Testing for random effects in the error component regression model with incomplete panels

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Abstract

This paper develops a new method to test for the existence of random effects in the error component model with incomplete panels. Based on the difference of variance estimators of the idiosyncratic errors at different levels, two statistics are constructed to test for the existence of individual and time effects, respectively. Some variants of the two statistics and joint tests for both the two effects are also discussed. Their asymptotic properties are obtained under some mild conditions. Monte Carlo simulation results show that our test statistics have better finite sample properties than the competitors in the existing literature at many aspects. A real data is analyzed for further illustration.

Keywords: Error component models; Incomplete panels; Moment estimation; Random effects; Test for random effects

JEL Subject Classification: C12, C23

1 Introduction

In the econometric analysis of panel data, one usually focus on statistical modeling analysis for complete data. In practice, however, it is common to encounter the missing observations in the collected data set. For example, in labor economics, some data on individual income may be dropped out after some time periods due to the retirement. Throughout this paper, the focused panel data sets can be allowed to be incomplete/unbalanced, that is, some data may be not observed in some time periods for some individuals. Actually, most panels encountered in practice are of the incomplete kinds (see, e.g, Baltagi (2008) or Baltagi and Song (2006)). And then statistical modeling analysis for incomplete panels has received more and more attention recently (see, e.g, Baltagi (2008)). Note that, misspecification of the existence of random effects in the error component will lead to seriously biased standard errors and even inefficient statistical inference, which is completely similar to that of complete panels. So, it is important and necessary to test for the existence of random effects in the error component regression models with incomplete panels.

Different from most of the existing literature, in this paper we focus mainly on testing for random effects in the error component model with incomplete panels as follows,

$$y_{it} = \alpha + X_{it}'\beta + u_{it}, \quad u_{it} = \mu_i + \eta_t + \nu_{it}, \qquad i = 1, \ldots, n, t = 1, \ldots, T_i, \tag{1}$$

where α is a scalar, X_{it} is the it-th observations on K observable regressors, β is the vector of coefficients of the regressors, and u_{it} is the error component including the two random effects μ_i and η_t and the idiosyncratic errors ν_{it}. The random effects μ_i and η_t are used to capture the heterogeneity of individual and time periods, respectively. Further, the individual effect μ_i is assumed to be independent and identically distributed with mean zero and finite variance σ_μ^2, and the idiosyncratic error ν_{it} is assumed to be independent and identically distributed with mean zero and finite variance σ_ν^2.

Till now, there are many relevant literature on testing for the existence of random effects in the error component regression model with incomplete panel data. In the following we give a simple reviews for some main literatures. Baltagi and Li (1990) extended the Breusch and Pagan (1980) LM test to the error component model with incomplete panels. Since these variance components can not be negative, the two-sided alternative hypotheses seem to be unreasonable. Moulton (1987) extended the uniformly mostly powerful test (UMPT) of Honda (1985) to the incomplete one-way error component model and illustrated this method by a hedonic housing price incomplete panel data model. However, as Moulton and Randolph (1989) argued, using the asymptotic critical values for the test of Moulton (1987) can lead to incorrect inference, especially when there is high correlations among the regressors or the number of regressors is very large. And then Moulton and Randolph (1989) suggested a standardized lagrange multiplier (SLM) test which had better critical value approximations. Some test statistics were similarly suggested for time effects, see, e.g, Baltagi et al. (1998), Honda (1985) and Moulton and Randolph (1989). However, these tests are based on the one-way error component models, i.e. the null hypotheses correspond to the case without any effects, and the sizes may be distorted due to the presence of the time (individual) effect when the individual (time) effect is tested (see, e.g, Wu and Li (2014)). Besides, although these LM-based methods have a simple form, all of them require the assumption of normality of idiosyncratic errors and independence among the regressors, the random effects and the idiosyncratic errors, which can not be guaranteed in practice. Wu and Li (2014) proposed several moment-based test statistics for the existence of random effects, which are shown to have many desired properties such as the simplicity, the robustness to the distribution assumptions and the possible dependency among the regressors, random effects and the idiosyncratic errors. However, their methods are only available for the case with complete panels.

The main purpose of this paper is to extend the moment-based test methods of Wu and Li (2014) to the case with incomplete panels. In Section 2, we outline the different forms of the original model and obtain an robust estimator of the coefficients β which is asymptotically normally distributed under some regular conditions. In Section 3, we construct the test for the individual effect. We first derive two estimators of the variance of the idiosyncratic error. One is the robust estimator which is consistent no matter of the existence of the random effects, and another one is consistent when the individual effect does not exist while inconsistent under the presence of the individual effect. Based on the difference of the two estimators, we construct the test statistic for individual effects, which can be shown to asymptotically normally distributed. And we can show our test statistic is more powerful than the traditional ANOVA F test when the regressors are correlated with the individual effect. In Section 4, we use the same method to construct statistics for testing time effects and study their asymptotic properties. In Section 5, we construct several joint test statistics for both the two random effects and study their asymptotic properties. Monte Carlo simulations are given in Section 6. Section 7 applies our methods to a real data example. Section 8 gives some conclusions and discussions. All proofs are provided in the Appendix.

For the sake of statements, we first introduce some notations as follows. We denote by A' the transpose of matrix A, by A^{-1} the inverse of matrix A, and by $\|A\| = [tr(A'A)]^{\frac{1}{2}}$ the norm of matrix A. $A \otimes B$ is the Kronecker product of matrices A and B, and $diag_L(A_l)$ is a block diagonal matrix with the diagonal elements $A_1, A_2, \ldots, A_L$. The symbol " $\Longrightarrow$ " stands for weak convergence. $a_n = o_p(b_n)$ means that a_n/b_n converges to zero in probability, and $a_n = O_p(b_n)$ means that a_n/b_n is bounded in probability. $\mathbb{E}X$ or $\mathbb{E}(X)$ stands for the mathematical expectation of the random variable X.

2 Model and notations

We assume that there are L disjoint subsets $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_L$ of $\{1, 2, \ldots, n\}$ such that the observed time length is identical for each $i \in \mathcal{N}_l$ with $l = 1, 2, \ldots, L$. The subjects with individuals in $\mathcal{N}_l$ form a balanced panel data set with the same time length T_l. And we denote the number of individuals in group $\mathcal{N}_l$ by n_l. For each group $\mathcal{N}_l$, model (1) can be rewritten into the vector form as

$$y_{li} = \alpha\iota_{T_l} + X_{li}\beta + \mu_{li}\iota_{T_l} + \eta_l + \nu_{li}, \quad i \in \mathcal{N}_l, \tag{2}$$

where ι_k is a vector of ones with dimension k, $y_{li} = (y_{li1}, y_{li2}, \ldots, y_{liT_l})'$, X_{li} and ν_{li} are defined similarly, and $\eta_l = (\eta_{l1}, \eta_{l2}, \ldots, \eta_{lT_l})'$.

For each group $\mathcal{N}_l$, we first eliminate the time effect by centering each term in model (2),

$$\tilde{y}_{li} = \tilde{X}_{li}\beta + \tilde{\mu}_{li}\iota_{T_l} + \tilde{\nu}_{li}, \quad i \in \mathcal{N}_l, \tag{3}$$

where $\tilde{y}_{li} = y_{li} - \frac{1}{n_l}\sum_{i=1}^{n_l} y_{li}$, $\tilde{X}_{li}$, $\tilde{\mu}_{li}$ and $\tilde{\nu}_{li}$ are defined similarly. Using the similar methods of Wu and Li (2014), we find a matrix Q_l such that $(\frac{\iota_{T_l}}{\sqrt{T_l}}, Q_l)$ is a $T_l \times T_l$

orthogonal matrix. Premultiplying the model (3) with the matrix Q_l' yields,

$$Q_l'\tilde{y}_{li} = Q_l'\tilde{X}_{li}\beta + Q_l'\tilde{\nu}_{li}. \tag{4}$$

The model (1) can be rewritten into the matrix form by stacking the observation of the L groups as

$$y = \alpha\iota_N + X\beta + Z_\mu\mu + Z_\eta\eta + \nu,$$

where $y = (y_1', y_2', \ldots, y_L')'$ with $y_l = (y_{l1}', y_{l2}', \ldots, y_{ln_l}')'$, X, μ, η and ν are defined similarly. Moreover, denote $N = \sum_{l=1}^{L} n_l T_l$, $Z_\mu = diag_L\{Z_{l\mu}\}$ with $Z_{l\mu} = I_{n_l} \otimes \iota_{T_l}$, and $Z_\eta = diag_L\{Z_{l\eta}\}$ with $Z_{l\eta} = \iota_{n_l} \otimes I_{T_l}$, where the symbol "$\otimes$" is the Kronecker product operator and I_k is an identity matrix of dimension k. From (3) and (4), we can show that

$$Qy = QX\beta + Q\nu, \tag{5}$$

where $Q = Q_{Z_\mu}Q_{Z_\eta}$, $Q_{Z_\mu} = diag_L\{Q_l'\}$ and $Q_{Z_\eta} = diag_L\{Q_{Z_{l\eta}}\}$ with $Q_{Z_{l\eta}} = I - Z_{l\eta}(Z_{l\eta}'Z_{l\eta})^{-1}Z_{l\eta}'$. Based on model (5), we can obtain a robust ordinary least squares estimator of β as follows,

$$\hat{\beta} = (\sum_{l=1}^{L}\sum_{i=1}^{n_l} \tilde{X}_{li}'P_l\tilde{X}_{li})^{-1}\sum_{l=1}^{L}\sum_{i=1}^{n_l} \tilde{X}_{li}'P_l\tilde{y}_{li},$$

where $P_l = I_{T_l} - \frac{1}{T_l}J_{T_l}$ with J_k denoting a $k \times k$ matrix of ones. Under some mild assumptions, we can show that

$$\sqrt{n}(\hat{\beta} - \beta) \Longrightarrow N(0, \Sigma_1^{-1}\Sigma_2\Sigma_1^{-1}),$$

where $\Sigma_1 = \sum_{l=1}^{L} m_l[\mathbb{E}(X_{li}'P_lX_{li}) - \mathbb{E}X_{li}'P_l\mathbb{E}X_{li}]$ and $\Sigma_2 = \sum_{l=1}^{L} m_l\mathbb{E}[(X_{li} - \mathbb{E}X_{li})'P_l\nu_{li}\nu_{li}'$ $(X_{li} - \mathbb{E}X_{li})]$. The asymptotic results in this paper are based on the setting that the individual number n goes to infinity and the time lengths T_i are fixed. Besides we assume that $\lim_{n\to\infty} \frac{n_l}{n} = m_l > 0$, which is a commonly-used setting in the literature, see, e.g, Shao et al. (2011) and Chowdhury (1991).

3 Testing for individual effect

In this section, we consider to construct a statistic to test for individual effects in model (1). The hypotheses of the individual effects test can be formalized as

$$H_0^\mu: \ \sigma_\mu^2 = 0 \qquad \text{vs} \qquad H_1^\mu: \ \sigma_\mu^2 > 0, \tag{6}$$

where σ_μ^2 is the variance of the individual effects μ_i. Denote the j-th column vector of the matrix Q_l by $q_{lj}, l = 1, \ldots, L, j = 1, \ldots, T_l - 1$, and then we have

$$\sum_{l=1}^{L} \sum_{i=1}^{n_l} \mathbb{E}\|Q_l' \tilde{\nu}_{li}\|^2 = \sum_{l=1}^{L} \sum_{i=1}^{n_l} \sum_{j=1}^{T_l-1} \mathbb{E}(q_{lj}' \tilde{\nu}_{li})^2 = c_1 \sigma_\nu^2,$$

where $c_1 = \sum_{l=1}^{L} c_{1l}$ with $c_{1l} = (n_l - 1)(T_l - 1)$. And then we obtain a feasible estimator of the variance of the idiosyncratic error,

$$\hat{\sigma}_{0\nu}^2 = \frac{1}{c_1} \sum_{l=1}^{L} \sum_{i=1}^{n_l} \|Q_l' \hat{\tilde{\nu}}_{li}\|^2 = \frac{1}{c_1} \sum_{l=1}^{L} \sum_{i=1}^{n_l} \|Q_l' (\tilde{y}_{li} - \tilde{X}_{li} \hat{\beta})\|^2. \tag{7}$$

Under some regular conditions, $\hat{\sigma}_{0\nu}^2$ can be shown to be a consistent estimator of σ_ν^2 no matter of the existence of the two random effects μ_i and η_t.

