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Research Article

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q-dual mixed volumes and L_p-intersection bodies

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Abstract: In this paper, we introduce the new notions of L_p-intersection and mixed intersection bodies. Inequalities for the q-dual volume sum of L_p-mixed intersection bodies are established.

Keywords: q-dual mixed volumes, L_p-intersection bodies, L_p-mixed intersection bodies

MSC 2010: 52A40

1 Introduction

The intersection operator and the class of intersection bodies were defined by Lutwak [28]. The closure of the class of intersection body operators was studied by Goody, Lutwak and Weil [11]. The intersection body operator and the class of intersection bodies played a crucial role in [36] and [5] for the solution of the famous Busemann–Petty problem (see also [10]).

Just as the period from the mid-60s to the mid-80s was a time of great advances in the understanding of the projection operator and the class of projection bodies, during the past 20 years significant advances have been made in our understanding of the intersection operator and the class of intersection bodies by Koldobsky, Zhang, Campi, Goodey, Gardner, Grinberg, Fallert, Weil, Ludwig and others (see, e.g., [1–9, 11–22, 24, 30, 32–37]).

As Lutwak [28] showed (and as is further elaborated in Gardner's book [8]), there is a duality between projection and intersection bodies (that at present is not yet understood). Consider the following illustrative example: It is well known that the projections (onto lower dimensional subspaces) of projection bodies are themselves projection bodies. Lutwak conjectured the "dual": When intersection bodies are intersected with lower dimensional subspaces, the results are intersection bodies (within the lower dimensional subspaces). This was proved by Fallert, Goodey and Weil [4]. In [26] (see also [27, 29]), Lutwak introduced mixed projection bodies and derived their fundamental inequalities.

In 2006, Haberl and Ludwig [15] introduced the L_p-intersection bodies ($p \in (0,1)$). For $K \in \mathcal{P}_0^n$, where $\mathcal{P}_0^n$ denotes the set of convex polytopes in $\mathbb{R}^n$ that contain the origin in their interiors, the star body $\mathbf{I}_p^+ K$ is defined for $u \in S^{n-1}$ by

$$\rho(\mathbf{I}_p^+ K, u)^p = \int\limits_{K \cap u^+} |u \cdot x|^{-p}\, dx, \tag{1.1}$$

where $u^+ = \{x \in \mathbb{R}^n : u \cdot x \geq 0\}$. For $p < 1$, the centrally symmetric star body $\mathbf{I}_p K = \mathbf{I}_p^+ K \,\tilde{+}_p\, \mathbf{I}_p^- K$, where $\tilde{+}_p$ denotes the *p-radial sum* and $\mathbf{I}_p^- K = \mathbf{I}_p^+(-K)$, is called the L_p*-intersection body* of K. So, for $u \in S^{n-1}$,

$$\rho^p(\mathbf{I}_p K, u) = \int\limits_K |u \cdot x|^{-p}\, dx.$$

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Note that

$$v(K \cap u^+) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{2} \int_K |u \cdot x|^{-1+\varepsilon}\, dx$$

and

$$\rho(\mathbf{I}K, u) = \lim_{p \to 1_-} \frac{1-p}{2} \rho^p(\mathbf{I}_p K, u),$$

that is, the intersection body of K is obtained as a limit of L_p-intersection bodies of K. Also note that a change to polar coordinates in (1.1) shows that, up to a normalization factor, $\rho^p(\mathbf{I}_p K, u)$ equals the L_p cosine transform of $\rho(K, u)^{n-p}$.

Following Haberl and Ludwig, this paper introduces a new notion of L_p-intersection bodies ($p \in (0, 1)$). The *L_p-mixed intersection bodies* of $K_1, \dots, K_{n-1}$ are written as $\mathbf{I}^p(K_1, \dots, K_{n-1})$, where $p \in (0, 1)$, whose radial function is defined by

$$\rho^p(\mathbf{I}^p(K_1, \dots, K_{n-1}), u) = \frac{2}{1-p} \tilde{v}_p(K_1 \cap E_u, \dots, K_{n-1} \cap E_u), \tag{1.2}$$

