An Interpretation of the Condition for Precautionary Saving:

The Case of Greater Higher-Order Interest Rate Risk *

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This paper shows that an increase in interest rate risk via (m,n)th-order stochastic dominance induces precautionary saving if, and only if, the measure of $(k+1)$th-degree relative risk aversion exceeds k for all $k=m$, ..., n. This result has the following interpretation. On the one hand, the measures of $(k+1)$th-degree relative risk aversion for all $k=m$, ..., n capture the prudence effect with respect to a risk increase via (m,n)th-order stochastic dominance, which favors precautionary saving. On the other hand, the thresholds, $k=m$, ..., n, measure the elasticity of the change in the kth moment of future income with respect to saving. The adverse changes in higher moments of future income for all $k=m$, ..., n when saving increases give rise to the risk aversion effect with respect to a risk increase via (m,n)th-order stochastic dominance, which limits precautionary saving. The necessary and sufficient condition for precautionary saving simply states that the prudence effect dominates the risk aversion effect, thereby making precautionary saving prevail.

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1. Introduction

In a recent paper in this Journal, Magnani (2017) offers a new interpretation of the necessary and sufficient condition for precautionary saving when interest rate risk prevails, i.e., the measure of relative prudence, $-xu'''(x)/u''(x)$, exceeds two for all relevant values of x, where $u(x)$ is a von Neumann-Morgenstern utility function. Magnani (2017) shows that the threshold of two comes from the elasticity of the variance of future income with respect to saving.

The purpose of this paper is twofold. First, we derive the necessary and sufficient conditions for precautionary saving when the increase in interest rate risk is characterized by a general notion of (m, n)th-order stochastic dominance, which is essentially nth-order stochastic dominance in that the first $m-1$ moments of the interest rate risk are preserved (Liu, 2014; Ebert et al., 2018).[1] Two extreme cases are in order. When $m = 1$, $(1, n)$th-order stochastic dominance reduces to the regular nth-order stochastic dominance. When $m = n$, (n, n)th-order stochastic dominance becomes nth-degree risk changes in the sense of Ekern (1980). Indeed, Eeckhoudt and Schlesinger (2008) examine these two extreme cases and show that precautionary saving prevails if, and only if, the measure of $(k+1)$th-degree relative risk aversion, $-xu^{(k+1)}(x)/u^{(k)}(x)$, exceeds k for all relevant values of x and for all $k = 1, \ldots, n$ in the former extreme case, and for $k = n$ in the latter extreme case, where $u^{(k)}(x) = \mathrm{d}^k u(x)/\mathrm{d}x^k$ is the kth derivative of $u(x)$.[2] In the general case of (m, n)th-order stochastic dominance, we show that the necessary and sufficient condition for precautionary saving is that the measure of $(k+1)$th-degree relative risk aversion exceeds k for all relevant values of x and for all $k = m, \ldots, n$, which includes the results of Eeckhoudt and Schlesinger (2008) for $m = 1$ and $m = n$.

Second, we offer an interpretation of our necessary and sufficient condition for pre-

[1] The two integers, n and m, satisfy that $n \geq m \geq 1$.

[2] Throughout the paper, we use the notation, $f^{(k)}(x) = \mathrm{d}^k f(x)/\mathrm{d}x^k$, to denote the kth derivative of the function, $f(x)$. For the first, second, and third derivatives of $f(x)$, we use the usual notation, $f'(x)$, $f''(x)$, and $f'''(x)$, respectively.

cautionary saving along the lines of Magnani (2017). As shown by Liu (2014), an individual whose utility function, $u(x)$, exhibits (m, n)th-degree mixed risk aversion, i.e., $(-1)^{k+1}u^{(k)}(x) > 0$ for all x and for all $k = m$, ..., n, dislikes any risk increases via (m, n)th-order stochastic dominance. The individual as such suffers a "pain" (in terms of reduction in expected utility) when there is an increase in interest rate risk via (m, n)th-order stochastic dominance. We refer to this as the risk aversion effect with respect to a risk increase via (m, n)th-order stochastic dominance, which induces the individual to save less in order to reduce his risk exposure. We show that this effect is proportional to the elasticity of the change in the kth moment of future income with respect to saving, which is a constant equal to k. On the other hand, given that $-u'(x)$ exhibits (m, n)th-degree mixed risk aversion, the pain would be smaller should the individual be wealthier.[3] We refer to this as the prudence effect with respect to a risk increase via (m, n)th-order stochastic dominance, which induces the individual to save more in order to increase his wealth level. We show that this effect is proportional to the elasticity of the kth-degree risk aversion with respect to saving, which is equal to the measure of $(k + 1)$th-degree relative risk aversion. Hence, for the individual who exhibits $(m, n + 1)$th-degree mixed risk aversion, the necessary and sufficient condition for precautionary saving requires that the prudence effect dominates the risk aversion effect with respect to a risk increase via (m, n)th-order stochastic dominance, which has to hold for all relevant values of x and for all $k = m$, ..., n given that the increase in interest rate risk via (m, n)th-order stochastic dominance is arbitrarily chosen.

