qs}(q,s) > (<)\ 0$ for all $\eta \in (0,1)$.

To characterize the optimal fixed fee, T^*, for all $\eta \in (0,1)$, we denote $\hat{T}(p)$ as the fixed fee given by

$$\hat{T}(p) = \eta\Big\{\mathrm{E}\{b[q(p,\tilde{s}),\tilde{s}]\} - \mathrm{E}[b(0,\tilde{s})] - p\mathrm{E}[q(p,\tilde{s})]\Big\} - (1-\eta)(p-c)\mathrm{E}[q(p,\tilde{s})]. \quad \text{(A.29)}$$

It follows from Eq. (36) and (A.30) that $T^0 = \hat{T}(c)$. Define the fixed fee, $\hat{T}^* = \hat{T}(p^*)$. Since $\partial\{b[q(p,s),s] - pq(p,s)\}/\partial p = -q(p,s) < 0$, we have

$$\begin{aligned} T^0 &> \eta\Big\{\mathrm{E}\{b[q(p^*,\tilde{s}),\tilde{s}]\} - \mathrm{E}[b(0,\tilde{s})] - p^*\mathrm{E}[q(p,\tilde{s})]\Big\} \\ &> \eta\Big\{\mathrm{E}\{b[q(p^*,\tilde{s}),\tilde{s}]\} - \mathrm{E}[b(0,\tilde{s})] - p^*\mathrm{E}[q(p,\tilde{s})]\Big\} - (1-\eta)(p^*-c)\mathrm{E}[q(p,\tilde{s})] \\ &= \hat{T}^*, \end{aligned} \quad \text{(A.30)}$$

if $p^* > c$. Differentiating Eq. (A.30) with respect to p yields

$$\hat{T}'(p) = -\mathrm{E}[q(p,\tilde{s})] - (1-\eta)(p-c)\mathrm{E}[q_p(p,\tilde{s})]. \quad \text{(A.31)}$$

Since $q_p(p,s) = 1/b_{qq}[q(p,s),s] < 0$, Eq. (A.31) implies that $\hat{T}'(p) < 0$ for all $p \le c$. Hence, if $p^* < c$, we have $T^0 < \hat{T}^*$. Evaluating the left-hand side of Eq. (31) at $T = \hat{T}^*$ yields

$$\begin{aligned} &\eta\Big\{\mathrm{E}\{u[x(I,\hat{T}^*,p^*,\tilde{s})]\} - \mathrm{E}\{u[b(0,\tilde{s}) + I]\}\Big\} \\ &-\eta\{\mathrm{E}[x(I,\hat{T}^*,p^*,\tilde{s})] - \mathrm{E}[b(0,\tilde{s}) + I]\}\mathrm{E}\{u'[x(I,\hat{T}^*,p^*,\tilde{s})]\}. \end{aligned} \quad \text{(A.32)}$$

If we can show that expression (A.32) is negative (positive) when $b_{qs}(q,s) > (<)\ 0$, it then follows from Eq. (31) and the second-order conditions for program (30) that $\hat{T}^* > (<)\ T^*$. If $b_{qs}(q,s) > (<)\ 0$, we have $p^* > (<)\ c$ so that $T^0 > (<)\ \hat{T}^*$. Hence, we conclude that $T^* < (>)\ T^0$ if $b_{qs}(q,s) > (<)\ 0$.

Consider first the case that $b_{qs}(q,s) > 0$. Since $u''(x) < 0$, we have

$$\frac{u[x(I,\hat{T}^*,p^*,s)] - u[b(0,s)+I]}{x(I,\hat{T}^*,p^*,s) - b(0,s) - I} > (<)\ u'[b(0,s)+I], \quad \text{(A.33)}$$

if $x(I, \hat{T}^*, p^*, s) < (>)\ b(0, s) + I$. Multiplying $x(I, \hat{T}^*, p^*, s) - b(0, s) - I$ to both sides of Eq. (A.33) and taking expectations with respect to $G(s)$ yields

$$\begin{aligned}
&\mathrm{E}\{u[x(I, \hat{T}^*, p^*, \tilde{s})]\} - \mathrm{E}\{u[b(0, \tilde{s}) + I]\} \\
&< \mathrm{E}\{u'[b(0, \tilde{s}) + I][x(I, \hat{T}^*, p^*, \tilde{s}) - b(0, \tilde{s}) - I]\} \\
&= \mathrm{E}\{u'[b(0, \tilde{s}) + I]\}\{\mathrm{E}[x(I, \hat{T}^*, p^*, \tilde{s})] - \mathrm{E}[b(0, \tilde{s}) + I]\} \\
&\quad + \mathrm{Cov}\{u'[b(0, \tilde{s}) + I], x(I, \hat{T}^*, p^*, \tilde{s}) - b(0, \tilde{s})\}. \qquad \text{(A.34)}
\end{aligned}$$