where $\tilde{v}_p(K_1 \cap E_u, \dots, K_{n-1} \cap E_u)$ denotes the p-dual mixed volumes of $K_1 \cap E_u, \dots, K_{n-1} \cap E_u$ in an $(n-1)$-dimensional space (see Section 2). E_u denotes the hyperplane, through the origin, that is orthogonal to u. If $K_1 = \dots = K_{n-i-1} = K$, $K_{n-i} = \dots = K_{n-1} = L$, then $\tilde{v}_p(K_1 \cap E_u, \dots, K_{n-1} \cap E_u)$ is written as $\tilde{v}_{p,i}(K \cap E_u, L \cap E_u)$. If $L = B$, then $\tilde{v}_{p,i}(K \cap E_u, L \cap E_u)$ is written as $\tilde{v}_{p,i}(K \cap E_u)$.

By the definition given above, we have

$$\begin{aligned} \lim_{p \to 1_-} \frac{1-p}{2} \rho^p(\mathbf{I}^p(K_1, \dots, K_{n-1}), u) &= \lim_{p \to 1_-} \tilde{v}_p(K_1 \cap E_u, \dots, K_{n-1} \cap E_u) \\ &= \tilde{v}(K_1 \cap E_u, \dots, K_{n-1} \cap E_u) \\ &= \rho(\mathbf{I}(K_1, \dots, K_{n-1}), u). \end{aligned}$$

The mixed intersection body $\mathbf{I}(K_1, \dots, K_{n-1})$ was defined by Leichtweiß [23] as

$$\rho(\mathbf{I}(K_1, \dots, K_{n-1}), u) = \frac{1}{n-1} \int_{S^{n-1} \cap E_u} \rho(K_1, u) \cdots \rho(K_{n-1}, u)\, dS(u).$$

For the L_p-mixed intersection bodies, $\mathbf{I}^p(K_1, \dots, K_{n-1})$, if $K_1 = \dots = K_{n-i-1} = K$, $K_{n-i} = \dots = K_{n-1} = L$, then $\mathbf{I}^p(K_1, \dots, K_{n-1})$ is written as $\mathbf{I}^p(K, L)_i$. If $L = B$, then $\mathbf{I}^p(K, L)_i$ is written as $\mathbf{I}^p K_i$ and called the ith L_p-intersection body of K. For $\mathbf{I}^p K_0$, we simply write $\mathbf{I}^p K$ and call it the L_p-intersection body of K.

The L_p-intersection body as defined by Haberl and Ludwig does not agree with the definition of a mixed L_p-Intersection body, see (1.2). Thus, the L_p-intersection body considered in this paper is different from the notion introduced by Haberl and Ludwig. Moreover, a quasi L_p-intersection body and a mixed quasi L_p-intersection body were introduced by Leng, Wu and Yu [31], for $p \geq 1$, by

$$\rho(\mathbf{I}_p(K_1, \dots, K_{n-1}), u)^p = \frac{1}{(n-1)\omega_{n-1}} \int_{S^{n-1} \cap E_u} \rho(K_1, u)^{\frac{n-p}{n-1}} \cdots \rho(K_{n-1}, u)^{\frac{n-p}{n-1}}\, dS(u).$$

Obviously, the L_p-intersection body considered here is also different from the notion introduced by Leng, Wu and Yu.

In this paper, the inequalities for the *q-dual volume sum* function of L_p-mixed intersection bodies are established. Our main results are stated as follows.

Theorem A. *Let K, L, D and D' be star bodies and D' be a dilated copy of D. If $0 \leq i < n$, $0 \leq j < n-1$, $i, j \in \mathbb{N}$, $q \geq 1$ and $p \in (0, 1)$, then*

$$S_{\tilde{v}_{q,i}}(\mathbf{I}^p(K, L)_j, \mathbf{I}^p(D, D')_j)^{n-1} \leq S_{\tilde{v}_{q,i}}(\mathbf{I}^p K, \mathbf{I}^p D)^{n-j-1} S_{\tilde{v}_{q,i}}(\mathbf{I}^p L, \mathbf{I}^p D')^j,$$

with the equality holding if and only if K and L are dilates.

