Periodic and Localized Wave Patterns for Coupled Ablowitz-Ladik Systems with Negative Cross Phase Modulation

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Abstract

A new system of coupled Ablowitz-Ladik equations is introduced where cubic nonlinearities from intensities of both waveguide arrays are included. The Hirota bilinear transform is formulated and is used to derive breathers periodic in space or time. One spatially periodic solution is utilized to verify the lowest order conservation laws. Algebraically localized rogue wave modes with pulsating properties are obtained from breathers in the limit of large wave periods. Incorporating additional modes of cubic nonlinearities, namely, cross phase modulations, in two arrays of oscillators on an integer lattice can further enhance the modeling capability in optical physics.

Keywords

Ablowitz-Ladik equations; Hirota bilinear method; Discrete breathers; Rogue waves.

Research Highlights

● A new system of Ablowitz-Ladik equations with cross phase modulation is studied.

● The Hirota bilinear form is deduced and breathers are derived exactly.

● Rogue wave (localized) solutions with pulsations are generated by a long wave limit.

● Spatially periodic breathers are utilized to verify conservation laws.

Declarations of Financial Interests

None

1. Introduction

The evolution and dynamics of wave packets governed by hierarchies of nonlinear Schrödinger equations are widely applicable to many branches of physics, e.g. fluid mechanics [1] and optics [2]. Discrete versions of these nonlinear equations have also been studied intensively, both for their intrinsic interest in theoretical physics as well as their description of realistic practical situations, e.g. spatially localized modes in a periodic array of optical waveguides [2]. One example of evolution equation which allows analytical progress is the Ablowitz-Ladik system for oscillators on an integer lattice (t = time, n = integer, * = complex conjugate) [3–8]:

$$iu_{n,t} + \beta(u_{n+1} + u_{n-1} - 2u_n) + \sigma u_n u_n^*(u_{n+1} + u_{n-1}) = 0, u_n = u_n(t) . \tag{1}$$

The real parameters β, σ represent a measure of second order dispersion and cubic nonlinearity, in ways similar to the continuous counterpart [1,2],

$$i\Psi_t + \beta_0\Psi_{xx} + 2\sigma\Psi^2\Psi^* = 0 . \tag{2}$$

The focus here is coupled Ablowitz-Ladik systems which provide models for two arrays of oscillators. Various versions with contrasting features have been studied in the literature [9–11]. Examples of novel properties treated include branched dispersion [9],

$$i(u_n)_t = (1 + |u_n|^2)(v_{n+1} + v_{n-1}) , \quad i(v_n)_t = (1 + |v_n|^2)(u_{n+1} + u_{n-1}) ,$$

linear coupling [10],

$$i(u_n)_t + (1 + |u_n|^2)(u_{n+1} + u_{n-1}) + (v_{n+1} + v_{n-1}) = 0,$$

$$i(v_n)_t + (1 + |v_n|^2)(v_{n+1} + v_{n-1}) + (u_{n+1} + u_{n-1}) = 0,$$

and self-attractive nonlinear terms [11],

$$i(u_n)_t = (1+|u_n|^2)(v_n^* + v_{n-1}^*), \quad i(v_n)_t = -(1+|v_n|^2)(u_{n+1}^* + u_n^*).$$

Terminology from optics will prove to be instructive. For plane wave solutions of coupled nonlinear Schrödinger equations (Φ_0, Ψ_0 = constants)

$$i\Phi_t + \Phi_{xx} + (|\Phi|^2 + |\Psi|^2)\Phi = 0, \qquad i\Psi_t + \Psi_{xx} + (|\Phi|^2 + |\Psi|^2)\Psi = 0,$$

$$\Phi = \Phi_0 \exp[i(|\Phi_0|^2 + |\Psi_0|^2)t], \qquad \Psi = \Psi_0 \exp[i(|\Phi_0|^2 + |\Psi_0|^2)t],$$

one can associate $|\Phi|^2\Phi$ and $|\Psi|^2\Phi$ with the phase change due to the intensity of the light beam (Φ) itself and the co-propagating light beam (Ψ) respectively. Hence the terms self-phase-modulation (SPM) and cross-phase-modulation (XPM) will be employed [2].

A remark on the physical applications of the Ablowitz-Ladik systems is in order. One issue is the competition between nonlinearity and randomness, where a random potential can destroy or degrade the stability property of a soliton [12]. Another application is the switching dynamics of discrete solitons moving along two coupled arrays forming a Möbius strip. The Ablowitz-Ladik system provides a realistic model for the potential of the topological switches and the monopole spectra in parameter space [13]. A linear interchain coupling can achieve a well defined switching time where soliton modes move from one array to the other.

The goal here is to propose theoretically yet another variant of coupled Ablowitz-Ladik systems which possesses both SPM and XPM. The Hirota bilinear form is formulated (Section 2). Exact periodic (breather) and localized (rogue wave) solutions are obtained for special cases (Section 3). One spatially periodic solution is employed to verify the existence of conservation laws. Discussions, conclusions and applications of the present work will be discussed in Section 4.