Similarly, it holds that

$$\sum_{l=1}^{L} \sum_{i=1}^{n_l} \sum_{j=1}^{T_l-1} \mathbb{E}(q_{lj}' \tilde{\nu}_{li})^4 = c_2 \gamma_\nu^4 + c_2 c_3 (\sigma_\nu^2)^2,$$

where $\gamma_\nu^4 = \mathbb{E}\nu_{it}^4$ is the fourth-order moment of ν_{it},

$$c_2 = \sum_{l=1}^{L} \sum_{j=1}^{T_l-1} \sum_{t=1}^{T_l} q_{ljt}^4 (n_l - 1)(n_l^2 - 3n_l + 3)/n_l^2,$$

and

$$c_3 = c_2^{-1} \sum_{l=1}^{L} 3(n_l - 1)^2 (T_l - 1)/n_l - 3.$$

Therefore we can obtain an estimator of γ_ν^4 by solving the above equation and using the empirical version of the solution as follows,

$$\hat{\gamma}_\nu^4 = c_2^{-1} \sum_{l=1}^{L} \sum_{i=1}^{n_l} \sum_{j=1}^{T_l-1} (q_{lj}' \hat{\tilde{\nu}}_{li})^4 - c_3 (\hat{\sigma}_{0\nu}^2)^2 = c_2^{-1} \sum_{l=1}^{L} \sum_{i=1}^{n_l} \sum_{j=1}^{T_l-1} [q_{lj}' (\tilde{y}_{li} - \tilde{X}_{li} \hat{\beta})]^4 - c_3 (\hat{\sigma}_{0\nu}^2)^2.$$

And we can show that $\hat{\gamma}_\nu^4$ is a consistent estimator of γ_ν^4 under some mild conditions no matter of the existence of the two random effects μ_i and η_t.

When we do have the information on the absence of individual effects, that is, under the null hypothesis without individual effect, model (2) turns out

$$y_{li} = \alpha \iota_{T_l} + X_{li} \beta + \eta_l + \nu_{li}, \quad i \in \mathcal{N}_l.$$

And then we only need to eliminate the time effect by the centering transformation,

$$\tilde{y}_{li} = \tilde{X}_{li} \beta + \tilde{\nu}_{li}.$$

It holds that

$$\sum_{l=1}^{L}\sum_{i=1}^{n_l}\mathbb{E}\|\tilde{\nu}_{li}\|^2 = c_4\sigma_\nu^2,$$

where $c_4 = \sum_{l=1}^{L} c_{4l}$ with $c_{4l} = (n_l - 1)T_l$. Similarly with equation (7), we can obtain another estimator of σ_ν^2,

$$\hat{\sigma}_{1\nu}^2 = \frac{1}{c_4}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\|\hat{\tilde{\nu}}_{li}\|^2 = \frac{1}{c_4}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\|\tilde{y}_{li} - \tilde{X}_{li}\hat{\beta}\|^2.$$

Under some regular conditions and the null hypothesis of H_0^μ, $\hat{\sigma}_{1\nu}^2$ can be shown to be a consistent estimator of σ_ν^2. However, $\hat{\sigma}_{1\nu}^2$ is not consistent anymore under the alternative hypotheses of H_1^μ. Therefore, a Hausman-type test (e.g, Hausman (1978)) can be proposed by the statistically significant difference between $\hat{\sigma}_{0\nu}^2$ and $\hat{\sigma}_{1\nu}^2$ to test for the individual effect. The test statistic can be constructed as follows,

$$\mathrm{T}_\mu^{IC} = \omega_n^{-\frac{1}{2}}\sqrt{n}(\hat{\sigma}_{1\nu}^2 - \hat{\sigma}_{0\nu}^2), \tag{8}$$

where the scalar $\omega_n = a_n\hat{\gamma}_\nu^4 + b_n(\hat{\sigma}_{0\nu}^2)^2$ is used to standardize the statistic with

$$a_n = \frac{1}{n}\sum_{l=1}^{L} n_l[\frac{n^2}{c_4^2}T_l + \frac{n^2}{c_1^2}(T_l + \frac{1}{T_l} - 2) - \frac{2n^2}{c_1c_4}(T_l - 1)],$$

and

$$b_n = \frac{1}{n}\sum_{l=1}^{L} n_l[\frac{n^2}{c_4^2}T_l(T_l - 1) + \frac{n^2}{c_1^2}(T_l - 1)(T_l + \frac{3}{T_l} - 2) - \frac{2n^2}{c_1c_4}(T_l - 1)^2].$$

Below, we state some assumptions for the study of the asymptotic properties of the proposed test statistics.

Assumption A: The individual effect μ_i is independent and identically distributed with mean zero and variance $\sigma_\mu^2 = n^{-\frac{1}{2}}\sigma_1^2$ with a constant $\sigma_1^2 \geq 0$.

Assumption B: $\mathbb{E}(\mu_i\nu_{it}) = 0$, $n^{\frac{1}{2}}\mathbb{E}(\mu_i^2\nu_{it}^2) < \infty$, $\mathbb{E}(n^{\frac{1}{4}}X_{it,k}\mu_i) < \infty$, for $i = 1, \ldots, n, t = 1, \ldots, T_i$.

Theorem 1. *For model (1), suppose that $\mathbb{E}\nu_{it}^5 < \infty$, $|\Sigma_1| > 0$, $\mathbb{E}X_{it,k}^4 < \infty$, and $\mathbb{E}(X_{it,k}\nu_{is}) = 0$, for $i = 1, \ldots, n; t = 1, \ldots, T_i; k = 1, \ldots, K$. If Assumptions A and B hold, we have that*

$$\mathrm{T}_\mu^{IC} \Longrightarrow \Phi^{-\frac{1}{2}}\sigma_1^2 + N(0,1),$$

where $\Phi = a\gamma_\nu^4 + b(\sigma_\nu^2)^2$ with $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$.

Under the null hypothesis of H_0^{μ}, we may find another consistent estimator as follows,

$$\tilde{\sigma}_{1\nu}^2 = \frac{1}{c_4} \sum_{l=1}^{L} \sum_{i=1}^{n_l} \|\tilde{y}_{li} - \tilde{X}_{li}\tilde{\beta}\|^2,$$

where

$$\tilde{\beta} = (\sum_{l=1}^{L} \sum_{i=1}^{n_l} \tilde{X}_{li}'\tilde{X}_{li})^{-1} \sum_{l=1}^{L} \sum_{i=1}^{n_l} \tilde{X}_{li}'\tilde{y}_{li}.$$

Similarly with equation (8), we can obtain another test statistic as follows,

$$\mathrm{T}_{\mu}^{IC*} = \omega_n^{-\frac{1}{2}} \sqrt{n}(\tilde{\sigma}_{1\nu}^2 - \hat{\sigma}_{0\nu}^2),$$

where ω_n is defined as in equation (8). Denote $\Sigma_3 = \sum_{l=1}^{L} m_l \mathbb{E}[(X_{li} - \mathbb{E}X_{li})'(X_{li} - \mathbb{E}X_{li})]$.

Corollary 1. *Suppose $|\Sigma_3| > 0$ and the conditions of Theorem 1 hold. For model (1), under Assumptions A and B, we have that*

$$\mathrm{T}_{\mu}^{IC*} \Longrightarrow \Phi^{-\frac{1}{2}}(\sigma_1^2 - \pi) + N(0, 1),$$

where $\pi = \frac{\Omega_1' \Sigma_3^{-1} \Omega_1}{\sum_{l=1}^{L} m_l T_l}$ with $\Omega_1 = \sum_{l=1}^{L} m_l n^{\frac{1}{4}} \mathbb{E}(X_{li}'\mu_{li})\iota_{T_l}$.

Actually, we can obtain the ANOVA F test for the individual effect,

$$\mathrm{F}_{\mu}^{IC} = \frac{(c_4\tilde{\sigma}_{1\nu}^2 - c_1\hat{\sigma}_{0\nu}^2)/\sum_{l=1}^{L}(n_l - 1)}{c_1\hat{\sigma}_{0\nu}^2/[\sum_{l=1}^{L}(n_l - 1)(T_l - 1) - K]}.$$

Denote

$$\mathrm{T}_{\mu}^{IC**} = \sigma_{\nu}^2 \Phi^{-\frac{1}{2}} \sqrt{n}(\frac{\tilde{\sigma}_{1\nu}^2 - \hat{\sigma}_{0\nu}^2}{\hat{\sigma}_{0\nu}^2}).$$

It then follows from the Slutsky's theorem that T_{μ}^{IC*} and T_{μ}^{IC**} have identical asymptotic distribution, where

$$\mathrm{T}_{\mu}^{IC**} = \theta_1 \mathrm{F}_{\mu}^{IC} - \vartheta_1,$$

with

$$\theta_1 = \sigma_{\nu}^2 \Phi^{-\frac{1}{2}} \frac{\sqrt{n}c_1(n - L)}{c_4(c_1 - K)}, \ \vartheta_1 = \sigma_{\nu}^2 \Phi^{-\frac{1}{2}} \frac{\sqrt{n}(c_4 - c_1)}{c_4}.$$

Therefore T_{μ}^{IC**} is a simple affine transformation of F_{μ}^{IC}. A test based on T_{μ}^{IC**} (or T_{μ}^{IC*}) and the critical value from the standard normal distribution is asymptotically equivalent to the ANOVA F test. It holds that $0 \leq \pi \leq \sigma_1^2$ and $\pi = 0$ if and only if $\mathbb{E}(\mu_i X_{it}) = 0$. We may conclude that our test T_{μ}^{IC} will be asymptotically more powerful than the ANOVA F test when the individual effect is correlated with regressors.

4 Testing for time effect

In this section, we consider to construct test statistics to test for time effects in model (1). The hypotheses of the time effects test can be formalized as

$$H_0^\eta: \ \mathrm{var}(\eta_1) = \cdots = \mathrm{var}(\eta_T) = 0 \qquad \text{vs} \qquad H_1^\eta: \ \text{at least one of them is nonzero.} \tag{9}$$

Under the null hypothesis of H_0^η, model (2) turns out

$$y_{li} = \alpha \iota_{T_l} + X_{li}\beta + \mu_{li}\iota_{T_l} + \nu_{li}, \quad i \in \mathcal{N}_l. \tag{10}$$

We only need to eliminate the individual effect by premultiplying the model (10) with matrix Q_l',

$$Q_l' y_{li} = Q_l' X_{li}\beta + Q_l'\nu_{li}.$$

It holds that

$$\sum_{l=1}^{L}\sum_{i=1}^{n_l} \mathbb{E}\|Q_l'\nu_{li}\|^2 = c_5\sigma_\nu^2,$$

where $c_5 = \sum_{l=1}^{L} n_l(T_l - 1)$. Similarly, we use the empirical version to replace the population one to obtain the estimator of σ_ν^2 as follows,

$$\hat{\sigma}_{2\nu}^2 = \frac{1}{c_5}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\|Q_l'\hat{\nu}_{li}\|^2 = \frac{1}{c_5}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\|Q_l'(y_{li} - X_{li}\hat{\beta})\|^2.$$

Note that $\hat{\sigma}_{2\nu}^2$ is consistent under the null hypothesis H_0^η of no time effect and inconsistent under the alternative hypotheses of H_1^η. Based on the difference between $\hat{\sigma}_{0\nu}^2$ and $\hat{\sigma}_{2\nu}^2$, we can construct a test statistic as follows,

$$\mathrm{T}_\eta^{IC} = \frac{c_5}{\hat{\sigma}_{0\nu}^2}(\hat{\sigma}_{2\nu}^2 - \hat{\sigma}_{0\nu}^2) + \sum_{l=1}^{L}(T_l - 1). \tag{11}$$

Below, we state some additional assumptions to study the asymptotic behavior of the proposed test T_η^{IC}.

Assumption C: The time effect $\eta_t = n^{-\frac{1}{2}}\eta_{0t}$, where η_{0t} is a random variable with mean zero and finite variance $\mathbb{E}\eta_{0t}^2 \geq 0$.