The rest of this paper is organized as follows. In the next section, we define (m, n)th-degree mixed risk aversion and (m, n)th-order stochastic dominance, where these two concepts are shown to be closely related. In Section 3, we derive the necessary and sufficient condition for precautionary saving when there is an increase in interest rate risk via (m, n)th-order stochastic dominance. In Section 4, we offer an interpretation of our necessary and sufficient condition for precautionary saving. We relate this interpretation to the choice

[3]It is worth pointing out that $-u'(x)$ having (m, n)th-degree mixed risk aversion corresponds to $u(x)$ having $(m + 1, n + 1)$th-degree mixed risk aversion.

between two 50-50 lotteries with a preference for harm disaggregation. The final section concludes.

2. Notation and preliminaries

Consider an individual who has random wealth, $\tilde{x}$, that takes on values in $[a, b]$, where $a < b$. The individual possesses a von Neumann-Morgenstern utility function, $u(x)$, defined over his wealth level, $x \in [a, b]$. We state the definition of (m, n)th-degree mixed risk aversion as follows, where n and m are two integers such that $n \geq m \geq 1$, and $u^{(k)}(x) = \mathrm{d}^k u(x)/\mathrm{d}x^k$ is the kth derivative of $u(x)$.

Definition 1. For any two integers, n and m, such that $n \geq m \geq 1$ and any utility function, $u(x)$, we say that $u(x)$ exhibits (m, n)th-degree mixed risk aversion if $(-1)^{k+1}u^{(k)}(x) > 0$ for all $x \in [a, b]$ and for all $k = m$, ..., n.

When $m = n$, the notion of (m, n)th-degree mixed risk aversion reduces to the regular nth-degree risk aversion in that $(-1)^{n+1}u^{(n)}(x) > 0$ for all $x \in [a, b]$. As shown by Eeckhoudt and Schlesinger (2006), mixed risk aversion characterizes the common preferences in that individuals prefer to disaggregate risks across different states of nature. If $u(x)$ satisfies $(1, n)$th-degree mixed risk aversion, letting n go to infinity allows $u(x)$ to have all odd derivatives positive and all even derivatives negative, thereby rendering $u(x)$ to be completely monotone (Brockett and Golden, 1987; Caballé and Pomansky, 1996).

Let $F(x)$ and $G(x)$ be the cumulative distribution functions (CDFs) of two random variables, $\tilde{x}^h$ and $\tilde{x}^\ell$, respectively, over support $[a, b]$, where $F(a) = G(a) = 0$ and $F(b) = G(b) = 1$. Denote $F_1(x) = F(x)$ and

$$F_k(x) = \int_a^x F_{k-1}(y)\mathrm{d}y, \tag{1}$$

for all $x \in [a, b]$ and for all $k = 2, ..., n$, where $n \geq 2$. Applying integration by parts to Eq. (1) yields

$$\begin{aligned} F_k(x) &= xF_{k-1}(x) - \int_a^x y\mathrm{d}F_{k-1}(y) = \int_a^x (x-y)F_{k-2}(y)\mathrm{d}y \\ &= -\frac{1}{2}\int_a^x F_{k-2}(y)\mathrm{d}(x-y)^2 = \frac{1}{2}\int_a^x (x-y)^2\mathrm{d}F_{k-2}(y) = \cdots \\ &= \frac{1}{(k-1)!}\int_a^x (x-y)^{k-1}\mathrm{d}F(y) = \frac{\mathrm{E}[\max(x-\tilde{x}^h, 0)^{k-1}]}{(k-1)!}, \end{aligned} \tag{2}$$

for all $x \in [a, b]$ and for all $k = 2, ..., n$, where $\mathrm{E}[\cdot]$ is the expectations operator. Similar notation applies to $G(x)$.

The following general characterization of (m, n)th-order stochastic dominance is adopted from Liu (2014) (see also Ebert et al., 2018).

Definition 2. For any two integers, n and m, such that $n \geq m \geq 1$ and any two random variables, $\tilde{x}^h$ and $\tilde{x}^\ell$, we say that $\tilde{x}^h$ is riskier than $\tilde{x}^\ell$ via (m, n)th-order stochastic dominance if $F_k(b) = G_k(b)$ for all $k = 2, ..., m$, $F_k(b) \geq G_k(b)$ for all $k = m+1, ..., n$, and $F_n(x) \geq G_n(x)$ for all $x \in [a, b]$.