2. Coupled Ablowitz-Ladik Systems

We shall investigate a discrete system which displays nonlinearities arising from both SPM and XPM [2]. Furthermore, we allow for SPM and XPM of different signs:

$$i(A_n)_t + \beta(A_{n+1} + A_{n-1} - 2A_n) + \sigma(|A_n|^2 - |B_n|^2)(A_{n+1} + A_{n-1}) = 0\,, \; A_n = A_n(t)\,, \tag{3a}$$

$$i(B_n)_t - \beta(B_{n+1} + B_{n-1} - 2B_n) - \sigma(|A_n|^2 - |B_n|^2)(B_{n+1} + B_{n-1}) = 0\,, \; B_n = B_n(t)\,. \tag{3b}$$

We first implement a change of variable:

$$A_n = \phi_n \exp(-2\beta i t)\,, \quad B_n = \psi_n \exp(2\beta i t)\,, \tag{4}$$

to derive

$$i(\phi_n)_t + [\beta + \sigma(|\phi_n|^2 - |\psi_n|^2)](\phi_{n+1} + \phi_{n-1}) = 0\,, \tag{5a}$$

$$i(\psi_n)_t - [\beta + \sigma(|\phi_n|^2 - |\psi_n|^2)](\psi_{n+1} + \psi_{n-1}) = 0\,. \tag{5b}$$

The plane wave or continuous wave is given by

$$\phi_n = i^n \rho_1 \exp[i(k_1 n - \omega_1 t)]\,, \; \psi_n = i^n \rho_2 \exp[i(k_2 n - \omega_2 t)]\,, \tag{6a}$$

$$\omega_1 = 2[\beta + \sigma(\rho_1^2 - \rho_2^2)]\sin k_1\,, \; \omega_2 = -2[\beta + \sigma(\rho_1^2 - \rho_2^2)]\sin k_2\,. \tag{6b}$$

The Hirota bilinear transform which proves to be effective for the single component case will now be generalized [5,14]:

$$\phi_n = i^n \frac{G_n}{f_n} \exp[i(k_1 n - \omega_1 t)]\,, \quad \psi_n = i^n \frac{H_n}{f_n} \exp[i(k_2 n - \omega_2 t)]\,. \tag{7}$$

The wavenumber and angular frequency of the plane wave in the background will still be basically Eq. (6b) but is written in a more instructive form

$$\omega_1 = -i[\beta + \sigma(\rho_1^2 - \rho_2^2)][\exp(ik_1) - \exp(-ik_1)],$$

$$\omega_2 = i[\beta + \sigma(\rho_1^2 - \rho_2^2)][\exp(ik_2) - \exp(-ik_2)],$$

together with the additional constraint

$$\beta + \sigma(\rho_1^2 - \rho_2^2) = -1. \tag{8}$$

The bilinear form is then given by

$$D_t G_n \cdot f_n = (G_{n+1} f_{n-1} - G_n f_n)\exp(ik_1) + (G_n f_n - G_{n-1} f_{n+1})\exp(-ik_1), \tag{9a}$$

$$D_t H_n \cdot f_n = (H_n f_n - H_{n+1} f_{n-1})\exp(ik_2) + (H_{n-1} f_{n+1} - H_n f_n)\exp(-ik_2), \tag{9b}$$

$$f_{n+1} f_{n-1} + \beta f_n^2 + \sigma(|G_n|^2 - |H_n|^2) = 0 . \tag{9c}$$

Following well established procedure in locating breather by bilinear transform, we adopt the expansion (M real, a_n, b_n, n = 1, 2, complex, η_1, η_2 are phase factors):

$$G_n = \rho_1\{1 + a_1\exp(pn - \Omega t + \eta_1) + a_2\exp(p^*n - \Omega^*t + \eta_2)$$
$$+ Ma_1a_2\exp[(p + p^*)n - (\Omega + \Omega^*)t + \eta_1 + \eta_2]\}, \tag{10a}$$

$$H_n = \rho_2\{1 + b_1\exp(pn - \Omega t + \eta_1) + b_2\exp(p^*n - \Omega^*t + \eta_2)$$
$$+ Mb_1b_2\exp[(p + p^*)n - (\Omega + \Omega^*)t + \eta_1 + \eta_2]\}, \tag{10b}$$

$$f_n = 1 + \exp(pn - \Omega t + \eta_1) + \exp(p^*n - \Omega^*t + \eta_2)$$
$$+ M\exp[(p + p^*)n - (\Omega + \Omega^*)t + \eta_1 + \eta_2] . \tag{10c}$$

The phase factors η_1, η_2 originates from the flexibility in setting the origins of the spatial coordinate n and time t. In subsequent calculations we shall select special values to obtain nonsingular rational solutions. The expansion Eq. (10) will satisfy bilinear equations (9) only if a set of constraints is satisfied. For arbitrary input parameters (k_1, k_2, β, σ, ρ_1, ρ_2, p) this remains an open question.

To reduce algebraic complexity in the remaining of this paper, we shall take for simplicity

$$k_1 = -k_2 = k,\ \omega_1 = \omega_2 = \omega = 2[\beta + \sigma(\rho_1^2 - \rho_2^2)]\sin k\ . \tag{11}$$

Under such assumption, the formulations simplify considerably:

$$a_1 = \frac{\Omega+\mu}{\Omega+\lambda},\ a_2 = \frac{1}{a_1^*}, b_1 = \frac{\Omega-\lambda}{\Omega-\mu},\ b_2 = \frac{1}{b_1^*}, \tag{12a}$$

$$\lambda = \exp(ik)[\exp(p) - 1] + \exp(-ik)[1 - \exp(-p)]\ , \tag{12b}$$

$$\mu = \exp(ik)[1 - \exp(-p)] + \exp(-ik)[\exp(p) - 1]\ , \tag{12c}$$

and the set of simplified constraints is listed in the Appendix.

If we further assume equal amplitude for the background waves:

$$\rho_1 = \rho_2 = \rho\ , \tag{13}$$

the dispersion relation ($\Omega = \Omega(p)$) for real p is given by a fourth order polynomial:

$$(\cosh p - 1)(\Omega^2 - \lambda^2)(\Omega^2 - \mu^2) = \sigma\rho^2(\lambda - \mu)^2(\lambda + \mu)\Omega\ . \tag{14}$$

We now proceed to calculate special exact solutions.