Here, for the star bodies K and L, $S_{\tilde{V}_{q,i}}(K,L)$ denotes the q-dual volume sum function of K and L, i.e.,

$$S_{\tilde{V}_{q,i}}(K,L) = \tilde{V}_{q,i}(K) + \tilde{V}_{q,i}(L)$$

and $\tilde{V}_{q,i}(K)$ denotes the ith *q-dual mixed volume*

$$\tilde{V}_q(\underbrace{K,\ldots,K}_{n-i},\underbrace{B,\ldots,B}_{i}),$$

which was defined in [39] (see also Section 2). If $q = 1$, $\tilde{V}_{q,i}(K)$ is the classical dual quermassintegral $\tilde{W}_i(K)$. Moreover, L_p-dual quermassintegral sums were introduced in [38].

Remark 1.1. In the special case where D and D' are single points, the inequality in Theorem A takes the form

$$\tilde{V}_{q,i}(\mathbf{I}^p(K,L)_j)^{n-1} \le \tilde{V}_{q,i}(\mathbf{I}^p K)^{n-j-1}\tilde{V}_{q,i}(\mathbf{I}^p L)^j,$$

with the equality holding if and only if K and L are dilates.

Let D and D' be single points and put $q = 1$ in the inequality in Theorem A. Then

$$\tilde{W}_i(\mathbf{I}^p(K,L)_j)^{n-1} \le \tilde{W}_i(\mathbf{I}^p K)^{n-j-1}\tilde{W}_i(\mathbf{I}^p L)^j,$$

with the equality holding if and only if K and L are dilates.

Theorem B. *Let $K_1,\ldots,K_{n-1}$ be star bodies, $p \in (0,1)$, $q \ge 1$, $1 < r \le n-1$, $0 \le i < n$, $0 \le j < n-1$, $i,j \in \mathbb{N}$. Let D_i ($i = 1,2,\ldots,n$) be the dilated copies of each other, respectively. Then*

$$\begin{aligned}&S_{\tilde{V}_{q,i}}(\mathbf{I}^p(K_1,\ldots,K_{n-1}),\mathbf{I}^p(D_1,\ldots,D_{n-1}))^r\\ &\quad\le \prod_{j=1}^{r} S_{\tilde{V}_{q,i}}(\mathbf{I}^p(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_{n-1}),\mathbf{I}^p(\underbrace{D_j,\ldots,D_j}_{r},D_{r+1},\ldots,D_{n-1})),\end{aligned}$$

with the equality holding if and only if $K_1,\ldots,K_{n-1}$ are all dilations of each other.

Remark 1.2. In the special case where $D_1,\ldots,D_{n-1}$ are single points, the inequality in Theorem B takes the form

$$\tilde{V}_{q,i}(\mathbf{I}^p(K_1,\ldots,K_{n-1}))^r \le \prod_{j=1}^{r}\tilde{V}_{q,i}(\mathbf{I}^p(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_{n-1})),$$

with the equality holding if and only if $K_1,\ldots,K_{n-1}$ are all dilations of each other.

Let $D_1,\ldots,D_{n-1}$ be single points and put $q = 1$ in the inequality in Theorem B. Then

$$\tilde{W}_i(\mathbf{I}^p(K_1,\ldots,K_{n-1}))^r \le \prod_{j=1}^{r}\tilde{W}_i(\mathbf{I}^p(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_{n-1})),$$

with the equality holding if and only if $K_1,\ldots,K_{n-1}$ are all dilations of each other.

2 q-dual mixed volume

The setting for this paper is the n-dimensional Euclidean space $\mathbb{R}^n$ ($n > 2$). We preserve the symbol u for unit vectors, and the symbol B is preserved for the unit ball centered at the origin. The surface of B is S^{n-1}. The volume of the unit n-ball is denoted by ω_n. Integration over S^{n-1} by the usual Borel measure on S^{n-1} is denoted by dS.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K,\cdot)\colon S^{n-1} \to \mathbb{R}$ defined for $u \in S^{n-1}$, by $\rho(K,u) = \max\{\lambda \ge 0 : \lambda u \in K\}$. If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Let φ^n denote the set of star bodies in $\mathbb{R}^n$. Let $\tilde{\delta}$ denote the radial Hausdorff metric defined as follows: if $K, L \in \varphi^n$, then

$$\tilde{\delta}(K, L) = |\rho(K, \cdot) - \rho(L, \cdot)|_\infty,$$

where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

The *q-dual mixed volume* $\tilde{V}_q(K_1, \dots, K_n)$ was defined in [39] as follows.

