MODULI SPACES OF TEN-LINE ARRANGEMENTS WITH DOUBLE AND TRIPLE POINTS

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Abstract. Two arrangements with the same combinatorial intersection lattice but whose complements have different fundamental groups are called a Zariski pair. This work finds that there are at most nine such pairs amongst all ten line arrangements whose intersection points are doubles or triples. This result is obtained by considering the moduli space of a given configuration table which describes the intersection lattice. A complete combinatorial classification is given of all arrangements of this type under a suitable assumption, producing a list of seventy-one described in a table, most of which do not explicitly appear in the literature. This list also includes other important counterexamples: nine combinatorial arrangements that are not geometrically realizable.

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1. Introduction

A *line arrangement* $\mathcal{A}$ in $\mathbb{CP}^2$ is a finite collection of projective lines. We define the complement of $\mathcal{A}$ as $\mathbb{CP}^2 \setminus \bigcup_{L\in\mathcal{A}} L$ and denote it as $M(\mathcal{A})$. The set $L(\mathcal{A}) = \{\bigcap_{i\in S} L_i | S \subseteq \{1, 2, \ldots, k\}\}$ partially ordered by reverse inclusion is called the *intersection lattice* of $\mathcal{A}$. A set with such an intersection lattice structure is often called a *configuration*. Two line arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ are lattice isomorphic, denoted as $\mathcal{A}_1 \sim \mathcal{A}_2$, if up to a permutation on the labels of the lines their lattices are the same. See the textbook [OT92] by Orlik and Terao for a brief introduction to the history and general theory of arrangements of hyperplanes.

In 1980, Orlik and Solomon [OS80] proved that the cohomology algebra of $M(\mathcal{A})$ is determined by the combinatorics of $L(\mathcal{A})$. This led to a "conjecture" that homotopy invariants of $M(\mathcal{A})$ are combinatorial invariants.

Conversely, Falk studied whether $L(\mathcal{A})$ is a homotopy invariant. In [Fal90], he presented a pair of central arrangements in $\mathbb{C}^3$ with different underlying lattices but homotopy equivalent complements. However, for line arrangements, Jiang and Yau [JY98] showed that homeomorphic equivalence implies lattice isomorphism. Towards the "conjecture," it was proved for line arrangements by Jiang and Yau [JY98] and for hyperplane arrangement by Randell [Ran89] independently that if two arrangements are lattice isotopic, i.e. they are connected by a one-parameter family with constant intersection lattice, then their complements are diffeomorphic. As applications, Jiang, Wang, and Yau [JY94], [WY05] showed that the intersection lattices of *nice arrangements* and *simple arrangements* determine the topology of the complements. Nazir and Yoshinaga [NY12] define *simple* C_3 line arrangements and show that their intersection lattices determine the topology of their complements.

However, the "conjecture" is not true in general. In 1998, Rybnikov [Ryb11] discovered a pair of two complex arrangements of thirteen lines with fifteen triple points. The arrangements are lattice isomorphic but the fundamental groups of complements are different. We call a pair of lattice isomorphic line arrangements a *Zariski pair* if the fundamental groups of complements are different. Our definition is stronger than the original definition, introduced by Artal Bartolo in [AB94], which is a pair of lattice isomorphic line arrangements with different embedding type. Rybnikov's example is the first and smallest Zariski pair so far. It is not known if there is a Zariski pair of arrangements of less than thirteen lines. It is not known if there is a Zariski pair of real line arrangements.

By studying fundamental groups, Garber, Teicher and Vishne [GTV03] proved that there is no Zariski pair of arrangements of up to eight real lines which covered a result of Fan [Fan97] on arrangements of six lines.

By studying moduli spaces, Nazir and Yoshinaga [NY12] proved that there is no Zariski pair of arrangements of up to eight complex lines and listed a classification of arrangements of nine lines without proof (later proved to be complete by Ye in [Ye13]). The classification implies that there is also no Zariski pair of arrangements of nine lines.

1.1. **The moduli space of an arrangement with fixed intersection lattice.** The earlier program classified the moduli spaces of arrangements of nine lines [Ye13], followed by arrangements of ten lines with some multiple points of order greater than three [ATY13]. The goal of this work is to classify the moduli spaces of arrangements of ten lines with only triple points and double points. One purpose of the classification is to search for Zariski pairs from arrangements of ten lines.

Comparing fundamental groups of complements of line arrangements is very hard. Rybnikov [Ryb11] distinguished the two fundamental groups of his pair of arrangements by delicate analysis of the lower central series quotients of the groups. Following Rybnikov's idea, Artal Bartolo, Carmona Ruber, Cogolludo Agustín, and Marco Buzunáriz present in [ABCRCAMB06] an alternative proof in detail by using the Alexander module and its truncations as well as certain combinatorial invariants developed only for line arrangements with only double and triple points. We hope that their method can be applied to the arrangements of ten lines with only double and triple points that we produce in this present paper.

Let $\mathcal{A}$ be a complex line arrangement. We define the *moduli space* of line arrangements with the fixed lattice $L(\mathcal{A})$ (or simply, the moduli space of $\mathcal{A}$) as

$$\mathcal{M}_{\mathcal{A}} = \{\mathcal{B} \in ((\mathbb{CP}^2)^*)^n | \mathcal{B} \sim \mathcal{A})\}/PGL(3, \mathbb{C}).$$

By Randell's Lattice-Isotopy Theorem in [Ran89] and Cohen and Suciu's Theorem 3.9 in [CS97], we know that two arrangements in the same connected component of the moduli space, or in two complex conjugate components, respectively, are not Zasiki pairs. See more in Section 3.

One very useful result on structure of moduli spaces of line arrangements is the following lemma of Nazir and Yoshinaga.

Lemma 1.1. [NY12, Lemma 2.4] *Let $\mathcal{A} = \{L_1, L_2, \ldots, L_n\}$ be a line arrangement in $\mathbb{CP}^2$. Assume that the line L_n passes through* ***at most two*** *multiple points of the arrangement $\mathcal{A}$. Set $\mathcal{A}' = \mathcal{A} \setminus \{L_n\}$. Then the moduli space $\mathcal{M}_{\mathcal{A}}$ is irreducible if $\mathcal{M}_{\mathcal{A}'}$ is irreducible.*

With the support of this lemma, we make the following reasonable assumption throughout, except for the general results in Sections 2 and 3.

Assumption 1.2. *Let $\mathcal{A}$ be a line arrangement with only double and triple points such that each line passes through at least* ***three*** *triple points.*

Departing from the previous work, the techniques presented in this paper prioritize the points instead of the lines. This perspective comes from matroid theory, which due to its more combinatorial nature serves as a better model for enumeration.

1.2. **The matroid perpective.** A hyperplane arrangement can be realized in more combinatorial language as an *oriented matroid*, an object well-documented in the textbook [BLVS+99] by Björner, Las Vergnas, Sturmfels, White, and Ziegler.

We investigate an arrangement of interest above by looking at its *underlying matroid*. Matroids, now stripped of their topology and appearing as purely combinatorial objects, are also heavily documented in the literature, with special introductory care taken in the textbook [Oxl92] by Oxley.

A *matroid* is the simultaneous generalization of a matrix and a graph: the common property is the notion of dependence, achieved by linear independence of column vectors in a matrix or by cycles of edges in a graph. A matroid obtained from a matrix (whose entries belong to some field) is called *representable* (over this field).

The notion of representability over the field $\mathbb{R}$ or $\mathbb{C}$ corresponds to the notion of geometrically realizable in the setting above over $\mathbb{R}$ or $\mathbb{C}$, respectively.

Thus the arrangements produced in the work below may be used to find *forbidden minors* (also called *excluded minors* or *minor-minimal obstructions*) studied in infinite classes of matroids. Although it is most natural to ask about forbidden minors for the class of $\mathbb{R}$-representable matroids, this list is infinite. Mayhew, Newman, and Whittle [MNW09] recently showed that for any infinite field $\mathbb{K}$ and any $\mathbb{K}$-representable matroid N, there is an excluded minor in $\mathbb{K}$ that has N as a minor.

However, as our list produces new, relatively larger matroids whose base sets have between nineteen and twenty-five elements, they might also be useful for the study of forbidden minors of finite fields, another topic of popular interest.

A *geometric matroid* represents elements from its base set by points, showing dependency geometrically: a circuit of size three by a line through three points, a circuit of size four by a plane through four points, etc. This is the setting that will be most useful for us below, as it naturally gives rise to line arrangements.

Remark 1.3. Assumption 1.2 is the right choice from the matroid perspective, as well. Every two distinct points in a geometric matroid are independent, but the lines between them are omitted to avoid confusion. The lines included are ones that give dependencies and so must contain at least three points.

In order to describe the matroids in our investigation one would have to enumerate all of the intersection points. However in this context we only enumerate the triple points. For this approach we use *configuration tables* following the textbook *Configurations of points and lines* by Grünbaum [Grü09]. Examples of configuration tables can be found in Table 2 and throughout the paper.

A configuration table is *geometrically realizable* if it is the configuration table associated to a geometrically realizable line arrangement. In Section 3, we illustrate how to determine whether a given configuration table is realizable.

1.3. **Results and organization.** The two main results are the classification of ten-line arrangements that satisfy Assumption 1.2 with only double and triple points given in Theorem 5.1 and Table 6 and also the discovery of nine potential Zariski pairs from this classification based on the moduli spaces of the arrangements given in Theorem 5.3 and Table 7.

Aside from these, other important counterexamples are given: nine non-geometrically realizable arrangements in Theorem 5.6 and Table 8.

This work is important for its own sake, as a complete list of such large arrangements cannot be found in the literature; but furthermore it will be instrumental in answering the question of whether there exist other small Zariski pairs of arrangements.

The paper is structured as follows. The combinatorics is discussed in Section 2, giving restrictions that will be used as lemmas throughout the main proofs. The geometric methodology is outlined in Section 3, highlighting intermediate steps and results that are part of Algorithm 3.11 used to classify the moduli spaces of the arrangements. Some examples are given that demonstrate the procedures we use.

In Section 4 earlier results are discussed for arrangements of nine lines with nine and ten triples. These appear as reductions in later proofs of our arrangements of ten lines.

Our main results are outlined in Section 5, with Theorem 5.1 giving a classification of our arrangements and Theorem 5.3 giving nine potential Zariski pairs.

The outline of the proof is then explained in Subsection 5.1. The proof is presented in four parts: Sections 6, 7, 8, 9 for ten, thirteen, twelve, and eleven triples, respectively. These are ordered based on the relative straightforwardness of the proofs, with the final section on eleven lines being the most fragmented via casework.

2. The combinatorial data

For the uninitiated reader, we review some background that sets up the proof of our classification Theorem 5.1 and Table 6. We then introduce some basic arrangement results and intuitions that will be used as lemmas in the main proofs below.

Let $\mathcal{A}$ be an arrangement of ten lines, and let n_i be the number of intersection points of order i. We assume throughout that $n_i = 0$ for $i > 3$, as the other cases have been handled already by the work of [ATY13], where $n_4 \neq 0$.

Lemma 2.1. *Assume that $\mathcal{A}$ satisfies Assumption 1.2. Let ℓ_i be the number of lines with exactly i triple points on it. Then we have*

$$\ell_3 + \ell_4 = 10 \tag{2.1}$$

$$3\ell_3 + 4\ell_4 = 3n_3, \tag{2.2}$$

yielding the results of Table 1.

n_3	*# of triples*	*10*	*11*	*12*	*13*
ℓ_3	*# of lines with three triples*	*10*	*7*	*4*	*1*
ℓ_4	*# of lines with four triples*	*0*	*3*	*6*	*9*

TABLE 1. The four cases for the number of triple points in a ten-line arrangement.

Proof. Assumption 1.2 gives that $\ell_i = 0$ for $i \leq 2$. For a line to have at least five triple points on it, there must be at least ten other lines. Because there are only ten lines total, we have $\ell_i = 0$ for $i \geq 5$. This gives Equation 2.1.

Since each triple point is on three lines, we need a total of $3n_3$ lines *counted with multiplicities* to form n_3 triple points. On the other hand, if a line passes through i triple points, then it contributes i lines. This gives Equation 2.2. □

This can also be expressed as a corollary of a more general result.

Lemma 2.2. *Let $\mathcal{A}$ be an arrangement of k lines with at most triple points. Denote by ℓ_i the number of lines that each passes through exactly i triple points. Then the following equalities hold.*

$$\begin{aligned} \ell_0 + \ell_1 + \ell_2 + \ldots + \ell_k &= k \\ \ell_1 + 2\ell_2 + \ldots + k\ell_k &= 3n_3, \end{aligned}$$

where n_3 is the number of triple points in the arrangement.

There are two other useful formulae involving the number of lines and the number of points of different multiplicities.

Fact 2.3 (see for instance Section 6 of [Hir86]). Let $\mathcal{A}$ be an arrangement of k projective lines. Then

$$\frac{k(k-1)}{2} = \sum_{i=2}^{k} \frac{i(i-1)n_i}{2}. \tag{2.3}$$

Theorem 2.4 (Hirzebruch [Hir86], Equation (9))**.** *Let $\mathcal{A}$ be an arrangement of k projective lines. Assume that $n_k = n_{k-1} = n_{k-2} = 0$. Then we have the following inequality*

$$n_2 + \frac{3}{4}n_3 \geq k + \sum_{i\geq 5}(2i-9)n_i. \tag{2.4}$$

We now include several more facts and lemmas that appear in the main proof below.

Fact 2.5. An arrangement containing two triples must contain at least five lines.

Fact 2.6. A line containing three triples must be part of an arrangement with at least six other lines.

Fact 2.7. A line containing four triples must be part of an arrangement with at least eight other lines.

Lemma 2.8. *Three triples that are not colinear must be part of an arrangement with at least six lines.*

Proof. Let $\mathcal{A} = \{L_1, L_2, \ldots, L_k\}$ be an arrangement in $\mathbb{CP}^2$, and recall that ℓ_i is the number of lines that each passes through exactly i triple points. Since no three triple points of $\mathcal{A}$ are collinear, then $\ell_i = 0$ for $i \geq 3$. By lemma 2.2, we have $\ell_0 + \ell_1 + \ell_2 = k$ and $\ell_1 + 2\ell_2 = 3n_3 = 9$. Since there are only three triple points and there exists a unique line passing through two distinct points, then $\ell_2 \leq 3$. Therefore, $k = 9 + \ell_0 - \ell_2 \geq 6$. □

Lemma 2.9. *A subarrangement of six lines can have at most four triple points.*

Proof. Since $n_i \geq 0$ for $i \geq 2$, using the Fact 2.3, we see that $n_3 = \frac{1}{3}(15 - n_2 - 6n_4 - 10n_5 - 15n_6) \leq 5$. If $n_3 = 5$, then $n_2 = n_4 = n_5 = n_6 = 0$. By Theorem 2.4, we would have that $n_2 + \frac{3}{4}n_3 = \frac{15}{4} \geq 6$, a contradiction. Therefore, $n_3 \leq 4$. □

Lemma 2.10. *There is a unique combinatorial arrangement $\mathcal{B}$ of six lines with exactly three triples as shown in Figure 1.*

Proof. Let $L_1 \cap L_2 \cap L_3$ be a triple point. We claim that the other two triple points must be on $L_1 \cup L_2 \cup L_3$. Otherwise, let $L_4 \cap L_5 \cap L_6$ be another triple point apart from $L_1 \cup L_2 \cup L_3$, then we need one more line to form the third triple point. Let $L_1 \cap L_4 \cap L_5$ be the second triple point. Since there is only one more line, then one of the intersections $(L_2 \cup L_3) \cap (L_4 \cup L_5)$ must be a triple point. Up to a permutation, we may assume that $L_2 \cap L_4 \cap L_6$ is the third triple points. It is not hard to check that up to a permutation, this is the unique arrangement of 6 line with exactly 3 triple points. □

Lemma 2.11. *There is a unique combinatorial arrangement $\mathcal{C}$ called the* Ceva arrangement *of six lines with exactly four triples as shown in Figure 1.*

Proof. Since there are four triple points, then each line must pass through at least one but no more than two triple points. By Lemma 2.1, we know that each line passes through exactly two triple points. Let $L_1 \cap L_2 \cap L_3$ be a triple point. Then all the other three triple points should be on $L_1 \cup L_2 \cup L_3$. Let $L_1 \cap L_4 \cap L_5$ be the second triple point on L_1. Then $L_2 \cap L_4$ or $L_2 \cap L_5$ should be a triple point. Without loss of generality, we assume that $L_2 \cap L_4 \cap L_6$ is a triple point. Then $L_3 \cap L_5 \cap L_6$ must be a triple point. Hence we get a unique arrangement of 6 lines with 4 triple points. □

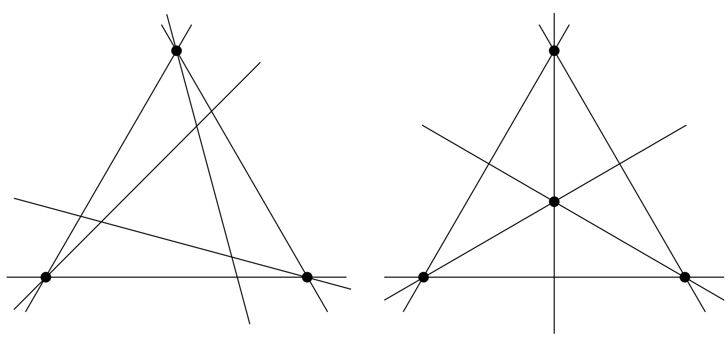


FIGURE 1. The unique arrangements $\mathcal{B}$ and $\mathcal{C}$ given in Lemmas 2.10 and 2.11, respectively.

3. THE GEOMETRIC METHODOLOGY

The two goals of this section are to determine the realizability of the configuration tables that are produced in the main proofs below and to determine the irreducibility of these spaces to rule out arrangements which cannot produce Zariski pairs.

To begin we must first convert the combinatorial data of the configuration tables to algebraic equations of geometric lines. From here we can study the moduli spaces of such arrangements to answer our main questions. What follows is a lead up to Algorithm 3.11 that we used in the main proofs below.

We fix some notations from projective geometry.

Definition 3.1. Let $P_1 = [a_1, b_1, c_1]$, $P_2 = [a_2, b_2, c_2]$, and $P_3 = [a_3, b_3, c_3]$ be three points in the projective plane. We define the determinant of P_1, P_2 and P_3 as

$$\det(P_1, P_2, P_3) := \det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

For a projective line L defined by $ax + by + cz = 0$, we denote by $L^* = [a, b, c]$ the point in the dual projective plane $(CP^2)^*$. We define the *determinant of the coefficient matrix* of three lines L_1, L_2 and L_3 to be $\det(L_1^*, L_2^*, L_3^*)$.

To study the geometry, especially irreducibility, of the moduli space $\mathcal{M}_{\mathcal{A}}$ of a projective line arrangement $\mathcal{A}$, the idea is to convert combinatorial data into polynomial equations.

Here is the first such idea.

Property 3.2. For any three projective lines L_i defined by $a_i x + b_i y + c_i z = 0$, $i = 1, 2, 3$, the intersection $L_1 \cap L_2 \cap L_3$ is nonempty if and only if the determinant of the coefficient matrix, which we notate as $\det(L_1^*, L_2^*, L_3^*)$, is zero.

We now use this to define spaces: the first by applying only one direction of this property to the triples that appear on the configuration tables; and the second by applying both directions to ensure that no other triples appear in arrangement except for those in the configuration tables.

Definition 3.3. Let C be a configuration table with n columns L_1, L_2, $\dots$, L_n. We define the total space $\mathcal{T}_C$ of the configuration table C to be the Zariski subset of $((\mathbb{CP}^2)^*)^n$ which consists of points $(P_1, P_2, \dots, P_n)$ such that P_1, P_2, $\dots$, P_n are distinct and satisfy the following condition:

(1) $\det(P_i, P_j, P_k) = 0$ whenever the three columns L_i, L_j and L_k share a point.