3. Breathers and Rogue Waves

Breathers are solutions periodic in the spatial variable n or time t. In the literature, those modes periodic in space and time are usually termed the Akhmediev and Kuznetsov-Ma breathers respectively [15].

3.1 Breathers quasi-periodic in time with rogue waves as special limits

If we take p as purely real, complex values of Ω will lead to wave pattern with a certain degree of periodic character due to the imaginary parts. We shall defer a detailed study on periodic modes to future works and instead focus the attention here on the case of long wavelength limit ($p \to 0$), where the solution now becomes a rogue wave mode (Figure 1). Rogue waves are localized in space and time and have received intensive attention [16,17]. The novel feature here is that the amplitude of the rogue wave pulsates instead of just simply grows to a maximum and then subsides.

Analytically, on taking the $p \to 0$ limit, the leading order angular frequency $\Omega_0 = a + ib$ will satisfy a reduced form of the dispersion relation Eq. (14), namely,

$\Omega = p\Omega_0 + p^2\Omega_1 + O(p^3)$,

$$\left(\Omega_0^2 - 4\cos^2 k\right)^2 + 32\sigma\rho^2 \cos k \sin^2 k\, \Omega_0 = 0 \qquad (15)$$

A remarkable property for subsequent and future discussion is the flexibility in changing the sign of the parameter σ. If $\Omega_0 = a + ib$ is a solution of Eq. (15) for a given σ, then $\Omega_{00} = -a + ib$ also solves the governing equations with σ replaced by $-\sigma$. Further calculations show that $\Omega_1 = 0$.

The parameters in the bilinear formulation defined by Eq. (12) can also be expanded in power series of p:

$$a_1 = 1 - \frac{2i\sin k}{\Omega_0 + \exp(ki) + \exp(-ki)} p - \frac{2\sin^2 k}{[\Omega_0 + \exp(ki) + \exp(-ki)]^2} p^2 + O(p^3),$$

$$b_1 = 1 - \frac{2i\sin k}{\Omega_0 - \exp(ki) - \exp(-ki)} p - \frac{2\sin^2 k}{[\Omega_0 - \exp(ki) - \exp(-ki)]^2} p^2 + O(p^3),$$

$$\mu = (2\cos k)p - i(\sin k)p^2 + O(p^3), \lambda = (2\cos k)p + i(\sin k)p^2 + O(p^3),$$

$$M = 1 + \left(\frac{4 - a^2 - b^2}{4b^2}\right) p^2 + O(p^4).$$

By taking the phase factors as $\exp(\eta_1) = \exp(\eta_2) = -1$ and defining $\theta = n - \Omega_0 t$, the long wave expansion of f_n gives algebraic expression

$$\begin{aligned} f_n &= 1 - \exp(p\theta) - \exp(p\theta^*) + \left[1 + \left(\frac{4 - a^2 - b^2}{4b^2}\right) p^2\right] \exp(p\theta + p\theta^*) + O(p^3) \\ &= p^2 \left(\frac{4 - a^2 - b^2}{4b^2} + \theta\theta^*\right) + O(p^3) \end{aligned},$$

and a similar mechanism works for g_n and h_n.

From these leading order terms, the exact, rational rogue wave solution for Eq. (3) is the set of Eqs. (4, 7) supplemented by Eqs. (8, 11) and

$$f_n^{rogue} = (n - at)^2 + b^2 t^2 + \frac{4 - a^2 - b^2}{4b^2},$$

$$G_n^{rogue} = \rho g_n^{rogue}, H_n^{rogue} = \rho h_n^{rogue},$$

$$g_n^{rogue} = f_n^{rogue} - \frac{4}{(2\cos k + a)^2 + b^2}\left\{\sin^2 k + in[a\sin k + \sin(2k)] + it[(b^2 - a^2)\sin k - a\sin(2k)]\right\},$$

$$h_n^{rogue} = f_n^{rogue} - \frac{4}{(2\cos k - a)^2 + b^2}\left\{\sin^2 k + in[a\sin k - \sin(2k)] + it[(b^2 - a^2)\sin k + a\sin(2k)]\right\}.$$

(16)

The exact rogue modes expressed in terms of the original Eqs. (3a, 3b) are thus

$$A_n^{rogue} = i^n \rho \exp[i(kn - \omega t + 2t)]\frac{g_n^{rogue}}{f_n^{rogue}}, B_n^{rogue} = i^n \rho \exp[-i(kn + \omega t + 2t)]\frac{h_n^{rogue}}{f_n^{rogue}}. \quad (17)$$

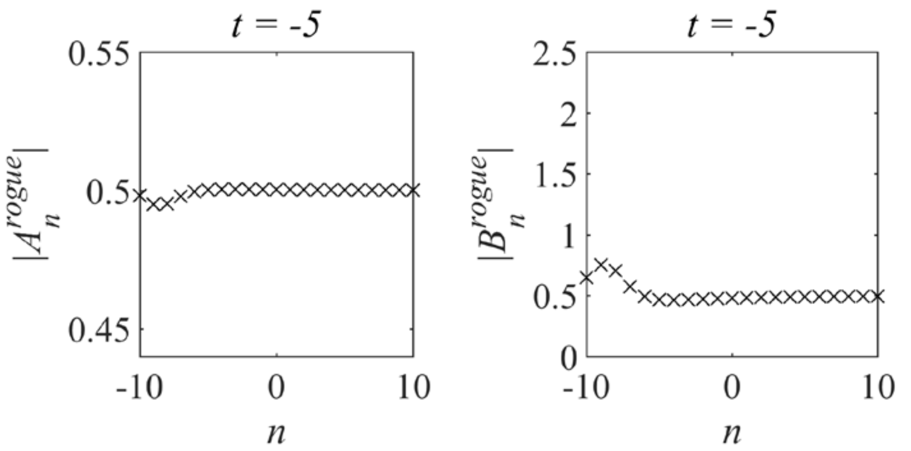
Furthermore, there are additional constraints for the existence of rogue waves. From Eq. (16), the necessary criteria are

(i) b is non-zero; (ii) $4 - a^2 - b^2 > 0$.