Theorem 2. *For model (1), suppose that $\mathbb{E}\nu_{it}^2 < \infty$, $|\Sigma_1| > 0$, $\mathbb{E}X_{it,k}^2 < \infty$, and $\mathbb{E}(X_{it,k}\nu_{is}) = 0$, for $i = 1, \ldots, n; t = 1, \ldots, T_i; k = 1, \ldots, K$. If Assumption C holds, we have that*

$$\mathrm{T}_\eta^{IC} \Longrightarrow \sum_{l=1}^{L}\| \ \sigma_\nu^{-1}Q_l'\eta_{1,T_l}^* + N(0, I_{T_l-1} - \sigma_\nu^{-2}\Sigma_{6l})\|^2,$$

where $\eta^*_{1,T_l} = (\eta_1, \ldots, \eta_{T_l})'$, $\Sigma_{6l} = \Sigma_{5l} + \Sigma'_{5l} - \Sigma_{4l}$ *with* $\Sigma_{4l} = m_l Q'_l \mathbb{E} X_{li} \Sigma_1^{-1} \Sigma_2 \Sigma_1^{-1} \mathbb{E} X'_{li} Q_l$ *and* $\Sigma_{5l} = m_l Q'_l \mathbb{E}[\nu_{li}\nu'_{li} P_l (X_{li} - \mathbb{E}X_{li})]\Sigma_1^{-1}\mathbb{E}X'_{li}Q_l$.

It is not difficult to obtain that $\Sigma_{6l} = \sigma^2_\nu I_{T_l-1}$ if $\mathbb{E}X_{it}$ is independent of t. Under the null hypothesis of no time effect, we can show that the asymptotic distribution of T^{IC}_η is the chi-square distribution with $\sum_{l=1}^L (T_l - 1)$ degrees of freedom, $\chi^2_{\sum_{l=1}^L (T_l-1)}$. In application, we need first to center the regressor X_{li} for each group $\mathcal{N}_l$, resulting in $\tilde{X}_{li}$, and then perform the test T^{IC}_η with p-values or critical values calculated from $\chi^2_{\sum_{l=1}^L (T_l-1)}$.

Under the null hypothesis of no time effect, we may find another consistent estimator as follows,

$$\tilde{\sigma}^2_{2\nu} = \frac{1}{c_5} \sum_{l=1}^L \sum_{i=1}^{n_l} \|Q'_l(y_{li} - X_{li}\tilde{\beta}_2)\|^2,$$

where

$$\tilde{\beta}_2 = (\sum_{l=1}^L \sum_{i=1}^{n_l} X'_{li} P_l X_{li})^{-1} \sum_{l=1}^L \sum_{i=1}^{n_l} X'_{li} P_l y_{li}.$$

Similarly with equation (11), we can obtain another test statistic as follows,

$$\mathrm{T}^{IC*}_\eta = \frac{c_5}{\hat{\sigma}^2_{0\nu}} (\tilde{\sigma}^2_{2\nu} - \hat{\sigma}^2_{0\nu}) + \sum_{l=1}^L (T_l - 1).$$

Corollary 2. *For model (1), suppose that* $\mathbb{E}\nu^2_{it} < \infty$, $|\Sigma_1| > 0$, $\mathbb{E}X^2_{it,k} < \infty$, $\mathbb{E}(X_{it,k}\nu_{is}) = 0$, $\mathbb{E}X_{it}$ *is independent of* t, *for* $i = 1, \ldots, n; t = 1, \ldots, T_i; k = 1, \ldots, K$. *If Assumption C holds, we have that*

$$\mathrm{T}^{IC*}_\eta \Longrightarrow \sum_{l=1}^L \|\sigma_\nu^{-1} Q'_l \eta^*_l + N(0, I_{T_l-1})\|^2.$$

Similarly, we can obtain the ANOVA F test for the time effect,

$$\mathrm{F}^{IC}_\eta = \frac{(c_5\tilde{\sigma}^2_{2\nu} - c_1\hat{\sigma}^2_{o\nu})/\sum_{l=1}^L (T_l - 1)}{c_1\hat{\sigma}^2_{o\nu}/[\sum_{l=1}^L (n_l - 1)(T_l - 1) - K]},$$

and we can further show that $\mathrm{T}^{IC*}_\eta = \mathrm{F}^{IC}_\eta \kappa$ with $\kappa = \frac{c_4 \sum_{l=1}^L (T_l-1)}{c_1 - K}$. From the above corollary, unlike the counterpart in the previous section, the test T^{IC}_η is not significantly more powerful than the ANOVA F test due to the small time length.

5 Test jointly for both individual and time effects

In this section we consider the joint test for individual and time effects. The hypotheses can be formalized as follows,

$$H_0^{\mu\eta} : \sigma_\mu^2 = \mathrm{var}(\eta_1) = \cdots = \mathrm{var}(\eta_T) = 0 \quad \text{vs} \quad H_1^{\mu\eta} : \text{at least one of them is nonzero.} \tag{12}$$

Under the null hypotheses of $H_0^{\mu\eta}$, model (2) turns out

$$y_{li} = \alpha\iota_{T_l} + X_{li}\beta + \nu_{li}, \quad i \in \mathcal{N}_l.$$

It holds that

$$\sum_{l=1}^{L}\sum_{i=1}^{n_l}\mathbb{E}\|\nu_{li}\|^2 = \sum_{l=1}^{L} n_l T_l \sigma_\nu^2 = N\sigma_\nu^2.$$

and we can obtain an estimator of σ_ν^2 as follows,

$$\hat{\sigma}_{3\nu}^2 = \frac{1}{N}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\|y_{li} - \hat{\alpha}\iota_{T_l} - X_{li}\hat{\beta}\|^2,$$

where

$$\hat{\alpha} = \frac{1}{N}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\iota_{T_l}'(y_{li} - X_{li}\hat{\beta}).$$

Clearly, $\hat{\sigma}_{3\nu}^2$ is consistent under the null hypothesis of $H_0^{\mu\eta}$ and inconsistent under the alternative hypotheses of $H_1^{\mu\eta}$. Based on the difference between $\hat{\sigma}_{0\nu}^2$ and $\hat{\sigma}_{3\nu}^2$, we can construct a test statistic as follows,

$$\mathrm{T}_{\mu\eta 1}^{IC} = \omega_n^{-\frac{1}{2}}\sqrt{n}(\hat{\sigma}_{3\nu}^2 - \hat{\sigma}_{0\nu}^2),$$

where ω_n is defined as in equation (8).

Denote that

$$\Sigma_7 = \sum_{l=1}^{L} m_l\mathbb{E}(X_{li}'X_{li}) - \frac{1}{\sum_{l=1}^{L} m_l T_l}(\sum_{l=1}^{L} m_l\mathbb{E}X_{li}'\iota_{T_l})(\sum_{l=1}^{L} m_l\mathbb{E}X_{li}'\iota_{T_l})',$$

and

$$\Omega_2 = \sum_{l=1}^{L} m_l\mathbb{E}X_{li}'\iota_{T_l}.$$

Theorem 3. *For model (1), suppose that* $\mathbb{E}\nu_{it}^5 < \infty$, $|\Sigma_1| > 0$, $|\Sigma_7| > 0$, $\mathbb{E}X_{it,k}^4 < \infty$, *and* $\mathbb{E}(X_{it,k}\nu_{is}) = 0$, *for* $i = 1, \ldots, n; t = 1, \ldots, T_i; k = 1, \ldots, K$. *If Assumptions A−C hold, we have that*

$$\mathrm{T}_{\mu\eta 1}^{IC} \Longrightarrow \Phi^{-\frac{1}{2}}\sigma_1^2 + N(0,1),$$

where $\Phi = a\gamma_\nu^4 + b(\sigma_\nu^2)^2$.

Under the null hypothesis of $H_0^{\mu\eta}$, we may find another consistent estimator as,

$$\tilde{\sigma}_{3\nu}^2 = \frac{1}{N}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\|y_{li} - \tilde{\alpha}_3\iota_{T_l} - X_{li}\tilde{\beta}_3\|^2,$$

where

$$(\tilde{\alpha}_3, \tilde{\beta}_3) = \mathrm{argmin}_{(\alpha,\beta)}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\|y_{li} - \alpha\iota_{T_l} - X_{li}\beta\|^2.$$

We may consider another test statistic

$$\mathrm{T}_{\mu\eta}^{IC*} = \omega_n^{-\frac{1}{2}}\sqrt{n}(\tilde{\sigma}_{3\nu}^2 - \hat{\sigma}_{0\nu}^2).$$

Corollary 3. *For model (1), suppose that* $\mathbb{E}\nu_{it}^5 < \infty$, $|\Sigma_1| > 0$, $|\Sigma_7| > 0$, $\mathbb{E}X_{it,k}^4 < \infty$, *and* $\mathbb{E}(X_{it,k}\nu_{is}) = 0$, *for* $i = 1, \ldots, n; t = 1, \ldots, T_i; k = 1, \ldots, K$. *If Assumptions A−C hold, we have that*

$$\mathrm{T}_{\mu\eta}^{IC*} \Longrightarrow \Phi^{-\frac{1}{2}}(\sigma_1^2 - \lambda) + N(0,1),$$

where $\lambda = \frac{2}{\sum_{l=1}^{L} m_l T_l}\Omega_1'\Sigma_7^{-1}\Omega_1 + \frac{1}{(\sum_{l=1}^{L} m_l T_l)^2}(\Omega_2\Sigma_7^{-1}\Omega_1)^2$.

And we can obtain the ANOVA F test for both individual and time effects,

$$\mathrm{F}_{\mu\eta}^{IC} = \frac{(N\tilde{\sigma}_{3\nu}^2 - c_1\hat{\sigma}_{0\nu}^2)/\sum_{l=1}^{L}(n_l + T_l - 2)}{c_1\hat{\sigma}_{0\nu}^2/(\sum_{l=1}^{L}(n_l - 1)(T_l - 1) - K)}.$$

Denote

$$\mathrm{T}_{\mu\eta}^{IC**} = \sigma_\nu^2\Phi^{-\frac{1}{2}}\sqrt{n}(\frac{\tilde{\sigma}_{3\nu}^2 - \hat{\sigma}_{0\nu}^2}{\hat{\sigma}_{0\nu}^2}),$$

and then we have that

$$\mathrm{T}_{\mu\eta}^{IC**} = \theta_2\mathrm{F}_{\mu\eta}^{IC} - \vartheta_2,$$

with

$$\theta_2 = \sigma_\nu^2\Phi^{-\frac{1}{2}}\sqrt{n}\frac{\sum_{l=1}^{L}(n_l + T_l - 2)c_1}{(c_1 - K)N}, \vartheta_2 = \sigma_\nu^2\Phi^{-\frac{1}{2}}\sqrt{n}\frac{\sum_{l=1}^{L}(n_l + T_l - 1)}{N}.$$

Furthermore, it follows from the Slutsky's theorem that $\mathrm{T}_{\mu\eta}^{IC*}$ and $\mathrm{T}_{\mu\eta}^{IC**}$ have the same asymptotic distribution. And then a test based on $\mathrm{T}_{\mu\eta}^{IC*}$ with the critical value from the standard normal distribution is asymptotically equivalent to the ANOVA F test $\mathrm{F}_{\mu\eta}^{IC}$.

As Wu and Li (2014) argued, $\mathrm{T}_{\mu\eta 1}^{IC}$ and $\mathrm{T}_{\mu\eta}^{IC*}$ fail asymptotically to detect the presence of the time effect. From the proof of Theorem 2, test statistics T_{μ}^{IC} and T_{η}^{IC} are asymptotically independent if $\{\nu_i\}$ is independent of $\{X_i\}$. Following the similar method of Wu and Li (2014), we propose a weighted test statistic as follows,

$$\mathrm{T}_{\mu\eta 2}^{IC} = w(\mathrm{T}_{\mu}^{IC})^2 + (1-w)\mathrm{T}_{\eta}^{IC},$$

where $w \in [0,1]$ can be specified by practitioners. Under the null hypothesis of $H_0^{\mu\eta}$, if $Q_l'\mathbb{E}X_{li} = 0$ holds and $\{\nu_i\}$ is independent of $\{X_i\}$, then

$$\mathrm{T}_{\mu\eta 2}^{IC} \Longrightarrow w\chi_1^2 + (1-w)\chi^2_{\sum_{l=1}^{L}(T_l-1)}.$$

As Wu and Li (2014) argued, if we have no preference about the weight in $\mathrm{T}_{\mu\eta 2}^{IC}$, we can simply set $w = 0.5$ in practice. And then the asymptotic distribution of $\mathrm{T}_{\mu\eta 2}^{IC}$ can be $0.5\chi^2_{\sum_{l=1}^{L}(T_l-1)+1}$ under the null hypothesis.