Let $\mathcal{A} = \{L_1, L_2, \dots, L_n\}$ be an arrangement of n projective lines associated to the configuration table C. We define the total space $\mathcal{T}_{\mathcal{A}}$ of the arrangement $\mathcal{A}$ to be the Zariski subset of $((\mathbb{CP}^2)^*)^n$ which consists of points $(P_1, P_2, \dots, P_n)$ such that P_1, P_2, $\dots$, P_n are distinct and satisfy the following two conditions:

(1) $\det(P_i, P_j, P_k) = 0$ if the intersection of L_i, L_j and L_k is non-empty, and
(2) $\det(P_r, P_s, P_t) \neq 0$ if the intersection of L_r, L_s, L_t is empty.

The quotient of $\mathcal{T}_C$ by the automorphism group $PGL(3, \mathbb{C})$ of the dual plane, denoted by $\mathcal{M}_C := \mathcal{T}_C / PGL(3, \mathbb{C})$ is called the *moduli space* of the configuration table C.

The quotient of $\mathcal{T}_{\mathcal{A}}$ by the automorphism group $PGL(3, \mathbb{C})$ of the dual plane, denoted by $\mathcal{M}_{\mathcal{A}} := \mathcal{T}_{\mathcal{A}} / PGL(3, \mathbb{C})$ is called the *moduli space* of the line arrangement $\mathcal{A}$.

In the above definition, the points P_i should be considered as duals of projective lines.

Proposition 3.4. *Let $\mathcal{A}$ be an arrangement of projective lines associated to a configuration table C. The moduli space $\mathcal{M}_{\mathcal{A}}$ is irreducible if the moduli space space $\mathcal{M}_C$ is irreducible.*

Proof. We see that $\mathcal{T}_{\mathcal{A}}$ is a Zariski open subset of $\mathcal{T}_C$. Hence, the moduli space $\mathcal{M}_{\mathcal{A}}$ is a Zariski open subset of $\mathcal{M}_C$. □

This prescribes Algorithm 3.11 that we present in the next subsection, where the reader can find more details.

First we present an important theorem from the literature that will be used to classify moduli spaces.

Theorem 3.5 (Randell's Lattice-Isotopy Theorem [Ran89])**.** *If $\mathcal{A}_t$ is a lattice-isotopy between two line arrangements $\mathcal{A}_0$ and $\mathcal{A}_1$, then the complement of $\mathcal{A}_0$ is diffeomorphic to the complement of $\mathcal{A}_1$.*

We denote by $\mathcal{M}_{\mathcal{A}}^{\mathbb{C}}$ the quotient of $\mathcal{M}_{\mathcal{A}}$ under complex conjugation. Using Randell's Lattice-Isotopy Theorem, we observe the following rigidity result on diffeomorphic types of complements of line arrangements.

Proposition 3.6. *If $\mathcal{M}_{\mathcal{A}}^{\mathbb{C}}$ is irreducible, then the complements of any two line arrangements in $\mathcal{M}_{\mathcal{A}}$ are diffeomorphic. In particular, the fundamental group of the complement of $\mathcal{A}$ is a combinatorial invariant.*

Proof. Let $[\mathcal{A}_1]$ and $[\mathcal{A}_2]$ be two points in $\mathcal{M}_{\mathcal{A}}$. If the two arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ are in the same irreducible component of $\mathcal{M}_{\mathcal{A}}$, then by Randell's Lattice-Isotopy Theorem [Ran89] the complements of $\mathcal{A}_1$ and $\mathcal{A}_2$ are diffeomorphic.

Suppose that the two arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ are in two different irreducible components of $\mathcal{M}_{\mathcal{A}}$. Since $\mathcal{M}_{\mathcal{A}}^{\mathbb{C}}$ is irreducible, the two irreducible components must be complex conjugated to each other. Therefore, there exist a line arrangement $\mathcal{A}_1'$ which is complex conjugated to $\mathcal{A}_1$ and in the same irreducible component of $[\mathcal{A}_2]$. Notice that complex conjugation is a diffeomorphism. Then the complements of $\mathcal{A}_1$ and $\mathcal{A}_1'$ are diffeomorphic. Since $\mathcal{A}_1'$ and $\mathcal{A}_2$ are in the same irreducible component, then their complements are diffeomorphic. Therefore, the complement of $\mathcal{A}_1$ is diffeomorphic to the complement of $\mathcal{A}_2$. □

Remark 3.7. Notice that in [CS97], Cohen-Suciu prove the following theorem which implies that the fundamental groups of complements of complex conjugated curves are isomorphic. Together with Randell's Lattice-Isotopy Theorem, this also implies that the fundamental group of the complement of $\mathcal{A}$ is combinatorially invariant if $\mathcal{M}_{\mathcal{A}}^{\mathbb{C}}$ is irreducible.

Theorem 3.8 (Cohen-Suciu [CS97], Theorem 3.9)**.** *The braid monodromies of complex conjugated curves are equivalent.*

3.1. **A practical algorithm and examples.** The highlight of this subsection is Algorithm 3.11, which uses a configuration table to obtain a line arrangement and its moduli space, and to determine whether this space is irreducible and whether the configuration table is geometrically realizable.

Several examples are given highlighting the problematic spots of this algorithm, and the reader should pay careful attention to Example 3.12 which produces a potential Zariski pair.

To set this up we begin with an important lemma that will allow us to use the same three-by-three grid of lines for all of our cases. Afterwards we mention several results and techniques that are used to discern the moduli spaces.

Definition 3.9. A *pencil* is an arrangement of k lines with a multiple point of multiplicity k.

We start from a basic observation that a line arrangement can be specified to a special position in the projective plane. This fact, without proof, has been used in [NY12], [Ye13], and [ATY13] to calculate moduli spaces of arrangements. We include the proof here for the sake of completion.

Note that we use Lemma 3.10 to draw pictures in the affine plane which is the complement of the line at infinity (not included in the arrangement) defined by $z = 0$ in $\mathbb{CP}^2$.

Lemma 3.10. *Let $\{L_1, L_2, L_3\}$ and $\{L_4, L_5, L_6\}$ be two pencils of lines which intersect transversally in 9 points. Then there is a unique automorphism of the dual projective plane such that the six lines under the automorphism are defined by $x = 0$, $x = z$, $x = t_1 z$, $y = 0$, $y = z$, $y = t_2 z$.*

Proof. We know that the automorphism group of the projective plane is $PGL(3, \mathbb{C})$. We can find an automorphism M_0 sending the line passing through the two triple points $L_1 \cap L_2 \cap L_3$ and $L_4 \cap L_5 \cap L_6$ to the line at infinity $z = 0$. Under this automorphism, the six lines can be defined by $ax + by + c_i z = 0$ for $i = 1, 2, 3$ and $dx + ey + f_{i-3} z = 0$ for $i = 4, 5, 6$, respectively, such that $D = \det \begin{pmatrix} a & b \\ d & e \end{pmatrix} \neq 0$ and $c_i \neq c_j$, $f_i \neq f_j$ if $i \neq j$.

In the next step we will find an automorphism fixing the line at infinity and changing the six lines to three horizontal and three vertical lines. Consider the general linear matrix

$$M_1 = -\frac{1}{D}\begin{pmatrix} b & -e & 0 \\ -a & d & 0 \\ 0 & 0 & -D \end{pmatrix}$$

which acts on the dual projective plane from right and fixes the point $[0,0,1]$. We see that M_1 sends the points $[a,b,c_i]$ and $[d,e,f_i]$ to $[0,1,c_i]$ and $[1,0,f_i]$, respectively, for $i=1,2,3$, as we expected.

To complete the proof, let us consider the matrix

$$M_2 = \begin{pmatrix} f_2-f_1 & 0 & -f_1 \\ 0 & c_2-c_1 & -c_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

One can check that M_2 sends $[0,1,c_1]$, $[0,1,c_2]$, $[1,0,f_1]$ and $[1,0,f_2]$ to $[0,1,0]$, $[0,1,1]$, $[1,0,0]$ and $[1,1,0]$, respectively. The automorphism $M_0M_1M_2$ is then the unique automorphism we want. □

We now present the full details of the procedures used throughout the proofs below. We conclude with examples that illustrate the different methods and situations that occur in the algorithm.

Algorithm 3.11. *This algorithm is used to determine the irreducibility and realizability of an arrangement obtained from a given configuration table.*

(A1) *Use the Grid Lemma 3.10 and its intersection points to determine equations for the lines.*
(A2) *Obtain special defining equations from triple points not used in (A1) above.*
(A3) *Determine the* ***irreducibility*** *of the moduli space $\mathcal{M}$ by checking the irreducibility of the equations from (A2) above.*
  (a) *True for* $\dim\mathcal{M}=0$.
  (b) *If* $\dim\mathcal{M}>0$, *either use Mathematica's irreducibility test given by the command*
  `IrreduciblePolynomialQ[poly, Extension->All]`
  and notated by $(*)$ or equations ending in $=0^*$ in the tables
  or check by hand, as in Example 3.13 for the case 11.B.3.a.iii.
(A4) *Determine the* ***realizability*** *of the configuration table.*
  (a) *Whenever the set $\mathcal{M}$ is empty, a contradiction arises, and it is noted in the tables.*
  (b) *True for* $\dim\mathcal{M}>0$.
  (c) *If* $\dim\mathcal{M}=0$, *look for a realization for one of the given points: if a real solution exists, draw the arrangement to check, as in Example 3.12 for the case 12.B.3.b.iii.;*
  if only complex solutions exist, as notated by $(\mathbb{C})$ or equations ending in $=0^{\mathbb{C}}$ in the tables, need to check, as in Example 3.14 for the case (9_3).ii.DFH, that no double points coincide in a triple.

Potential Zariski paris are notated by $(\mathcal{Z})$ or equations ending in $=0^{\mathcal{Z}}$ in the tables.

Example 3.12 (The moduli space of the arrangement 12.B.3.b.iii.: two real points)**.** Using Lemma 3.10, we may assume that the lines L_7, L_5, L_4, L_8, L_9 and L_3 are defined by equations $y=0$, $y=z$, $y=bz$, $x=0$, $x=z$, and $x=az$, respectively, where a and b are complex numbers in $\mathbb{C}\setminus\{0,1\}$.

Since the three triples $e_{11}=L_4\cap L_8=[0,b,1]$, $e_6=L_3\cap L_7=[a,0,1]$ and $e_{12}=L_5\cap L_9=[1,1,1]$ are on L_6, the equation of the line L_6 can be written as $y=-\frac{b}{a}x+bz$ such that the first defining equation $1=b-\frac{b}{a}$, or equivalently, $ab=a+b$, holds.

Now we determine the equations for the rest of lines, L_1, L_2, and L_{10}. Since the line L_1 passes though the two triples $e_2=L_3\cap L_5=[a,1,1]$ and $e_4=L_7\cap L_9=[1,0,1]$, the line L_1 is defined by $y=\frac{1}{a-1}(x-z)$. Therefore, the triple $e_1=L_1\cap L_4$ is given by $[b(a-1)+1,b,1]=[a+1,b,1]$ by the equality $ab=a+b$. The line L_2 passes through the two triples $e_5=L_7\cap L_8=[0,0,1]$ and

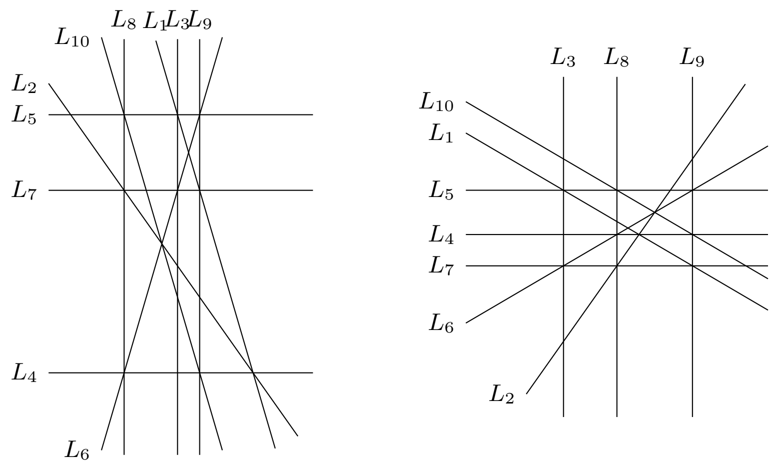
$e_1 = [a+1, b, 1]$, and so its equation can be written as $y = \frac{b}{a+1}x$. The last line L_{10} passes through the triples $e_8 = L_4 \cap L_9 = [1, b, 1]$ and $e_9 = L_5 \cap L_8 = [0, 1, 1]$, and so its equation is given by $y = (b+1)x + z$.

Note that the triple $e_3 = L_2 \cap L_6 \cap L_{10}$ did not yet enter into the equations of the lines; thus it will become the second defining equation

$$\det(L_2, L_6, L_{10}) = \det \begin{pmatrix} 1 & \frac{b}{a+1} & 0 \\ 1 & -\frac{b}{a} & b \\ 1 & b-1 & 1 \end{pmatrix} = 0.$$

Simplifying this second defining equation, we obtain the equation $a^2b - a^2 + a + 1 = 0$. Together with the first defining equation $ab = a + b$, we solve for a and b to obtain $a = \pm\frac{\sqrt{2}}{2}$ and $b = \frac{1}{1 \mp \sqrt{2}}$.

We then have two real line arrangements shown in Figure 2.



$a = \sqrt{2}/2$ $\qquad$ $a = -\sqrt{2}/2$

FIGURE 2. The two real realizations of the arrangement 12.B.2.iv

From the figure, we see that there are exactly twelve triple points and no points of higher multiplicities. Therefore, the moduli space of the arrangement 12.B.2.iv consists of two distinct real points.

Example 3.13 (The moduli space of the arrangement 11.B.3.a.iii.)**.** Again using Lemma 3.10, we may assume that the lines L_1, L_2, L_4, L_3, L_{10} and L_9 are defined by equations $y = 0$, $y = z$, $y = bz$, $x = 0$, $x = z$, and $x = az$, respectively, where a and b are complex numbers in $\mathbb{C} \setminus \{0, 1\}$.

Since the line L_6 passes through the triples $e_3 = L_2 \cap L_3 = [0, 1, 1]$ and $e_4 = L_1 \cap L_9 = [a, 0, 1]$, the defining equation of the line L_6 can be written as $y = -\frac{1}{a}x + z$. Therefore the triple $e_{10} = L_4 \cap L_5 \cap L_6$ can be given by $[a(1-b), b, 1]$. Notice that the triple $e_2 = L_2 \cap L_5 \cap L_{10}$ can be given by $[0, 0, 1]$.

The defining equation of the line L_5 is $y = \frac{b}{a(1-b)}x$. Then we can find the triples $e_6 = L_2 \cap L_5 \cap L_7 = [\frac{a(1-b)}{b}, 1, 1]$.

Since L_7 passes through e_6 and $e_5 = L_1 \cap L_{10} = [1, 0, 1]$, its defining equation can be written as $y = \frac{b}{a-b-ab}(x-1)$.

Now only the defining equation of the line L_8 was not written down yet. Since L_8 passes through the points $e_7 = L_2 \cap L_8 \cap L_{10} = [1, 1, 1]$ and $e_8 = L_3 \cap L_4 \cap L_8 = [0, b, 1]$, we can write its defining equations as $y = (1-b)x + bz$.

We have used all but $e_{11} = L_7 \cap L_8 \cap L_9$ to calculate the defining equations. The parameters a and b must satisfy Property 3.2 for the determinant of the coefficient matrix of L_4, L_5 and L_6, i.e.

$$\det \begin{pmatrix} 1 & \frac{b}{a-b-ab} & -\frac{b}{a-b-ab} \\ 1 & 1-b & b \\ 0 & -a & 1 \end{pmatrix} = 0.$$

Simplify this equation to get $a^2b^2 - 2a^2b + a^2 - ab - b^2 + b = 0$. So we see the moduli space is a curve defined by $a^2b^2 - 2a^2b + a^2 - ab - b^2 + b = 0$.

We claim that this equation defines an irreducible curve in $\mathbb{C}^2$. Assume contrarily that the polynomial $p(a, b) = a^2b^2 - 2a^2b + a^2 - ab - b^2 + b = a^2(b-1)^2 - ab - b(b-1)$ is reducible. Then by the definition of reducibility it must be factored as $p(a, b) = [f_1(b)a + g_1(b)][f_2(b)a + g_2(b)]$, where $f_1(b)$, $f_2(b)$, $g_1(b)$ and $g_2(b)$ are polynomials in $\mathbb{C}[b]$. Viewing $p(a, b)$ as a quadratic polynomial of a, we see that the discriminant $b^2 + 4b(b-1)^3 = (f_1(b)g_2(b) - g_1(b)f_2(b))^2$ must be a perfect square. We check that $b^2 + 4b(b-1)^3$ is not a perfect square. In fact, it has no double root at all.

Therefore, the moduli space is irreducible.

Choose the two evaluations $a = 4$ and $b = 0.62404$ to get the arrangement pictured in Figure 3.

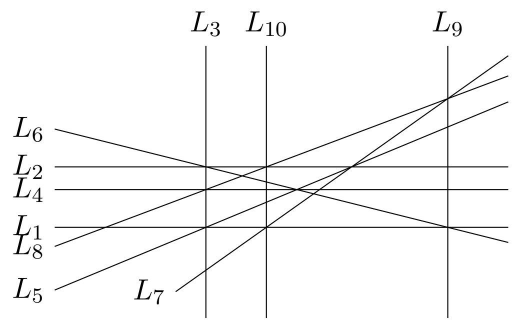


FIGURE 3. The arrangement 11.B.3.a.iii.

Example 3.14 (A moduli space consists of two complex conjugate points)**.** Consider the arrangement in Lemma 8.3 obtained by adding the line DFH to the (9_3).ii. arrangement (see Table 15), i.e. add a line passing through the intersection points $L_2 \cap L_5$, $L_3 \cap L_6$, and $L_4 \cap L_9$.

Assume that the lines L_3, L_2, L_1, L_4, L_5, and L_6 are defined by equations $y = 0$, $y = z$, $y = bz$, $x = 0$, $x = z$, and $x = az$, respectively, as depicted in Figure 4.

By the same method used in the previous examples, we can write down the equations of the rest of lines: $L_7 : y = -x + z$, $L_8 : y = \frac{b}{a}x$, $L_9 : y = \frac{b(b-1)}{b-a}x - \frac{b(1-a)}{b-a}z$ and $L_{10} : y = \frac{1}{1-a}(x-1)$, where a and b satisfy the following equations $a = b^2 - b + 1$ and $b^2 + 1 = 0$.

Solving the system of the two equations, we get two pairs of solutions $(a = -i, b = i)$ and $(a = i, b = -i)$. Replacing a and b in the defining equations of the ten lines by $\mp i$ and $\pm i$, respectively, we see that the line L_{10} given by the equation $y = \frac{1}{1+i}(x - z)$ is indeed distinct from the other nine lines.

Now we check whether there will be extra multiple points forced to appear when adding the tenth line. If there is an extra multiple point, then it must be on the line L_{10}. By checking the Figure 4 or Figure 10, we see that only the points $P = L_1 \cap L_7 = [1 - i, i, 1]$ and $Q = L_7 \cap L_8 = [-1, 1, 0]$ may fall on the line L_{10} accidentally. However, by plugging the two points into the equations of L_{10}, we get contradictions.

Therefore, P and Q cannot be on the line L_{10}, which means that adding the line L_{10} introduces no extra multiple points.