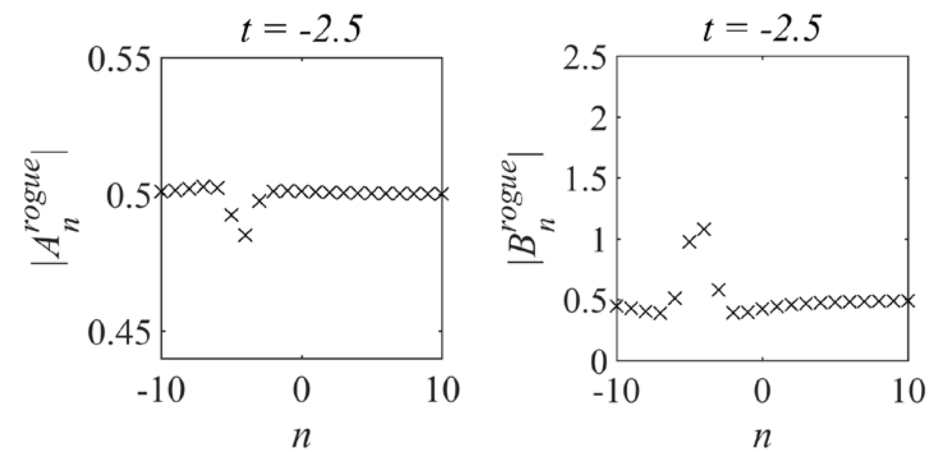
For the second condition (ii), the dispersion relation now leads to a constraint on k for the rogue wave to exist:

$$2 - a^2 + 2\cos^2 k - \frac{4\sigma\rho^2 \cos k \sin^2 k}{a} > 0 \quad . \quad (18)$$

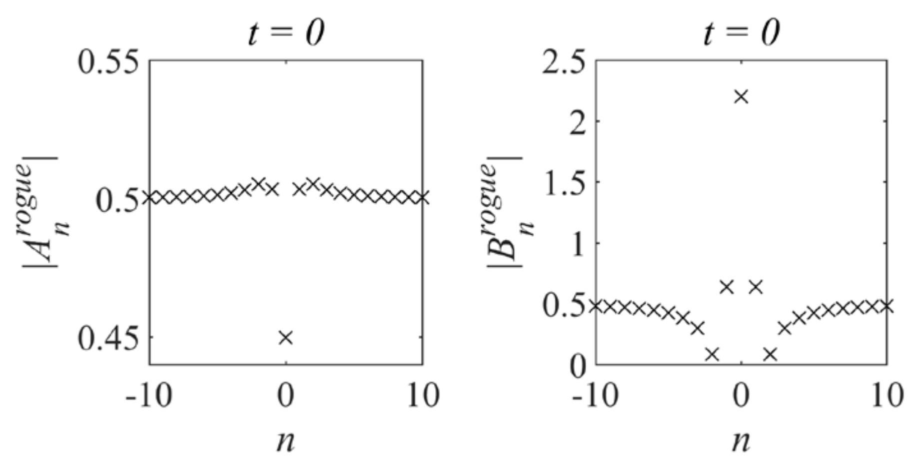
(a)


t = -5
t = -5
$|A_n^{rogue}|$
$|B_n^{rogue}|$
0.55
0.5
0.45
2.5
2
1.5
1
0.5
0
-10
0
10
n
n

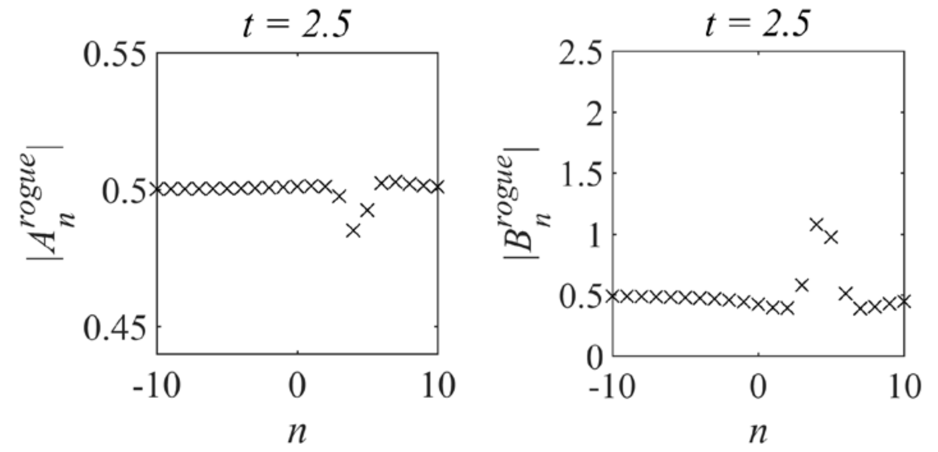
(b)


t = -2.5
t = -2.5
$|A_n^{rogue}|$
$|B_n^{rogue}|$
0.55
0.5
0.45
2.5
2
1.5
1
0.5
0
-10
0
10
n
n

(c)


t = 0
t = 0
$|A_n^{rogue}|$
$|B_n^{rogue}|$
0.55
0.5
0.45
2.5
2
1.5
1
0.5
0
-10
0
10
n
n

(d)



(e)