6 Simulation study

In this section, we conduct several Monte Carlo simulation experiments to evaluate the performance of the proposed test statistics ($\mathrm{T}_{\mu}^{IC}, \mathrm{T}_{\eta}^{IC}, \mathrm{T}_{\mu\eta 1}^{IC}, \mathrm{T}_{\mu\eta 2}^{IC}$). Several commonly-used tests are also given to compare the finite sample properties with our test statistics. They are Breusch and Pagan (1980)'s tests ($\mathrm{BP}_{\mu}^{IC}, \mathrm{BP}_{\eta}^{IC}, \mathrm{BP}_{\mu\eta}^{IC}$), Honda (1985)'s tests ($\mathrm{H}_{\mu}^{IC}, \mathrm{H}_{\eta}^{IC}, \mathrm{H}_{\mu\eta}^{IC}$), Moulton and Randolph (1989)'s tests ($\mathrm{SLM}_{\mu}^{IC}, \mathrm{SLM}_{\eta}^{IC}, \mathrm{SLM}_{\mu\eta}^{IC}$) and ANOVA F's tests ($\mathrm{F}_{\mu}^{IC}, \mathrm{F}_{\eta}^{IC}, \mathrm{F}_{\mu\eta}^{IC}$), respectively. The empirical sizes and powers of these test statistics are obtained based on 1000 replications.

The first experiment is to check the performance of the test statistic T_{μ}^{IC}. The data generating process is

$$y_{it} = 0.5 + X_{it1} + 2X_{it2} + \mu_i + \eta_t + \nu_{it}, \qquad i = 1,2,\ldots,n, t = 1,2,\ldots,T_i, \tag{13}$$

where the numbers of time length T_i's are randomly taken from three different time periods 4, 8 and 12, X_{it1} follow the normal distributions with mean zero and $\mathrm{var}(X_{it1}) =$

1, $X_{it2} \overset{i.i.d.}{\sim} N(0,1)$, $\mu_i \overset{i.i.d.}{\sim} \sigma_\mu N(0,1)$, $\eta_t \overset{i.i.d.}{\sim} \sigma_\eta N(0,1)$. Besides, we let $\mathrm{corr}(X_{it1}, \mu_i) = \rho$ and $\mathrm{corr}(X_{it1}, X_{is1}) = \rho^2$ for $t \neq s$. Clearly, $\rho = 0$ means that X_{it1} and μ_i are uncorrelated with each other. And we choose $n = 50, 100, 200$ to observe the change of the performance of these statistics as the number of the individuals increases. Moreover, $\sigma_\eta = 0$ or > 0 corresponds repectively to the absence or the presence of the time effect, and $\sigma_\mu = 0$ or > 0 the size or the power. We compare these test statistics on the case of two different distributions of ν_{it}: (i) $\nu_{it} \overset{i.i.d.}{\sim} N(0,1)$ and (ii): $\nu_{it} \overset{i.i.d.}{\sim} \sqrt{\frac{1}{2}}(\chi_1^2 - 1)$, where χ_1^2 means the chi-square distribution with 1 degree of freedom.

We first compare these test statistics for the existence of individual effects in the case with $\rho = 0$. When the time effect is not present, all of these tests perform well although they still have some small difference. The test H_μ^{IC} has a distorted empirical size even when the number of individual is large enough, which was also illustrated by Moulton and Randolph (1989). The power of BP_μ^{IC} is smaller than those of the other tests as the alternative hypotheses of BP_μ^{IC} test are two-sided. We can see that these tests are robust to the distributions of the idiosyncratic error even though the distributions of the test statistics BP_μ^{IC}, H_μ^{IC}, SLM_μ^{IC} and F_μ^{IC} are inferred under assumption of normality. When the time effect is present, the performances of BP_μ^{IC}, H_μ^{IC} and SLM_μ^{IC} are worse since their empirical sizes are distorted, however, F_μ^{IC} and T_μ^{IC} are comparable and both keep the desired performance. The details on the simulation results for the case with $\rho = 0$ are listed in Tables 1-2 as follows.

Table 1: Empirical sizes and powers of the test T_{μ}^{IC} and other four tests in case of $\nu_{it} \sim N(0,1)$ and $\rho = 0$. The nominal level is 5%.

n	σ_η	σ_μ	BP_{μ}^{IC}	H_{μ}^{IC}	SLM_{μ}^{IC}	F_{μ}^{IC}	T_{μ}^{IC}
	0.0	0.0	0.056	0.109	0.056	0.041	0.058
		0.1	0.072	0.139	0.124	0.092	0.118
		0.2	0.264	0.397	0.423	0.366	0.425
		0.3	0.678	0.833	0.851	0.802	0.817
50	0.5	0.0	0.056	0.044	0.004	0.057	0.068
		0.1	0.043	0.144	0.070	0.103	0.130
		0.2	0.093	0.185	0.366	0.382	0.379
		0.3	0.413	0.529	0.787	0.791	0.803
	1.0	0.0	0.348	0.592	0.001	0.052	0.066
		0.1	0.263	0.537	0.005	0.095	0.111
		0.2	0.106	0.306	0.026	0.347	0.401
		0.3	0.132	0.214	0.176	0.792	0.801
	0.0	0.0	0.044	0.108	0.047	0.050	0.055
		0.1	0.090	0.153	0.156	0.136	0.144
		0.2	0.445	0.624	0.639	0.569	0.585
		0.3	0.931	0.967	0.980	0.959	0.965
100	0.5	0.0	0.111	0.266	0.043	0.050	0.044
		0.1	0.067	0.185	0.097	0.120	0.144
		0.2	0.145	0.264	0.518	0.561	0.602
		0.3	0.639	0.784	0.950	0.965	0.964
	1.0	0.0	0.625	0.797	0.002	0.049	0.058
		0.1	0.492	0.706	0.003	0.125	0.145
		0.2	0.269	0.431	0.021	0.565	0.601
		0.3	0.198	0.344	0.219	0.961	0.970
	0.0	0.0	0.065	0.094	0.048	0.060	0.043
		0.1	0.103	0.212	0.220	0.189	0.184
		0.2	0.749	0.873	0.866	0.839	0.836
		0.3	0.997	0.999	0.999	0.999	1.000
200	0.5	0.0	0.248	0.429	0.025	0.050	0.044
		0.1	0.134	0.282	0.118	0.183	0.185
		0.2	0.234	0.391	0.760	0.848	0.816
		0.3	0.873	1.000	0.834	1.000	1.000
	1.0	0.0	0.813	0.904	0.001	0.047	0.049
		0.1	0.719	0.840	0.003	0.176	0.202
		0.2	0.428	0.573	0.046	0.825	0.821
		0.3	0.332	0.459	0.279	1.000	0.999

Table 2: Empirical sizes and powers of the test T_{μ}^{IC} and other four tests in case of $\nu_{it} \sim \sqrt{\frac{1}{2}}(\chi_1^2 - 1)$ and $\rho = 0$. The nominal level is 5%.

n	σ_η	σ_μ	BP_{μ}^{IC}	H_{μ}^{IC}	SLM_{μ}^{IC}	F_{μ}^{IC}	T_{μ}^{IC}
	0.0	0.0	0.057	0.091	0.053	0.060	0.061
		0.1	0.085	0.117	0.116	0.109	0.061
		0.2	0.249	0.426	0.454	0.389	0.389
		0.3	0.701	0.837	0.842	0.777	0.772
50	0.5	0.0	0.051	0.169	0.034	0.070	0.049
		0.1	0.05	0.137	0.087	0.121	0.113
		0.2	0.098	0.179	0.390	0.370	0.387
		0.3	0.405	0.575	0.779	0.772	0.809
	1.0	0.0	0.365	0.613	0.005	0.058	0.048
		0.1	0.252	0.550	0.005	0.113	0.111
		0.2	0.125	0.308	0.032	0.389	0.375
		0.3	0.111	0.237	0.156	0.778	0.779
	0.0	0.0	0.048	0.095	0.054	0.074	0.062
		0.1	0.091	0.156	0.139	0.161	0.132
		0.2	0.465	0.622	0.658	0.560	0.547
		0.3	0.938	0.973	0.980	0.953	0.952
100	0.5	0.0	0.115	0.260	0.034	0.062	0.062
		0.1	0.065	0.170	0.105	0.153	0.127
		0.2	0.160	0.265	0.550	0.549	0.555
		0.3	0.644	0.808	0.944	0.951	0.955
	1.0	0.0	0.625	0.776	0.002	0.061	0.054
		0.1	0.540	0.682	0.007	0.137	0.147
		0.2	0.257	0.433	0.029	0.577	0.57
		0.3	0.189	0.328	0.218	0.961	0.949
	0.0	0.0	0.040	0.091	0.050	0.074	0.057
		0.1	0.122	0.183	0.212	0.184	0.178
		0.2	0.736	0.859	0.875	0.826	0.796
		0.3	0.998	1.000	0.999	1.000	0.998
200	0.5	0.0	0.228	0.422	0.027	0.059	0.056
		0.1	0.119	0.246	0.098	0.181	0.180
		0.2	0.203	0.376	0.740	0.811	0.800
		0.3	0.869	0.935	0.997	1.000	0.998
	1.0	0.0	0.816	0.912	0.002	0.061	0.055
		0.1	0.721	0.815	0.006	0.184	0.185
		0.2	0.439	0.579	0.027	0.821	0.782
		0.3	0.337	0.473	0.275	0.998	0.997

And we let $\rho > 0$ to observe the performance of the test statistics when the regressors are related to the individual effects. Here we only consider the case with $\nu_{it} \stackrel{i.i.d.}{\sim} N(0,1)$ and the number of individuals $n = 200$. The results show further that our test T_{μ}^{IC} has better performance than the competitors including the ANOVA F test F_{μ}^{IC}, see more details in the following table.

Table 3: Empirical powers of the test T_{μ}^{IC} and other four tests in case of $\nu_{it} \sim N(0,1)$ and $\rho > 0$. The nominal level is 5%.

ρ	σ_η	σ_μ	BP_{μ}^{IC}	H_{μ}^{IC}	SLM_{μ}^{IC}	F_{μ}^{IC}	T_{μ}^{IC}
	0.0	0.1	0.104	0.174	0.156	0.147	0.205
		0.2	0.453	0.658	0.678	0.719	0.843
		0.3	0.968	0.988	0.990	0.999	0.999
0.4	0.5	0.1	0.051	0.110	0.088	0.159	0.202
		0.2	0.318	0.458	0.527	0.690	0.816
		0.3	0.904	0.974	0.970	0.996	0.999
	1.0	0.1	0.759	0.855	0.002	0.153	0.178
		0.2	0.527	0.659	0.015	0.730	0.823
		0.3	0.291	0.411	0.125	0.998	1.000
	0.0	0.1	0.073	0.101	0.109	0.119	0.210
		0.2	0.165	0.332	0.345	0.533	0.808
		0.3	0.681	0.838	0.868	0.965	1.000
0.6	0.5	0.1	0.048	0.124	0.049	0.122	0.203
		0.2	0.100	0.214	0.209	0.557	0.814
		0.3	0.489	0.682	0.688	0.966	0.996
	1.0	0.1	0.764	0.886	0.002	0.123	0.193
		0.2	0.645	0.769	0.006	0.533	0.809
		0.3	0.454	0.583	0.027	0.969	0.997
	0.0	0.1	0.052	0.084	0.065	0.097	0.236
		0.2	0.072	0.140	0.103	0.266	0.791
		0.3	0.114	0.203	0.242	0.691	0.995
0.8	0.5	0.1	0.065	0.134	0.028	0.082	0.231
		0.2	0.051	0.105	0.056	0.272	0.804
		0.3	0.073	0.132	0.125	0.715	0.995
	1.0	0.1	0.825	0.888	0.002	0.094	0.255
		0.2	0.785	0.852	0.002	0.254	0.790
		0.3	0.701	0.815	0.002	0.700	0.995

The second experiment is to check the performance of the test statistic T_{η}^{IC}. The data generating process is the same as equation (13) with $n = 200$ and $\rho = 0$. We also consider two different kinds of the distributions of ν_{it}: (i) $\nu_{it} \overset{i.i.d.}{\sim} N(0,1)$ and (ii): $\nu_{it} \overset{i.i.d.}{\sim} \sqrt{\frac{1}{2}}(\chi_1^2 - 1)$, and the corresponding results are as follows.