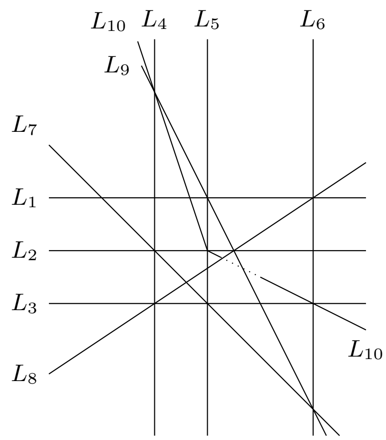


FIGURE 4. The arrangement (9_3).ii.DFH

Remark 3.15. This process outlined in Example 3.14 for the case (9_3).ii.DFH also occurs for the cases (9_3).iii.ACG, (9_3).iii.AEG, and (9_3).iii.BEG.

4. Arrangements of nine lines with nine and ten triples

The purpose of this section is to highlight previous results for nine lines that will be used in reduction arguments for ten lines in Sections 7 and 8.

Definition 4.1. [NY12, Definition 3.1] Let $k \in \mathbb{N}$. We say a line arrangement $\mathcal{A}$ is of type C_k if k is the minimum number of lines of $\mathcal{A}$ containing all points of multiplicity at least three.

Definition 4.2. [NY12, Definition 3.4] Let $\mathcal{A}$ be an arrangement of type C_3. Then $\mathcal{A}$ is a *simple* C_3 arrangement if there are three lines $L_1, L_2, L_3 \in \mathcal{A}$ such that all points of multiplicity at least three are contained in $L_1 \cup L_2 \cup L_3$ and one of the following holds:

- $L_1 \cap L_2 \cap L_3 = \emptyset$ and one of the lines contains only one point of multiplicity at least three apart from the other two lines; or
- $L_1 \cap L_2 \cap L_3 \neq \emptyset$.

From these definitions, Nazir and Yoshinaga proved the following results.

Theorem 4.3 (Theorem 3.2 of [NY12])**.** *Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a line arrangement in $\mathbb{P}^2_{\mathbb{C}}$ of class $C_{\leq 2}$ (i.e., either C_0, C_1 or C_2). Then the realization space $R(I(A))$ is irreducible.*

Theorem 4.4 (Theorem 3.5 of [NY12])**.** *Let $\mathcal{A}$ be an arrangement of C_3 of simple type. Then the realization space $R(I(A))$ is irreducible.*

The following is a more historical definition.

Definition 4.5. An (n_3) arrangement is one with n lines, n triples, and no points of higher multiplicity.

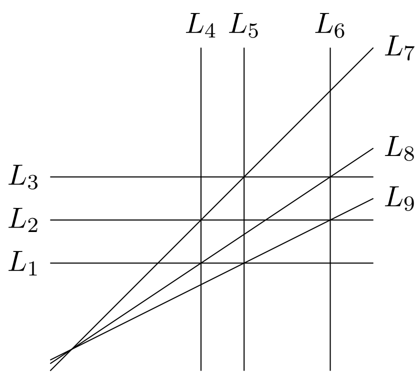
The textbook *Configurations of points and lines* by Grünbaum [Grü09] gives a table summarizing the numbers of such combinatorial and geometric arrangements for small n in Theorem 2.2.1. and Table 2.2.1. on p.69.

In the next proposition we consider the case $n = 9$, first appearing as early as the 1880's, which Grünbaum attributes to Kantor [Kan81], Martinetti [Mar87], Schröter [Sch88], and again to Levi [Lev29, p.103], Hilbert and Cohn-Vossen [HCV52], and Gropp [Gro97].

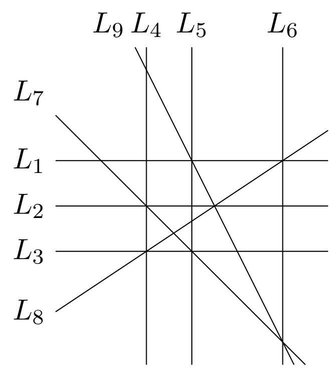
Proposition 4.6 (see Theorem 2.2.1. of [Grü09] or Proposition 3.7 of [Ye13])**.** *Let $\mathcal{A}$ be an arrangement of nine projective lines with nine triple points and no higher multiplicity points. If each line of $\mathcal{A}$ passes through exactly three triple points, then $\mathcal{A}$ is lattice isomorphic to one of the three arrangements given in Figure 5.*

We provide configuration tables in Table 2 (that differ from those in [Grü09]*) and our own equations of these lines in Table 3.*

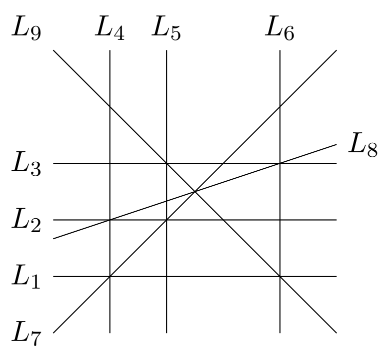
Note that this result holds for combinatorial as well as geometric arrangements.



The arrangement (9_3).i.



The arrangement (9_3).ii.



The arrangement (9_3).iii.

FIGURE 5. The three (9_3) arrangements of nine lines with nine triples and no higher multiplicity points as appearing in [Ye13].

Two of these three arrangements are C_3: all triple points lie on three lines. For (9_3).i. three such lines are L_3, L_4, L_9, and for (9_3).iii. three such lines are L_1, L_5, L_8.

These can be used to produce arrangements of nine lines with ten triples by making the three specified lines meet at a tenth triple. In the first case this results in the Pappus arrangement (and so (9_3).i. can be thought of a degeneration of the Pappus arrangement), and in the second case this results in a non-geometric arrangement.

Remark 4.7. The Pappus arrangement is named after Pappus of Alexandria, one of the last great Greek mathematicians of Antiquity, whose hexagon theorem states that given two sets of three collinear points, the three intersection points of six lines are collinear. The arrangement comes from the geometry of this theorem.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9
e_1	e_1	e_1	e_8	e_8	e_8	e_9	e_9	e_9
e_2	e_4	e_6	e_2	e_3	e_5	e_4	e_2	e_3
e_3	e_5	e_7	e_4	e_6	e_7	e_6	e_7	e_5

The arrangement (9_3).i.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9
e_1	e_1	e_1	e_8	e_8	e_8	e_4	e_3	e_2
e_2	e_4	e_6	e_4	e_2	e_3	e_7	e_5	e_5
e_3	e_5	e_7	e_6	e_7	e_9	e_9	e_6	e_9

The arrangement (9_3).ii.

e_1	e_1	e_1	e_8	e_8	e_8	e_9	e_9	e_9
e_2	e_4	e_6	e_2	e_5	e_3	e_2	e_4	e_3
e_3	e_5	e_7	e_4	e_6	e_7	e_5	e_7	e_6

The arrangement (9_3).iii.

TABLE 2. The arrangements (9_3).i., (9_3).ii., and (9_3).iii. given as configuration tables.

(9_3). Nine lines with nine triples					
Arr.	$y=0,z,bz$	$x=0,z,az$	$y=Ax$	$y=Bx+z$	$y=C(x-z)$
i.	L_1,L_2,L_3	L_4,L_5,L_6	L_8: $\frac{b}{a}$	L_7: $b-1$	L_9: $\frac{1}{a-1}$
ii.	L_3,L_2,L_1	L_4,L_5,L_6	L_8: $\frac{b}{a}$	L_7: -1	
	with $L_9: y=\frac{b(b-1)}{b-a}(x+\frac{b(1-a)}{b-a}z)$ and satisfying $a-(b^2-b+1)=0$				
iii.	L_1,L_2,L_3	L_4,L_5,L_6	L_7: 1	L_8: $\frac{b-1}{a}$	
	with $L_9: y=\frac{b}{1-a}(x-az)$ and satisfying $a-(b+1)=0$				

TABLE 3. Equations for geometric arrangements (9_3).

Proposition 4.8. *There are three combinatorial arrangements of nine lines with ten triples and no points of higher multiplicity. We provide configuration tables in Table 4 and equations of these lines in Table 5. Just two are geometric: we show these in Figure 6.*

Proof. There are nine lines with ten triples, and so exactly three lines L_1, L_2, L_3 must each contain four triples.

If these three lines form a central subarrangement, they intersect at a triple, say e_1, and the arrangement is in fact simple C_3. In this case, the triple e_1 can be degenerated to obtain an arrangement of nine lines with nine triples that is also simple C_3. Therefore it must be either (9_3).i. or (9_3).iii., as discussed above. In the first case our arrangement of nine lines with ten triples is the Pappus arrangement; in the second case we obtain a non-geometrically realizable one.

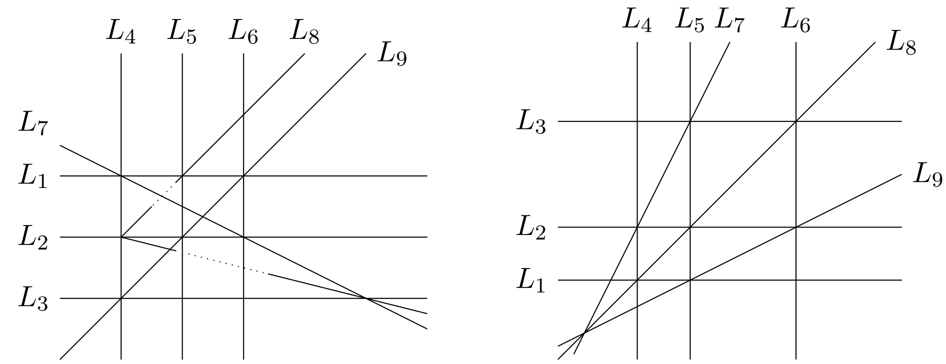
Otherwise these three lines form a generic subarrangement. We proceed according to how many of these three doubles are in fact triples in the original arrangement.

Suppose none are triples in the arrangement. Then there must be four distinct triples on each of the three lines, totalling twelve triples, a contradiction.

Suppose one is a triple in the arrangement. Then there must be three other distinct triples on each of the two lines meeting at this double, along with four distinct triples on the third line, totalling eleven triples, a contradiction.

Suppose two are triples in the arrangement. Then although there are exactly ten triples accounted for here, one of these three lines passes through just one of these two triples that contains three additional triples. However, this contradicts Fact 2.6 because there are just five other lines for these triples to appear.

Suppose all three are triples e_1, e_2, e_3 in the arrangement. Then the three additional lines passing through these points must also intersect in e_{10}, the only point not on these first three lines. This gives exactly one arrangement: the Nazir-Yoshinaga arrangement given in Example 5.3 of [NY12] and proved to be the only geometric arrangement of nine lines with ten triples which is not simple C_3 in Proposition 3.8 of [Ye13]. □



(a) The Nazir-Yoshinaga arrangement. (b) The Pappus arrangement.

FIGURE 6. The two geometric arrangements with nine lines and ten triples as appearing in [Ye13].

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9
e_1	e_1	e_1	e_{10}	e_{10}	e_{10}	e_2	e_3	e_4
e_2	e_5	e_8	e_2	e_3	e_4	e_7	e_5	e_6
e_3	e_6	e_9	e_5	e_6	e_7	e_9	e_9	e_8
e_4	e_7		e_8					

The Nazir-Yoshinaga arrangement.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9
e_1	e_1	e_1	e_9	e_9	e_9	e_{10}	e_{10}	e_{10}
e_2	e_4	e_7	e_2	e_3	e_6	e_4	e_2	e_3
e_3	e_5	e_8	e_4	e_5	e_8	e_7	e_5	e_6
	e_6			e_7			e_8	

The Pappus arrangement.

e_1	e_1	e_1	e_8	e_8	e_8	e_9	e_9	e_9
e_2	e_4	e_6	e_2	e_5	e_3	e_2	e_4	e_3
e_3	e_5	e_7	e_4	e_6	e_7	e_5	e_7	e_6
e_{10}				e_{10}			e_{10}	

Non-geometric degenerated (9_3).iii.

TABLE 4. The arrangements of Proposition 4.8 given as configuration tables.

Arrangements of nine lines with ten triples.					
Arr.	$y = 0, z, bz$	$x = 0, z, az$	$y = Ax$	$y = Bx + z$	$y = C(x - z)$
NY.	L_3, L_2, L_1	L_4, L_5, L_6	L_9: 1	L_8: $a - 1$	
	with $L_7 : y = \frac{1-a}{a}x + az$ satisfying $a - b = 0$ and $a^2 + 1 = 0$				
Pappus.	L_1, L_2, L_3	L_4, L_5, L_6	L_8: 1	L_7: $a - 1$	L_9: $\frac{1}{a-1}$
	satisfying $a - b = 0$				
Non-geom.	L_1, L_2, L_3	L_4, L_5, L_6	L_7: 1	L_8: $\frac{b-1}{a}$	
	with $L_9 : y = \frac{b}{1-a}(x - az)$ and satisfying $a - (b + 1) = 0$ and $a + b - 1 = 0$, a contradiction				

TABLE 5. Equations for arrangements of nine lines and ten triples.

In this present work, whose focus is arrangements of ten lines, we are not concerned with combinatorial arrangements of nine lines with more than ten triple points, as these do not arise in our reduction arguments below. So we conclude this discussion here. However, one may obviously ask for a complete list of combinatorial arrangements for nine lines.

Question 4.9. What other combinatorial (but not geometric) line arrangements of ten lines and no points of multiplicity higher than three arise with some other number of triples $n_3 \notin [9, 10]$?

5. Main Results

The main work of this paper gives a classification of non-trivially constructed geometric arrangements of ten lines with only triple points. We ignore those arrangements that are constructed trivially by adding lines which do not contain at least three triple points.

Theorem 5.1 (Classification)**.** *The number of arrangements of ten lines with only triple points are as given in Table 6 and as follows: there are* ***seventy-one*** *combinatorial arrangements and sixty-two geometric, with fifty-four of these having either irreducible or complex conjugate moduli spaces. In particular, this classification gives exactly nine non-geometric arrangements and exactly* ***nine*** *potential Zariski pairs.*

Here the arrangements considered in the last column either have irreducible moduli spaces or have moduli spaces of two components which are complex conjugate.

We are primarily interested in geometric line arrangements here, and so the only cases considered are arrangements of ten lines with some number of triples in order for a geometric arrangement to occur: $n_3 \in [10, 13]$ following Lemma 2.1 and Table 1. Since some combinatorial but not geometric arrangements arose in these situations, we have included them. However, this is not the complete list.

Case by # triples	# comb	# non-geom: Tab 8	# geom	# ($\mathcal{Z}$): Tab 7	# irred or ($\mathbb{C}$)	Result	Config Table	Table of Eqns
10.	10	1	9		9	Thm 6.1-6.3	9	10-11
13.	2	2				Thm 7.1	12	13
(9_3).i.	5		5		5	Lem 8.2		14
(9_3).ii.	4		4	2	2	Lem 8.3		15
(9_3).iii.	5	2	3	2	1	Lem 8.4		16
12.B.3.	4	1	3	3		Lem 8.6	17	18
12.B.2.	4	2	2	1	1	Lem 8.7	19	20
12. total	22	5	17	8	9	Thm 8.1		
11.A.	10		10		10	Lem 9.2	21	22-23
11.B.2.	4		4		4	Lem 9.3	24	25
11.B.3.a.	3		3		3		26	27
11.B.3.b.2.	7		7	1	6	Lem 9.4	28	29
11.B.3.b.1.	13	1	12		12	Lem 9.5	30-31	32-33
11. total	37	1	36	1	35	Thm 9.1		
Total	71	9	62	9	53	Thm 5.1+5.3		

Table 6. A summary of results with hyperlinks.

Question 5.2. What combinatorial but not geometric line arrangements of ten lines and no points of multiplicity higher than three arise with some other number of triples $n_3 \notin [10, 13]$?

In particular, this classification gives rise to several arrangements of particular interest: those whose combinatorics do not determine their topology.

Theorem 5.3 (Main Theorem)**.** *This present classification of arrangements of ten complex lines with only triple points gives a list of nine potential Zariski pairs listed in Table 7.*

Remark 5.4. It is interesting to note that all but one of the nine potential Zariski pairs listed in Table 7 from Theorem 5.3 have exactly twelve triple points, with this last one having exactly eleven triples.

This finishes the classification of geometric arrangements of ten lines begun in previous work by three of the authors.

Corollary 5.5. *Together with the classification of ten lines with quadruple points given in* [ATY13]*, which provides a list of nine potential Zariski pairs, this gives a total of eighteen such pairs for all complex line arrangements of ten lines satisfying Assumption 1.2.*

This work has produced a list of further counterexamples: arrangements which are combinatorial but not geometric.

Theorem 5.6. *This present classification of arrangements of ten complex lines with only triple points gives a list of nine non-geometric arrangements listed in Table 8.*

5.1. **A note about the proofs that follow.** For the sake of organization, the proofs of these results are split into Sections 6, 7, 8, and 9 according to the cases $n_3 = 10, 13, 12, 11$, respectively, as ordered by straightforwardness of the proofs. Each section begins with the local classification result, followed by its proof. The nature of these proofs are as follows.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_1	e_8	e_8	e_8	e_4	e_3	e_2	D
e_2	e_4	e_6	e_4	e_2	e_3	e_7	e_5	e_5	F
e_3	e_5	e_7	e_6	e_7	e_9	e_9	e_6	e_9	I
I	D	F	I	D	F				

The arrangement (9_3).ii.DFI.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_1	e_8	e_8	e_8	e_4	e_3	e_2	C
e_2	e_4	e_6	e_4	e_2	e_3	e_7	e_5	e_5	F
e_3	e_5	e_7	e_6	e_7	e_9	e_9	e_6	e_9	I
I		F	I	C	F		C		

The arrangement (9_3).ii.CFI.

e_1	e_1	e_1	e_8	e_8	e_8	e_9	e_9	e_9	B
e_2	e_4	e_6	e_2	e_5	e_3	e_2	e_4	e_3	D
e_3	e_5	e_7	e_4	e_6	e_7	e_5	e_7	e_6	F
	F	B	B		D	D		F	

The arrangement (9_3).iii.BDF.

e_1	e_1	e_1	e_8	e_8	e_8	e_9	e_9	e_9	A
e_2	e_4	e_6	e_2	e_5	e_3	e_2	e_4	e_3	C
e_3	e_5	e_7	e_4	e_6	e_7	e_5	e_7	e_6	G
G		C	A	G		C		A	

The arrangement (9_3).iii.ACG.

L_1	L_2	L_3	L_{10}	L_4	L_5	L_6	L_7	L_8	L_9
e_1	e_1	e_6	e_6	e_{11}	e_2	e_{11}	e_{11}	e_{12}	e_{12}
e_2	e_4	e_7	e_9	e_{12}	e_4	e_3	e_8	e_3	e_5
e_3	e_5	e_8	e_{10}	e_1	e_7	e_5	e_{10}	e_{10}	e_8
				e_7	e_9	e_6	e_2	e_4	e_9

The arrangement 12.B.2.iv.