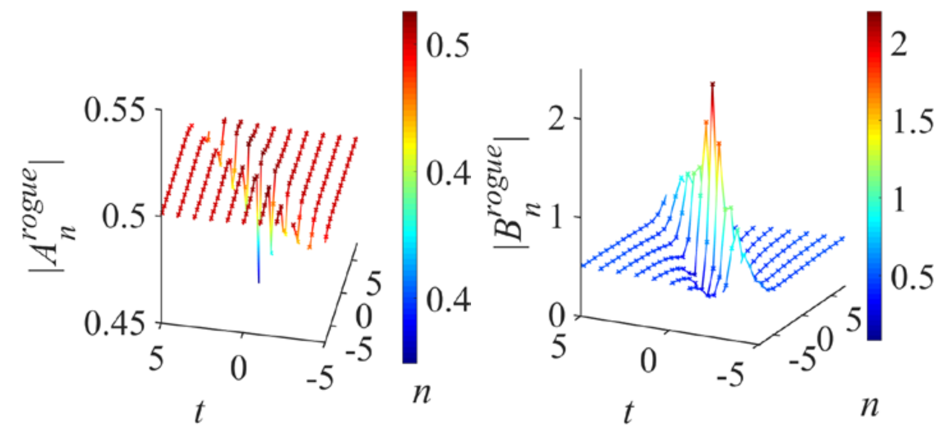
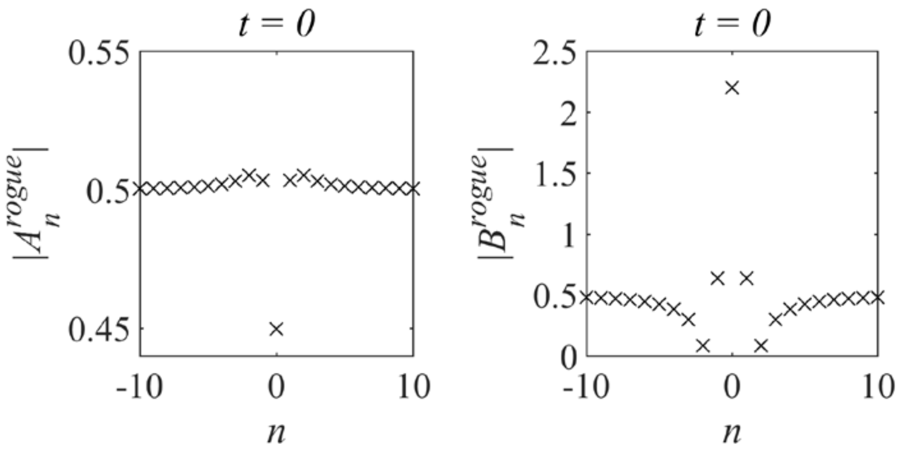
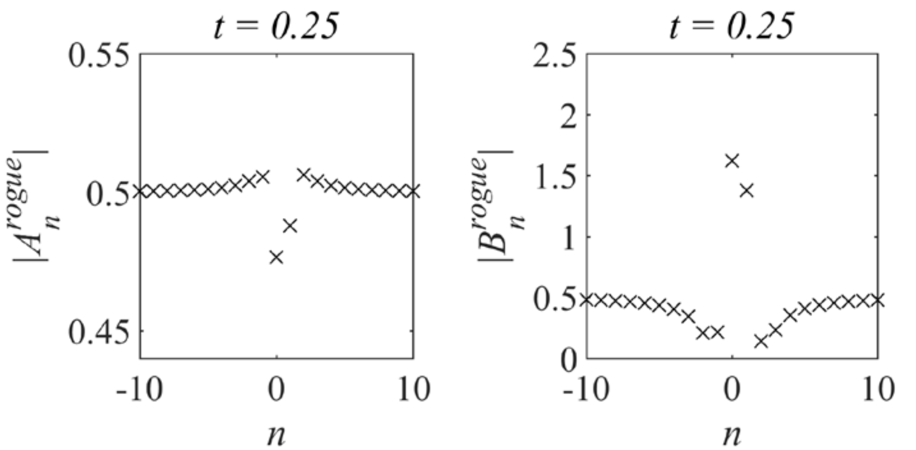


Figure 1: A rogue wave [Eqs. (4, 7, 8, 11, 15, 16, 17)] propagating from left to right with a burst of maximum amplitude around $t = 0$ for parameters σ = 1, ρ = 0.5, $k = 0.5$; (a) $t = –5$; (b) $t = –2.5$; (c) $t = 0$; (d) $t = 2.5$; (e) in the time interval [–5, 5]. [Left (Right) panel: $|A_n|$ ($|B_n|$)]

(a)



(b)



(c)

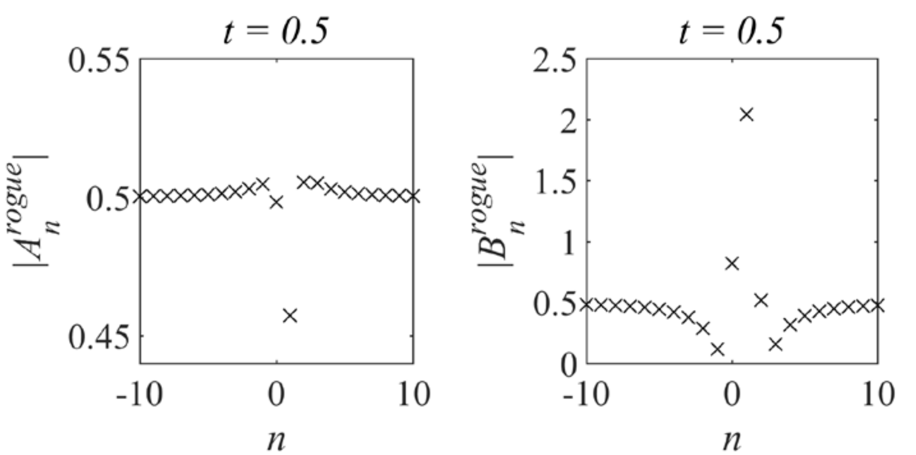


Figure 2: A sequence of wave profiles showing the pulsating nature of the rogue wave (same parameters as in Fig. 1): (a) t = 0; (b) t = 0.25; (c) t = 0.5 (the displacement from the background in (b) is less than those of (a) and (c)). [Left (Right) panel: $|A_n|$ ($|B_n|$)]