Table 4: Empirical sizes and powers of the test T_η^{IC} and other four tests in case of $\rho = 0$. The nominal level is 5%.

	σ_μ	σ_η	BP_η^{IC}	H_η^{IC}	SLM_η^{IC}	F_η^{IC}	T_η^{IC}
	0.0	0.0	0.046	0.057	0.051	0.045	0.048
		0.2	0.092	0.123	0.160	0.093	0.102
		0.4	0.923	0.947	0.947	0.943	0.927
$\nu_{it} \sim N(0,1)$	0.5	0.0	0.032	0.060	0.030	0.047	0.048
		0.2	0.052	0.079	0.078	0.094	0.088
		0.4	0.805	0.862	0.912	0.936	0.935
	1.0	0.0	0.011	0.127	0.001	0.053	0.053
		0.2	0.014	0.072	0.020	0.108	0.105
		0.4	0.560	0.639	0.715	0.934	0.941
	0.0	0.0	0.045	0.066	0.060	0.054	0.044
		0.2	0.100	0.148	0.164	0.103	0.104
		0.4	0.923	0.933	0.950	0.912	0.943
$\nu_{it} \sim \sqrt{\frac{1}{2}}(\chi_1^2 - 1)$	0.5	0.0	0.021	0.052	0.017	0.050	0.048
		0.2	0.051	0.070	0.079	0.092	0.108
		0.4	0.799	0.869	0.918	0.913	0.937
	1.0	0.0	0.008	0.136	0.003	0.055	0.052
		0.2	0.012	0.086	0.013	0.088	0.122
		0.4	0.530	0.639	0.684	0.936	0.943

From Table 4, we can see that, when the individual effect is present, the empirical sizes of BP_η^{IC}, H_η^{IC} and SLM_η^{IC} are all distorted, and in contrast, the empirical sizes and powers of F_η^{IC} and T_η^{IC} are comparable and desired. In the experiment we also compare our test with the competitors in the case that the regressors are related to the focused random effects. The unreported results show that our test statistics have slightly higher power than the competitors including the robust ANOVA F test statistics. We guess that one main reason is that the time length in this paper is set to be fixed and not large.

The third experiment is to check the performance of the joint tests. We consider the same data generating process as equation (13). And we let $\nu_{it} \overset{i.i.d.}{\sim} N(0,1)$. The number of the individuals is set to $n = 100$ or $n = 200$. Besides, we also consider the case $\rho = 0$ and the case $\rho = 0.6$. The results are listed in Table 5.

Table 5: Empirical sizes and powers of the tests $T^{IC}_{\mu\eta 1}$, $T^{IC}_{\mu\eta 2}$ and other four tests. The nominal level is 5%.

ρ	σ_η	σ_μ	$BP^{IC}_{\mu\eta}$	$H^{IC}_{\mu\eta}$	$SLM^{IC}_{\mu\eta}$	$F^{IC}_{\mu\eta}$	$T^{IC}_{\mu\eta 1}$	$T^{IC}_{\mu\eta 2}$
				$n = 100$				
	0.0	0.0	0.042	0.164	0.056	0.058	0.060	0.055
		0.1	0.068	0.225	0.117	0.143	0.148	0.068
		0.2	0.358	0.656	0.451	0.578	0.588	0.206
		0.3	0.900	0.968	0.914	0.956	0.959	0.678
0	0.2	0.0	0.064	0.174	0.094	0.086	0.089	0.086
		0.1	0.067	0.216	0.161	0.143	0.147	0.084
		0.2	0.366	0.644	0.487	0.578	0.593	0.221
		0.3	0.900	0.962	0.917	0.962	0.963	0.708
	0.4	0.0	0.248	0.433	0.660	0.326	0.335	0.635
		0.1	0.265	0.462	0.709	0.487	0.491	0.682
		0.2	0.518	0.753	0.906	0.838	0.840	0.773
		0.3	0.921	0.982	0.992	0.993	0.994	0.950
	0.0	0.1	0.042	0.165	0.076	0.103	0.172	0.064
		0.2	0.083	0.285	0.159	0.326	0.552	0.195
		0.3	0.300	0.590	0.425	0.783	0.945	0.686
0.6	0.2	0.1	0.047	0.173	0.120	0.112	0.171	0.104
		0.2	0.094	0.256	0.215	0.363	0.577	0.250
		0.3	0.275	0.547	0.433	0.782	0.936	0.694
	0.4	0.1	0.242	0.458	0.680	0.411	0.498	0.672
		0.2	0.283	0.509	0.756	0.738	0.852	0.799
		0.3	0.415	0.693	0.898	0.943	0.990	0.950
				$n = 200$				
	0.0	0.0	0.039	0.167	0.055	0.047	0.045	0.046
		0.1	0.082	0.250	0.145	0.189	0.186	0.067
		0.2	0.659	0.849	0.699	0.835	0.829	0.338
		0.3	0.996	0.998	0.997	1.000	1.000	0.941
0	0.2	0.0	0.065	0.174	0.116	0.067	0.062	0.105
		0.1	0.089	0.273	0.239	0.211	0.204	0.120
		0.2	0.647	0.864	0.777	0.853	0.844	0.432
		0.3	0.996	0.998	0.995	0.999	0.999	0.964
	0.4	0.0	0.548	0.707	0.887	0.519	0.507	0.913
		0.1	0.583	0.728	0.935	0.718	0.706	0.935
		0.2	0.837	0.952	0.994	0.979	0.977	0.974
		0.3	0.997	1.000	1.000	1.000	1.000	1.000
	0.0	0.1	0.056	0.204	0.091	0.125	0.197	0.065
		0.2	0.157	0.383	0.229	0.523	0.779	0.358
		0.3	0.554	0.825	0.616	0.964	0.996	0.931
0.6	0.2	0.1	0.065	0.199	0.166	0.171	0.243	0.134
		0.2	0.170	0.420	0.335	0.605	0.830	0.451
		0.3	0.562	0.827	0.723	0.969	0.999	0.941
	0.4	0.1	0.553	0.728	0.911	0.646	0.711	0.926
		0.2	0.637	0.817	0.961	0.917	0.971	0.978
		0.3	0.825	0.947	0.989	0.995	1.000	0.998

Table 5 gives the empirical sizes and powers of these tests. Clearly, the tests $\mathrm{T}_{\mu\eta1}^{IC}$ and $\mathrm{T}_{\mu\eta2}^{IC}$ are immune to the correlations between the possible individual effects and the regressors. Moreover, the powers of $\mathrm{T}_{\mu\eta1}^{IC}$ are larger than those of $\mathrm{T}_{\mu\eta2}^{IC}$ as the time effect is not present, in contrast, $\mathrm{T}_{\mu\eta2}^{IC}$ seems more powerful than $\mathrm{T}_{\mu\eta1}^{IC}$ as the time effect is present.

7 A real example

In this section we apply the proposed tests to a data set which was used by Munnell (1990) and Baltagi and Pinnoi (1995). To investigate the productivity of public capital in private production, Munnell (1990) proposed the following Cobb-Douglas production function,

$$\ln Y = \alpha + \beta_1 \ln K_1 + \beta_2 \ln K_2 + \beta_3 \ln L + \beta_4 Unemp + u,$$

where Y is gross state product, K_1 is public capital which includes highways and streets, water and sewer facilities and other public buildings and structures, K_2 is the private capital stock based on the Bureau of Economic Analysis national stock estimates, L is labor input measured as employment in nonagricultural payrolls, $Unemp$ is the state unemployment rate included to capture business cycle effects. This panel data consists of annual observations of 48 contiguous states covering the period 1970-1986. In order to illustrate our method clearly, we choose three subsets of this data set as follows: Data (1) contains 16 states observed over 2 years, 16 states observed over 4 years and the other 16 states observed over 6 years, simply denoted by, 16(2), 16(4) and 16(6), respectively. Similarly, Data 2 has 16(6), 16(8) and 16(10). Finally, Data 3 has 16(10), 16(12) and 16(14). And then we use the artificially incomplete panel to demonstrate the performances of our tests. And the other tests are also performed for comparisons.

Table 6: Values of several statistics for individual effects.

	BP_{μ}^{IC}	H_{μ}^{IC}	SLM_{μ}^{IC}	F_{μ}^{IC}	T_{μ}^{IC}
Data 1	203.14	14.25	15.24	87.12	3115.14
Data 2	913.42	30.22	31.82	104.37	633.73
Data 3	2214.94	47.06	49.40	116.60	643.37

Table 6 gives the values of the test statistics for the individual effects, and the corrersponding p-values are all less than 0.0001 for the three data sets. Clearly, for the three data sets, all the tests mentioned above are verified that the individual effect is present. However, with no information of whether the time effect is present, the inferences made from BP_{μ}^{IC}, H_{μ}^{IC} and SLM_{μ}^{IC} can not be convinced.

Table 7: Values of several statistics for time effects.

	BP_{η}^{IC}	H_{η}^{IC}	SLM_{η}^{IC}	F_{η}^{IC}	T_{η}^{IC}
Data 1	0.03 (0.49)	0.18 (0.43)	0.61 (0.27)	11.52 (0.00)	718.43 (0.00)
Data 2	6.29 (0.00)	2.51 (0.006)	3.12 (0.0009)	4.19 (0.00)	1717.84 (0.00)
Data 3	0.43 (0.33)	0.66 (0.25)	0.97 (0.17)	3.22 (0.00)	2127.01 (0.00)

Note: The values in the parentheses $(\cdot)$ *are the* p*-values of the corresponding test statistics.*

Table 7 gives the results of the tests for the time effects. Clearly, for the three data sets, the null hypothesis is rejected by F_{η}^{IC} and T_{η}^{IC}, and their p-values are all very small for the three data sets. However except for Data 2, the null hypothesis can not be rejected by BP_{η}^{IC}, H_{η}^{IC} and SLM_{η}^{IC} even as the significant size is 0.05. It may be caused by the fact that the individual effect is present and this affects the inferences by BP_{μ}^{IC}, H_{μ}^{IC} and SLM_{μ}^{IC}. Finally, we also consider the joint tests for the presence of both effects. The corresponding p-values of these tests are all less than 0.0001 for the three data sets mentioned above.

Table 8: Values of several statistics for both the two effects.

	$\mathrm{BP}_{\mu\eta}^{IC}$	$\mathrm{H}_{\mu\eta}^{IC}$	$\mathrm{SLM}_{\mu\eta}^{IC}$	$\mathrm{F}_{\mu\eta}^{IC}$	$\mathrm{T}_{\mu\eta 1}^{IC}$	$\mathrm{T}_{\mu\eta 2}^{IC}$
Data 1	203.18	10.21	12.02	77.50	3044.41	9704789.00
Data 2	919.70	23.14	25.79	77.49	611.52	403335.30
Data 3	2215.37	33.74	36.56	71.02	621.48	416053.90

8 Conclusion and discussion

In this paper, we develop some methods to test for the existence of random effects in the error component regression models with incomplete panels. Some asymptotic properties are obtained under some mild conditions. Comparing with the work in the existing literature, the test statistics in this paper have the desired properties as follows. They are simple and easy to compute; they are robust to the misspecification of the distribution assumptions; they are robust to the misspecification of another effects while testing for one effects; they have desired performance when there are dependency among the regressors, random effects and the idiosyncratic errors.

In this paper, we adopt the same setting in missing form of data as Shao et al. (2011) and Chowdhury (1991) and then obtain the asymptotic properties. However, the missing data in practice may be more general than the setting in this paper. So, how to handle the more general missing data deserves further study.

Appendix

This appendix contains the proofs of theorems in the previous sections. Before that, we also state several lemmas which will be used in the process of the proofs.

Lemma 1. *Suppose that* $\mathbb{E}\nu_{it}^2 < \infty$, $|\Sigma_1| > 0$, $\mathbb{E}X_{it,k}^2 < \infty$, *and* $\mathbb{E}(X_{it,k}\nu_{is}) = 0$, *for* $i = 1, \ldots, n; t = 1, \ldots, T_i; k = 1, \ldots, K$, $\hat{\beta}$ *is a* $\sqrt{n}$ *consistent estimator of the parameter* β.