Note that $\partial u'[b(0, s) + I]/\partial s = u''[b(0, s) + I]b_s(0, s) < 0$. Note also that $\partial[x(I, \hat{T}^*, p^*, s) - b(0, s)]/\partial s = b_s[q(p^*, s), s] - b_s(0, s) > 0$ since $b_{qs}(q, s) < 0$. Thus, the covariance term on the right-hand side of Eq. (A.34) is negative. It then follows from Eq. (A.34) that expression (A.32) is less than

$$\begin{aligned}
&\eta\{\mathrm{E}[x(I, \hat{T}^*, p^*, \tilde{s})] - \mathrm{E}[b(0, \tilde{s}) + I]\} \\
&\times\Big\{\mathrm{E}\{u'[b(0, \tilde{s}) + I]\} - \mathrm{E}\{u'[x(I, \hat{T}^*, p^*, \tilde{s})]\}\Big\}. \qquad \text{(A.35)}
\end{aligned}$$

Define the two-part pricing contract, (T^c, p^c), that solves the following system of equations:

$$\mathrm{E}[x(I, T^c, p^c, \tilde{s})] = \mathrm{E}[b(0, \tilde{s}) + I], \qquad \text{(A.36)}$$

and

$$\mathrm{E}\{u'[x(I, T^c, p^c, \tilde{s})]\} = \mathrm{E}\{u'[x(I, \hat{T}^*, p^*, \tilde{s})]\}. \qquad \text{(A.37)}$$

Using similar arguments as in the proof of Proposition 3, it follows from Eq. (A.36) that the distribution of $x(I, T^c, p^c, \tilde{s})$ is a mean-preserving-spread of the distribution of $b(0, \tilde{s}) + I$ in the sense of Rothschild and Stiglitz (1970) since $b_{qs}(q, s) > 0$. Given prudence, i.e., $u'''(x) > 0$ for all $x \geq 0$, it follows immediately from Rothschild and Stiglitz (1971) that

$$\mathrm{E}\{u'[b(0, \tilde{s}) + I]\} < \mathrm{E}\{u'[x(I, T^c, p^c, \tilde{s})]\} = \mathrm{E}\{u'[x(I, \hat{T}^*, p^*, \tilde{s})]\}. \qquad \text{(A.38)}$$

where the equality follows from Eq. (A.37). It then follows from expression (A.35) and Eq. (A.38) that expression (A.32) is unambiguously negative when $b_{qs}(q, s) > 0$.

We now consider the case that $b_{qs}(q,s)<0$. Since $u''(x)<0$, we have

$$\frac{u[x(I,\hat{T}^*,p^*,s)]-u[b(0,s)+I]}{x(I,\hat{T}^*,p^*,s)-b(0,s)-I} < (>)\ u'[x(I,\hat{T}^*,p^*,s)], \tag{A.39}$$

if $x(I,\hat{T}^*,p^*,s) < (>)\ b(0,s)+I$. Multiplying $x(I,\hat{T}^*,p^*,s)-b(0,s)-I$ to both sides of Eq. (A.39) and taking expectations with respect to $G(s)$ yields

$$\begin{aligned}
&\mathrm{E}\{u[x(I,\hat{T}^*,p^*,\tilde{s})]\}-\mathrm{E}\{u[b(0,\tilde{s})+I]\}\\
&>\mathrm{E}\{u'[x(I,\hat{T}^*,p^*,\tilde{s})][x(I,\hat{T}^*,p^*,\tilde{s})-b(0,\tilde{s})-I]\}\\
&=\mathrm{E}\{u'[x(I,\hat{T}^*,p^*,\tilde{s})]\}\{\mathrm{E}[x(I,\hat{T}^*,p^*,\tilde{s})]-\mathrm{E}[b(0,\tilde{s})+I]\}\\
&\quad+\mathrm{Cov}\{u'[x(I,\hat{T}^*,p^*,\tilde{s})],x(I,\hat{T}^*,p^*,\tilde{s})-b(0,\tilde{s})\}.
\end{aligned} \tag{A.40}$$

Note that $\partial u'[x(I,\hat{T}^*,p^*,s)]/\partial s = u''[x(I,\hat{T}^*,p^*,s)]b_s[q(p^*,s),s] < 0$. Note also that $\partial[x(I,\hat{T}^*,p^*,s)-b(0,s)]/\partial s = b_s[q(p^*,s),s]-b_s(0,s) < 0$ if $b_{qs}(q,s)<0$. Thus, the covariance term on the right-hand side of Eq. (A.40) is positive. It then follows from Eq. (A.40) that expression (A.32) is unambiguously positive when $b_{qs}(q,s)<0$. This completes our proof.