Definition 2.1. Let $K_1, \dots, K_n \in \varphi^n$ and $q \neq 0$. The *q-dual mixed volume* is defined by

$$\tilde{V}_q(K_1, \dots, K_n) = \omega_n \Big(\frac{1}{n\omega_n} \int_{S^{n-1}} \rho^q(K_1, u) \cdots \rho^q(K_n, u)\, dS(u) \Big)^{\frac{1}{q}}. \tag{2.1}$$

By Definition 2.1, $\tilde{V}_q$ is a map

$$\tilde{V}_q : \varphi^n \times \cdots \times \varphi^n \to \mathbb{R}.$$

Taking $q = 1$ in (1.1), we have

$$\tilde{V}_1(K_1, \dots, K_n) = \tilde{V}(K_1, \dots, K_n),$$

where $\tilde{V}(K_1, \dots, K_n)$ is the classical dual mixed volume which was defined by Lutwak [25]. Moreover, we will write $\tilde{V}_q(\underbrace{K, \dots, K}_{n-i}, \underbrace{L, \dots, L}_{i})$ as $\tilde{V}_{q,i}(K, L)$, and $\tilde{V}_q(\underbrace{K, \dots, K}_{n-i}, \underbrace{B, \dots, B}_{i})$ as $\tilde{V}_{q,i}(K)$.

3 Auxiliary results

For the q-dual mixed volume, we obtain a new inequality, *an Aleksandrov–Fenchel type inequality between q-dual mixed volumes*, defined as follows.

Lemma 3.1. *If $K_1, \dots, K_n \in \varphi^n$, $1 < r \leq n$, and $q \geq 0$, then*

$$\tilde{V}_q(K_1, \dots, K_n)^r \leq \prod_{j=1}^{r} \tilde{V}_q(\underbrace{K_j, \dots, K_j}_{r}, K_{r+1}, \dots, K_n), \tag{3.1}$$

with the equality holding if and only if $K_1, \dots, K_n$ are all dilations of each other.

The inequality is reversed for $q < 0$.

Proof. Hölder's inequality for integrals states that for positive continuous functions $f_1, \dots, f_m, g\colon S^{n-1} \to \mathbb{R}$ and positive numbers $p_1, \dots, p_m$ with $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1$, we have

$$\int_{S}^{n-1} f_1 \cdots f_m g\, dS \leq \prod_{j=1}^{m} \Big(\int_{S}^{n-1} f_j^{p_j} g\, dS \Big)^{\frac{1}{p_j}}.$$

Let $q \neq 0$ and set $m = r$, $g = \frac{1}{n\omega_n} \rho(K_{r+1}, \cdot)^q \cdots \rho(K_n, \cdot)^q$ and $f_j = \rho(K_j, \cdot)^q$, $p_j = \frac{1}{r}$ for $j = 1, \dots, r$. Then

$$\frac{1}{n\omega_n} \int_{S}^{n-1} \rho(K_1, u)^q \cdots \rho(K_n, u)^q\, dS(u) \leq \prod_{j=1}^{r} \Big(\frac{1}{n\omega_n} \int_{S}^{n-1} \rho(K_j, u)^{qr} \rho(K_{r+1}, u)^q \cdots \rho(K_n, u)^q\, dS(u) \Big)^{\frac{1}{r}},$$

and therefore

$$\tilde{V}_q(K_1, \dots, K_n)^q \leq \prod_{j=1}^{r} \tilde{V}_q(\underbrace{K_j, \dots, K_j}_{r}, K_{r+1}, \dots, K_n)^{\frac{q}{r}}.$$

Hence, for $q > 0$, we have that

$$\tilde{V}_q(K_1, \dots, K_n)^r \leq \prod_{j=1}^{r} \tilde{V}_q(\underbrace{K_j, \dots, K_j}_{r}, K_{r+1}, \dots, K_n),$$

and, for $q < 0$, we have that

$$\tilde{V}_q(K_1, \ldots, K_n)^r \geq \prod_{j=1}^{r} \tilde{V}_q(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_n).$$

Taking $q = 1$ in (1.2), we have

$$\tilde{V}(K_1, \ldots, K_n)^r \leq \prod_{j=1}^{r} \tilde{V}(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_n),$$

with the equality holding if and only if $K_1, \ldots, K_n$ are all dilations of each other. □

This is just the Aleksandrov–Fenchel inequality between dual mixed volumes, which is due to Lutwak [25].