The proof of Lemma 1. Since the sequences $\{X_{li}\}$ and $\{\nu_{li}\}$ are both *i.i.d.* for each $l = 1, 2, \ldots, L$ and the conditions $\mathbb{E}\nu_{it}^2 < \infty$ and $\mathbb{E}X_{it,k}^2 < \infty$ hold, it gives

$$\frac{1}{\sqrt{n_l}}\sum_{i=1}^{n_l}(X_{li} - \mathbb{E}X_{li}) = O_p(1), \quad \frac{1}{\sqrt{n_l}}\sum_{i=1}^{n_l}\nu_{li} = O_p(1).$$

And we can show

$$\begin{aligned}
\frac{1}{n}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde{X}_{li}'P_l\tilde{X}_{li} &= \sum_{l=1}^{L}\frac{n_l}{n}\frac{1}{n_l}\sum_{i=1}^{n_l}(X_{li} - \frac{1}{n_l}\sum_{i=1}^{n_l}X_{li})'P_l(X_{li} - \frac{1}{n_l}\sum_{i=1}^{n_l}X_{li}) \\
&= \sum_{l=1}^{L}m_l[\mathbb{E}(X_{li}'P_lX_{li}) - \mathbb{E}X_{li}'P_l\mathbb{E}X_{li}] + o_p(1) \\
&= \Sigma_1 + o_p(1),
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\sqrt{n}}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde{X}_{li}'P_l\tilde{\nu}_{li} &= \sum_{l=1}^{L}\sqrt{\frac{n_l}{n}}\frac{1}{\sqrt{n_l}}\sum_{i=1}^{n_l}(X_{li} - \frac{1}{n_l}\sum_{i=1}^{n_l}X_{li})'P_l(\nu_{li} - \frac{1}{n_l}\sum_{i=1}^{n_l}\nu_{li})' \\
&= \sum_{l=1}^{L}\sqrt{m_l}\frac{1}{\sqrt{n_l}}\sum_{i=1}^{n_l}(X_{li} - \mathbb{E}X_{li})'P_l\nu_{li} + o_p(1).
\end{aligned}$$

Since $\sum_{i=1}^{n_l}\tilde{X}_{li}'P_l\tilde{\nu}_{li}$, $l = 1, 2, \ldots, L$, are independent,

$$\frac{1}{\sqrt{n}}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde{X}_{li}'P_l\tilde{\nu}_{li} \Longrightarrow N(0, \Sigma_2),$$

where $\Sigma_2 = \sum_{l=1}^L m_l \mathbb{E}[(X_{li} - \mathbb{E}X_{li})' P_l \nu_{li} \nu_{li}' P_l (X_{li} - \mathbb{E}X_{li})]$. Therefore

$$\begin{aligned}
\sqrt{n}(\hat{\beta} - \beta) &= (\frac{1}{n}\sum_{l=1}^{L}\sum_{i=1}^{n_l} \tilde{X}_{li}' P_l \tilde{X}_{li})^{-1} (\frac{1}{\sqrt{n}}\sum_{l=1}^{L}\sum_{i=1}^{n_l} \tilde{X}_{li}' P_l \tilde{\nu}_{li}) \\
&\Longrightarrow N(0, \Sigma_1^{-1}\Sigma_2\Sigma_1^{-1}),
\end{aligned}$$

and then $\hat{\beta} - \beta = O_p(n^{-\frac{1}{2}})$. The proof of Lemma 1 is completed. $\square$

Lemma 2. *Suppose that* $\mathbb{E}\nu_{it}^4 < \infty$*,* $|\Sigma_1| > 0$*,* $\mathbb{E}X_{it,k}^2 < \infty$*, and* $\mathbb{E}(X_{it,k}\nu_{is}) = 0$*, for* $i = 1, \ldots, n; t = 1, \ldots, T_i; k = 1, \ldots, K$*,* $\hat{\sigma}_{0\nu}^2$ *is a consistent estimator of* σ_ν^2 *and* $\hat{\gamma}_\nu^4$ *is a consistent estimator of* γ_ν^4*.*

The proof of Lemma 2. From Lemma 1, we can show

$$\begin{aligned}
\hat{\sigma}_{0\nu}^2 &= \frac{1}{c_1}\sum_{l=1}^{L}\sum_{i=1}^{n_l} \hat{\tilde{\nu}}_{li}' P_l \hat{\tilde{\nu}}_{li} \\
&= \frac{1}{c_1}\sum_{l=1}^{L}\sum_{i=1}^{n_l} [\tilde{\nu}_{li} - \tilde{X}_{li}(\hat{\beta} - \beta)]' P_l [\tilde{\nu}_{li} - \tilde{X}_{li}(\hat{\beta} - \beta)] \\
&= \frac{1}{c_1}\sum_{l=1}^{L}\sum_{i=1}^{n_l} \nu_{li}' P_l \nu_{li} + o_p(1) \\
&= \sum_{l=1}^{L} \frac{n_l}{c_1}\frac{1}{n_l}\sum_{i=1}^{n_l} \nu_{li}' P_l \nu_{li} + o_p(1) \\
&= \sigma_\nu^2 + o_p(1).
\end{aligned}$$

To study the consistence of $\hat{\gamma}_\nu^4$, we first note that,

$$\begin{aligned}
& c_2^{-1}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\sum_{j=1}^{T_l-1} (q_{lj}'\hat{\tilde{\nu}}_{li})^4 = c_2^{-1}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\sum_{j=1}^{T_l-1} \{q_{lj}'[\tilde{\nu}_{li} - \tilde{X}_{li}(\hat{\beta} - \beta)]\}^4 \\
= {} & c_2^{-1}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\sum_{j=1}^{T_l-1} (q_{lj}'\tilde{\nu}_{li})^4 - 4c_2^{-1}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\sum_{j=1}^{T_l-1} (q_{lj}'\tilde{\nu}_{li})^3 q_{lj}'\tilde{X}_{li}(\hat{\beta} - \beta) + o_p(\hat{\beta} - \beta) \\
= {} & c_2^{-1}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\sum_{j=1}^{T_l-1} (q_{lj}'\tilde{\nu}_{li})^4 + o_p(1).
\end{aligned}$$

Hence,

$$
\begin{aligned}
\hat{\gamma}_\nu^4 &= c_2^{-1}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\sum_{j=1}^{T_l-1}(q_{lj}'\hat{\tilde{\nu}}_{li})^4 - c_3(\hat{\sigma}_{0\nu}^2)^2 \\
&= c_2^{-1}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\sum_{j=1}^{T_l-1}(q_{lj}'\tilde{\nu}_{li})^4 - c_3(\hat{\sigma}_{0\nu}^2)^2 + o_p(1) \\
&= c_2^{-1}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\sum_{j=1}^{T_l-1}(q_{lj}'\nu_{li})^4 - c_3(\hat{\sigma}_{0\nu}^2)^2 + o_p(1) \\
&= \sum_{l=1}^{L}\frac{n_l}{c_2}\frac{1}{n_l}\sum_{i=1}^{n_l}\sum_{j=1}^{T_l-1}(q_{lj}'\nu_{li})^4 - c_3(\hat{\sigma}_{0\nu}^2)^2 + o_p(1) \\
&= \sum_{l=1}^{L}\frac{n_l}{c_2}\mathbb{E}[\sum_{j=1}^{T_l-1}(q_{lj}'\nu_{li})^4] - c_3(\hat{\sigma}_{0\nu}^2)^2 + o_p(1) \\
&= \sum_{l=1}^{L}\frac{n_l}{c_2}\sum_{j=1}^{T_l-1}\sum_{t=1}^{T_l}q_{ljt}^4\gamma_\nu^4 + [3\sum_{l=1}^{L}\frac{n_l}{c_2}(T_l-1) - 3\sum_{l=1}^{L}\frac{n_l}{c_2}\sum_{j=1}^{T_l-1}\sum_{t=1}^{T_l}q_{ljt}^4](\sigma_\nu^2)^2 - c_3(\hat{\sigma}_{0\nu}^2)^2 + o_p(1).
\end{aligned}
$$

Since

$$\lim_{n\to\infty}\sum_{l=1}^{L}\frac{n_l}{c_2}\sum_{j=1}^{T_l-1}\sum_{t=1}^{T_l}q_{ljt}^4 = 1,$$

and

$$\lim_{n\to\infty}[3\sum_{l=1}^{L}\frac{n_l}{c_2}(T_l-1) - 3\sum_{l=1}^{L}\frac{n_l}{c_2}\sum_{j=1}^{T_l-1}\sum_{t=1}^{T_l}q_{ljt}^4 - c_3] = 0,$$

together with the consistence of $\hat{\sigma}_{0\nu}^2$, we obtain $\hat{\gamma}_\nu^4 = \gamma_\nu^4 + o_p(1)$. The proof of Lemma 2 is completed. □

The proof of Theorem 1. From Lemma 1, we can show that

$$
\begin{aligned}
\sqrt{n}\hat{\sigma}_{0\nu}^2 &= \frac{\sqrt{n}}{c_1}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\|Q_l'\hat{\tilde{\nu}}_{li}\|^2 \\
&= \frac{\sqrt{n}}{c_1}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde{\nu}_{li}'P_l\tilde{\nu}_{li} - 2\frac{\sqrt{n}}{c_1}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde{\nu}_{li}'P_l\tilde{X}_{li}(\hat{\beta}-\beta) \\
&+ \frac{\sqrt{n}}{c_1}\sum_{l=1}^{L}\sum_{i=1}^{n_l}(\hat{\beta}-\beta)'\tilde{X}_{li}'P_l\tilde{X}_{li}(\hat{\beta}-\beta) \\
&= \frac{\sqrt{n}}{c_1}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde{\nu}_{li}'P_l\tilde{\nu}_{li} + o_p(1) \\
&= \frac{\sqrt{n}}{c_1}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\nu_{li}'P_l\nu_{li} + o_p(1),
\end{aligned}
$$

and

$$\begin{aligned}
\sqrt{n}\hat{\sigma}^2_{1\nu} &= \frac{\sqrt{n}}{c_4}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\|\hat{\tilde{\nu}}_{li}\|^2 \\
&= \frac{\sqrt{n}}{c_4}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\|\tilde{\nu}_{li}+\tilde{\mu}_{li}\iota_{T_l}-\tilde{X}_{li}(\hat{\beta}-\beta)\|^2.
\end{aligned}$$