We have verified the accuracy of the rogue wave solution by direct substitution into the bilinear equations. One remarkable feature of the rogue wave which contrasts sharply with similar solutions of the continuous case is the ‘pulsating’ nature. As an illustrative example (Figure 2), the amplitude oscillates during the time evolution to and from maximum displacement.

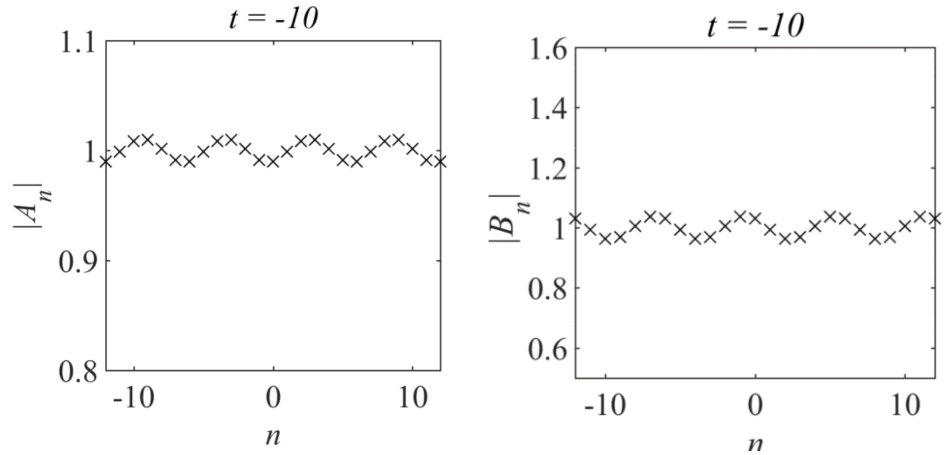
3.2 Breathers periodic in the lattice coordinate n

Another class of exact solutions periodic in the coordinate n can be derived by taking the parameter p to be purely imaginary. Indeed Eq. (10) will be of period 6 by setting $p = i\pi/3$. However, due to the i^n term and exponential factors in Eq. (7), the actual spatial period of the complex valued array of oscillators will be the lowest common multiple of 4 and 6, i.e. a period of 12, provided that $k = \pi/3$ in Eq. (11). This amplitude will still generally be a localized function of time. Typical wave profiles for maximum amplitudes versus those away from the peak displacements are illustrated in Figure 3. We conjecture that solutions of spatial period of $4N$ (odd N) or $2N$ (even N) (N = a positive integer greater than 3) can be attained by setting $p = i\pi/N$ and $k = \pi/N$, but details of the verification will be left for future studies.

We should also remark that the assumption of equal amplitude for the background plane waves (Eq. (13)) will imply $\beta = -1$ through Eq. (8). In other words, the quadratic dispersion (Eq. (2)) as represented by the second order central difference

(Eq. (3)) must be negative. The implication on the range of cubic nonlinearity (the parameter σ) deserves further studies in the future.

(a)



(b)

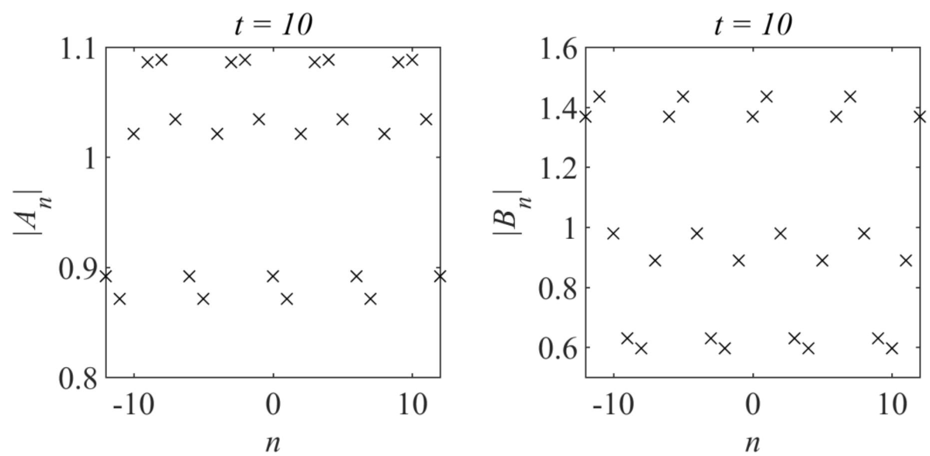


Figure 3: Typical wave profiles of the breather solutions [Eqs. (4, 7, 8, 10, 11, 13)] with spatial period 12 for different values of time t with input parameters $\sigma = 0.2$, $\rho = 1$, $k = \pi/3$, $p = i\pi/3$, $\Omega = 0.161+i$: Top row: $t = -10$; Bottom row: $t = 10$; Left (Right) panel: $|A_n|$ ($|B_n|$).