Given that $F_k(b) = G_k(b)$ for all $k = 2, ..., m$, it is evident from Eq. (2) that the first $m-1$ moments of $\tilde{x}^h$ and $\tilde{x}^\ell$ must be the same, i.e., $\mathrm{E}[(\tilde{x}^h)^{k-1}] = \mathrm{E}[(\tilde{x}^\ell)^{k-1}]$ for all $k = 2, ..., m$. Following the approach of Fishburn (1980) and O'Brien (1984), we can show that a necessary condition for $F(x)$ to be riskier than $G(x)$ via (m, n)th-order stochastic dominance is that there exists an integer, $k \in \{m, ..., n\}$, such that $\mathrm{E}[(\tilde{x}^h)^j] = \mathrm{E}[(\tilde{x}^\ell)^j]$ for all $j = m, ..., k-1$, and $(-1)^k\mathrm{E}[(\tilde{x}^h)^k] > (-1)^k\mathrm{E}[(\tilde{x}^\ell)^k]$. For example, if $F(x)$ is riskier than $G(x)$ via $(1, 2)$th-order stochastic dominance, then either $\tilde{x}^h$ has a smaller mean than $\tilde{x}^\ell$, or they have the same mean but the variance of $\tilde{x}^h$ is larger than that of $\tilde{x}^\ell$.

Many well-known definitions of increased risk are included as special cases. For example, $(1, n)$th-order stochastic dominance reduces to the regular nth-order stochastic dominance for all $n \geq 1$, and $(2, n)$th-order stochastic dominance defines mean-preserving nth-order stochastic dominance as in Denuit and Eeckhoudt (2013) for all $n \geq 2$. When $m = n$, Definition 2 becomes the notion of more nth-degree risk in the sense of Ekern (1980) for all $n \geq 1$. In this case, more second-degree risk refers to mean-preserving spreads in the sense of Rothschild and Stiglitz (1970). More third-degree risk is equivalent to an increase in downside risk *à la* Menezes et al. (1980), which moves risk from right to left while keeping the mean and variance intact. More fourth-degree risk is an increase in outer risk (Menezes and Wang, 2005) that has higher peaks and longer tails (i.e., more kurtotic) while keeping the mean, variance, and third central moment constant.

The concept of (m, n)th-degree mixed risk aversion and that of (m, n)th-order stochastic dominance are closely related, as stated in the following lemma, where a proof can be found in Liu (2014).

Lemma 1. For any two integers, n and m, such that $n \geq m \geq 1$ and any two random variables, $\tilde{x}^h$ and $\tilde{x}^\ell$, all individuals who exhibit (m, n)th-degree mixed risk aversion prefer $\tilde{x}^\ell$ to $\tilde{x}^h$, i.e., $\mathrm{E}[u(\tilde{x}^\ell)] > \mathrm{E}[u(\tilde{x}^h)]$, if, and only if, $\tilde{x}^h$ is riskier than $\tilde{x}^\ell$ via (m, n)th-order stochastic dominance.

Following Courbage et al. (2018) and Wong (2018), we define the (m, n)th-degree utility premium as follows:

$$\pi_u(w, \alpha) = \mathrm{E}[u(w + \alpha\tilde{x}^\ell)] - \mathrm{E}[u(w + \alpha\tilde{x}^h)], \tag{3}$$

where w is the wealth level and $\alpha > 0$ is a scalar. As is evident from Eq. (3), $\pi_u(w, \alpha)$ measures the "pain" associated with facing the passage from the more favorable risk, $\alpha\tilde{x}^\ell$, to the less favorable risk, $\alpha\tilde{x}^h$, where the risk increase is specified by means of (m, n)th-order stochastic dominance. Given that $u(x)$ exhibits (m, n)th-degree mixed risk aversion, it follows from Lemma 1 that $\pi_u(w, \alpha) > 0$ for all w and for all $\alpha > 0$.

Kimball (1990, 1993) refers to $u'''(x) > 0$ as prudence or preferences for bearing a zero-mean risk in the wealthier states of nature. The prudence utility premium, introduced by Crainich and Eeckhoudt (2008), measures the increase in pain of facing a zero-mean risk in the presence of a sure loss, $\rho > 0$. Following Courbage et al. (2018) and Wong (2018), we extend the definition of Crainich and Eeckhoudt (2008) to the (m, n)th-degree prudence utility premium as follows:

$$\pi_p(w, \alpha) = \pi_u(w - \rho, \alpha) - \pi_u(w, \alpha). \tag{4}$$

Given than $u(x)$ exhibits (m, n)th-degree mixed risk aversion, $\pi_p(w, \alpha)$ measures the additional "pain" associated with facing the passage from the more favorable risk, $\alpha\tilde{x}^\ell$, to the less favorable risk, $\alpha\tilde{x}^h$, when the individual suffers a sure loss, $\rho > 0$, where the risk increase is specified by means of (m, n)th-order stochastic dominance. It follows from Eq. (4) that $\pi_p(w, \alpha) > 0$ for all w and for all $\alpha > 0$ if, and only if, $\mathrm{E}[u'(w+\alpha\tilde{x}^\ell)] < \mathrm{E}[u'(w+\alpha\tilde{x}^h)]$, which, from Lemma 1, is true if, and only if, $u(x)$ exhibits $(m+1, n+1)$th-degree mixed risk aversion. Hence, we can interpret $(m+1, n+1)$th-degree mixed risk aversion as "prudence with respect to risk increases via (m, n)th-order stochastic dominance" or preferences for bearing risk increases via (m, n)th-order stochastic dominance in the wealthier states of nature (Courbage et al., 2018; Wong, 2018).