Besides, it holds that

$$\begin{aligned}
\frac{\sqrt{n}}{c_4}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde{X}'_{li}\tilde{\nu}_{li} &= \sum_{l=1}^{L}\frac{n}{c_4}\sqrt{\frac{n_l}{n}}\frac{1}{\sqrt{n_l}}\sum_{i=1}^{n_l}(X_{li}-\frac{1}{n_l}\sum_{i=1}^{n_l}X_{li})'(\nu_{li}-\frac{1}{n_l}\sum_{i=1}^{n_l}\nu_{li}) \\
&= \sum_{l=1}^{L}\frac{\sqrt{m_l}}{\sum_{l=1}^{L}m_lT_l}\frac{1}{\sqrt{n_l}}\sum_{i=1}^{n_l}(X_{li}-\mathbb{E}X_{li})'\nu_{li}+o_p(1) \\
&\Longrightarrow N(0,\Sigma_4), \qquad (14)
\end{aligned}$$

where $\Sigma_4=\sum_{l=1}^{L}\frac{m_l}{(\sum_{l=1}^{L}m_lT_l)^2}\mathbb{E}[(X_{li}-\mathbb{E}X_{li})'\nu_{li}\nu'_{li}(X_{li}-\mathbb{E}X_{li})]$. Under Assumption B and the condition $\mathbb{E}X^2_{it,k}<\infty$ hold, it gives

$$\begin{aligned}
& n^{\frac{1}{4}}\frac{\sqrt{n}}{c_4}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde{\nu}'_{li}\tilde{\mu}_{li}\iota_{T_l}=O_p(1), \\
& \frac{1}{c_4}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde{X}'_{li}\tilde{X}_{li}=\frac{1}{\sum_{l=1}^{L}m_lT_l}\Sigma_3+o_p(1), \\
& n^{\frac{1}{4}}\frac{1}{c_4}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde{X}'_{li}\tilde{\mu}_{li}\iota_{T_l}=\frac{1}{\sum_{l=1}^{L}m_lT_l}\Omega_1+o_p(1), \qquad (15)
\end{aligned}$$

where $\Sigma_3=\sum_{l=1}^{L}m_l\mathbb{E}[(X_{li}-\mathbb{E}X_{li})'(X_{li}-\mathbb{E}X_{li})]$ and $\Omega_1=\sum_{l=1}^{L}m_ln^{\frac{1}{4}}\mathbb{E}(X'_{li}\mu_{li})\iota_{T_l}$. Under Assumption A and from equations (14-15), we can show that

$$\begin{aligned}
\sqrt{n}\hat{\sigma}^2_{1\nu} &= \frac{\sqrt{n}}{c_4}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde{\nu}'_{li}\tilde{\nu}_{li}+\frac{\sqrt{n}}{c_4}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde{\mu}^2_{li}T_l+o_p(1) \\
&= \frac{\sqrt{n}}{c_4}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\nu'_{li}\nu_{li}+\sigma^2_1+o_p(1).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sqrt{n}(\hat{\sigma}^2_{1\nu}-\hat{\sigma}^2_{0\nu}) &= \frac{\sqrt{n}}{c_4}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\nu'_{li}\nu_{li}-\frac{\sqrt{n}}{c_1}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\nu'_{li}P_l\nu_{li}+\sigma^2_1+o_p(1) \\
&= \frac{1}{\sqrt{n}}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\xi_{li}+\sigma^2_1+o_p(1).
\end{aligned}$$

where $\xi_{li}=\frac{n}{c_4}\nu'_{li}\nu_{li}-\frac{n}{c_1}\nu'_{li}P_l\nu_{li}$. Note that ξ_{li}, $l=1,2,\ldots,L$, $i=1,2,\ldots,n_l$, are mutually independent, and it then holds that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\mathbb{E}\xi_{li}=0,$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\mathbb{E}(\xi_{2i})^2 = a\gamma_\nu^4 + b(\sigma_\nu^2)^2 =: \Phi,$$

where

$$a = \lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{L} n_l[\frac{n^2}{c_2^2}T_l + \frac{n^2}{c_1^2}(T_l+\frac{1}{T_l}-2) - \frac{2n^2}{c_1c_2}(T_l-1)],$$

and

$$b = \lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{L} n_l[\frac{n^2}{c_2^2}T_l(T_l-1) + \frac{n^2}{c_1^2}(T_l-1)(T_l+\frac{3}{T_l}-2) - \frac{2n^2}{c_1c_2}(T_l-1)^2].$$

Note that $\lim_{n\to\infty}\sum_{i=1}^{n}\mathbb{E}\xi_{2i}^2 \to \infty$ and $\mathbb{E}\nu_{it}^5 < \infty$ hold, and the Liapounov condition is satisfied. Therefore $\Phi^{-\frac{1}{2}}\sqrt{n}(\hat\sigma_{1\nu}^2 - \hat\sigma_{0\nu}^2) \xrightarrow{D} N(0,1) + \Phi^{-\frac{1}{2}}\sigma_1^2$. From the Slutsky's theorem and Lemma 2, we can easily complete the proof of Theorem 1. □

The proof of Corollary 1. It holds that

$$\begin{aligned}
\tilde\beta &= (\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde X_{li}'\tilde X_{li})^{-1}(\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde X_{li}'\tilde y_{li}) \\
&= \beta + (\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde X_{li}'\tilde X_{li})^{-1}(\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde X_{li}'\tilde\nu_{li}) + (\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde X_{li}'\tilde X_{li})^{-1}(\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde X_{li}'\tilde\mu_{li}\iota_{T_l}).
\end{aligned}$$

From (14)-(15), we can show that

$$(\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde X_{li}'\tilde X_{li})^{-1}(\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde X_{li}'\tilde\nu_{li}) = O_p(\frac{1}{\sqrt{n}}),$$

and

$$\begin{aligned}
& n^{\frac{1}{4}}(\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde X_{li}'\tilde X_{li})^{-1}(\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde X_{li}'\tilde\mu_{li}\iota_{T_l}) \\
=\ & (\frac{1}{n}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde X_{li}'\tilde X_{li})^{-1}(n^{-\frac{3}{4}}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde X_{li}'\tilde\mu_{li}\iota_{T_l}) \\
=\ & [\sum_{l=1}^{L} m_l\mathbb{E}(X_{li}-\mathbb{E}X_{li})'(X_{li}-\mathbb{E}X_{li})]^{-1}[\sum_{l=1}^{L} m_l n^{\frac{1}{4}}\mathbb{E}(X_{li}'\mu_{li})\iota_{T_l}] + o_p(1) \\
=:\ & \Sigma_3^{-1}\Omega_1 + o_p(1).
\end{aligned}$$

By the proof of Theorem 1, it gives

$$\begin{aligned}
\sqrt{n}\hat\sigma_{1\nu}^2 &= \frac{\sqrt{n}}{c_4}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\|\hat{\tilde\nu}_{li}\|^2 \\
&= \frac{\sqrt{n}}{c_4}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\|\tilde\nu_{li}\|^2 + \|\tilde\mu_{li}\iota_{T_l} - \tilde X_{li}(\tilde\beta-\beta)\|^2 \\
&= \frac{\sqrt{n}}{c_4}\sum_{l=1}^{L}\sum_{i=1}^{n_l}[\nu_{li}'\nu_{li} + (\tilde\beta-\beta)'\tilde X_{li}'\tilde X_{li}(\tilde\beta-\beta) - 2(\tilde\beta-\beta)'\tilde X_{li}'\tilde\mu_{li}\iota_{T_l} + T_l\tilde\mu_{li}^2] + o_p(1).
\end{aligned}$$

Since

$$\begin{aligned}
& \frac{\sqrt{n}}{c_4}\sum_{l=1}^{L}\sum_{i=1}^{n_l}(\tilde{\beta}-\beta)'\tilde{X}_{li}'\tilde{X}_{li}(\tilde{\beta}-\beta) \\
= \;& n^{\frac{1}{4}}(\tilde{\beta}-\beta)'\frac{1}{c_4}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde{X}_{li}'\tilde{X}_{li}n^{\frac{1}{4}}(\tilde{\beta}-\beta) \\
= \;& \pi+o_p(1),
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\sqrt{n}}{c_4}\sum_{l=1}^{L}\sum_{i=1}^{n_l}(\tilde{\beta}-\beta)'\tilde{X}_{li}'\tilde{\mu}_{li}\iota_{T_l} \\
= \;& n^{\frac{1}{4}}(\tilde{\beta}-\beta)'\sum_{l=1}^{L}\sum_{i=1}^{n_l}n^{\frac{1}{4}}\tilde{X}_{li}'\tilde{\mu}_{li}\iota_{T_l} \\
= \;& \pi+o_p(1),
\end{aligned}$$

where $\pi=\frac{1}{\sum_{l=1}^{L}m_lT_l}\Omega_1'\Sigma_3^{-1}\Omega_1$. Hence

$$\sqrt{n}(\tilde{\sigma}_{1\nu}^2-\hat{\sigma}_{0\nu}^2)\Longrightarrow N(0,\Phi)+\sigma_1^2-\pi.$$

From the Slutsky's theorem, we can easily complete the proof of Corollary 1.

The proof of Theorem 2. We first note that

$$\sum_{l=1}^{L}\sum_{i=1}^{n_l}\hat{\tilde{\nu}}_{li}'P_l\hat{\tilde{\nu}}_{li}=\sum_{l=1}^{L}\sum_{i=1}^{n_l}\hat{\nu}_{li}'P_l\hat{\nu}_{li}-\sum_{l=1}^{L}\|\frac{1}{\sqrt{n_l}}\sum_{i=1}^{n_l}Q_l'\hat{\nu}_{li}\|^2.$$

For each $l=1,2,\ldots,L$, we can show

$$\begin{aligned}
\frac{1}{\sqrt{n_l}}\sum_{i=1}^{n_l}Q_l'\hat{\nu}_{li} &= \frac{1}{\sqrt{n_l}}\sum_{i=1}^{n_l}[Q_l'\nu_{li}-Q_l'X_{li}(\hat{\beta}-\beta)] \\
&= \frac{1}{\sqrt{n_l}}\sum_{i=1}^{n_l}Q_l'\nu_{li}-\frac{1}{\sqrt{n_l}}\sum_{i=1}^{n_l}Q_l'X_{li}(\hat{\beta}-\beta)+\frac{1}{\sqrt{n_l}}\sum_{i=1}^{n_l}Q_l'\eta_l.
\end{aligned}$$

Since

$$\frac{1}{\sqrt{n_l}}\sum_{i=1}^{n_l}Q_l'\nu_{li}\Longrightarrow N(0,\sigma_\nu^2I_{T_l-1}),$$

and

$$\begin{aligned}
\frac{1}{\sqrt{n_l}}\sum_{i=1}^{n_l}Q_l'X_{li}(\hat{\beta}-\beta) &= \sqrt{\frac{n_l}{n}}\frac{1}{n_l}\sum_{i=1}^{n_l}Q_l'X_{li}\sqrt{n}(\hat{\beta}-\beta) \\
&= \sqrt{m_l}Q_l'\mathbb{E}X_{li}\Sigma_1^{-1}\sum_{l=1}^{L}\sqrt{m_l}\frac{1}{\sqrt{n_l}}\sum_{i=1}^{n_l}(X_{li}-\mathbb{E}X_{li})'P_l\nu_{li}+o_p(1) \\
&\Longrightarrow N(0,\Sigma_{8l}),
\end{aligned}$$

where $\Sigma_{8l} = m_l Q_l' \mathbb{E} X_{li} \Sigma_1^{-1} \Sigma_2 \Sigma_1^{-1} \mathbb{E} X_{li}' Q_l$. Together with the fact that

$$\begin{aligned}
& \mathbb{E}\{\frac{1}{\sqrt{n_l}} \sum_{i=1}^{n_l} Q_l' \hat{\nu}_{li} [\sqrt{m_l} Q_l' \mathbb{E} X_{li} \Sigma_1^{-1} \sum_{l=1}^{L} \sqrt{m_l} \frac{1}{\sqrt{n_l}} \sum_{i=1}^{n_l} (X_{li} - \mathbb{E} X_{li})' P_l \nu_{li}]'\} \\
= \;& m_l Q_l' \mathbb{E}[\nu_{li} \nu_{li}' P_l (X_{li} - \mathbb{E} X_{li})] \Sigma_1^{-1} \mathbb{E} X_{li}' Q_l \\
= \;& \Sigma_{5l},
\end{aligned}$$

we have

$$\frac{1}{\sqrt{n_l}} \sum_{i=1}^{n_l} Q_l' \nu_{li} - \frac{1}{\sqrt{n_l}} \sum_{i=1}^{n_l} Q_l' X_{li} (\hat{\beta} - \beta) \xrightarrow{D} N(0, \Sigma_{6l}),$$

where $\Sigma_{6l} = \sigma_\nu^2 I_{T_l - 1} + \Sigma_{4l} - \Sigma_{5l} - \Sigma_{5l}'$. Therefore

$$\frac{1}{\sqrt{n_l}} \sum_{i=1}^{n_l} Q_l' \hat{\nu}_{li} \xrightarrow{D} N(0, \Sigma_{6l}) + Q_l' \eta_l^*.$$

Hence

$$\begin{aligned}
\mathrm{T}_\eta^{IC} &= \frac{1}{\hat{\sigma}_{0\nu}^2} \sum_{l=1}^{L} \| \frac{1}{\sqrt{n_l}} \sum_{i=1}^{n_l} Q_l' \hat{\nu}_{li} \|^2 \\
&\Longrightarrow \sum_{l=1}^{L} \| \, N(0, \sigma_\nu^{-2} \Sigma_{6l}) + \sigma_\nu^{-1} Q_l' \eta_l^* \|^2.
\end{aligned}$$