L_1	L_2	L_3	L_{10}	L_4	L_5	L_6	L_7	L_8	L_9
e_1	e_1	e_2	e_3	e_1	e_2	e_3	e_{10}	e_{11}	e_{12}
e_2	e_4	e_6	e_8	e_{10}	e_{10}	e_{11}	e_4	e_5	e_4
e_3	e_5	e_7	e_9	e_{11}	e_{12}	e_{12}	e_7	e_6	e_6
				e_9	e_5	e_7	e_8	e_8	e_9

The arrangement 12.B.3.a.i.

e_1	e_1	e_2	e_3	e_1	e_2	e_3	e_{10}	e_{11}	e_{12}
e_2	e_3	e_6	e_8	e_{10}	e_{10}	e_{11}	e_4	e_7	e_7
e_4	e_5	e_7	e_9	e_{11}	e_{12}	e_{12}	e_5	e_9	e_8
				e_8	e_9	e_6	e_6	e_4	e_5

The arrangement 12.B.3.b.ii.

e_1	e_1	e_2	e_3	e_1	e_2	e_3	e_{10}	e_{11}	e_{12}
e_2	e_3	e_6	e_8	e_{10}	e_{10}	e_{11}	e_4	e_7	e_7
e_4	e_5	e_7	e_9	e_{11}	e_{12}	e_{12}	e_5	e_9	e_8
				e_8	e_9	e_6	e_6	e_5	e_4

The arrangement 12.B.3.b.iii.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_2	e_1	e_2	e_3	e_4	e_5	e_5	e_7
e_2	e_3	e_3	e_{10}	e_6	e_4	e_7	e_8	e_6	e_9
e_4	e_6	e_8	e_{11}	e_{10}	e_{11}	e_8	e_{10}	e_9	e_{11}
e_5	e_7	e_9							

The arrangement 11.B.3.b.2.v.

TABLE 7. The nine potential Zariski pairs that arise from this present classification.

The case for ten triples is already classified in [Grü09]. We display this result in Section 6 and simply check these arrangements to determine their moduli spaces.

The case for thirteen triples comes next in Section 7 because it is the next easiest. Here a single line is distinguished by the number of triples on it. We delete this line leaving an arrangement of nine lines with ten triples, and so we can apply Proposition 4.8 to obtain only three arrangements. We consider casework by adding back the tenth line passing through three of the nine double points.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_1	e_4	e_5	e_8	e_2	e_3	e_2	e_3
e_2	e_4	e_6	e_6	e_7	e_9	e_4	e_6	e_5	e_7
e_3	e_5	e_7	e_{10}	e_{10}	e_{10}	e_8	e_8	e_9	e_9

The non-geometric arrangement (10_3).iv.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_2	e_1	e_2	e_3	e_4	e_5	e_5	e_7
e_2	e_3	e_3	e_8	e_6	e_4	e_9	e_7	e_6	e_{10}
e_4	e_6	e_8	e_{11}	e_{10}	e_{10}	e_{11}	e_9	e_8	e_{11}
e_5	e_7	e_9							

The non-geometric arrangement 11.B.3.b.1.vii.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_1	e_2	e_3	e_4	e_2	e_3	e_4	A
e_2	e_5	e_8	e_5	e_6	e_7	e_7	e_5	e_6	C
e_3	e_6	e_9	e_8	e_{10}	e_{10}	e_9	e_9	e_8	E
e_4	e_7	C	e_{10}	C	A	E	A	E	

The non-geometric arrangement 13.i.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_1	e_2	e_2	e_3	e_3	e_4	e_4	e_8
e_2	e_5	e_{11}	e_7	e_6	e_7	e_5	e_6	e_5	e_9
e_3	e_6	e_{12}	e_8	e_{10}	e_{10}	e_9	e_9	e_8	e_{10}
e_4	e_7	e_{13}	e_{13}	e_{11}	e_{12}	e_{13}	e_{12}	e_{11}	

The non-geometric arrangement 13.ii.

e_1	e_1	e_1	e_8	e_8	e_8	e_9	e_9	e_9	B
e_2	e_4	e_6	e_2	e_5	e_3	e_2	e_4	e_3	E
e_3	e_5	e_7	e_4	e_6	e_7	e_5	e_7	e_6	G
G	E	B	B	G	E				

The non-geometric arrangement (9_3).iii.BEG.

e_1	e_1	e_1	e_8	e_8	e_8	e_9	e_9	e_9	A
e_2	e_4	e_6	e_2	e_5	e_3	e_2	e_4	e_3	D
e_3	e_5	e_7	e_4	e_6	e_7	e_5	e_7	e_6	G
G			A	G	D	D		A	

The non-geometric arrangement (9_3).iii.ADG.

L_1	L_2	L_3	L_{10}	L_4	L_5	L_6	L_7	L_8	L_9
e_1	e_1	e_6	e_6	e_{11}	e_{11}	e_{11}	e_{12}	e_{12}	e_{12}
e_2	e_4	e_7	e_9	e_1	e_2	e_3	e_2	e_3	e_4
e_3	e_5	e_8	e_{10}	e_7	e_4	e_5	e_5	e_7	e_{10}
				e_{10}	e_9	e_8	e_6	e_9	e_8

The non-geometric arrangement 12.B.2.i.

L_1	L_2	L_3	L_{10}	L_4	L_5	L_6	L_7	L_8	L_9
e_1	e_1	e_6	e_6	e_{11}	e_2	e_{11}	e_{11}	e_{12}	e_{12}
e_2	e_4	e_7	e_9	e_{12}	e_4	e_3	e_8	e_3	e_5
e_3	e_5	e_8	e_{10}	e_1	e_7	e_5	e_{10}	e_{10}	e_8
				e_6	e_9	e_7	e_2	e_4	e_9

The non-geometric arrangement 12.B.2.iii.

e_1	e_1	e_2	e_3	e_1	e_2	e_3	e_{10}	e_{11}	e_{12}
e_2	e_3	e_6	e_8	e_{10}	e_{10}	e_{11}	e_4	e_5	e_4
e_4	e_5	e_7	e_9	e_{11}	e_{12}	e_{12}	e_7	e_6	e_6
				e_9	e_5	e_7	e_8	e_8	e_9

The non-geometric arrangement 12.B.3.b.i.

TABLE 8. The nine non-geometric arrangements that arise from this present classification.

The next easiest case is for twelve triples, as some of these arrangements can be considered by a reduction argument similar to that above taken care of in Subsection 8.2. The first subsection of Section 8 produces no arrangements, and all remaining arrangements are determined by casework in Subsection 8.3.

The remaining case for eleven triples contains the most laborious casework; this is handled in Section 9.

Remark 5.7. It may be useful here to explicitly describe the notation used for the cases, as in 11.B.3.b.1.iv., that appears throughout.

The first number stands for the number of triples in the arrangement. The second letter refers to the first subarrangement considered: it is (A) when this subarrangement is central and (B) when it is generic. The number that follows gives the number of doubles of the generic subarrangement that are taken as triples in the original arrangement. The lowercase letters (a) and (b) refer to second subarrangement considered, similarly central and generic, respectively. The number that follows is again the number of doubles of the second generic subarrangement that are taken as triples in the original arrangement. The roman numeral that concludes it represents the specific case number within these confines.

Thus in Figure 1, the arrangement $\mathcal{B}$ may be thought of as having its first three outer lines being generic (B) and forming a triangle with its next three lines also being generic (b) and forming an inner triangle.

On the other hand the Ceva arrangement $\mathcal{C}$ in Figure 1 may be thought of as having its first three outer lines being generic (B) and forming a triangle with its next three lines being central (a) and meeting at a single point.

Furthermore, the notations $(*)$, $(\mathbb{C})$, and $(\mathcal{Z})$ that appear in the tables below are as described in Algorithm 3.11: cases for which we use Mathematica's irreducibility test, for which only complex solutions exist, and cases which yield potential Zariski pairs.

6. Arrangements of ten lines with ten triples

For completion we first present the results already known about the (10_3) arrangements: those with ten lines and ten triples.

Theorem 6.1. [Grü09, Theorem 2.2.1.] *For ten lines and ten triples, there are ten combinatorial configurations and nine geometric configurations, as shown in Table 9.*

Remark 6.2. The arrangement (10_3).i. is called the Desargues arrangement, named after Girard Desargues, a French mathematician of the seventeenth century, whose theorem in projective geometry can be stated: two triangles are in perspective axially if and only if they are in perspective centrally. The arrangement comes from the geometry of this theorem.

We perform geometric checks on these nine arrangements and arrive at the following conclusion.

Theorem 6.3. *For ten lines and ten triples, the nine geometric configurations have irreducible moduli spaces. Thus there are no potential Zariski pairs of ten lines and ten triples.*

Proof. The equations of the lines for each of the ten cases are given in Tables 10 and 11. These were produced in the manner described above in Section 3. □

7. Arrangements of ten lines with thirteen triples

Theorem 7.1. *There are two combinatorial configurations of ten lines and thirteen triples, given as configuration tables in Table 12. Neither of these are geometric, as shown by contradicting equations in Table 13. Thus there are no potential Zariski pairs of ten lines and thirteen triples.*

Proof. Let $\mathcal{A}$ be an arrangement of ten lines with thirteen triples. Nine of the lines pass through exactly four triples, and one line, say L_{10}, passes through exactly three triples. Let $\mathcal{A}' := \mathcal{A}\backslash\{L_{10}\}$ be the deletion of L_{10} from the original arrangement. Then $\mathcal{A}'$ has nine lines with ten triples, each line passing through at least three triples.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_1	e_2	e_3	e_8	e_2	e_3	e_4	e_5
e_6	e_2	e_4	e_4	e_5	e_9	e_6	e_7	e_6	e_7
e_7	e_3	e_5	e_8	e_8	e_{10}	e_9	e_9	e_{10}	e_{10}

The arrangement (10_3).i.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_1	e_4	e_5	e_8	e_2	e_3	e_2	e_3
e_2	e_4	e_6	e_6	e_7	e_9	e_4	e_7	e_6	e_5
e_3	e_5	e_7	e_{10}	e_{10}	e_{10}	e_8	e_8	e_9	e_9

The arrangement (10_3).ii.

e_1	e_1	e_1	e_4	e_5	e_8	e_2	e_3	e_2	e_3
e_2	e_4	e_6	e_6	e_7	e_9	e_4	e_6	e_7	e_5
e_3	e_5	e_7	e_{10}	e_{10}	e_{10}	e_8	e_8	e_9	e_9

The arrangement (10_3).iii.

e_1	e_1	e_1	e_4	e_5	e_8	e_2	e_3	e_2	e_3
e_2	e_4	e_6	e_6	e_7	e_9	e_4	e_6	e_5	e_7
e_3	e_5	e_7	e_{10}	e_{10}	e_{10}	e_8	e_8	e_9	e_9

The non-geometric arrangement (10_3).iv.

e_1	e_1	e_1	e_2	e_3	e_8	e_2	e_4	e_3	e_5
e_2	e_4	e_6	e_4	e_7	e_9	e_5	e_6	e_6	e_7
e_3	e_5	e_7	e_8	e_8	e_{10}	e_9	e_9	e_{10}	e_{10}

The arrangement (10_3).v.

e_1	e_1	e_1	e_2	e_3	e_8	e_2	e_5	e_3	e_4
e_2	e_4	e_6	e_4	e_7	e_9	e_6	e_7	e_5	e_6
e_3	e_5	e_7	e_8	e_8	e_{10}	e_9	e_9	e_{10}	e_{10}

The arrangement (10_3).vi.

e_1	e_1	e_1	e_2	e_4	e_5	e_2	e_3	e_7	e_6
e_2	e_4	e_6	e_8	e_8	e_7	e_4	e_5	e_3	e_9
e_3	e_5	e_7	e_9	e_{10}	e_8	e_6	e_9	e_{10}	e_{10}

The arrangement (10_3).vii.

e_1	e_1	e_1	e_3	e_5	e_7	e_2	e_6	e_4	e_2
e_2	e_4	e_6	e_8	e_8	e_9	e_7	e_5	e_3	e_4
e_3	e_5	e_7	e_9	e_{10}	e_{10}	e_8	e_9	e_{10}	e_6

The arrangement (10_3).viii.

e_1	e_1	e_1	e_2	e_4	e_6	e_5	e_3	e_2	e_3
e_2	e_4	e_6	e_8	e_8	e_9	e_7	e_5	e_7	e_4
e_3	e_5	e_7	e_9	e_{10}	e_{10}	e_8	e_9	e_{10}	e_6

The arrangement (10_3).ix.

e_1	e_1	e_1	e_3	e_2	e_7	e_5	e_6	e_4	e_2
e_2	e_4	e_6	e_8	e_8	e_9	e_7	e_5	e_3	e_4
e_3	e_5	e_7	e_9	e_{10}	e_{10}	e_8	e_9	e_{10}	e_6

The arrangement (10_3).x.

Table 9. The (10_3) arrangements of ten lines with ten triples as found in [Grü09]. Nine of the ten are geometric, and each of these nine has an irreducible moduli space.

According to Proposition 4.8, the arrangement $\mathcal{A}'$ is either the Nazir-Yoshinaga arrangement, the Pappus arrangement, or a non-geometrically realizable arrangement.

The Nazir-Yoshinaga arrangement $\mathcal{A}^{\pm i}$ has six doubles, and so when the tenth line is added back, it must pass through exactly three of these. The six doubles can be arranged as the vertices of a hexagon with lines as edges passing between them in this cyclic manner: $L_3, L_5, L_7, L_9, L_8, L_6$. Thus for a line to pass through three of these doubles, it must skip every other one in this cycle. This should give two possibilities, but the symmetry $(L_1L_2)(L_5L_6)(L_7L_8)$ sends one to the other.

The combinatorial arrangement here is given as 13.i. in Table 12 and Figure 7, but the equations given in Table 13 preclude it from being geometric.

The Pappus arrangement has six doubles, and so when the tenth line is added back, it must pass through exactly three of these. The six doubles can be arranged as the vertices of two disjoint triangles with lines as edges passing between them in this (disjoint) cyclic manner: L_1, L_7, L_6 and

(10_3).						
Arr.	$y=0,z,bz$	$x=0,z,az$	$y=Ax$	$y=Bx+z$	$y=C(x-z)$	$y=D(x-z)+z$
(10_3).i.	L_3,L_2,L_1	L_4,L_5,L_6	L_9: $\frac{c}{a}$	L_7: $\frac{c(b-1)}{ab}$	L_{10}: $\frac{c}{a-1}$	L_8: $\frac{c(b-1)}{b(a-1)}$
	with $e_{10}=(a,c)$					
(10_3).ii.	L_3,L_2,L_1	L_4,L_5,L_6	L_9: $\frac{b(c-1)}{a(b-1)}$	L_7: $\frac{c-1}{a}$	L_8: $\frac{c}{a-1}$	L_{10}: $\frac{(a-1)(b-1)}{c(b-1)-(a-1)}$
	with $e_8=(a,c)$ and satisfying $2b-1=0$					
(10_3).iii.	L_3,L_2,L_1	L_4,L_5,L_6	L_8: $\frac{c}{a}$	L_7: $\frac{c-1}{a}$	L_9: $\frac{b(c-1)}{ab-a-c+1}$	L_{10}: $\frac{c(b-1)}{ab-c}$
	with $e_8=(a,c)$ and satisfying $1-a-b-c+2ab=0^*$					
(10_3).iv.	L_3,L_2,L_1	L_4,L_5,L_6	L_8: $\frac{c}{a}$	L_7: $\frac{c-1}{a}$	L_{10}: $\frac{bc}{ab-c}$	L_9: $\frac{(b-1)(c-1)}{ab-a-c+1}$
	with $e_8=(a,c)$ and satisfying $c-b=0$, a contradiction					
(10_3).v.	L_1,L_2,L_3	L_4,L_5,L_6	L_7: $\frac{c}{a}$	L_8: $\frac{c-1}{a}$	L_9: $\frac{b(c-1)}{ab-a-c+1}$	
	with $e_9=(a,c)$, $L_{10}: y=\frac{c(b-1)}{c-a}(x-z)+bz$ and satisfying $b(c-1)(a-1)(c-a)-(ab-a-c+1)(abc-ab-ac+c)=0^*$					
(10_3).vi.	L_1,L_2,L_3	L_4,L_5,L_6	L_7: $\frac{c}{a}$	L_{10}: $\frac{c(b-1)}{ab}$	L_9: $\frac{b-c}{(a-1)(b-1)}$	
	with $e_9=(a,c)$, $L_8: y=\frac{c-b}{a-1}(x-z)+bz$ and satisfying $c(b^2-b+1)-b=0$					

TABLE 10. Equations for arrangements with ten triples.

L_2, L_8, L_5. Thus no line may pass through three of these, and so there is not even a combinatorial arrangement that arises in this way.

The non-geometrically realizable arrangement of nine lines with ten triples has six doubles, and so when the tenth line is added back, it must pass through exactly three of these. The six doubles can be arranged as the vertices of a hexagon with lines as edges passing between them in this cyclic manner: $L_2, L_9, L_4, L_3, L_7, L_6$. Thus for a line to pass through three of these doubles, it must skip every other one in this cycle. This should give two possibilities, but symmetry sends one to the other.

The combinatorial arrangement here is given as 13.ii. in Table 12 and Figure 8, but the equations given in Table 13 preclude it from being geometric. □

(10_3). continued						
Arr.	$y = 0, z, bz$	$x = 0, z, az$	$y = Ax$	$y = Bx + z$	$y = C(x - z)$	$y = D(x - z) + z$
(10_3).vii.	L_2, L_3, L_1	L_6, L_5, L_4	L_8: $\frac{b(c-1)}{b-1}$	L_9: $c - 1$	L_7: $\frac{b}{a-1}$	
	with $e_{10} = (1, c)$, $L_{10} : y = \frac{b(1-c)}{a-1}(x - z) + cz$ and satisfying $(ab + (b-1)^2)c - (a + b - a)b = 0^*$					
(10_3).viii.	L_{10}, L_1, L_7	L_2, L_8, L_5	L_9: $\frac{c}{a}$	L_3: -1		
	with $e_{10} = (a, c)$, $L_4 : y = \frac{c(b-1)}{a(c-1)}(x - az) + bz$, $L_6 : y = \frac{c-b}{a+b-1}(x - az) + cz$ and satisfying $c(b - a)(1 - a)(a + b - 1) + ab(b - c)(c - 1) = 0^*$					
(10_3).ix.	L_2, L_3, L_1	L_7, L_5, L_4	L_8: $\frac{b}{bc-b+1}$	L_7: $\frac{b-1}{a}$	L_{10}: $\frac{1}{c-1}$	
	with $e_6 = (c, 1)$, $L_6 : y = \frac{b-1}{a(1-c)}(x - cz) + z$ and satisfying $a^2b(c-1) + (b(c-1)+1)((b-1)(a-c) + a(1-c)) = 0$, irreducible by hand					
(10_3).x.	L_1, L_2, L_3	L_9, L_5, L_6	L_4: c	L_{10}: -1		
	with $e_8 = (1, c)$, $L_7 : y = \frac{b-c}{a-1}(x - z) + cz$, $L_8 : y = \frac{b-ac}{1-a-b}(x - z + bz) + bz$ and satisfying $(b - ac)((1-c)(a-1) + b(b-c)) + (b-1)(b-c)(1-b-a) = 0$, irreducible by hand					

TABLE 11. Equations for arrangements with ten triples.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_1	e_2	e_3	e_4	e_2	e_3	e_4	A
e_2	e_5	e_8	e_5	e_6	e_7	e_7	e_5	e_6	C
e_3	e_6	e_9	e_8	e_{10}	e_{10}	e_9	e_9	e_8	E
e_4	e_7	C	e_{10}	C	A	E	A	E	

The non-geometric arrangement 13.i.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_1	e_2	e_2	e_3	e_3	e_4	e_4	e_8
e_2	e_5	e_{11}	e_7	e_6	e_7	e_5	e_6	e_5	e_9
e_3	e_6	e_{12}	e_8	e_{10}	e_{10}	e_9	e_9	e_8	e_{10}
e_4	e_7	e_{13}	e_{13}	e_{11}	e_{12}	e_{13}	e_{12}	e_{11}	

The non-geometric arrangement 13.ii.

TABLE 12. The two combinatorial (but not geometric) arrangements of ten lines with thirteen triples.