3. Changes in interest rate risk and precautionary saving

Consider the model of Eeckhoudt and Schlesinger (2008) wherein an individual lives for two periods. The individual has initial income, $w_\circ > 0$, that has to be divided into saving, s, and consumption, $w_\circ - s$, in the first period. Any amount of saving earns a gross rate of interest, $\tilde{x}^\ell$, so that the individual consumes $s\tilde{x}^\ell$ in the second period, where $\tilde{x}^\ell$ is a positive random variable distributed according to the CDF, $G(x)$, over support $[a, b]$ with $0 < a < b$.

The individual has preferences that are intertemporally separable, and are represented by a von Neumann-Morgenstern utility function, $u(w)$, and a personal rate of discounting for delaying the utility of future consumption, δ. We assume that $u'(w) > 0$ and $u''(w) < 0$ for all $w \geq 0$, and that $\lim_{w\to 0} u'(w) = \infty$. The individual's ex-ante decision problem is to choose an amount of saving, s, so as to maximize his lifetime utility of consumption:

$$\max_{s\in[0,w_\circ]} \; u(w_\circ - s) + \frac{1}{1+\delta}\int_a^b u(sx)\mathrm{d}G(x). \tag{5}$$

The first-order condition for program (5) is given by

$$u'(w_\circ - s^\ell) = \frac{1}{1+\delta}\int_a^b u'(s^\ell x)x\mathrm{d}G(x), \tag{6}$$

where s^ℓ is the optimal amount of saving. The second-order condition for program (5) is satisfied given that $u''(w) < 0$ for all $w \geq 0$. Since $\lim_{w\to 0} u'(w) = \infty$, it follows from Eq. (6) that $0 < s^\ell < w_\circ$.

Consider now that the gross rate of interest, $\tilde{x}^\ell$, changes to $\tilde{x}^h$, where $\tilde{x}^h$ is riskier than $\tilde{x}^\ell$ via (m,n)th-order stochastic dominance. Let $F(x)$ be the CDF of $\tilde{x}^h$ over support $[a,b]$. The first-order condition becomes

$$u'(w_\circ - s^h) = \frac{1}{1+\delta}\int_a^b u'(s^h x)x\mathrm{d}F(x), \tag{7}$$

where s^h is the optimal amount of saving. Since $\lim_{w\to 0} u'(w) = \infty$, it follows from Eq. (7) that $0 < s^h < w_\circ$.

Define the following function:

$$\phi(s) = u(w_\circ - s^\ell - s) + \frac{1}{1+\delta}\int_a^b u\Big((s^\ell + s)x\Big)\mathrm{d}F(x). \tag{8}$$

Differentiating Eq. (8) twice with respect to s yields

$$\phi''(s) = u''(w_\circ - s^\ell - s) + \frac{1}{1+\delta}\int_a^b u''\Big((s^\ell + s)x\Big)x^2\mathrm{d}F(x) < 0, \tag{9}$$

for all $s \in [-s^\ell, w_\circ - s^\ell]$. Hence, it follows from Eqs. (7) and (9) that precautionary saving prevails, i.e., $s^h > s^\ell$, if, and only if, $\phi'(0) > 0$. We state and prove our main result as follows.

Proposition 1. For any two integers, n and m, such that $n \geq m \geq 1$, and any gross rate of interest, $\tilde{x}^\ell$, such that the individual who exhibits $(m, n+1)$th-degree mixed risk aversion optimally saves $s^\ell \in (0, w_\circ)$, the following two statements are equivalent:

(i) The individual's utility function, $u(w)$, has the measure of $(k+1)$th-degree relative risk aversion that satisfies

$$-w\frac{u^{(k+1)}(w)}{u^{(k)}(w)} > k, \tag{10}$$

for all $w \in [s^\ell a, s^\ell b]$ and for all $k = m$, ..., n.

(ii) The individual optimally saves more, i.e., $s^h \in (s^\ell, w_\circ)$ whenever the gross rate of interest, $\tilde{x}^\ell$, changes to $\tilde{x}^h$ such that $\tilde{x}^h$ is riskier than $\tilde{x}^\ell$ via (m, n)th-order stochastic dominance.

Proof. See Appendix A. □

Eeckhoudt and Schlesinger (2008) examine the two extreme cases wherein $m = 1$ and $m = n$. Proposition 1 as such generalizes their results by providing the necessary and sufficient condition under which the individual optimally saves more in response to an increase in interest rate risk via (m, n)th-order stochastic dominance, where $n \geq m \geq 1$.