3.3 Conservation laws

Finally, the existence of spatially periodic solutions also facilitates the search for conservation laws [6], which are usually associated with the underlying physics and desirable features for the case of continuous evolution equations. Indeed momentum conservation law could be derived by a symplectic integrator for a nonlinear Schrödinger-type equation, and had been verified numerically earlier in the literature [18]. Here similar conservation laws for the coupled Ablowitz-Ladik system will be established directly from the governing system of Eq. (5). We shall in fact confirm two classes of conservation laws, the first type involves one component alone, while the second type incorporates both arrays.

In terms of derivation, an auxiliary equation can be obtained by multiplying Eq. (5a) by ϕ_{n+1}^*, while another auxiliary equation would be computed by multiplying the complex conjugate of Eq. (5a) by ϕ_{n+1}. On subtracting these two equations, we obtain

$$i\left[(\phi_n)_t\phi_{n+1}^* + (\phi_n^*)_t\phi_{n+1}\right] = \left[\beta + \sigma\left(|\phi_n|^2 - |\psi_n|^2\right)\right]\left(\phi_{n-1}^*\phi_{n+1} - \phi_{n-1}\phi_{n+1}^*\right).$$

The same procedure is applied again with the lattice point at n+1 taken as the center. Addition of these intermediate equations will yield a time derivative:

$$\begin{aligned} i\left(\phi_n\phi_{n+1}^* + \phi_n^*\phi_{n+1}\right)_t &= \left[\beta + \sigma\left(|\phi_n|^2 - |\psi_n|^2\right)\right]\left(\phi_{n-1}^*\phi_{n+1} - \phi_{n-1}\phi_{n+1}^*\right) \\ &\quad + \left[\beta + \sigma\left(|\phi_{n+1}|^2 - |\psi_{n+1}|^2\right)\right]\left(\phi_n^*\phi_{n+2} - \phi_n\phi_{n+2}^*\right). \end{aligned}$$

By the assumption of spatial periodicity, we can deduce that the quantity

$$J_1 = \sum\left(\phi_n\phi_{n+1}^* + \phi_n^*\phi_{n+1}\right) \tag{19}$$

is an invariant in time, provided that the summation is taken over one period (Figure 4). Similarly, the corresponding counterpart for the second component ψ_n is given by

$$J_2 = \sum \left(\psi_n \psi_{n+1}^* + \psi_n^* \psi_{n+1} \right). \tag{20}$$

Eqs. (19) and (20) thus constitute conservation laws of the governing system which involve only one component.

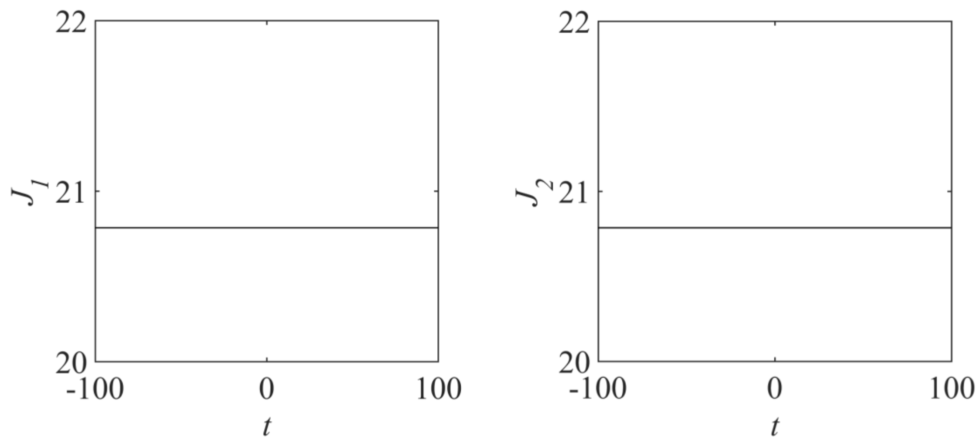


Figure 4: The invariants J_1 and J_2 as given in Eqs. (19, 20) are conserved over time. The input parameters are the same as those in Fig. 3. [Left (Right) panel: J_1 (J_2)]

The second class of conservation principles can be obtained by roughly similar algebraic manipulations. The contrast is that products from cross phase modulations can only be cancelled with a summation involving both components. The quantity

$$J_3 = \sum \left(\phi_n \phi_{n+1}^* - \phi_n^* \phi_{n+1} + \psi_n \psi_{n+1}^* - \psi_n^* \psi_{n+1} \right) \tag{21}$$

is then conserved for a summation over one period (Figure 5).

We conjecture that more sophisticated conservation laws involving more neighboring grids can be derived [18]. Here only one particular case will be tested numerically. The quantity J_4 defined by

$$J_4 = i\sum\left[8\left(\phi_n\phi_{n+1}^* - \phi_n^*\phi_{n+1} + \psi_n\psi_{n+1}^* - \psi_n^*\psi_{n+1}\right) - \left(\phi_n\phi_{n+2}^* - \phi_n^*\phi_{n+2} + \psi_n\psi_{n+2}^* - \psi_n^*\psi_{n+2}\right)\right]. \tag{22}$$

is invariant in time if the summation is taken over one period (Figure 5).