13.						
Arr.	$y=0,z,bz$	$x=0,z,az$	$y=Ax$	$y=Bx+z$	$y=C(x-z)$	$y=Ex+bz$
13.i.	L_3,L_2,L_1	L_4,L_5,L_6	L_9: 1	L_8: $a-1$	L_{10}: $\frac{a^2-a+1}{a-1}$	L_7: $\frac{1-a}{a}$
	satisfying $a-b=0$, $a^2+1=0$, and $a^3-3a^2+2a-1=0$, a contradiction					
13.ii.	L_1,L_2,L_3	L_5,L_6,L_{10}	L_4: 1	L_8: $b-1$	L_7: $\frac{b}{b-1}$	L_9: $\frac{(a-b)}{a}$
	satisfying $a(b^2+b-1)-b(2b-1)=0$, $a(-b^2+b+1)-b=0$, and $a(b-(b-1)^2)-(2b-1)=0$, a contradiction					

TABLE 13. Equations for (non-geometric) arrangements with thirteen triples.

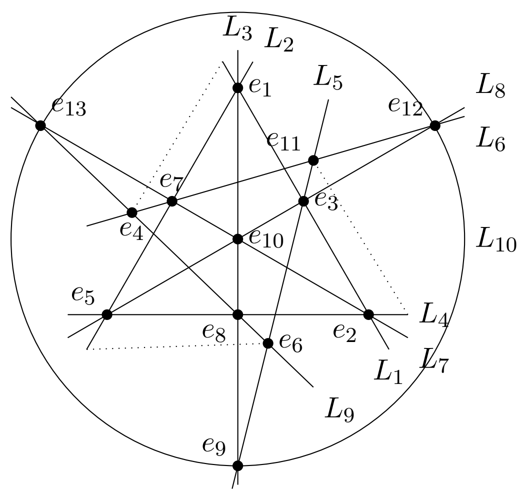


FIGURE 7. The combinatorial arrangement 13.i. of ten lines with thirteen triples.

8. Arrangements of ten lines with twelve triples

Theorem 8.1. *There are twenty-two combinatorial configurations of ten lines and twelve triples. All but three of these are geometric, and four of these remaining nineteen are potential Zariski pairs.*

Proof. Let $\mathcal{A}$ be arrangement of ten lines with twelve triples. There are exactly four lines that have exactly three triple points on each of them; call these L_1, L_2, L_3, and L_{10}. We consider the possible subarrangements of just these four lines. Observe that there can be no central subarrangement of these four lines because we do not consider quadruple points. Other than the generic subarrangement with six doubles, there is only one other subarrangement: $\mathcal{E}$ with one triple and three doubles, resembling an artist's easel.

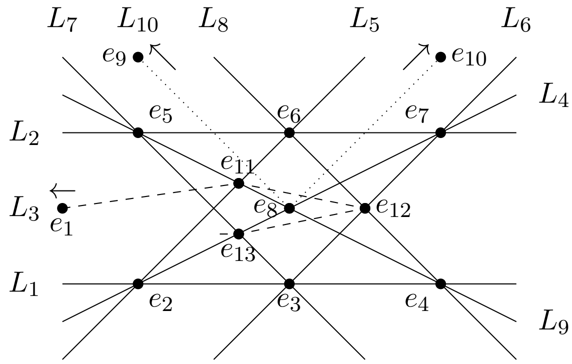


FIGURE 8. The combinatorial arrangement 13.ii. of ten lines with thirteen triples.

8.1. **Easel subarrangement.** Let e_1 be this triple on L_1, L_2, and L_3, and consider the three double points on the subarrangement $\mathcal{E}$ of these lines with L_{10}.

If all three of these doubles in $\mathcal{E}$ are indeed triples in $\mathcal{A}$, then the lines L_1, L_2, L_3, and L_{10} contribute a total of seven triples, leaving five triples left among the remaining six lines. However, this contradicts Lemma 2.9.

If exactly two of these three doubles in $\mathcal{E}$ are indeed triples in $\mathcal{A}$, then the lines L_1, L_2, L_3, and L_{10} contribute a total of eight triples, leaving four triples left among the remaining six lines. By Lemma 2.11, there is a unique subarrangement $\mathcal{C}$ of these six lines with exactly four triples. However, this subarrangement $\mathcal{C}$ has only three doubles, and since $\mathcal{E}$ has no more doubles that can become triples in $\mathcal{A}$, there are not enough triples to account for the remaining five in $\mathcal{A}$, a contradiction.

If exactly one of these three doubles in $\mathcal{E}$ is indeed a triple in $\mathcal{A}$, then the lines L_1, L_2, L_3, and L_{10} contribute a total of nine triples, leaving three triples left among the remaining six lines. By Lemma 2.10, there is a unique subarrangement $\mathcal{B}$ of these six lines with exactly three triples. However, this subarrangement $\mathcal{B}$ has only six doubles, and since $\mathcal{E}$ has no more doubles that can become triples in $\mathcal{A}$, there are not enough triples to account for the remaining seven in $\mathcal{A}$, a contradiction.

If none of these three doubles in $\mathcal{E}$ is indeed a triple in $\mathcal{A}$, then we use a reduction argument as in the case of thirteen triples. Let $\mathcal{A}' := \mathcal{A}\backslash\{L_{10}\}$ be the deletion of L_{10} from the original arrangement. Then $\mathcal{A}'$ has nine lines with nine triples, each line passing through *exactly* three triples, and we consider this case below.

8.2. **A reduction as in thirteen triples.** Consider the four lines L_1, L_2, L_3, and L_{10} with exactly three triples on each, and suppose that in general the line L_{10} intersects the other three lines at points which are only doubles in the original arrangment $\mathcal{A}$. Let $\mathcal{A}' := \mathcal{A}\backslash\{L_{10}\}$ be the deletion of L_{10} from the original arrangement. Then $\mathcal{A}'$ has nine lines with nine triples, each line passing through *exactly* three triples.

By Proposition 4.6 there are only three such combinatorial or geometric arrangements: (9_3).i., (9_3).ii., and (9_3).iii., as appearing in Figure 5.

Lemma 8.2. *For the reduction of twelve triples giving case* (9_3)*.i., we have five combinatorial configurations, each with a one-parameter family of geometric configurations as given in Table 14, and so each has an irreducible moduli space. Thus there are no potential Zariski pairs.*

Proof. We begin by considering the combinatorial configuration without worrying about its geometric realization as in Figure 9. The nine doubles $A, \ldots, I$ are labelled; observe that these are arranged in three disjoint triangles using the nine lines: A, C, E; B, D, F; and G, H, I. Thus for the line L_{10} to pass through three of these, it must pass through exactly one from each triangle.

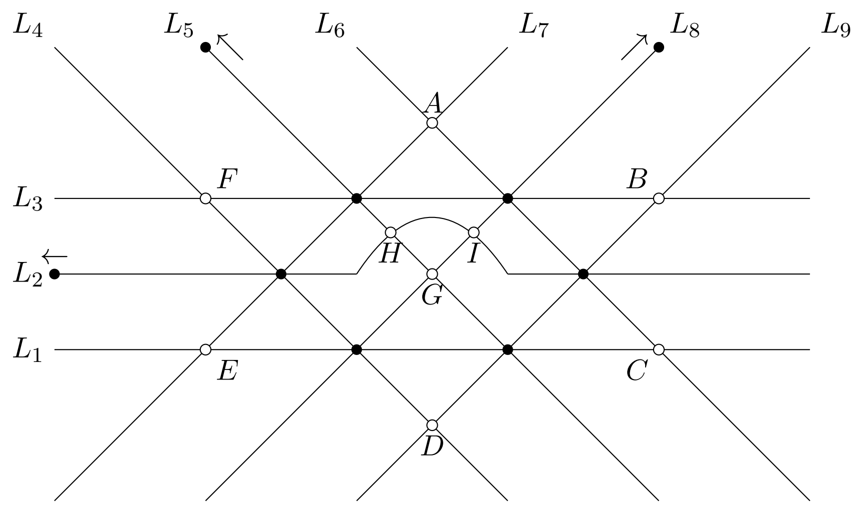


FIGURE 9. A combinatorial arrangement for (9_3).i. showing its symmetry.

Without loss of generality we may assume the line passes through C. Up to symmetry there are just two choices for the next triangle: $D(= B)$ and F. In the first case there is no symmetry left so there are three choices (G, H, I), but in the second case there is still symmetry leaving two choices $G(= I)$ and H.

Equations for a geometric realization of (9_3).i. given in Table 3 give coordinates for these doubles, and the equations given in Table 14 demonstrate that all five of these combinatorial arrangements are geometric with irreducible moduli spaces. □

Adding a tenth line to (9_3).i.		
10^{th} line	Equation	satisfying
CDG	$y = \frac{1}{a(a-1)}(x - az)$	$b + 1 = 0$
CDH	$y = \frac{1}{a(a-1)}(x - az)$	$a + 1 = 0$
CDI	$y = \frac{1}{a(a-1)}(x - az)$	$ab - 1 = 0$
CFG	$y = \frac{-b}{a}x + bz$	$a - 2 = 0$
CFH	$y = \frac{-b}{a}x + bz$	$a(b - 1) - b = 0$

TABLE 14. Equations for (geometric) arrangements from (9_3).i.

Lemma 8.3. *For the reduction of twelve triples giving case* (9_3)*.ii., we have four combinatorial configurations, all geometric as given in Table 15: one with one real solution, one with irreducible moduli space by complex conjugation, and two potential Zariski pairs.*

Proof. We begin by considering the combinatorial configuration without worrying about its geometric realization as in Figure 10. The nine doubles $A, \ldots, I$ are labelled; observe that these are arranged in a nine-gon. The line L_{10} must pass through three of these but cannot pass through any two that are adjacent.

Without loss of generality we may assume the line passes through $F = (a, 0)$. There are four possible cyclic partitions of the six remaining doubles A, B, C, D, H, I into three nonempty sets: the partition (1,1,4) giving D, F, H, the partition (1,2,3) giving D, F, I, the partition (1,3,2) giving D, F, A, and the partition (2,2,2) giving C, F, I.

Equations for a geometric realization of (9_3).ii. given in Table 3 give coordinates for these doubles, and the equations given in Table 15 demonstrate that all four of these combinatorial arrangements are geometric. The first gives complex conjugate solutions, the second is a potential Zariski pair, the third gives a single (real) solution, and the last gives two real (Galois conjugate) solutions and thus is also a potential Zariski pair. Thus two of these give potential Zariski pairs that appear on the final list of Theorem 5.3. □

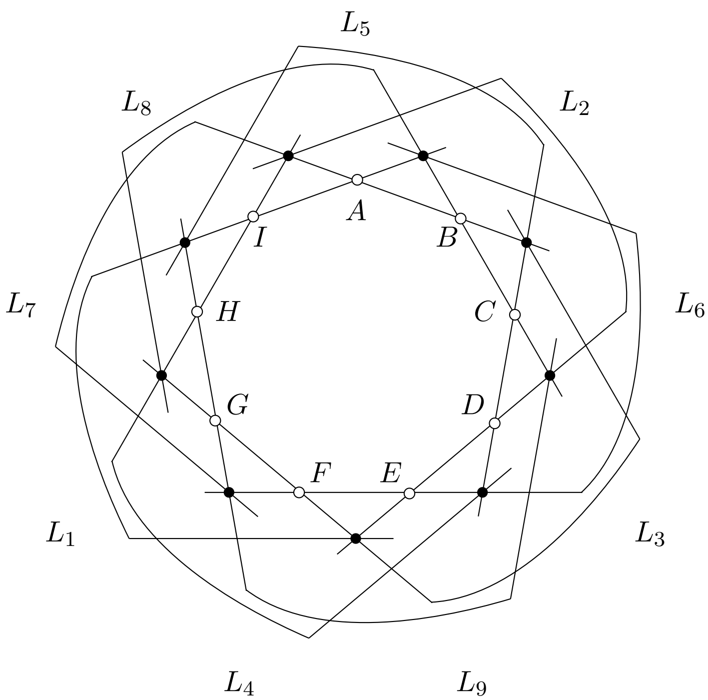


FIGURE 10. A combinatorial arrangement for (9_3).ii. showing its symmetry.

Lemma 8.4. *For the reduction of twelve triples giving case* (9_3)*.iii., we have five combinatorial configurations, three of which are geometric as given in Table 16: one with irreducible moduli space by complex conjugation and two with two real solutions, giving two potential Zariski pairs.*

Proof. We begin by considering the combinatorial configuration without worrying about its geometric realization as in Figure 11. The nine doubles $A, \ldots, I$ are labelled; observe that these are

Adding a tenth line to (9_3).ii.		
10^{th} line	Equation	satisfying
DFH	$y = \frac{1}{1-a}(x-a)$	$a-(b^2-b+1)=0,\ b^2+1=0^{\mathbb{C}}$
DFI	$y = \frac{1}{1-a}(x-a)$	$a-(b^2-b+1)=0,\ b^3-2b^2+b-1=0^{\mathcal{Z}}$
DFA	$y = \frac{1}{1-a}(x-a)$	$a-(b^2-b+1)=0,\ b-2=0$
CFI	$y = \frac{-b}{a}(x-a)$	$a-(b^2-b+1)=0,\ b^2-b-1=0^{\mathcal{Z}}$

TABLE 15. Equations for (geometric) arrangements from (9_3).ii.

arranged in a hexagon A, B, C, D, E, F and a triangle G, H, I. The line L_{10} must pass through three of these points, and so it can pass through at most one point on the triangle.

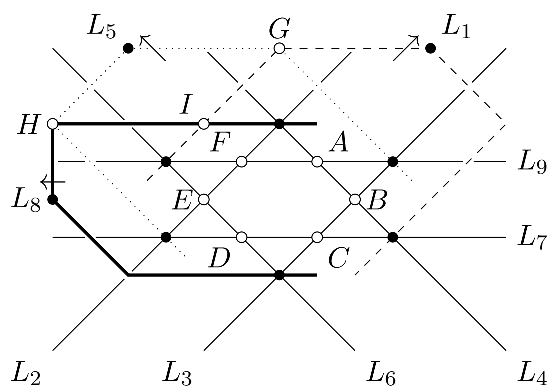


FIGURE 11. A combinatorial arrangement for (9_3).iii. showing its symmetry.

If the line passes through none of the points on the triangle, this gives just one choice up to symmetry: B, D, F. If the line passes through one point on the triangle (say G) and two on the hexagon, these two on the hexagon are either one apart, in which case there are two choices up to symmetry, $AC(=DF)$ or $AE(=FB=DB=CE)$, or two apart, in which case there are two choices up to symmetry, BE or $AD(=FC)$.

Equations for a geometric realization of (9_3).iii. given in Table 3 give coordinates for these doubles, and the equations given in Table 16 demonstrate that only three of these five combinatorial arrangements are geometric: one with an irreducible moduli space by complex conjugation and two with two real solutions, which give potential Zariski pairs. $\square$

Remark 8.5. Observe that geometrically the line BEG must pass through the triple $L_7 \cap L_8 \cap L_9$, making it a quadruple. This arrangement already appears in [ATY13, Proposition 5.1], as it has ten lines and a quadruple. However in the non-geometric setting we need not worry about this quadruple.

Adding a tenth line to (9_3).iii.		
10^{th} line	Equation	satisfying
BDF	$y = \frac{a-b}{a}x + bz$	$a-b-1=0,\ a^2-a-1=0^{\mathbb{Z}}$
ACG	$y = \frac{1}{1-a}(x-bz)+bz$	$a-b-1=0,\ a^2-a-1=0^{\mathbb{Z}}$
AEG	$y = \frac{a-1-ab}{a(a-1)}(x-az)+z$	$a-b-1=0,\ b^2+b+1=0^{\mathbb{C}}$
ADG	$y = \frac{a-b-1}{a-1}(x-az)+az$	$a-b-1=0,\ a-1=0$, a contradiction
BEG	$y = \frac{1-b}{a}x + bz$	$a-b-1=0,\ b^2+1=0$, a contradiction (see Remark 8.5)

TABLE 16. Equations for arrangements from (9_3).iii.

8.3. **Remaining generic non-reduction subarrangements.** Lastly we consider the remaining cases when the four initial lines form a generic subarrangement but there does not exist a line of it whose three doubles all fail to be triples in $\mathcal{A}$, the original arrangement.

We proceed according to how many of the six doubles are in fact triples in $\mathcal{A}$.

Suppose all six doubles are triples in $\mathcal{A}$. Then the remaining six lines must form a subarrangement with exactly six triples, contradicting Lemma 2.9.

Suppose five of the six doubles are triples in $\mathcal{A}$; then two additional triples must lie on this subarrangement. However the remaining six lines must form another subarrangement with exactly five triples, contradicting Lemma 2.9.

Suppose four of the six doubles are triples in $\mathcal{A}$; then four additional triples must lie on this subarrangement. The remaining six lines must form another subarrangement with exactly four triples. By Lemma 2.11 there is a unique subarrangement $\mathcal{C}$ satisfying this. However this subarrangement $\mathcal{C}$ has only three double points, fewer than the four needed to produce triples on the original subarrangement, a contradiction.

Suppose three of the six doubles are triples in $\mathcal{A}$. Some cases already accounted for are those in which the intersections of a given line with the other three lines are not triple points; these were handled in Subsection 8.2. Those not accounted for are:

(a) those in which these three triples are colinear and
(b) those in which these three triples are not colinear but do not form a triangle.

Here six additional triples must lie on this subarrangement. Then the remaining six lines must form another subarrangement with exactly three triples. By Lemma 2.10 there is a unique subarrangement $\mathcal{B}$ satisfying this.

Lemma 8.6 (12.B.3.)**.** *For the non-reduction of twelve triples where three of the six doubles are triples, there are four combinatorial configurations, three of which are geometric configurations. Three of these give potential Zariski pair. Configuration tables are given in Table 17 and equations are given in Table 18.*

Proof. Let e_1, e_2, e_3 be the three triples coming from doubles as mentioned above. The two remaining cases not handled by the reduction above are (a) when these three points are colinear and (b) when they are not but do not form a triangle. In both we may assume e_1 is the intersection $L_1 \cap L_2$ and e_2 is the intersection $L_1 \cap L_3$. In (a) we take e_3 to be the intersection $L_1 \cap L_{10}$, and in (b) we take e_3 to be the intersection $L_2 \cap L_{10}$.

By re-ordering the remaining triple points, we may say that in both the line L_2 contains e_5, the line L_3 contains e_6 and e_7, and the line L_{10} contains e_8 and e_9 as in Table 17. Note that the element e_4 is contained in line L_2 for (a) but line L_1 for (b).

Note that there are now three remaining triple points e_{10}, e_{11}, e_{12} unaccounted for in the remaining six lines. By Lemma 2.10, there is a unique subarrangement $\mathcal{B}$ of six lines with these three triples. We take the lines L_4, L_5, L_6 to be those with exactly two of these three triples as in Table 17.

On the subarrangement $\mathcal{B}$ there are six doubles. Each of the lines L_7, L_8, L_9 contains three of these, so that all four triples are accounted for by multiple points of $\mathcal{B}$. However, each of the lines L_4, L_5, L_6 contains just one of these doubles, and so the three triples e_1, e_2, e_3 from the initial subarrangement must appear on these lines. Without loss of generality we may assume e_1 is on L_4, e_2 is on L_5, and e_3 is on L_6.

We now consider the three remaining triples on the lines L_4, L_5, L_6.

In (a) there is only one choice up to the symmetries (e_4e_5), (e_6e_7), and (e_8e_9), as well as the symmetries between the lines L_2, L_3, L_{10}. This gives only one arrangement, the arrangement 12.B.3.a.i. in Table 17.

A similar arrangement holds for (b) with no overlapping of elements e_4 and e_5 despite them not being colinear anymore, giving the arrangement 12.B.3.b.i. in Table 17.