Note that

$$\begin{aligned}
& \mathrm{cov}(\frac{1}{\sqrt{n}} \sum_{l=1}^{L} \sum_{i=1}^{n_l} \xi_{li}, \frac{1}{\sqrt{n_l}} \sum_{i=1}^{n_l} Q_l' \nu_{li} + \sqrt{m_l} Q_l' \mathbb{E} X_{li} \Sigma_1^{-1} \sum_{l=1}^{L} \sqrt{m_l} \frac{1}{\sqrt{n_l}} \sum_{i=1}^{n_l} (X_{li} - \mathbb{E} X_{li})' P_l \nu_{li}) \\
= \;& \mathbb{E}(\frac{1}{\sqrt{n}} \sum_{l=1}^{L} \sum_{i=1}^{n_l} \xi_{li})(\frac{1}{\sqrt{n_l}} \sum_{i=1}^{n_l} Q_l' \nu_{li}) + \mathbb{E}(\frac{1}{\sqrt{n}} \sum_{l=1}^{L} \sum_{i=1}^{n_l} \xi_{li})(\sqrt{m_l} Q_l' \mathbb{E} X_{li} \Sigma_1^{-1} \sum_{l=1}^{L} \sqrt{m_l} \\
& \frac{1}{\sqrt{n_l}} \sum_{i=1}^{n_l} (X_{li} - \mathbb{E} X_{li})' P_l \nu_{li}).
\end{aligned}$$

And if $\{\nu_i\}$ is independent of $\{X_i\}$, then it holds that

$$\mathbb{E}(\frac{1}{\sqrt{n}} \sum_{l=1}^{L} \sum_{i=1}^{n_l} \xi_{li})(\frac{1}{\sqrt{n_l}} \sum_{i=1}^{n_l} Q_l' \nu_{li}) \to 0,$$

and

$$\mathbb{E}(\frac{1}{\sqrt{n}} \sum_{l=1}^{L} \sum_{i=1}^{n_l} \xi_{li})(\sqrt{m_l} Q_l' \mathbb{E} X_{li} \Sigma_1^{-1} \sum_{l=1}^{L} \sqrt{m_l} \frac{1}{\sqrt{n_l}} \sum_{i=1}^{n_l} (X_{li} - \mathbb{E} X_{li})' P_l \nu_{li}) \to 0.$$

when n tends to infinity. Hence, under the hypothesis of no individual and time effects, T_μ^{IC} and T_η^{IC} are asymptotically independent. The proof of Theorem 2 is completed. □

The proof of Corollary 2. Since the condition $Q_l'\mathbb{E}X_{li}=0$ holds, it gives

$$\begin{aligned}
\sqrt{n}(\hat\beta-\beta) &= (\frac{1}{n}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde X_{li}'P_l\tilde X_{li})^{-1}(\frac{1}{\sqrt{n}}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\tilde X_{li}'P_l\tilde\nu_{li})\\
&= [\sum_{l=1}^{L}m_l\mathbb{E}(X_{li}'P_lX_{li})]^{-1}(\sum_{l=1}^{L}\sqrt{m_l}\frac{1}{\sqrt{n_l}}\sum_{i=1}^{n_l}X_{li}'P_l\nu_{li})+o_p(1)\\
&= O_p(1).
\end{aligned}$$

It holds that

$$\tilde\beta_2-\beta = (\sum_{l=1}^{L}\sum_{i=1}^{n_l}X_{li}'P_lX_{li})^{-1}(\sum_{l=1}^{L}\sum_{i=1}^{n_l}X_{li}'P_l\nu_{li})+(\sum_{l=1}^{L}\sum_{i=1}^{n_l}X_{li}'P_lX_{li})^{-1}(\sum_{l=1}^{L}\sum_{i=1}^{n_l}X_{li}'P_l\eta_l).$$

So, we have that

$$\begin{aligned}
\sqrt{n}(\tilde\beta_2-\beta) &= [\sum_{l=1}^{L}m_l\mathbb{E}(X_{li}'P_lX_{li})]^{-1}(\frac{1}{\sqrt{n}}\sum_{l=1}^{L}\sum_{i=1}^{n_l}X_{li}'P_l\nu_{li})+o_p(1)\\
&= O_p(1),
\end{aligned}$$

and

$$\begin{aligned}
n(\tilde\beta_2-\hat\beta) &= n[\tilde\beta_2-\beta-(\hat\beta-\beta)]\\
&= [\sum_{l=1}^{L}m_l\mathbb{E}(X_{li}'P_lX_{li})]^{-1}(\sum_{l=1}^{L}\sum_{i=1}^{n_l}X_{li}'P_l\eta_l+\sum_{l=1}^{L}\sum_{i=1}^{n_l}(X_{li}'P_l\frac{1}{n_l}\sum_{i=1}^{n_l}\nu_{li})+o_p(1)\\
&= O_p(1).
\end{aligned}$$

Using these results, we can show that

$$\begin{aligned}
&\sum_{l=1}^{L}\sum_{i=1}^{n_l}\|Q_l'(y_{li}-X_{li}\tilde\beta)\|^2\\
=&\sum_{l=1}^{L}\sum_{i=1}^{n_l}\|Q_l'(y_{li}-X_{li}\hat\beta)-Q_l'X_{li}(\tilde\beta-\hat\beta)\|^2\\
=&\sum_{l=1}^{L}\sum_{i=1}^{n_l}\|Q_l'(y_{li}-X_{li}\hat\beta)\|^2+\sum_{l=1}^{L}\sum_{i=1}^{n_l}\|Q_l'X_{li}(\tilde\beta-\hat\beta)\|^2-2\sum_{l=1}^{L}\sum_{i=1}^{n_l}(y_{li}-X_{li}\hat\beta)'P_lX_{li}(\tilde\beta-\hat\beta)\\
=&\sum_{l=1}^{L}\sum_{i=1}^{n_l}\|Q_l'(y_{li}-X_{li}\hat\beta)\|^2+o_p(1).
\end{aligned}$$

Following the proof of Theorem 2, we finish the proof of Corollary 2. □

The proof of Theorem 3. It holds that

$$\begin{aligned}
\sqrt{n}(\hat\alpha-\alpha) &= \sum_{l=1}^{L}\frac{n_l}{\sum_{l=1}^{L}n_lT_l}\frac{1}{n_l}\sum_{i=1}^{n_l}\iota_{T_l}'X_{li}\sqrt{n}(\beta-\hat\beta)+\sum_{l=1}^{L}\frac{n_l}{\sum_{l=1}^{L}n_lT_l}\sqrt{\frac{n_l}{n}}\sum_{i=1}^{n_l}T_l\mu_{li}\\
&= \sum_{l=1}^{L}\frac{n_l}{\sum_{l=1}^{L}n_lT_l}\iota_{T_l}'\sqrt{n}\eta+\sum_{l=1}^{L}\frac{n_l}{\sum_{l=1}^{L}n_lT_l}\sqrt{\frac{n_l}{n}}\sum_{i=1}^{n_l}\iota_{T_l}'\nu_{li}\\
&= O_p(1).
\end{aligned}$$

Hence

$$\sqrt{n}\hat{\sigma}_{3\nu}^2 = \sum_{l=1}^{L} \frac{n_l}{\sum_{l=1}^{L} n_l T_l} \sqrt{\frac{n_l}{n}} \sum_{i=1}^{n_l} \nu_{li}'\nu_{li} + \sigma_1^2 + o_p(1).$$

Together with the proof of Theorem 1, we can easily complete the proof of Theorem 3.

□

The proof of Corollary 3. It holds that

$$\tilde{\alpha}_3 = \frac{1}{N} \sum_{l=1}^{L} \sum_{i=1}^{n_l} \iota_{T_l}(y_{li} - X_{li}\tilde{\beta}_3),$$

and

$$\begin{aligned}\tilde{\beta}_3 \;&=\; (\sum_{l=1}^{L}\sum_{i=1}^{n_l} X_{li}'X_{li} - \frac{1}{N}\sum_{l=1}^{L}\sum_{i=1}^{n_l} X_{li}'\iota_{T_l}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\iota_{T_l}'X_{li})^{-1}\\ &\quad(\sum_{l=1}^{L}\sum_{i=1}^{n_l} X_{li}'y_{li} - \frac{1}{N}\sum_{l=1}^{L}\sum_{i=1}^{n_l} X_{li}'\iota_{T_l}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\iota_{T_l}'y_{li}).\end{aligned}$$

Therefore

$$\begin{aligned}\tilde{\beta}_3 - \beta \;&=\; (\sum_{l=1}^{L}\sum_{i=1}^{n_l} X_{li}'X_{li} - \frac{1}{N}\sum_{l=1}^{L}\sum_{i=1}^{n_l} X_{li}'\iota_{T_l}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\iota_{T_l}'X_{li})^{-1}\\ &\quad(\sum_{l=1}^{L}\sum_{i=1}^{n_l} X_{li}'\mu_{li}\iota_{T_l} - \frac{1}{N}\sum_{l=1}^{L}\sum_{i=1}^{n_l} X_{li}'\iota_{T_l}\sum_{l=1}^{L}\sum_{i=1}^{n_l} T_l\mu_{li}\\ &\quad+\sum_{l=1}^{L}\sum_{i=1}^{n_l} X_{li}'\eta_l - \frac{1}{N}\sum_{l=1}^{L}\sum_{i=1}^{n_l} X_{li}'\iota_{T_l}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\iota_{T_l}'\eta_l\\ &\quad+\sum_{l=1}^{L}\sum_{i=1}^{n_l} X_{li}'\nu_{li} - \frac{1}{N}\sum_{l=1}^{L}\sum_{i=1}^{n_l} X_{li}'\iota_{T_l}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\iota_{T_l}'\nu_{li}),\end{aligned}$$

and

$$\tilde{\alpha}_3 - \alpha = \frac{1}{N}\sum_{l=1}^{L}\sum_{i=1}^{n_l}[\iota_{T_l}'X_{li}(\beta - \tilde{\beta}) + T_l\mu_{li} + \iota_{T_l}'\eta_l + \iota_{T_l}'\nu_{li}].$$

And it is not difficult to show that

$$n^{\frac{1}{4}}(\tilde{\beta}_3 - \beta) = \Sigma_7^{-1}\Omega_1 + o_p(1),$$

and

$$n^{\frac{1}{4}}(\tilde{\alpha}_3 - \alpha) = -\frac{1}{\sum_{l=1}^{L} m_l T_1}\Omega_2\Sigma_7^{-1}\Omega_1 + o_p(1),$$

where

$$\Sigma_7 = \sum_{l=1}^{L} m_l\mathbb{E}(X_{li}'X_{li}) - \frac{1}{\sum_{l=1}^{L} m_l T_l}(\sum_{l=1}^{L} m_l\mathbb{E}X_{li}'\iota_{T_l})(\sum_{l=1}^{L} m_l\mathbb{E}X_{li}'\iota_{T_l})',$$

and

$$\Omega_2 = \sum_{l=1}^{L} m_l \mathbb{E} X'_{li} \iota_{T_l}.$$

Hence, we have

$$\begin{aligned}
\sqrt{n}\tilde{\sigma}_{3\nu} &= \frac{\sqrt{n}}{\sum_{l=1}^{L} n_l T_l} \sum_{l=1}^{L} \sum_{i=1}^{n_l} \|y_{li} - \tilde{\alpha}\iota_{T_l} - X_{li}\tilde{\beta}\|^2 \\
&= \frac{\sqrt{n}}{\sum_{l=1}^{L} n_l T_l} \sum_{l=1}^{L} \sum_{i=1}^{n_l} \|(\alpha - \tilde{\alpha})\iota_{T_l} - X_{li}(\tilde{\beta} - \beta) + \mu_{li}\iota_{T_l} + \eta_l + \nu_{li}\|^2 \\
&= \frac{\sqrt{n}}{\sum_{l=1}^{L} n_l T_l} \sum_{l=1}^{L} \sum_{i=1}^{n_l} \|\nu_{li}\|^2 + \sigma_1^2 - \lambda + o_p(1),
\end{aligned}$$

where $\lambda = \frac{2}{\sum_{l=1}^{L} m_l T_l} \Omega'_1 \Sigma_7^{-1} \Omega_1 + \frac{1}{(\sum_{l=1}^{L} m_l T_l)^2} (\Omega_2 \Sigma_7^{-1} \Omega_1)^2$. Together with the proof of Theorem 1, we derive the asymptotic distribution of $\mathrm{T}_{\mu\eta}^{IC*}$. The proof of Corollary 3 is completed. □

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