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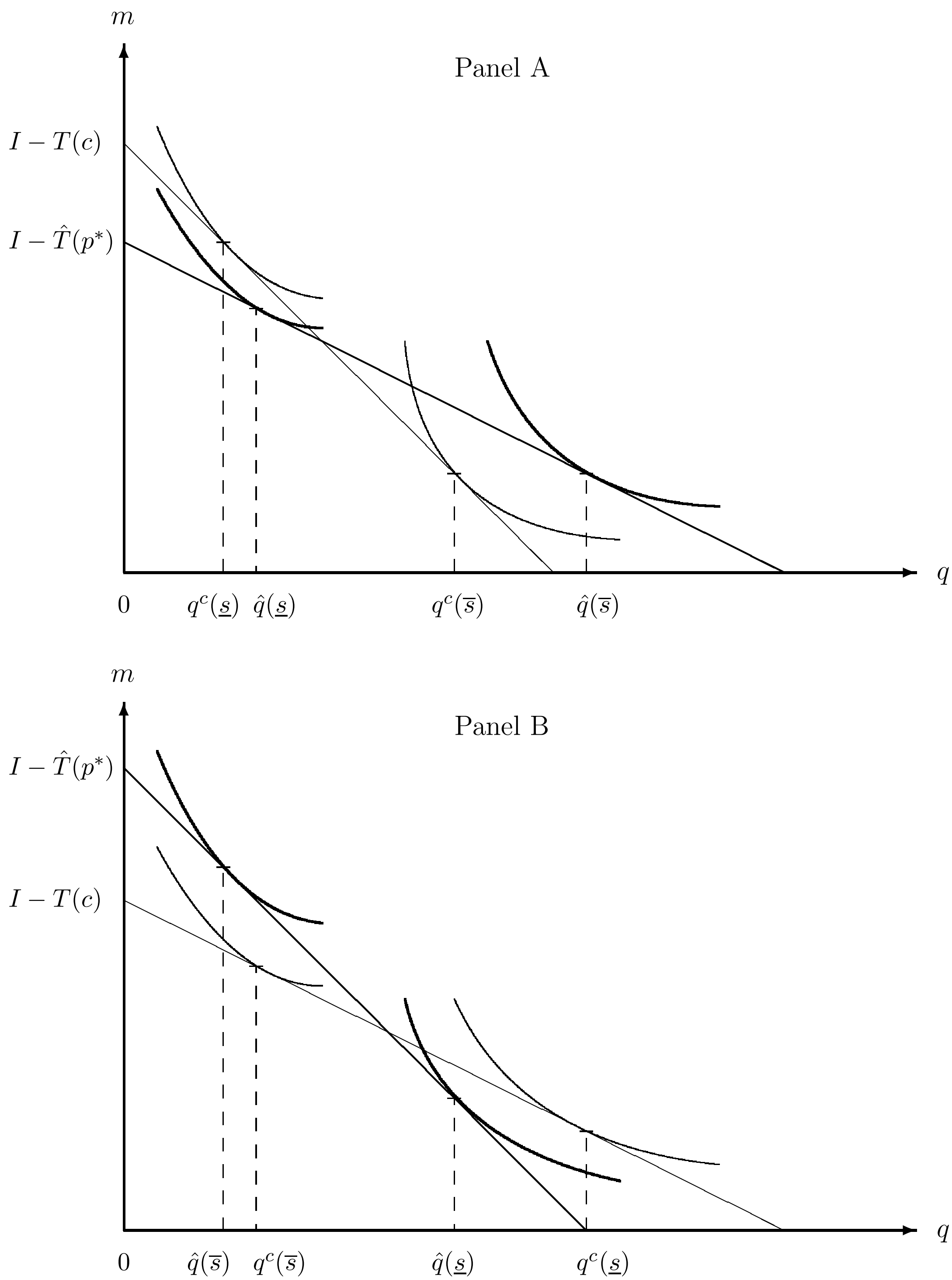

Figure 1: Buyers' optimal consumption bundles for difference two-part pricing contracts and states. Panel A depicts the case wherein $p^* < c$ and $\hat{T}(p^*) > T(c)$ given that $q_s(I, T, p, s) > 0$. Panel B depicts the case wherein $p^* > c$ and $\hat{T}(p^*) < T(c)$ given that $q_s(I, T, p, s) < 0$.