Lemma 3.2. *If $K, L \in \varphi^n$, $0 \leq i < n$, $0 \leq j < n-1$, $i, j \in \mathbb{N}$, $q \geq 1$ and $p < 1$, then*

$$\tilde{V}_{q,i}(\mathbf{I}^p(K, L)_j) = \omega_n \left(\frac{1}{n\omega_n} \left(\frac{2}{1-p} \right)^{\frac{(n-i)q}{p}} \int_{S^{n-1}} \tilde{v}_{p,j}(K \cap E_u, L \cap E_u)^{\frac{(n-i)q}{p}} \, dS(u) \right)^{\frac{1}{q}}.$$

From (1.2) and (2.1), Lemma 3.2 easily follows.

We shall need the following elementary inequality.

Lemma 3.3. *If $a_i \geq 0$, $b_i > 0$ ($i = 1, 2, \ldots, n$), then*

$$\left(\prod_{i=1}^{n} (a_i + b_i) \right)^{\frac{1}{n}} \geq \left(\prod_{i=1}^{n} a_i \right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} b_i \right)^{\frac{1}{n}}, \tag{3.2}$$

with the equality holding if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

Obviously, a special case of (3.2) is the following result.

If $a, b \geq 0$ and $c, d > 0$, then, for $0 < m < 1$,

$$(a+b)^m (c+d)^{1-m} \geq a^m c^{1-m} + b^m d^{1-m}, \tag{3.3}$$

with the equality holding if and only if $ad = bc$.

4 Inequalities for the q-dual volume sum of L_p-intersection bodies

4.1 L_p-Minkowski inequality for the q-dual volume sum

The following Minkowski inequality for q-dual mixed volumes of L_p-mixed intersection bodies will be established: If $K, L \in \varphi^n$, $p \in (0, 1)$, $q \geq 1$ and $0 < j < n-1$, $j \in \mathbb{N}$, then

$$\tilde{V}_q(\mathbf{I}^p(K, L)_j)^{n-1} \leq \tilde{V}_q(\mathbf{I}^p K)^{n-j-1} \tilde{V}_q(\mathbf{I}^p L)^j, \tag{4.1}$$

with the equality holding if and only if K and L are dilates.

This is just a special case of the following result.

Theorem 4.1. *Let $K, L, D, D' \in \varphi^n$, let D' be a dilated copy of D, and let $0 \leq i < n$, $0 \leq j < n-1$, $i, j \in \mathbb{N}$, $q \geq 1$ and $p \in (0, 1)$. Then*

$$S_{\tilde{V}_{q,i}}(\mathbf{I}^p(K, L)_j, \mathbf{I}^p(D, D')_j)^{n-1} \leq S_{\tilde{V}_{q,i}}(\mathbf{I}^p K, \mathbf{I}^p D)^{n-j-1} S_{\tilde{V}_{q,i}}(\mathbf{I}^p L, \mathbf{I}^p D')^j, \tag{4.2}$$

with the equality holding if and only if K and L are dilates.

Proof. Taking $K_1 = \cdots = K_{n-j} = K, K_{n-j+1} = \cdots = K_n = L$ and $r = n$ in (3.1), we obtain

$$\tilde{V}_{q,j}(K,L)^n \le \tilde{V}_q(K)^{n-j}\tilde{V}_q(L)^j,$$

with the equality holding if and only if K and L are dilates.

Hence, in an $(n-1)$-dimensional space, we have

$$\tilde{v}_{q,j}(K\cap E_u, L\cap E_u) \le \tilde{v}_q(K\cap E_u)^{\frac{n-j-1}{n-1}}\tilde{v}_q(L\cap E_u)^{\frac{j}{n-1}}, \tag{4.3}$$

with the equality holding if and only if $K\cap E_u$ and $L\cap E_u$ are dilates, which is true if and only if K and L are dilates.