4. An interpretation of the condition for precautionary saving

Following Magnani (2017), we offer an interpretation of condition (10). To this end, we write

$$\phi'(0) = -\lim_{s\to 0}\frac{u(w_\circ - s^\ell) - u(w_\circ - s^\ell - s)}{s}$$

$$+\frac{1}{1+\delta}\lim_{s\to 0}\frac{\mathrm{E}\Big[u\Big((s^\ell + s)\tilde{x}^h\Big) - u(s^\ell\tilde{x}^h)\Big]}{s}. \tag{11}$$

Using Eq. (6), we have

$$\lim_{s\to 0}\frac{u(w_\circ - s^\ell) - u(w_\circ - s^\ell - s)}{s} = \frac{1}{1+\delta}\lim_{s\to 0}\frac{\mathrm{E}\Big[u\Big((s^\ell + s)\tilde{x}^\ell\Big) - u(s^\ell\tilde{x}^\ell)\Big]}{s}. \tag{12}$$

Substituting Eq. (12) into Eq. (11) yields

$$\phi'(0) = \frac{1}{1+\delta}\lim_{s\to 0}\frac{\mathrm{E}\Big[u\Big((s^\ell + s)\tilde{x}^h\Big) - u(s^\ell\tilde{x}^h)\Big] - \mathrm{E}\Big[u\Big((s^\ell + s)\tilde{x}^\ell\Big) - u(s^\ell\tilde{x}^\ell)\Big]}{s}. \tag{13}$$

Consider the following two 50-50 lotteries: (i) lottery A gives either $s^\ell\tilde{x}^\ell$ or $(s^\ell + s)\tilde{x}^h$, and (ii) lottery B gives either $s^\ell\tilde{x}^h$ or $(s^\ell + s)\tilde{x}^\ell$, where $s > 0$. As such, lottery A is preferred to lottery B if, and only if,

$$\frac{1}{2}\mathrm{E}[u(s^\ell\tilde{x}^\ell)] + \frac{1}{2}\mathrm{E}\Big[u\Big((s^\ell + s)\tilde{x}^h\Big)\Big] > \frac{1}{2}\mathrm{E}[u(s^\ell\tilde{x}^h)] + \frac{1}{2}\mathrm{E}\Big[u\Big((s^\ell + s)\tilde{x}^\ell\Big)\Big]. \tag{14}$$

It follows from Eqs. (13) and (14) that $\phi'(0) > 0$ if, and only if, lottery A is preferred to lottery B for small enough s.

To facilitate the exposition, we fix a point, $x^\circ \in [a, b]$. Lottery A is preferred to lottery B if, and only if, Eq. (14) holds, which can be written as

$$\begin{aligned}&\mathrm{E}[u(s^\ell\tilde{x}^\ell)] - \mathrm{E}[u(s^\ell\tilde{x}^h)] - \{\mathrm{E}[u(sx^\circ + s^\ell\tilde{x}^\ell)] - \mathrm{E}[u(sx^\circ + s^\ell\tilde{x}^h)]\}\\ &> \mathrm{E}\Big[u\Big((s^\ell + s)\tilde{x}^\ell\Big)\Big] - \mathrm{E}\Big[u\Big((s^\ell + s)\tilde{x}^h\Big)\Big] - \{\mathrm{E}[u(sx^\circ + s^\ell\tilde{x}^\ell)] - \mathrm{E}[u(sx^\circ + s^\ell\tilde{x}^h)]\}.\end{aligned} \tag{15}$$

Let $\rho = sx^\circ > 0$. We can use Eqs. (3) and (4) to write Eq. (15) as

$$\pi_p(sx^\circ, s^\ell) > \pi_u(0, s^\ell + s) - \pi_u(sx^\circ, s^\ell). \tag{16}$$

Holding the scale of the risk constant at s^ℓ, lottery A has better risk apportionment than lottery B in the sense that the less favorable risk, $\tilde{x}^h$, is assigned to the state with a positive wealth level, i.e., sx°, and the more favorable risk, $\tilde{x}^\ell$, is assigned to the state with zero wealth level. Given that $u(w)$ exhibits $(m, n+1)$th-degree mixed risk aversion, the prudence effect with respect to a risk increase via (m, n)th-order stochastic dominance works in favor of lottery A over lottery B, which is captured by $\pi_p(sx^\circ, s^\ell) > 0$. On the other hand, holding the risk exposure constant at $s^\ell + s$, lottery B has the more favorable risk, $\tilde{x}^\ell$, with the full exposure, $s^\ell + s$, and the less favorable risk, $\tilde{x}^h$, with only the partial exposure, s^ℓ, as compared to lottery A. This difference in risk exposure gives rise to the risk aversion effect with respect to a risk increase via (m, n)th-order stochastic dominance, which is captured by $\pi_u(0, s^\ell + s) - \pi_u(sx^\circ, s^\ell)$ and works in favor of lottery B over lottery A. Hence, lottery A is preferred to lottery B if, and only if, the prudence effect dominates the risk aversion effect with respect to a risk increase via (m, n)th-order stochastic dominance, as shown by Eq. (16).