Finally for (b) we take the triples e_4 and e_5 to be colinear on the line L_7, and we may assume up to symmetry that the final triple on the line is e_6. Then up to symmetry of the triples (e_8e_9) and $(e_{11}e_{12})$, this forces the positions of the triples e_8 and e_9 on the remaining lines. It must be the case that the line L_6 also contains the triple e_6, and this gives two choices for the positions of the triples e_4 and e_5. Up to symmetry this gives two arrangements, the arrangements 12.B.3.b.ii. and 12.B.3.b.iii., which differ only by the placement of e_4, e_5 on lines L_8, L_9, in Table 17. ☐

L_1	L_2	L_3	L_{10}	L_4	L_5	L_6	L_7	L_8	L_9
e_1	e_1	e_2	e_3	e_1	e_2	e_3	e_{10}	e_{11}	e_{12}
e_2	e_4	e_6	e_8	e_{10}	e_{10}	e_{11}	e_4	e_5	e_4
e_3	e_5	e_7	e_9	e_{11}	e_{12}	e_{12}	e_7	e_6	e_6
				e_9	e_5	e_7	e_8	e_8	e_9

The $(\mathcal{Z})$ arrangement 12.B.3.a.i.

L_1	L_2	L_3	L_{10}	L_4	L_5	L_6	L_7	L_8	L_9
e_1	e_1	e_2	e_3	e_1	e_2	e_3	e_{10}	e_{11}	e_{12}
e_2	e_3	e_6	e_8	e_{10}	e_{10}	e_{11}	e_4	e_5	e_4
e_4	e_5	e_7	e_9	e_{11}	e_{12}	e_{12}	e_7	e_6	e_6
				e_9	e_5	e_7	e_8	e_8	e_9

The non-geometric arrangement 12.B.3.b.i.

e_1	e_1	e_2	e_3	e_1	e_2	e_3	e_{10}	e_{11}	e_{12}
e_2	e_3	e_6	e_8	e_{10}	e_{10}	e_{11}	e_4	e_7	e_7
e_4	e_5	e_7	e_9	e_{11}	e_{12}	e_{12}	e_5	e_9	e_8
				e_8	e_9	e_6	e_6	e_4	e_5

The $(\mathcal{Z})$ arrangement 12.B.3.b.ii.

e_1	e_1	e_2	e_3	e_1	e_2	e_3	e_{10}	e_{11}	e_{12}
e_2	e_3	e_6	e_8	e_{10}	e_{10}	e_{11}	e_4	e_7	e_7
e_4	e_5	e_7	e_9	e_{11}	e_{12}	e_{12}	e_5	e_9	e_8
				e_8	e_9	e_6	e_6	e_5	e_4

The $(\mathcal{Z})$ arrangement 12.B.3.b.iii.

TABLE 17. Arrangements with twelve triples where three of six doubles are triples, not arising from the reduction.

12.B.3.						
Arr.	$y = 0, z, bz$	$x = 0, z, az$	$y = Ax$	$y = Bx + z$	$y = C(x - z)$	$y = D(x - z) + z$
a.i.	L_7, L_5, L_4	L_8, L_9, L_3	L_{10}: b	L_2: -1		L_6: $\frac{-b}{a}$
	with $L_1 : y = \frac{1-b}{a+b-1}(x - az) + z$ and satisfying $a(b-1) - b = 0$ and $b^3 - 3b^2 + 2b - 1 = 0^{\mathcal{Z}}$					
b.i.	L_7, L_5, L_4	L_8, L_9, L_3	L_{10}: b	L_2: $\frac{b-1}{ab-b+1}$	L_1: $\frac{1}{a-1}$	L_6: $-\frac{b}{a}$
	and satisfying $a(b-1) - b = 0$ and $b - 1 = 0$, a contradiction					
b.ii.	L_7, L_5, L_4	L_8, L_9, L_3	L_1: $\frac{1}{a}$	L_{10}: $b - 1$	L_2: $\frac{b}{ab-1}$	L_6: $-\frac{b}{a}$
	and satisfying $a(b-1) - b = 0$ and $b^3 + b^2 - b + 1 = 0^{\mathcal{Z}}$					
b.iii.	L_7, L_5, L_4	L_8, L_9, L_3	L_2: $\frac{b}{a+1}$	L_{10}: $b - 1$	L_1: $\frac{1}{a-1}$	L_6: $\frac{-b}{a}$
	and satisfying $b(a-1) - a = 0$ and $2a^2 - 1 = 0^{\mathcal{Z}}$					

Table 18. Equations for (geometric) arrangements 12.B.3.

Suppose two of the six doubles are triples in $\mathcal{A}$. Some of these cases were handled in Subsection 8.2; the ones that were not have these two doubles not colinear.

Lemma 8.7 (12.B.2.). *For the non-reduction of twelve triples where two of the six doubles are triples, there are four combinatorial configurations, two of which are geometric, and one of these represents a potential Zariski pair. Configuration tables are given in Table 19 and equations are given in Table 20.*

Proof. Let e_1 be the intersection $L_1 \cap L_2$ with e_2, e_3 on L_1 and e_4, e_5 on L_2. Similarly, let e_6 be the intersection $L_3 \cap L_{10}$ with e_7, e_8 on L_3 and e_9, e_{10} on L_{10}. Then the remaining triples e_{11} and e_{12} must lie on the six remaining lines.

First suppose that these triples e_{11} and e_{12} are not colinear. Then we may assume that the first three remaining lines L_4, L_5, L_6 contain the triple e_{11} and that the last three remaining lines L_7, L_8, L_9 contain the triple e_{12}. Each of these remaining lines contains three additional triples.

None of these six remaining lines may contain more than two of the triples e_1, e_2, e_3, e_4, e_5 or more than two of the triples $e_6, e_7, e_8, e_9, e_{10}$, and so each line must contain one triple from the first set and two triples from the second set or vice versa.

Claim. *Up to symmetry this gives only one arrangement, the arrangement 12.B.2.i. in Table 19.*

Proof. We may assume that the line L_9 does not contain the triples e_1, e_2, e_3 by the symmetry $(e_{11}e_{12})$ and by re-ordering the lines L_4 through L_9. Furthermore we may assume the line L_4 contains the triple e_1 and that $e_2 = L_5 \cap L_7$ and $e_3 = L_6 \cap L_8$.

By the assertion just above the claim, the line L_9 must contain one of the triples e_4 and e_5, say e_4. Then since this line also contains the triple e_{12}, the other line that contains the triple e_4 must be L_5 or L_6. By the symmetry (e_2e_3) we may assume it is L_5. This leaves three lines L_6, L_7, L_8, two of which must contain the triple e_5. However we already have $e_3 = L_6 \cap L_8$ and $e_{12} = L_7 \cap L_8$, and so the lines L_6 and L_7 must contain the triple e_5.

This gives three lines L_4, L_8, L_9 that contain single triples and three lines L_5, L_6, L_7 that contain pairs of triples.

A similar argument with the triples $e_6, e_7, e_8, e_9, e_{10}$ gives three lines containing the pair e_7, e_{10}, the pair e_7, e_9, and the pair e_8, e_{10} and three lines containing the single triples e_6, e_8, e_9. Since these pairs overlap in both e_7 and e_{10} and since $e_{12} = L_8 \cap L_9$, the line L_4 must contain e_7 and e_{10}.

We may assume up to the symmetry $(e_7e_{10})(e_8e_9)$ that the line L_8 contains the triple e_7 (and therefore e_9) and the line L_9 contains the triple e_{10} (and therefore e_8). The line L_7 must contain the triple e_6 since $e_{12} = L_8 \cap L_9$, and finally the line L_5 must contain e_9 since $e_3 = L_6 \cap L_8$ and the line L_6 must contain e_8 since $e_4 = L_5 \cap L_9$. ☐

Now supposing that there is a line L_4 containing both e_{11} and e_{12}, then there must be some other of the six lines L_5 not containing either. Since any line can contain at most two elements from each the first two multisets $\{e_1, e_2, e_2, e_3, e_3, e_4, e_4, e_5, e_5\}$ and $\{e_6, e_7, e_7, e_8, e_8, e_9, e_9, e_{10}, e_{10}\}$, and since this last line L_5 must contain four elements, we may assume that e_2, e_4, e_7, e_9 are on L_5.

None of the triples e_3, e_5, e_8, e_{10} can be on the line L_4 because all the remaining lines contain one of the triples e_{11} and e_{12}. At most one of the triples e_2, e_4, e_7, e_9 can be on this line because they are colinear on the line L_5. This gives two possibilities for the remaining elements on the line L_4: either both e_1, e_6 or just one of these, say e_1, with one of the other set, say e_7, up to the appropriate symmetry.

First suppose that both the triples e_1, e_6 are on the line L_4. Again, since the triples e_2, e_4, e_7, e_9 are already colinear on the line L_5, they must appear on the remaining four lines L_6, L_7, L_8, L_9. Up to symmetry of the triples $(e_{11}e_{12})$ we have only two possibilities: when both e_2, e_4 are colinear with the same triple of e_{11}, e_{12} or when they are not. Each of these gives just a single arrangement, giving two arrangements in total: the arrangements 12.B.2.ii. and 12.B.2.iii. in Table 19.

If on the other hand the triples e_1, e_7 are on the line L_4, then the triple e_6 must appear on one of the remaining four lines, and we may assume it is L_6 with the triple e_{11} up to symmetry. These four lines must contain exactly two copies of each of the triples $e_3, e_5, e_8, e_{10}, e_{11}, e_{12}$, and since this accounts for all $\binom{4}{2} = 6$ possibilities, the line L_4 must contain three of these including e_{11}. Thus the remaining two triples on this line must be e_3 and e_5 since the triples e_8 and e_{10} are already colinear with e_6. Up to the symmetry $(e_2e_4)(e_3e_5)$ and (L_1L_2) this gives just one arrangement, the arrangement 12.B.2.iv. in Table 19.

Two of these four arrangements are non-geometric. The equations given in Table 20 demonstrate that one of the two remaining geometric arrangements has an irreducible moduli space while the other is a potential Zariski pair, having three roots and appearing on the final list of Theorem 5.3. ☐

Remark 8.8. A geometric restriction to 12.B.2.iii. forces a quintuple point to occur. However in the non-geometric setting we need not worry about this quintuple.

Finally suppose that only one or none of the six doubles are triples in $\mathcal{A}$; then all cases were handled already in the reduction argument of Subsection 8.2.

This completes the proof of Theorem 8.1 for arrangements of ten lines with twelve triples. ☐

L_1	L_2	L_3	L_{10}	L_4	L_5	L_6	L_7	L_8	L_9
e_1	e_1	e_6	e_6	e_{11}	e_{11}	e_{11}	e_{12}	e_{12}	e_{12}
e_2	e_4	e_7	e_9	e_1	e_2	e_3	e_2	e_3	e_4
e_3	e_5	e_8	e_{10}	e_7	e_4	e_5	e_5	e_7	e_{10}
				e_{10}	e_9	e_8	e_6	e_9	e_8

The non-geometric arrangement 12.B.2.i.

L_1	L_2	L_3	L_{10}	L_4	L_5	L_6	L_7	L_8	L_9
e_1	e_1	e_6	e_6	e_{11}	e_2	e_{11}	e_{11}	e_{12}	e_{12}
e_2	e_4	e_7	e_9	e_{12}	e_4	e_3	e_5	e_3	e_5
e_3	e_5	e_8	e_{10}	e_1	e_7	e_8	e_{10}	e_{10}	e_8
				e_6	e_9	e_4	e_2	e_7	e_9

The arrangement 12.B.2.ii.

e_1	e_1	e_6	e_6	e_{11}	e_2	e_{11}	e_{11}	e_{12}	e_{12}
e_2	e_4	e_7	e_9	e_{12}	e_4	e_3	e_8	e_3	e_5
e_3	e_5	e_8	e_{10}	e_1	e_7	e_5	e_{10}	e_{10}	e_8
				e_6	e_9	e_7	e_2	e_4	e_9

The non-geometric arrangement 12.B.2.iii.

e_1	e_1	e_6	e_6	e_{11}	e_2	e_{11}	e_{11}	e_{12}	e_{12}
e_2	e_4	e_7	e_9	e_{12}	e_4	e_3	e_8	e_3	e_5
e_3	e_5	e_8	e_{10}	e_1	e_7	e_5	e_{10}	e_{10}	e_8
				e_7	e_9	e_6	e_2	e_4	e_9

The ($\mathcal{Z}$) arrangement 12.B.2.iv.

TABLE 19. Arrangements with twelve triples where two of six doubles are triples, not arising from the reduction.

12.B.2.						
Arr.	$y=0,z,bz$	$x=0,z,az$	$y=Ax$	$y=Bx+z$	$y=C(x-z)$	$y=D(x-z)+z$
i.	L_6,L_5,L_4	L_8,L_9,L_7	L_1: $\frac{1}{a}$	L_{10}: $b-1$	L_3: $-b$	L_2: $\frac{1}{1-a}$
	satisfying $2ab-a-b+1=0$ and $2ab-a-b=0$, a contradiction					
ii.	L_4,L_8,L_9	L_1,L_5,L_7	L_2: $\frac{b}{a}$	L_6: $-\frac{1}{a}$		L_3: $\frac{b-1}{a-ab-1}$
	with $L_{10}: y=\frac{1-b}{a-1}(x-z)+bz$ and satisfying $a-(b+1)=0$ and $2b-1=0$					
iii.	L_{10},L_5,L_9	L_4,L_6,L_7	L_3: 1			
	$L_8: y=-x+a$, $L_1: y=\frac{a}{1-a}(x-a)+1$, $L_2: y=\frac{1-a}{a}(x-1)+a$ and satisfying $a^2+a-1=0$, a contradiction (see Remark 8.8)					
iv.	L_4,L_7,L_6	L_5,L_9,L_{10}	L_3: 1	L_1: $\frac{a-1}{a^2-a+1}$	L_8: $\frac{1}{a-1}$	
	with $L_2: y=\frac{a^2-a+1}{a-1}(x-z)+az$ and satisfying $a-b=0$ and $a^3-2a^2+3a-1=0^{\mathcal{Z}}$					

TABLE 20. Equations for arrangements 12.B.2.

9. Arrangements of ten lines with eleven triples

Theorem 9.1. *There are thirty-eight combinatorial configurations of ten lines and eleven triples, given as configuration tables in Tables 21, 24, 26, 28, 30 and 31. All but two of these are geometric, as shown in Tables 22, 23, 25, 27, 29, 30, and 31, but there are no potential Zariski pairs.*

Proof. Let $\mathcal{A}$ be arrangement of ten lines with eleven triples. Here there are exactly three lines that have exactly four triple points on them; call these L_1, L_2, and L_3. We consider the possible subarrangements of just these three lines: either they are central or generic.

9.1. **Central subarrangement.** Suppose that the three lines L_1, L_2, L_3 form a central subarrangement, and let e_1 be the name of this common triple.

Lemma 9.2 (11.A.). *For the central subarrangement of L_1, L_2, L_3 for eleven triples, there are ten combinatorial configurations, all of which are geometric with irreducible moduli spaces. Thus there are no potential Zariski pairs. Configuration tables are given in Table 21 and equations are given in Tables 22 and 23.*

Proof. These three lines L_1, L_2, L_3 contain an additional nine triples. We may assume that the triples e_1, e_2, e_3, e_4 are on L_1, the triples e_1, e_5, e_6, e_7 are on L_2, and the triples e_1, e_8, e_9, e_{10} are on L_3. This leaves one additional triple e_{11} not on these lines; we may assume it is on the lines L_4, L_5, L_6. By Lemma 3.10 we may consider the first three lines as horizontal and the second three lines as vertical with the triples e_1 and e_{11} at infinity.

The three horizontal lines contain the remaining nine triples $e_2, \ldots, e_{10}$, three on each line. However the vertical lines only contain two additional triples each, and so only six of the nine points of the three-by-three grid from Lemma 3.10 can be triples.

Suppose by way of contradiction that four of these form a rectangle on the grid; by symmetry we may suppose that these lie in a close square on the lines $L_1 \cup L_2$ with the triple e_2, e_5 on the line L_4 and the triples e_3, e_6 on the line L_5. Then the line L_3 will have only one of the grid points as a triple. However, this contradicts Fact 2.7 with L_3 containing e_1, e_8, e_9, e_{10} needing eight additional lines amongst L_1, L_2 and $L_6, \ldots, L_{10}$.

Instead we must have, up to symmetry, the triples e_2, e_5 on the line L_4, the triples e_3, e_8 on the line L_5, and the triples e_6, e_9 on the line L_6. Unaccounted for are the triples e_4, e_7, e_{10}.

If just three of the four remaining lines $L_7, \ldots, L_{10}$ contain these last three points, then up to symmetry there is just one arrangement: the arrangement 11.A.i. in Table 21.

Otherwise these three points fall on all four remaining lines. Suppose that two of these three points are not colinaer, say e_4 and e_{10}. Then up to symmetry we have the triple e_4 on the line L_7, the triples e_4, e_7 on the line L_8, the triples e_7, e_{10} on the line L_9, and the triple e_{10} on L_{10}. Supposing that the triples e_3 and e_5 are on the line L_{10}, there is just one arrangement: the arrangement 11.A.ii. in Table 21. Supposing that the triples e_3 and e_6 are on the line L_{10}, there are two arrangements: the arrangements 11.A.iii. and 11.A.iv. in Table 21. Supposing that the triple e_3 is not on the line L_{10}, there are two arrangements: the arrangements 11.A.v. and 11.A.vi. in Table 21.

Suppose that one of these lines, say L_7, contains all three of these points. Then up to symmetry we have the triple e_4 on the line L_8, the triple e_7 on the line L_9, and the triple e_{10} on L_{10}. Supposing that the triples e_3 and e_6 are on the line L_{10}, there are two arrangements: the arrangements 11.A.vii. and 11.A.viii. in Table 21. Supposing that the triples e_3 and e_5 are on the line L_{10}, there is just one arrangement: the arrangement 11.A.ix. in Table 21. Supposing that the triple e_3 is not on the line L_{10}, there is just one arrangement: the arrangement 11.A.x. in Table 21. □

9.2. **Generic subarrangement.** Now suppose the three lines L_1, L_2, L_3 that each contain four triples are not central but form a generic subarrangement giving three doubles. We proceed according to how many of these three doubles are in fact triples in $\mathcal{A}$, the original arrangement.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_1	e_2	e_3	e_6	e_4	e_4	e_7	e_2
e_2	e_5	e_8	e_5	e_8	e_9	e_7	e_{10}	e_{10}	e_6
e_3	e_6	e_9	e_{11}	e_{11}	e_{11}	e_9	e_5	e_3	e_8
e_4	e_7	e_{10}							

The arrangement 11.A.i.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_1	e_2	e_3	e_6	e_4	e_4	e_2	e_3
e_2	e_5	e_8	e_5	e_8	e_9	e_6	e_7	e_7	e_5
e_3	e_6	e_9	e_{11}	e_{11}	e_{11}	e_8	e_9	e_{10}	e_{10}
e_4	e_7	e_{10}							

The arrangement 11.A.ii.

e_1	e_1	e_1	e_2	e_3	e_6	e_4	e_4	e_2	e_3
e_2	e_5	e_8	e_5	e_8	e_9	e_5	e_7	e_7	e_6
e_3	e_6	e_9	e_{11}	e_{11}	e_{11}	e_8	e_9	e_{10}	e_{10}
e_4	e_7	e_{10}							

The arrangement 11.A.iii.

e_1	e_1	e_1	e_2	e_3	e_6	e_4	e_4	e_2	e_3
e_2	e_5	e_8	e_5	e_8	e_9	e_5	e_7	e_7	e_6
e_3	e_6	e_9	e_{11}	e_{11}	e_{11}	e_9	e_8	e_{10}	e_{10}
e_4	e_7	e_{10}							

The arrangement 11.A.iv.

e_1	e_1	e_1	e_2	e_3	e_6	e_4	e_4	e_3	e_2
e_2	e_5	e_8	e_5	e_8	e_9	e_5	e_7	e_7	e_6
e_3	e_6	e_9	e_{11}	e_{11}	e_{11}	e_8	e_9	e_{10}	e_{10}
e_4	e_7	e_{10}							

The arrangement 11.A.v.

e_1	e_1	e_1	e_2	e_3	e_6	e_4	e_4	e_3	e_2
e_2	e_5	e_8	e_5	e_8	e_9	e_5	e_7	e_7	e_6
e_3	e_6	e_9	e_{11}	e_{11}	e_{11}	e_9	e_8	e_{10}	e_{10}
e_4	e_7	e_{10}							

The arrangement 11.A.vi.

e_1	e_1	e_1	e_2	e_3	e_6	e_4	e_4	e_2	e_3
e_2	e_5	e_8	e_5	e_8	e_9	e_7	e_5	e_7	e_6
e_3	e_6	e_9	e_{11}	e_{11}	e_{11}	e_{10}	e_8	e_9	e_{10}
e_4	e_7	e_{10}							

The arrangement 11.A.vii.

e_1	e_1	e_1	e_2	e_3	e_6	e_4	e_4	e_2	e_3
e_2	e_5	e_8	e_5	e_8	e_9	e_7	e_5	e_7	e_6
e_3	e_6	e_9	e_{11}	e_{11}	e_{11}	e_{10}	e_9	e_8	e_{10}
e_4	e_7	e_{10}							

The arrangement 11.A.viii.

e_1	e_1	e_1	e_2	e_3	e_6	e_4	e_4	e_2	e_3
e_2	e_5	e_8	e_5	e_8	e_9	e_7	e_6	e_7	e_5
e_3	e_6	e_9	e_{11}	e_{11}	e_{11}	e_{10}	e_8	e_9	e_{10}
e_4	e_7	e_{10}							

The arrangement 11.A.ix.

e_1	e_1	e_1	e_2	e_3	e_6	e_4	e_4	e_3	e_2
e_2	e_5	e_8	e_5	e_8	e_9	e_7	e_5	e_7	e_6
e_3	e_6	e_9	e_{11}	e_{11}	e_{11}	e_{10}	e_8	e_9	e_{10}
e_4	e_7	e_{10}							

The arrangement 11.A.x.