From Lemma 3.2, (4.3), the Minkowski inequality for an integral, and in view of the following fact:

$$\tilde{V}_{q,i}(\mathbf{I}^pM) = \omega_n\bigg(\frac{1}{n\omega_n}\Big(\frac{2}{1-p}\Big)^{\frac{(n-i)q}{p}}\int_{S^{n-1}}\tilde{v}_p(M\cap E_u)^{\frac{(n-i)q}{p}}\,dS(u)\bigg)^{\frac{1}{q}},$$

where $M = K$ or L, we obtain

$$\begin{aligned}
\tilde{V}_{q,i}(\mathbf{I}^p(K,L)_j) &= \omega_n\bigg(\frac{1}{n\omega_n}\Big(\frac{2}{1-p}\Big)^{\frac{(n-i)q}{p}}\int_{S^{n-1}}\tilde{v}_{p,j}(K\cap E_u, L\cap E_u)^{\frac{(n-i)q}{p}}\,dS(u)\bigg)^{\frac{1}{q}}\\
&\le \omega_n\bigg(\frac{1}{n\omega_n}\Big(\frac{2}{1-p}\Big)^{\frac{(n-i)q}{p}}\int_{S^{n-1}}\Big(\tilde{v}_p(K\cap E_u)^{\frac{q(n-j-1)}{n-1}}\tilde{v}_p(L\cap E_u)^{\frac{qj}{n-1}}\Big)^{\frac{n-i}{p}}\,dS(u)\bigg)^{\frac{1}{q}}\\
&\le \omega_n\bigg(\frac{1}{n\omega_n}\Big(\frac{2}{1-p}\Big)^{\frac{(n-i)q}{p}}\Big(\int_{S^{n-1}}\tilde{v}_p(K\cap E_u)^{\frac{(n-i)q}{p}}\,dS(u)\Big)^{\frac{n-j-1}{n-1}}\bigg)^{\frac{1}{q}}\bigg(\int_{S^{n-1}}\tilde{v}_p(L\cap E_u)^{\frac{(n-i)q}{p}}\,dS(u)\bigg)^{\frac{j}{q(n-1)}}\\
&= \omega_n\bigg(\frac{1}{n\omega_n}\Big(\frac{2}{1-p}\Big)^{\frac{(n-i)q}{p}}\int_{S^{n-1}}\tilde{v}_p(K\cap E_u)^{\frac{(n-i)q}{p}}\,dS(u)\bigg)^{\frac{n-j-1}{q(n-1)}}\\
&\quad\times\bigg(\frac{1}{n\omega_n}\Big(\frac{2}{1-p}\Big)^{\frac{(n-i)q}{p}}\int_{S^{n-1}}\tilde{v}_p(L\cap E_u)^{\frac{(n-i)q}{p}}\,dS(u)\bigg)^{\frac{j}{q(n-1)}}\\
&= \tilde{V}_{q,i}(\mathbf{I}^pK)^{\frac{n-j-1}{n-1}}\tilde{V}_{q,i}(\mathbf{I}^pL)^{\frac{j}{n-1}}.
\end{aligned}$$

In view of the equality conditions in (4.3) and the Minkowski inequality for integrals, it follows that the equality holds if and only if K and L are dilates.

Hence,

$$\tilde{V}_{q,i}(\mathbf{I}^p(K,L)_j)^{n-1} \le \tilde{V}_{q,i}(\mathbf{I}^pK)^{n-j-1}\tilde{V}_{q,i}(\mathbf{I}^pL)^j,$$

with the equality holding if and only if K and L are dilates.