Using Eq. (16), we can write Eq. (13) as

$$\phi'(0) = \frac{1}{1+\delta}\left[\lim_{s\to 0}\frac{\pi_p(sx^\circ, s^\ell)}{s} - \lim_{s\to 0}\frac{\pi_u(0, s^\ell + s) - \pi_u(sx^\circ, s^\ell)}{s}\right]. \tag{17}$$

Since $\tilde{x}^h$ is riskier than $\tilde{x}^\ell$ via (m, n)th-order stochastic dominance, there exists an integer, $k \in \{m, ..., n\}$, such that $\mathrm{E}[(\tilde{x}^h)^j] = \mathrm{E}[(\tilde{x}^\ell)^j]$ for all $j = 1, ..., k-1$, and $(-1)^k\mathrm{E}[(\tilde{x}^h)^k] > (-1)^k\mathrm{E}[(\tilde{x}^\ell)^k]$ (Fishburn, 1980; O'Brien, 1984). Let $\Delta_j(\alpha) = \mathrm{E}[(\alpha\tilde{x}^\ell - \alpha x^\circ)^j] - \mathrm{E}[(\alpha\tilde{x}^h - \alpha x^\circ)^j]$ be the difference between the jth (non-central) moments of $\alpha\tilde{x}^\ell$ and $\alpha\tilde{x}^h$ about αx° for any scalar $\alpha > 0$. Then, we have $\Delta_j(\alpha) = 0$ for all $j = 1, ..., k-1$ and $\Delta_k(\alpha) = \alpha^k\{\mathrm{E}[(\tilde{x}^\ell)^k] - \mathrm{E}[(\tilde{x}^h)^k]\}$ so that $(-1)^{k+1}\Delta_k(\alpha) > 0$. Applying kth-order Taylor expansions to the right-hand side of Eq. (17) around $x = x^\circ$ yields[4]

$$\begin{aligned}\phi'(0) &\approx \frac{u^{(k)}(s^\ell x^\circ)\Delta_k(s^\ell)}{k!(1+\delta)s^\ell} \\ &\quad\times\left[-\frac{s^\ell}{u^{(k)}(s^\ell x^\circ)}\frac{\partial u^{(k)}\Big((s^\ell + s)x^\circ\Big)}{\partial s}\bigg|_{s=0} - \frac{s^\ell}{\Delta_k(s^\ell)}\frac{\partial \Delta_k(s^\ell + s)}{\partial s}\bigg|_{s=0}\right] \\ &= \frac{u^{(k)}(s^\ell x^\circ)\Delta_k(s^\ell)}{k!(1+\delta)s^\ell} \times \left[-s^\ell x^\circ\frac{u^{(k+1)}(s^\ell x^\circ)}{u^{(k)}(s^\ell x^\circ)} - k\right],\end{aligned} \tag{18}$$

[4]See Appendix B for the derivation.

where $u^{(k)}(s^\ell x^\circ)\Delta_k(s^\ell) > 0$. The first term on the right-hand side of Eq. (18) shows that the prudence effect with respect to a risk increase via (m,n)th-order stochastic dominance is proportional to the elasticity of the kth-degree risk aversion, $(-1)^{k+1}u^{(k)}\Big((s^\ell + s)x^\circ\Big)$, with respect to s, which is equal to the measure of $(k+1)$th-degree relative risk aversion for small enough s. On the other hand, the second term on the right-hand side of Eq. (18) shows that the risk aversion effect with respect to a risk increase via (m,n)th-order stochastic dominance is proportional to the elasticity of the change in the kth (non-central) moment of future income, $(s^\ell + s)\tilde{x}$, about $(s^\ell + s)x^\circ$ with respect to s, which is a constant equal to k for small enough s. Inspection of Eq. (18) reveals that $\phi'(0) > 0$, i.e., $s^h > s^\ell$, should the prudence effect dominate the risk aversion effect with respect to a risk increase via (m,n)th-order stochastic dominance (Chiu et al., 2012; Eeckhoudt et al., 2009). Since $F(x)$ can be any CDF that is riskier than $G(x)$ via (m,n)th-order stochastic dominance, condition (10) must hold for all $w \in [s^\ell a, s^\ell b]$ and for all $k = m$, ..., n.