TABLE 21. Arrangements with eleven triples whose distinguished three lines form a central subarrangement. All of these have irreducible moduli spaces.

Suppose none of the three doubles are triples in $\mathcal{A}$; then there are four distinct triples on each of the three lines, giving a total of twelve triples, a contradiction.

Suppose one of the three doubles is a triple in $\mathcal{A}$; then these three lines contain an additional ten triples, giving a total of eleven triples. However, the line L_4 also passing through this first triple must pass through two more but can only pass through one, a contradiction.

Suppose two of the three doubles are triples in $\mathcal{A}$, say $L_1 \cap L_2 = e_1$ and $L_1 \cap L_3 = e_2$; then these three lines contain an additional eight triples. We may assume that the triples e_3, e_4 are on L_1, the triples e_5, e_6, e_7 are on L_2, and the triples e_8, e_9, e_{10} are on L_3. This leaves one additional triple e_{11} not on these lines. Let L_4 and L_5 be the last lines passing through the triples e_1 and

11.A.						
Arr.	$y=0,z,bz$	$x=0,z,az$	$y=Ax$	$y=Bx+z$	$y=C(x-z)$	$y=D(x-z)+z$
i.	L_3, L_2, L_1	L_5, L_6, L_4	L_{10}: 1		L_7: $\frac{a}{ac-a-c}$	
	with $e_{10} : (c,0)$ and $L_8 : y = \frac{1}{a-c}(x-cz)$, $L_9 : y = \frac{-a}{c}x + az$ and satisfying $a-b=0$ and $2c(a-1)-(a^2+a-1)=0$					
ii.	L_1, L_2, L_3	L_4, L_5, L_6	L_9: $\frac{b}{a+ab-1}$	L_{10}: -1		
	with $L_7 : y = \frac{b-1}{1-a}(x-az)+z$, $L_8 : y = \frac{b(b-1)}{1-a}(x-az)+bz$ and satisfying $a(1+b)-(2-b)=0$					
iii.	L_1, L_2, L_3	L_4, L_5, L_6	L_9: $\frac{b}{a-1}$	L_7: $b-1$	L_{10}: $\frac{1}{a-1}$	
	with $L_8 : y = \frac{b(1-b)}{a-ab-1}(x-az)+bz$ and satisfying $a(1-b)-(2-b)=0$					
iv.	L_1, L_2, L_3	L_4, L_5, L_6	L_9: $\frac{b}{1-a}$	L_7: $\frac{b-1}{a}$	L_{10}: $\frac{1}{a-1}$	
	with $L_8 : y = \frac{b(1-b)}{1-a-b}(x-z)+bz$ and satisfying $a(1+b)-b=0$					
v.	L_1, L_2, L_3	L_4, L_5, L_6	L_{10}: $\frac{1}{a}$	L_7: $b-1$	L_9: $\frac{b}{a-b-1}$	
	with $L_8 : y = \frac{b(1-b)}{a-ab-1}(x-az)+bz$ and satisfying $a(1-b)-b=0$					

TABLE 22. Equations for (geometric) arrangements 11.A.

e_2, respectively. Then they must intersect at e_{11} or else they could not contain three triples, and we may assume up to symmetry of the triples above that the triple e_5 is on the line L_5 with triple e_2 and that the triple e_8 is on the line L_4 with triple e_1. We may then assume that the triples e_5, e_6, e_6, e_7, e_7 lie on the lines $L_6, L_7, L_8, L_9, L_{10}$, respectively.

Lemma 9.3 (11.B.2.)**.** *For the generic subarrangement of L_1, L_2, L_3 for eleven triples where two of the three doubles are triples, there are four combinatorial configurations, all of which are geometric and have irreducible or complex conjugate moduli spaces. Thus there are no potential Zariski pairs. Configuration tables are given in Table 24 and equations are given in Table 25.*

Proof. Suppose first that the triples e_5 and e_8 are colinear on the line L_6. Then without loss of generality we may assume that the triples e_9, e_{10}, e_9, e_{10} are on the lines L_7, L_8, L_9, L_{10}, respectively. Since the triple e_{11} is already colinear with triples e_5 and e_8 it cannot be on the line L_6, and up to symmetry of triples e_3 and e_4 we may assume that the triple e_3 is on the line L_6. Then up to symmetry of triples e_9 and e_{10} we may assume that the triple e_4 is on both lines L_7 and L_{10}. This gives just one arrangement, the arrangements 11.B.2.i.. in Table 24.

11.A. continued						
Arr.	$y=0,z,bz$	$x=0,z,az$	$y=Ax$	$y=Bx+z$	$y=C(x-z)$	$y=D(x-z)+z$
vi.	L_1,L_2,L_3	L_4,L_5,L_6	L_{10}: $\frac{1}{a}$	L_7: $\frac{b-1}{a}$	L_9: $\frac{b}{1-a-b}$	
	with $L_8 : y = \frac{b(1-b)}{1-a-b}(x-z)+bz$ and satisfying $a(b+1)-(2-b)=0$					
vii.	L_1,L_2,L_3	L_4,L_5,L_6	L_9: $\frac{b}{a}$	L_8: $b-1$	L_{10}: $\frac{1}{a-1}$	
	with $L_7 : y = \frac{b(1-b)}{a-b-ab}(x-\frac{1}{1-b}z)$ and satisfying $a(b-1)-b=0$					
viii.	L_1,L_2,L_3	L_4,L_5,L_6	L_9: b	L_8: $\frac{b-1}{a}$	L_{10}: $\frac{1}{a-1}$	
	with $L_7 : y = \frac{b(1-b)}{1-b-ab}(x-\frac{a}{1-b}z)$ and satisfying $a(b-1)-b=0$					
ix.	L_1,L_2,L_3	L_4,L_5,L_6	L_9: $\frac{b}{a}$	L_{10}: -1		
	with $L_7 : y = \frac{b(b-1)}{b-b^2-a}(x-\frac{1-ab}{1-b}z)$, $L_8 : y = \frac{1-b}{a-1}(x-z)+bz$ and satisfying $a(b-1)-b=0$					
x.	L_1,L_2,L_3	L_4,L_5,L_6	L_{10}: $\frac{1}{a}$	L_8: $b-1$	L_9: $\frac{b}{a-1}$	
	with $L_7 : y = \frac{b(1-b)}{ab-ab^2-1}(x-\frac{1}{1-b}z)$ and satisfying $a(b-1)-b=0$					

Table 23. Equations for (geometric) arrangements 11.A.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_2	e_1	e_2	e_5	e_6	e_6	e_7	e_7
e_2	e_5	e_8	e_8	e_5	e_8	e_9	e_{10}	e_9	e_{10}
e_3	e_6	e_9	e_{11}	e_{11}	e_3	e_4	e_{11}	e_3	e_4
e_4	e_7	e_{10}							

The arrangement 11.B.2.i.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_2	e_1	e_2	e_5	e_6	e_6	e_7	e_7
e_2	e_5	e_8	e_8	e_5	e_{10}	e_{10}	e_9	e_9	e_8
e_3	e_6	e_9	e_{11}	e_{11}	e_3	e_{11}	e_4	e_3	e_4
e_4	e_7	e_{10}							

The (∗) arrangement 11.B.2.ii.

e_1	e_1	e_2	e_1	e_2	e_5	e_6	e_6	e_7	e_7
e_2	e_5	e_8	e_8	e_5	e_{10}	e_{10}	e_9	e_9	e_8
e_3	e_6	e_9	e_{11}	e_{11}	e_3	e_4	e_{11}	e_3	e_4
e_4	e_7	e_{10}							

The non-geometric arrangement 11.B.2.iii.

e_1	e_1	e_2	e_1	e_2	e_5	e_6	e_6	e_7	e_7
e_2	e_5	e_8	e_8	e_5	e_{10}	e_{10}	e_9	e_9	e_8
e_3	e_6	e_9	e_{11}	e_{11}	e_3	e_4	e_{11}	e_4	e_3
e_4	e_7	e_{10}							

The ($\mathbb{C}$) arrangement 11.B.2.iv.

Table 24. Arrangements with eleven triples whose distinguished three lines form a generic subarrangement where two of the three doubles are triples.

11.B.2.						
Arr.	$y=0,z,bz$	$x=0,z,az$	$y=Ax$	$y=Bx+z$	$y=C(x-z)$	$y=D(x-z)+z$
i.	L_1,L_2,L_4	L_7,L_9,L_3	L_{10}: 1	L_8: $\frac{a-1}{a}$	L_6: $\frac{b}{a-1}$	
	with $L_{10}: y=\frac{b}{a+b-ab-1}(x-az)$ and satisfying $b(a^2-a+1)-(2a^2-2a+1)=0$					
ii.	L_1,L_2,L_4	L_8,L_9,L_3	L_{10}: 1	L_7: $\frac{a-1}{a(c-a+1)}$	L_6: $\frac{1}{c-1)}$	
	with $e_5=(c,1)$ and $L_7: y=\frac{1}{c-a}(x-az)$ and satisfying $a-b=0$ and $a^2-(c+2)a+c^2+1=0$, irreducible by hand					
iii.	L_1,L_2,L_4	L_6,L_7,L_3	L_{10}: $\frac{b}{a}$	L_5: $-\frac{1}{a}$	L_9: $\frac{b}{a+b-1}$	L_8: $\frac{1-b}{ab-a+1}$
	satisfying $a+1=0$					
iv.	L_1,L_2,L_4	L_6,L_7,L_3	L_{10}: $\frac{b}{a}$	L_5: $-\frac{1}{a}$	L_9: $\frac{b}{a-b}$	L_8: $\frac{1-b}{ab-a+1}$
	satisfying $a^2-a+1=0^{\mathbb{C}}$					

TABLE 25. Equations for arrangements 11.B.2.

Next suppose that the triples e_5 and e_8 are not colinear with each other; and further suppose that they are not colinear with the same triple on the line L_1: then we may assume the triple e_3 is with triple e_5 on L_6 and the triple e_4 with trople e_8 on L_{10}. Up to symmetry of triples (e_6, e_7) and (e_9, e_{10}) we may assume that triples $e_{10}, e_{10}, e_9, e_9, e_8$ lie on lines $L_6, L_7, L_8, L_9, L_{10}$, respectively. Then the line L_7 cannot contain the triple e_3 and the line L_9 cannot contain the triple e_4. Up to symmetry of triples (e_3, e_4) with $(e_5, e_8)(e_6, e_{10})(e_7, e_9)$, this gives just two arrangements, as in Table 24: the arrangements 11.B.2.ii. and 11.B.2.iii., as determined by whether e_{11} is on the line L_7 (which is the same as when it is on the line L_9) or on the line L_8, respectively.

Finally suppose that the triples e_5 and e_8 are not colinear with each other but that they are each colinear with the same triple, say e_3, on the line L_1. As before we may assume that triples $e_{10}, e_{10}, e_9, e_9, e_8$ lie on lines $L_6, L_7, L_8, L_9, L_{10}$, respectively. Then the triple e_4 must be on lines L_7 and L_9, leaving the triple e_{11} on line L_8 as in the arrangement 11.B.2.iv. in Table 24. □

Suppose finally that all three doubles are triples in $\mathcal{A}$, say the triple e_1 is the intersection $L_1 \cap L_2$, the triple e_2 is the intersection $L_1 \cap L_3$, and the triple e_3 is the intersection $L_2 \cap L_3$; then these three lines contain an additional six triples. We may assume that the triples e_4, e_5 are on L_1, the triples e_6, e_7 are on L_2, and the triples e_8, e_9 are on L_3. This leaves two additional triples e_{10}, e_{11} not on these lines. Let L_4, L_5, L_6 be the last lines passing through the triples e_1, e_2, e_3, respectively.

The proof concludes with the consideration of a further subarrangement of the next three lines L_4, L_5, L_6: first forming a central subarrangement and then a generic one. In the latter instance, this leads to another case-by-case consideration of how many of the three doubles formed by this subarrangement are indeed triples in the original arrangement.

9.2.1. *Central subarrangement.* First suppose these three lines L_4, L_5, L_6 form a central subarrangement, and let e_{10} be the name of this triple.

If the last triple e_{11} is not on these three lines, then up to the symmetry of triples $(e_4, e_5)(e_6, e_7)(e_8, e_9)$, we may assume that the triple e_8 must be on the line L_4, the triple e_6 must be on the line L_5, and the triple e_4 must be on the line L_6, with the triples e_5, e_7, e_9 on the line L_{10}. This gives just one arrangement: the arrangement 11.B.3.a.i. in Table 26.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_2	e_1	e_2	e_3	e_5	e_7	e_9	e_5
e_2	e_3	e_3	e_8	e_6	e_4	e_6	e_8	e_4	e_7
e_4	e_6	e_8	e_{10}	e_{10}	e_{10}	e_{11}	e_{11}	e_{11}	e_9
e_5	e_7	e_9							

The arrangement 11.B.3.a.i.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_2	e_1	e_2	e_3	e_5	e_7	e_5	e_4
e_2	e_3	e_3	e_{10}	e_6	e_4	e_6	e_8	e_9	e_7
e_4	e_6	e_8	e_{11}	e_{10}	e_{10}	e_8	e_{11}	e_{11}	e_9
e_5	e_7	e_9							

The arrangement 11.B.3.a.ii.

e_1	e_1	e_2	e_1	e_2	e_3	e_5	e_7	e_4	e_5
e_2	e_3	e_3	e_{10}	e_6	e_4	e_6	e_8	e_9	e_7
e_4	e_6	e_8	e_{11}	e_{10}	e_{10}	e_8	e_{11}	e_{11}	e_9
e_5	e_7	e_9							

The arrangement 11.B.3.a.iii.

TABLE 26. Arrangements with eleven triples whose distinguished three lines form a generic subarrangement where two of the three doubles are triples.

If the last triple e_{11} is on one of these three lines, say L_4, then it cannot be on the lines L_5 or L_6 because of the triple e_{10}, and so we may assume the triple e_6 is on the line L_5 and that the triple e_4 is on the line L_6 by the symmetry of triples $(e_4, e_5)(e_6, e_7)$. Then up to symmetry the last four lines L_7, L_8, L_9, L_{10} contain the triples e_8, e_8, e_9, e_9, respectively.

If the triples e_5 and e_7 are not colinear, this gives just one arrangement up to symmetry: the arrangement 11.B.3.a.ii. in Table 26. If they are colinear, this gives just one arrangement up to symmetry, as well: the arrangement 11.B.3.a.iii. in Table 26.

9.2.2. *Generic subarrangement.* Now suppose the three lines L_4, L_5, L_6 form a generic subarrangement, forming three doubles. At most two of these can indeed be triples, as the first three lines already contain nine triples. At least one of these must be a triple because a subarrangement of the last four lines cannot contain two triples by Fact 2.5.

Lemma 9.4 (11.B.3.b.2.)**.** *For the generic subarrangement of L_1, L_2, L_3 for eleven triples where all three of the doubles are triples, with the generic subarrangement of L_4, L_5, L_6 where two of these doubles are triples, there are seven combinatorial configurations, all of which are geometric and have irreducible moduli spaces. Thus there are no potential Zariski pairs. Configuration tables are given in Table 28 and equations are given in Table 29.*

Proof. We assume that two of the three doubles formed by the generic subarrangement of L_4, L_5, L_6 are indeed triples, say the triple e_{10} is the intersection $L_4 \cap L_5$ and the triple e_{11} is the triple $L_4 \cap L_6$. By the symmetry of the triples $(e_4, e_5)(e_6, e_7)$, we may assume that the triple e_6 is on the line L_5

Arr.	$y = 0, z, bz$	$x = 0, z, az$	$y = Ax$	$y = Bx + z$	$y = C(x - z)$	$y = D(x - z) + z$
11.B.3.a.						
i.	L_1, L_2, L_4	L_3, L_{10}, L_9	L_6: $-\frac{1}{a}$	L_5: $\frac{b}{a(1-b)}$	L_7: $\frac{b}{a-b-ab}$	
	with $L_8 : y = (1 - b)x + bz$ and satisfying $a^2(b^2 - 2b + 1) - ab - (b^2 - b) = 0$ (see Example 3.13)					
ii.	L_1, L_2, L_4	L_3, L_{10}, L_9	L_5: $\frac{b}{1-b}$	L_6: -1		L_8: $\frac{b-1}{a-1}$
	with $L_7 : y = \frac{b}{1-b-ab}(x - az)$ and satisfying $a(b - 1) + b = 0$					
iii.	L_1, L_2, L_4	L_3, L_{10}, L_9	L_7: -$\frac{1}{a}$	L_4: $\frac{b}{a(1-b)}$	L_7: $\frac{b}{a-b-ab}$	L_8: $\frac{b-1}{a-1)}$
	satisfying $b(a - 1) - a^2 = 0$					

TABLE 27. Equations for (geometric) arrangements 11.B.3.a..

and that the triple e_4 is on the line L_6. Then up to the symmetry of triples (e_8, e_9) we may assume that the last four lines L_7, L_8, L_9, L_{10} contain the triples e_8, e_8, e_9, e_9, respectively, as well as the triples $e_4, e_5, e_5, -$, respectively.

If the triple e_6 is on the line L_7, there are two arrangements: the arrangements 11.B.3.b.2.i. and 11.B.3.b.2.ii. in Table 28. If the triple e_6 is on the line L_8, there are two arrangements: the arrangements 11.B.3.b.2.iii. and 11.B.3.b.2.iv. in Table 28. If the triple e_6 is on the line L_9, there are two arrangements: the arrangements 11.B.3.b.2.v. and 11.B.3.b.2.vi. in Table 28. If the triple e_6 is on the line L_{10}, there is just one arrangement: the arrangement 11.B.3.b.2.vii. in Table 28.