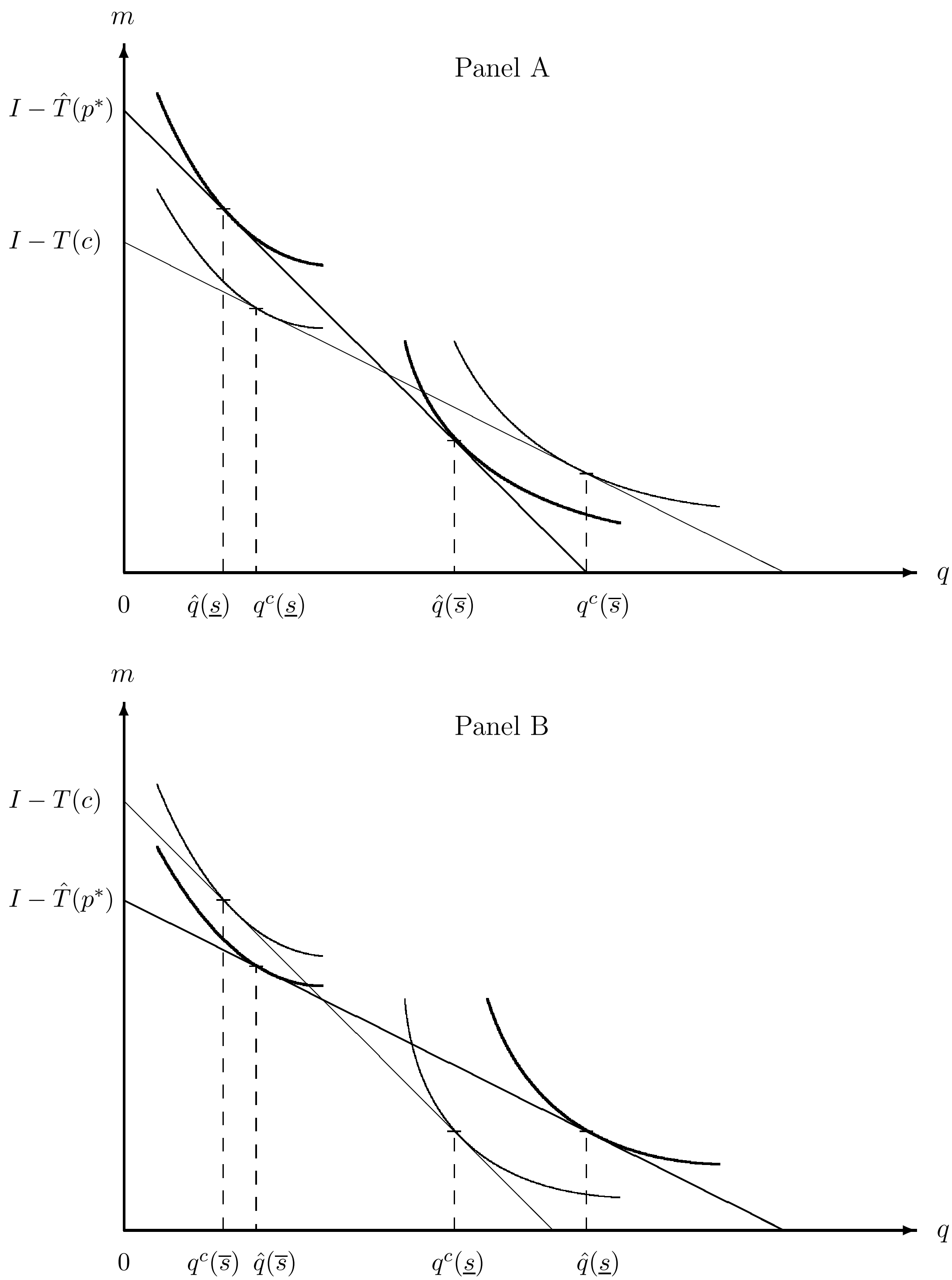

Figure 2: Buyers' optimal consumption bundles for difference two-part pricing contracts and states. Panel A depicts the case wherein $p^* > c$ and $\hat{T}(p^*) < T(c)$ given that $q_s(I, T, p, s) > 0$. Panel B depicts the case wherein $p^* < c$ and $\hat{T}(p^*) > T(c)$ given that $q_s(I, T, p, s) < 0$.