5. Conclusion

In this paper, we generalize an interpretation of the condition for precautionary saving offered by Magnani (2017) to the case wherein there is an increase in interest rate risk via (m,n)th-order stochastic dominance, where n and m are two integers such that $n \geq m \geq 1$. We first show that the necessary and sufficient condition for precautionary saving, i.e., $s^h > s^\ell$, in this case is that the measure of $(k+1)$th-degree relative risk aversion, $-wu^{(k+1)}(w)/u^{(k)}(w)$, exceeds k for all $w \in [s^\ell a, s^\ell b]$ and for all $k = m$, ..., n. Indeed, Eeckhoudt and Schlesinger (2008) derive similar necessary and sufficient conditions when $m = 1$ and $m = n$. We further show that there are two countervailing effects that jointly determine the precautionary saving motive. On the one hand, the measures of $(k+1)$th-degree relative risk aversion for all $k = m$, ..., n are shown to capture the prudence effect with respect to a risk increase via (m,n)th-order stochastic dominance, which favors precautionary saving.

On the other hand, the threshold of k is shown to come from the elasticity of the change in the kth moment of future income with respect to saving, which includes the finding of Magnani (2017) as a special case wherein $k = 2$. Such adverse changes in higher moments of future income for all $k = m$, ..., n when saving increases give rise to the risk aversion effect with respect to a risk increase via (m, n)th-order stochastic dominance, which limits precautionary saving. As such, the necessary and sufficient condition for precautionary saving describes this trade-off such that the prudence effect dominates the risk aversion effect with respect to a risk increase via (m, n)th-order stochastic dominance, making precautionary saving optimal.

While this paper focuses on the precautionary saving motive, the analysis is completely general and should be readily applicable to decision problems under uncertainty such as portfolio allocations (Hadar and Seo, 1990; Choi et al., 2001; De Donno et al., 2017), labor supply (Chiu and Eeckhoudt, 2010; Wang and Gong, 2013), prevention and cure in health decisions (Brianti et al., 2018), self-protection and effort choices (Wong, 2016, 2017), and many others. We leave these applications for future research.

Appendix

A. Proof of Proposition 1

To show that statement (i) implies statement (ii), we differentiate Eq. (8) with respect to s and evaluate the resulting derivative at $s = 0$ to yield

$$\begin{aligned}
\phi'(0) &= -u'(w_\circ - s^\ell) + \frac{1}{1+\delta}\int_a^b u'(s^\ell x)x\mathrm{d}F(x) \\
&= \frac{1}{1+\delta}\int_a^b u'(s^\ell x)x\mathrm{d}[F(x) - G(x)] \\
&= \frac{(s^\ell)^{-2}}{1+\delta}\Bigg\{\sum_{k=m}^{n-1}(-s^\ell)^{k+1}u^{(k)}(s^\ell b)\Bigg[-s^\ell b\frac{u^{(k+1)}(s^\ell b)}{u^{(k)}(s^\ell b)} - k\Bigg][F_{k+1}(b) - G_{k+1}(b)]
\end{aligned}$$

$$+\int_a^b (-s^\ell)^{n+1} u^{(n)}(s^\ell x)\left[-s^\ell x \frac{u^{(n+1)}(s^\ell x)}{u^{(n)}(s^\ell x)} - n\right][F_n(x) - G_n(x)]\mathrm{d}x\Big\}, \qquad \text{(A.1)}$$

where the second equality follows from Eq. (6), and the last equality follows from applying integration by parts and $F_k(b) = G_k(b)$ for all $k = 2, ..., m$. Since $F_k(b) \geq G_k(b)$ for all $k = m+1, ..., n$ and $F_n(x) \geq G_n(x)$ for all $x \in [a, b]$, it follows from Eq. (10) that the right-hand side of Eq. (A.1) is positive, rendering that $s^h > s^\ell$.

To show that statement (ii) implies statement (i), we suppose the contrary that there exist an integer, $k \in \{m, ..., n\}$, and a point, $w' \in [s^\ell a, s^\ell b]$, such that Eq. (10) does not hold. By continuity, we have

$$-w\frac{u^{(k+1)}(w)}{u^{(k)}(w)} \leq k, \qquad \text{(A.2)}$$

for all $w \in [w' - \varepsilon_1, w' + \varepsilon_2]$, where ε_1 and ε_2 are two small non-negative numbers such that $w' - \varepsilon_1 \geq s^\ell a$ and $w' + \varepsilon_2 \leq s^\ell b$. Let $x_1 = (w' - \varepsilon_1)/s^\ell$ and $x_2 = (w' + \varepsilon_2)/s^\ell$. It follows that $a \leq x_1 < x_2 \leq b$. Construct $F(x)$ such that $F(x) = G(x)$ for all $x \in [a, x_1] \bigcup [x_2, b]$, $F_j(x_2) = G_j(x_2)$ for all $j = 2, ..., k$, and $F_k(x) \geq G_k(x)$ for all $x \in [x_1, x_2]$. $F(x)$ as such has more kth-degree risk than $G(x)$ and thus $F(x)$ is riskier than $G(x)$ via (m, n)th-order stochastic dominance. In this case, Eq. (A.1) becomes

$$\phi'(0) = \frac{(s^\ell)^{k-1}}{1+\delta}\int_{x_1}^{x_2} (-1)^{k+1} u^{(k)}(s^\ell x)\left[-s^\ell x\frac{u^{(k+1)}(s^\ell x)}{u^{(k)}(s^\ell x)} - k\right][F_k(x) - G_k(x)]\mathrm{d}x. \qquad \text{(A.3)}$$