The equations for these arrangements are given in Table 29. □

Lastly we consider when just one of the three double points of the generic subarrangement of L_4, L_5, L_6 is a triple.

Lemma 9.5 (11.B.3.b.1.). *For the generic subarrangement of L_1, L_2, L_3 for eleven triples where all three of the doubles are triples, with the generic subarrangement of L_4, L_5, L_6 where just one of these doubles is a triple, there are thirteen combinatorial configurations, twelve of which are geometric and have irreducible moduli spaces. Thus there are no potential Zariski pairs. Configuration tables are given in Tables 30 and 31 and equations are given in Tables 32 and 33.*

Proof. We assume that only one of the three doubles formed by the generic subarrangement of L_4, L_5, L_6 is indeed a triple, say the triple e_{10} is the intersection of the lines $L_5 \cap L_6$. The line L_4 must contain three triples: since it already contains the triple e_1, it cannot contain any of the triples $e_2, \ldots, e_7$; it cannot contain more than one of the triples e_8, e_9 since these are colinear on the line L_3; it cannot contain the triple e_{10} since we have assumed that the lines L_4, L_5, L_6 are generic.

Thus the line L_4 must contain the last triple e_{11} not already accounted for. Then assuming that $L_4 \cap L_5$ and $L_4 \cap L_6$ are not triples and up to symmetries (e_4, e_5), (e_6, e_7), and (e_8, e_9), this forces

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_2	e_1	e_2	e_3	e_4	e_5	e_5	e_7
e_2	e_3	e_3	e_{10}	e_6	e_4	e_6	e_7	e_9	e_9
e_4	e_6	e_8	e_{11}	e_{10}	e_{11}	e_8	e_8	e_{10}	e_{11}
e_5	e_7	e_9							

The arrangement 11.B.3.b.2.i.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_2	e_1	e_2	e_3	e_4	e_5	e_5	e_7
e_2	e_3	e_3	e_{10}	e_6	e_4	e_6	e_7	e_9	e_9
e_4	e_6	e_8	e_{11}	e_{10}	e_{11}	e_8	e_8	e_{11}	e_{10}
e_5	e_7	e_9							

The arrangement 11.B.3.b.2.ii.

e_1	e_1	e_2	e_1	e_2	e_3	e_4	e_5	e_5	e_7
e_2	e_3	e_3	e_{10}	e_6	e_4	e_7	e_6	e_9	e_9
e_4	e_6	e_8	e_{11}	e_{10}	e_{11}	e_8	e_8	e_{10}	e_{11}
e_5	e_7	e_9							

The ($\mathbb{C}$) arrangement 11.B.3.b.2.iii.

e_1	e_1	e_2	e_1	e_2	e_3	e_4	e_5	e_5	e_7
e_2	e_3	e_3	e_{10}	e_6	e_4	e_7	e_6	e_9	e_9
e_4	e_6	e_8	e_{11}	e_{10}	e_{11}	e_8	e_8	e_{11}	e_{10}
e_5	e_7	e_9							

The ($\mathbb{C}$) arrangement 11.B.3.b.2.iv.

e_1	e_1	e_2	e_1	e_2	e_3	e_4	e_5	e_5	e_7
e_2	e_3	e_3	e_{10}	e_6	e_4	e_7	e_8	e_6	e_9
e_4	e_6	e_8	e_{11}	e_{10}	e_{11}	e_8	e_{10}	e_9	e_{11}
e_5	e_7	e_9							

The ($\mathcal{Z}$) arrangement 11.B.3.b.2.v.

e_1	e_1	e_2	e_1	e_2	e_3	e_4	e_5	e_5	e_7
e_2	e_3	e_3	e_{10}	e_6	e_4	e_7	e_8	e_6	e_9
e_4	e_6	e_8	e_{11}	e_{10}	e_{11}	e_8	e_{11}	e_9	e_{10}
e_5	e_7	e_9							

The arrangement 11.B.3.b.2.vi.

e_1	e_1	e_2	e_1	e_2	e_3	e_4	e_5	e_5	e_6
e_2	e_3	e_3	e_{10}	e_6	e_4	e_7	e_8	e_7	e_9
e_4	e_6	e_8	e_{11}	e_{10}	e_{11}	e_8	e_{10}	e_9	e_{11}
e_5	e_7	e_9							

The arrangement 11.B.3.b.2.vii.

TABLE 28. Arrangements with eleven triples whose distinguished three lines form a generic subarrangement where all three of the doubles are triples and whose next three lines form a generic subarrangement where two of the three doubles are triples.

the triple e_8 to be on the line L_4, the triple e_6 to be on the line L_5, and the triple e_4 to be on the line L_6.

Because the triple e_8 is already colinear with each of the triples e_9 and e_{11}, and because there are only four remaining lines, the triples e_9 and e_{11} must be colinear, say on the line L_7.

Suppose that the triple e_5 (which up to the symmetry of triples $(e_2, e_3)(e_4, e_6)$ is equivalent to the triple e_7) is also on this line L_7. Then the line that also contains the triple e_5, say L_8, must contain the triple e_8, as well. If the last triple on this line L_8 is the triple e_6, then the last two lines L_9 and L_{10} each have e_7 as well as one of e_9, e_{11} and one of e_4, e_{10}. This gives two arrangements up to symmetry: the arrangements 11.B.3.b.1.i. and 11.B.3.b.1.ii. in Table 30. If the last triple on this line L_8 is the triple e_7, then since the triple e_{10} cannot be again contained on the same line as either triples e_4 or e_6, it must be with the triple e_7. This gives two arrangements up to symmetry: the arrangements 11.B.3.b.1.iii. and 11.B.3.b.1.iv. in Table 30.

Finally suppose that the line L_7 does not contain the triples e_5 or e_7. Then up to symmetry the final three lines L_8, L_9, L_{10} must contain the triples e_5 and e_7 together, e_5, and e_7, respectively. If the last triple on the line L_7 is the triple e_4 (which up to the symmetry above is equivalent to e_6),

11.B.3.b.2.						
Arr.	$y=0,z,bz$	$x=0,z,az$	$y=Ax$	$y=Bx+z$	$y=C(x-z)$	$y=D(x-z)+z$
i.	L_1, L_2, L_4	L_3, L_7, L_8	L_5: 1	L_6: -1		
	with $L_9 : y = \frac{b}{b-a}(x-az)$, $L_{10} : y = \frac{1-b}{a+b-1}(x-az)+z$ and satisfying $a(2b^2-2b-1)-(b-b^2)=0$					
ii.	L_1, L_2, L_4	L_3, L_7, L_8	L_5: 1	L_6: -1		
	with $L_9 : y = \frac{b}{1-a-b}(x-az)$, $L_{10} : y = \frac{1-b}{a-b}(x-az)+z$ and satisfying $2a-1=0$					
iii.	L_1, L_2, L_4	L_3, L_7, L_8	L_5: $\frac{1}{a}$	L_6: -1		L_{10}: $\frac{1-b}{b}$
	with $L_9 : y = \frac{b}{ab-a}(x-az)$ and satisfying $3b^2-3b+1=0^{\mathbb{C}}$					
iv.	L_1, L_2, L_4	L_3, L_7, L_8	L_5: $\frac{1}{a}$	L_6: -1		L_{10}: $\frac{b-1}{ab-1}$
	with $L_9 : y = \frac{b}{1-a-b}(x-az)$ and satisfying $a^2-a+1=0^{\mathbb{C}}$					
v.	L_1, L_2, L_4	L_3, L_7, L_8	L_5: $\frac{b}{a}$	L_6: -1		L_{10}: $\frac{1-b}{b}$
	with $L_9 : y = \frac{b}{a(1-b)}(x-az)$ and satisfying $b^2-3b+1=0^{\mathcal{Z}}$					
vi.	L_1, L_2, L_4	L_7, L_3, L_8	L_6: 1	L_{10}: $\frac{a-1}{c}$	L_5: $\frac{a}{c-1}$	
	with $e_{10} : (c,b)$ and $L_9 : y = \frac{a(1-a)}{c(c-1)}(x-az)$ and satisfying $a-b=0$ and $c^2+(a-2)c-(a-1)(a^2-a+1)=0^*$					
vii.	L_1, L_2, L_4	L_3, L_7, L_8	L_5: $\frac{b}{a}$	L_6: -1		L_9: $\frac{1}{1-a}$
	with $L_{10} : y = \frac{a+b-ab}{(a-1)(b-1)}x + \frac{a}{a-1}z$ and satisfying $b^2-b(a^2-a+1)+a^2=0$, irreducible by hand					

TABLE 29. Equations for (geometric) arrangements 11.B.3.b.2..

then the triple e_6 must be on L_9 and the triple e_{10} must be on L_{10}. This gives six arrangements, following the six permutations of the remaining triples e_8, e_9, e_{11} on the three lines L_8, L_9, L_{10}: the arrangements 11.B.3.b.1.v. through 11.B.3.b.1.x. in Tables 30 and 31.

If the last triple on the line L_7 is the triple e_{10}, then the triple e_6 must be on the line L_9 with e_5 and the triple e_4 must be on the line L_{10} with e_7. Notice, however, that there still is symmetry exchanging these two last lines. Thus this gives only three arrangements, following the choice of one of the remaining triples e_8, e_9, e_{11} to be on the line L_8: the arrangements 11.B.3.b.1.xi. through 11.B.3.b.1.xiii. in Table 31.

The equations for these geometric arrangements are given in Tables 32 and 33. □

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_2	e_1	e_2	e_3	e_5	e_5	e_4	e_7
e_2	e_3	e_3	e_8	e_6	e_4	e_9	e_6	e_7	e_{10}
e_4	e_6	e_8	e_{11}	e_{10}	e_{10}	e_{11}	e_8	e_9	e_{11}
e_5	e_7	e_9							

The arrangement 11.B.3.b.1.i.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_2	e_1	e_2	e_3	e_5	e_5	e_7	e_4
e_2	e_3	e_3	e_8	e_6	e_4	e_9	e_6	e_9	e_7
e_4	e_6	e_8	e_{11}	e_{10}	e_{10}	e_{11}	e_8	e_{10}	e_{11}
e_5	e_7	e_9							

The arrangement 11.B.3.b.1.ii.

e_1	e_1	e_2	e_1	e_2	e_3	e_5	e_5	e_7	e_4
e_2	e_3	e_3	e_8	e_6	e_4	e_9	e_7	e_9	e_6
e_4	e_6	e_8	e_{11}	e_{10}	e_{10}	e_{11}	e_8	e_{10}	e_{11}
e_5	e_7	e_9							

The $(*)$ arrangement 11.B.3.b.1.iii.

e_1	e_1	e_2	e_1	e_2	e_3	e_5	e_5	e_7	e_4
e_2	e_3	e_3	e_8	e_6	e_4	e_9	e_7	e_{10}	e_6
e_4	e_6	e_8	e_{11}	e_{10}	e_{10}	e_{11}	e_8	e_{11}	e_9
e_5	e_7	e_9							

The arrangement 11.B.3.b.1.iv.

e_1	e_1	e_2	e_1	e_2	e_3	e_4	e_5	e_5	e_7
e_2	e_3	e_3	e_8	e_6	e_4	e_9	e_7	e_6	e_{10}
e_4	e_6	e_8	e_{11}	e_{10}	e_{10}	e_{11}	e_8	e_9	e_{11}
e_5	e_7	e_9							

The arrangement 11.B.3.b.1.v.

e_1	e_1	e_2	e_1	e_2	e_3	e_4	e_5	e_5	e_7
e_2	e_3	e_3	e_8	e_6	e_4	e_9	e_7	e_6	e_{10}
e_4	e_6	e_8	e_{11}	e_{10}	e_{10}	e_{11}	e_8	e_{11}	e_9
e_5	e_7	e_9							

The arrangement 11.B.3.b.1.vi.

TABLE 30. Arrangements with eleven triples whose distinguished three lines form a generic subarrangement where all three of the doubles are triples and whose next three lines form a generic subarrangement where one of the three doubles is a triple.

This completes the proof for ten lines with eleven triples. □

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L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_2	e_1	e_2	e_3	e_4	e_5	e_5	e_7
e_2	e_3	e_3	e_8	e_6	e_4	e_9	e_7	e_6	e_{10}
e_4	e_6	e_8	e_{11}	e_{10}	e_{10}	e_{11}	e_9	e_8	e_{11}
e_5	e_7	e_9							

The non-geometric arrangement 11.B.3.b.1.vii.

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
e_1	e_1	e_2	e_1	e_2	e_3	e_4	e_5	e_5	e_7
e_2	e_3	e_3	e_8	e_6	e_4	e_9	e_7	e_6	e_{10}
e_4	e_6	e_8	e_{11}	e_{10}	e_{10}	e_{11}	e_9	e_{11}	e_8
e_5	e_7	e_9							

The arrangement 11.B.3.b.1.viii.

e_1	e_1	e_2	e_1	e_2	e_3	e_4	e_5	e_5	e_7
e_2	e_3	e_3	e_8	e_6	e_4	e_9	e_7	e_6	e_{10}
e_4	e_6	e_8	e_{11}	e_{10}	e_{10}	e_{11}	e_{11}	e_8	e_9
e_5	e_7	e_9							

The arrangement 11.B.3.b.1.ix.

e_1	e_1	e_2	e_1	e_2	e_3	e_4	e_5	e_5	e_7
e_2	e_3	e_3	e_8	e_6	e_4	e_9	e_7	e_6	e_{10}
e_4	e_6	e_8	e_{11}	e_{10}	e_{10}	e_{11}	e_{11}	e_9	e_8
e_5	e_7	e_9							

The $(*)$ arrangement 11.B.3.b.1.x.

e_1	e_1	e_2	e_1	e_2	e_3	e_9	e_5	e_5	e_7
e_2	e_3	e_3	e_8	e_6	e_4	e_{10}	e_7	e_6	e_4
e_4	e_6	e_8	e_{11}	e_{10}	e_{10}	e_{11}	e_8	e_9	e_{11}
e_5	e_7	e_9							

The arrangement 11.B.3.b.1.xi.

e_1	e_1	e_2	e_1	e_2	e_3	e_9	e_5	e_5	e_7
e_2	e_3	e_3	e_8	e_6	e_4	e_{10}	e_7	e_6	e_4
e_4	e_6	e_8	e_{11}	e_{10}	e_{10}	e_{11}	e_9	e_8	e_{11}
e_5	e_7	e_9							

The arrangement 11.B.3.b.1.xii.

e_1	e_1	e_2	e_1	e_2	e_3	e_9	e_5	e_5	e_7
e_2	e_3	e_3	e_8	e_6	e_4	e_{10}	e_7	e_6	e_4
e_4	e_6	e_8	e_{11}	e_{10}	e_{10}	e_{11}	e_{11}	e_8	e_9
e_5	e_7	e_9							

The arrangement 11.B.3.b.1.xiii.

TABLE 31. Arrangements with eleven triples whose distinguished three lines form a generic subarrangement where all three of the doubles are triples and whose next three lines form a generic subarrangement where one of the three doubles is a triple (continued).

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11.B.3.b.1.						
Arr.	$y=0,z,bz$	$x=0,z,az$	$y=Ax$	$y=Bx+z$	$y=C(x-z)$	$y=D(x-z)+z$
i.	L_2,L_1,L_4	L_9,L_3,L_7	L_{10}: $\frac{b}{a}$	L_6: -1		L_5: $\frac{b-1}{b(1-a)}$
	with $L_8: y=\frac{b-1}{1-a}(x-z)+bz$ and satisfying $b-(1+a-a^2)=0$					
ii.	L_2,L_1,L_4	L_9,L_3,L_7	L_{10}: $\frac{b}{a}$		L_6: $\frac{b}{a-b}$	L_5: $\frac{a}{a-b)}$
	with $L_8: y=\frac{b-1}{1-a}(x-az)+z$ and satisfying $a^2b-a-b^2+b=0^*$					
iii.	L_1,L_3,L_5	L_8,L_2,L_9	L_7: $\frac{1}{a}$	L_4: -1		L_6: $\frac{b-1}{a-1)}$
	with $L_{10}: y=\frac{b(b-1)}{a-1}(x-z)+bz$ and satisfying $a-a^2b+b^2-1=0^*$					
iv.	L_2,L_1,L_4	L_{10},L_3,L_7	L_5: 1	L_6: -1		
	with $L_8: y=\frac{b-1}{1-a}(x-az)+z$ and $L_9: y=\frac{b(b-1)}{1-a}(x-az)+bz$ and satisfying $a(2b^2-1)-(b^2+b-1)=0$					
v.	L_2,L_1,L_4	L_9,L_3,L_7	L_5: 1	L_8: $b-1$	L_6: $\frac{1}{a-1}$	
	with $L_{10}: y=\frac{b(1-b)}{a-ab-1}(x-\frac{1}{1-b}z)$ and satisfying $a(2b-1)-(b^2+b-1)=0$					
vi.	L_4,L_1,L_2	L_7,L_3,L_{10}	L_9: $\frac{b}{a+b-1}$	L_6: $b-1$	L_8: $\frac{b}{a-1}$	L_5: $\frac{a(b-1)}{a-1}$
	satisfying $b(a)+(a^2-3a+1)=0$					
vii.	L_2,L_1,L_4	L_6,L_5,L_{10}	L_3: 1	L_7: $\frac{b-1}{a}$	L_9: $\frac{b}{b-1}$	
	with $L_8: y=\frac{1}{b-a}(x-az)$ and satisfying $b-1=0$, a contradiction					
viii.	L_2,L_1,L_4	L_6,L_5,L_{10}	L_3: 1		L_9: $\frac{c}{a(c-1)}$	
	with $e_9: (c,c)$, and $L_7: y=\frac{c-1}{c}x+1$, $L_8: y=\frac{c}{c-a}(x-az)$ and satisfying $a-b=0$ and $a^2c^2-2a^2c+a^2-ac^2+2c^2-c=0^*$					

TABLE 32. Equations for arrangements 11.B.3.b.1..

11.B.3.b.1. continued						
Arr.	$y=0,z,bz$	$x=0,z,az$	$y=Ax$	$y=Bx+z$	$y=C(x-z)$	$y=D(x-z)+z$
ix.	L_1,L_4,L_2	L_3,L_7,L_{10}	L_5: $\frac{b(1-a)}{a}$	L_9: $\frac{b-1}{a-b}$	L_6: $-b$	L_8: $\frac{b-1}{a-1}$
	satisfying $b(a-1)-(a^2)=0$					
x.	L_2,L_1,L_4	L_6,L_5,L_{10}	L_3: 1	L_7: $\frac{c-1}{c}$	L_9: $\frac{c}{c-1}$	
	with $e_9:(c,c)$, and $L_8: y=\frac{c}{2c-1-ac}(x-az)$ and satisfying $a-b=0$ and $c^2+a^2c+2ac^2-a^2c^2-4ac+a=0^*$					
xi.	L_2,L_1,L_4	L_9,L_3,L_7	L_5: 1	L_8: $b-1$	L_6: $\frac{a}{a-1}$	
	with $L_{10}: y=\frac{b(1-b)}{a-ab-1}(x-az)+bz$ and satisfying $b(1-2a)+(a^2-a)=0$					
xii.	L_2,L_1,L_4	L_8,L_3,L_7	L_{10}: $\frac{b}{a}$	L_9: $b-1$	L_6: $\frac{b}{a-b}$	L_5: $\frac{b-1}{b}$
	satisfying $b(1-2a)+(a^2-a)=0$					
xiii.	L_2,L_1,L_4	L_{10},L_3,L_7	L_8: $\frac{b}{a}$	L_6: -1		L_5: $\frac{a}{1-a}$
	with $L_9: y=\frac{b(1-b)}{a-b}(x-z)+bz$ and satisfying $b(1-2a)+(a^2+a-1)=0$					

TABLE 33. Equations for arrangements 11.B.3.b.1..

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