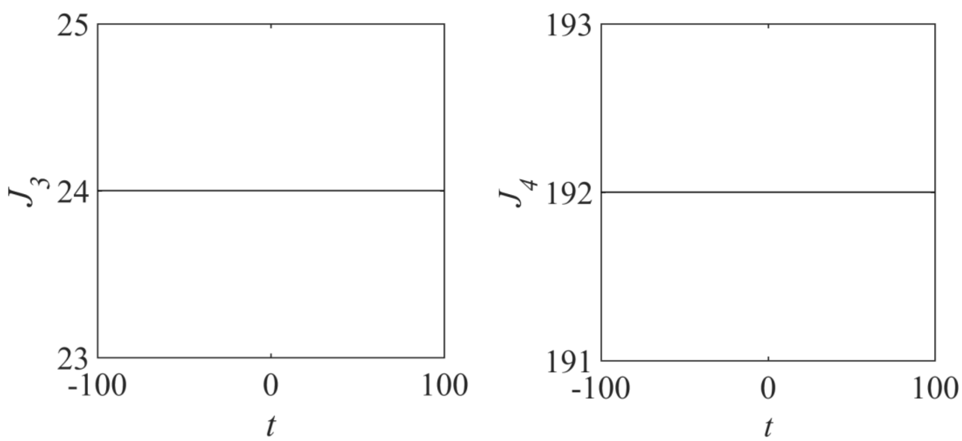


Figure 5: The invariants J_3 and J_4 as given by Eqs. (21, 22) are conserved over time. The input parameters are the same as those in Fig. 3 [Left (Right) panel: J_3 (J_4)].

These two classes of conservation principles might be associated with the physical quantities like momentum [6,12,18]. For boundary conditions other than spatial periodicity, derivation of invariant quantities might need additional considerations.

Nevertheless future efforts to extract further conservation laws of the system will enhance confidence that Eq. (3) or Eq. (5) is probably 'integrable' [6,18].

4. Discussions and Conclusions

A new system of coupled discrete (Ablowitz-Ladik) evolution equations with both self phase and cross phase modulations is introduced. The Hirota bilinear form is constructed and is utilized to obtain breathers periodic in space or time [19]:

● Rogue waves with pulsating growth phase are generated in the long wave limit;

● Low order conservation laws are verified for spatially periodic solutions.

Although versions of coupled Ablowitz-Ladik equations have been considered in the literature, the present formulation incorporates for the first time cubic nonlinearities due to intensities from both arrays of waveguides, and in particular SPM and XPM of different signs are permitted. We conjecture that the system is 'integrable' but the details are left for future works.

Ablowitz-Ladik systems are applicable to many issues of physical importance, e.g. nonlinearity versus randomness [12], and practical applications, e.g. switching of discrete soliton pulses in arrays of optical waveguides [13]. Indeed the time evolution of arrays of oscillators can model a variety of other scenarios, e.g. a nonlinear electrical transmission line consisting of a suitable combination of capacitors and band pass filters [20]. With the present formulation, the effect of cross phase modulation in these physical settings can be elucidated in subsequent works. Further investigations like modulation instability of the plane wave background [19] and extension to higher spatial dimensions [21] would be performed analytically as well as computationally in the near future.

Indeed discrete breathers have received tremendous attention in the literature, as such modes can be applied in the descriptions of molecular crystals, Josephson junctions and localization of electromagnetic waves in photonic crystals [22]. Hence further efforts on discrete systems along the direction of the present work would definitely be fruitful for theoretical physics and practical applications.

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Author Contributions

KWC proposed the coupled Ablowitz-Ladik system and initiated the extension of the bilinear forms. HNC conducted the analytical formulation and verified the accuracy of the new solutions. She also performed the computational tests.

Appendix

The criteria for the expansion Eq. (10) to satisfy the bilinear forms Eq. (9), with assumptions Eq. (11) but for general $\rho_1 \neq \rho_2$, can be formulated by these constraints:

$$M\{\exp(p+p^*)+\exp[-(p+p^*)]\}+\exp(p-p^*)+\exp(p^*-p)+2\beta(M+1)$$
$$+\sigma\left\{M\left[\rho_1^2\left(\frac{a_1}{a_1^*}+\frac{a_1^*}{a_1}\right)-\rho_2^2\left(\frac{b_1}{b_1^*}+\frac{b_1^*}{b_1}\right)\right]+\rho_1^2\left(a_1a_1^*+\frac{1}{a_1a_1^*}\right)-\rho_2^2\left(b_1b_1^*+\frac{1}{b_1b_1^*}\right)\right\}=0;$$

$$M(\Omega+\Omega^*)\left(1-\frac{a_1}{a_1^*}\right)-(\Omega-\Omega^*)\left(a_1-\frac{1}{a_1^*}\right)$$
$$=\exp(-ik)\left\{\begin{array}{l}M\left[1-\exp(p+p^*)+\dfrac{a_1}{a_1^*}(1-\exp(-p-p^*))\right]\\+a_1[1-\exp(p^*-p)]+\dfrac{1}{a_1^*}[1-\exp(p-p^*)]\end{array}\right\}$$
$$+\exp(ik)\left\{\begin{array}{l}M\left[\exp(-p-p^*)-1+\dfrac{a_1}{a_1^*}(\exp(p+p^*)-1)\right]\\+a_1[\exp(p-p^*)-1]+\dfrac{1}{a_1^*}[\exp(p^*-p)-1]\end{array}\right\};$$

$$M(\Omega+\Omega^*)\left(1-\frac{b_1}{b_1^*}\right)-(\Omega-\Omega^*)\left(b_1-\frac{1}{b_1^*}\right)$$

$$=\exp(-ik)\left\{\begin{array}{l} M\left[1-\exp(-p-p^*)+\dfrac{b_1}{b_1^*}(1-\exp(p+p^*))\right] \\ +b_1[1-\exp(p-p^*)]+\dfrac{1}{b_1^*}[1-\exp(p^*-p)] \end{array}\right\}$$

$$+\exp(ik)\left\{\begin{array}{l} M\left[\exp(p+p^*)-1+\dfrac{b_1}{b_1^*}(\exp(-p-p^*)-1)\right] \\ +b_1[\exp(p^*-p)-1]+\dfrac{1}{b_1^*}[\exp(p-p^*)-1] \end{array}\right\}.$$

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