Since $F_k(x) \geq G_k(x)$ for all $x \in [x_1, x_2]$, it follows from Eq. (A.2) that the right-hand side of Eq. (A.3) is non-positive, rendering that $s^h \leq s^\ell$, a contradiction. Hence, Eq. (10) must hold for all $w \in [s^\ell a, s^\ell b]$ and for all $k = m, ..., n$.

B. Derivation of Eq. (18)

Applying kth-order Taylor expansions to $\mathrm{E}[u(s^\ell \tilde{x}^\ell)]$ and $\mathrm{E}[u(s^\ell \tilde{x}^h)]$ around $x = x^\circ$ yields

$$\mathrm{E}[u(s^\ell \tilde{x}^\ell)] - \mathrm{E}[u(s^\ell \tilde{x}^h)] \approx \sum_{j=0}^{k} \frac{1}{j!} u^{(j)}(s^\ell x^\circ)\{\mathrm{E}[(s^\ell \tilde{x}^\ell - s^\ell x^\circ)^j] - \mathrm{E}[(s^\ell \tilde{x}^h - s^\ell x^\circ)^j]\}$$

$$= \frac{1}{k!} u^{(k)}(s^\ell x^\circ)\Delta_k(s^\ell), \tag{A.4}$$

where the equality follows from $\mathrm{E}[(\tilde{x}^\ell)^j] = \mathrm{E}[(\tilde{x}^h)^j]$ for all $j = 1, \ldots, k-1$ and $\Delta_k(s^\ell) = (s^\ell)^k\{\mathrm{E}[(\tilde{x}^\ell)^k] - \mathrm{E}[(\tilde{x}^h)^k]\}$. Likewise, we have

$$\mathrm{E}[u(sx^\circ + s^\ell \tilde{x}^\ell)] - \mathrm{E}[u(sx^\circ + s^\ell \tilde{x}^h)] \approx \frac{1}{k!} u^{(k)}\Big((s^\ell + s)x^\circ\Big)\Delta_k(s^\ell), \tag{A.5}$$

and

$$\mathrm{E}\Big[u\Big((s^\ell + s)\tilde{x}^\ell\Big)\Big] - \mathrm{E}\Big[u\Big((s^\ell + s)\tilde{x}^h\Big)\Big] \approx \frac{1}{k!} u^{(k)}\Big((s^\ell + s)x^\circ\Big)\Delta_k(s^\ell + s). \tag{A.6}$$

Using Eqs. (A.4) and (A.5), we have

$$\lim_{s\to 0} \frac{\pi_p(sx^\circ, s^\ell)}{s} \approx \frac{\Delta_k(s^\ell)}{k!} \lim_{s\to 0} \frac{u^{(k)}(s^\ell x^\circ) - u^{(k)}\Big((s^\ell + s)x^\circ\Big)}{s}$$

$$= \frac{u^{(k)}(s^\ell x^\circ)\Delta_k(s^\ell)}{k!s^\ell} \times -\frac{s^\ell}{u^{(k)}(s^\ell x^\circ)} \frac{\partial u^{(k)}\Big((s^\ell + s)x^\circ\Big)}{\partial s}\bigg|_{s=0}$$

$$= \frac{u^{(k)}(s^\ell x^\circ)\Delta_k(s^\ell)}{k!s^\ell} \times -s^\ell x^\circ \frac{u^{(k+1)}(s^\ell x^\circ)}{u^{(k)}(s^\ell x^\circ)}. \tag{A.7}$$

Using Eqs. (A.5) and (A.6), we have

$$\lim_{s\to 0} \frac{\pi_u(0, s^\ell + s) - \pi_u(sx^\circ, s^\ell)}{s} \approx \frac{u^{(k)}(s^\ell x^\circ)}{k!} \lim_{s\to 0} \frac{\Delta_k(s^\ell + s) - \Delta_k(s^\ell)}{s}$$

$$= \frac{u^{(k)}(s^\ell x^\circ)\Delta_k(s^\ell)}{k!s^\ell} \times \frac{s^\ell}{\Delta_k(s^\ell)} \frac{\partial \Delta_k(s^\ell + s)}{\partial s}\bigg|_{s=0}$$

$$= \frac{u^{(k)}(s^\ell x^\circ)\Delta_k(s^\ell)}{k!s^\ell} \times k. \tag{A.8}$$

Substituting Eqs. (A.7) and (A.8) into the right-hand side of Eq. (17) yields Eq. (18).

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