In view of the fact that D' is a dilated copy of D, we have

$$\tilde{V}_{q,i}(\mathbf{I}^p(D,D')_j)^{n-1} = \tilde{V}_{q,i}(\mathbf{I}^pD)^{n-j-1}\tilde{V}_{q,i}(\mathbf{I}^pD')^j.$$

Therefore, from inequality (3.3), we have

$$\begin{aligned}
S_{\tilde{V}_{q,i}}(\mathbf{I}^p(K,L)_j, \mathbf{I}^p(D,D')_j) &\le \tilde{V}_{q,i}(\mathbf{I}^pK)^{\frac{n-j-1}{n-1}}\tilde{V}_{q,i}(\mathbf{I}^pL)^{\frac{j}{n-1}} + \tilde{V}_{q,i}(\mathbf{I}^pD)^{\frac{n-j-1}{n-1)}}\tilde{V}_{q,i}(\mathbf{I}^pD')^{\frac{j}{n-1}}\\
&\le S_{\tilde{V}_{q,i}}(\mathbf{I}^pK, \mathbf{I}^pD)^{\frac{n-j-1}{n-1}}S_{\tilde{V}_{q,i}}(\mathbf{I}^pL, \mathbf{I}^pD')^{\frac{j}{n-1}}.
\end{aligned}$$

In view of the equality conditions in inequality (3.3), it follows that the equality holds if and only if K and L are dilates.

The proof of Theorem 4.1 is complete. □

Remark 4.2. Taking $q = 1$ in (4.2), we have

$$S_{\tilde{W}_i}(\mathbf{I}^p(K,L)_j, \mathbf{I}^p(D,D')_j)^{n-1} \le S_{\tilde{W}_i}(\mathbf{I}^pK, \mathbf{I}^pD)^{n-j-1}S_{\tilde{W}_i}(\mathbf{I}^pL, \mathbf{I}^pD')^j.$$

with the equality holding if and only if K and L are dilates.

Here $S_{\tilde{W}_i}$ denotes the i-dual quermassintegral sum function.

Let D and D' be single points. Taking $i = 0$ in (4.2), (4.2) transforms to (4.1).

4.2 L_p-Aleksandrov–Fenchel inequality for the q-dual volume sum

The following Aleksandrov–Fenchel type inequality for the q-dual volume sum function of L_p-mixed intersection bodies will be proved: If $K_1, \ldots, K_{n-1} \in \varphi^n$, $0 \le j < n-1$, $j \in \mathbb{N}$, $p \in (0,1)$ and $q \ge 1$, then

$$S_{\tilde{V}_q}(\mathbf{I}^p(K_1,\ldots,K_{n-1}),\mathbf{I}^p(D_1,\ldots,D_{n-1}))^r \le \prod_{j=1}^{r} S_{\tilde{V}_q}(\mathbf{I}^p(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_{n-1}),\mathbf{I}^p(\underbrace{D_j,\ldots,D_j}_{r},D_{r+1},\ldots,D_{n-1}))$$

with the equality holding if and only if $K_1, \ldots, K_{n-1}$ are all dilations of each other.

This is just a special case of the following result:

Theorem 4.3. *Let $K_1, \ldots, K_{n-1} \in \varphi^n$, $p \in (0,1)$, $q \ge 1$, $1 < r \le n-1$, $0 \le i < n$, $0 \le j < n-1$, $i, j \in \mathbb{N}$ and let D_i $(i = 1, 2, \ldots, n)$ be the dilated copies of each other, respectively. Then*

$$S_{\tilde{V}_{q,i}}(\mathbf{I}^p(K_1,\ldots,K_{n-1}),\mathbf{I}^p(D_1,\ldots,D_{n-1}))^r \le \prod_{j=1}^{r} S_{\tilde{V}_{q,i}}(\mathbf{I}^p(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_{n-1}),\mathbf{I}^p(\underbrace{D_j,\ldots,D_j}_{r},D_{r+1},\ldots,D_{n-1})), \tag{4.4}$$

with the equality holding if and only if $K_1, \ldots, K_{n-1}$ are all dilations of each other.

Proof. From (1.2) and (2.1), we have that

$$\tilde{V}_{q,i}(\mathbf{I}^p(K_1,\ldots,K_{n-1})) = \omega_n\bigg(\frac{1}{n\omega_n}\Big(\frac{2}{1-p}\Big)^{\frac{(n-i)q}{p}}\int_{S^{n-1}}\tilde{v}_p(K_1\cap E_u,\ldots,K_{n-1}\cap E_u)^{\frac{(n-i)q}{p}}\,dS(u)\bigg)^{\frac{1}{q}}. \tag{4.5}$$

By using inequality (3.1), we easily get that

$$\tilde{v}_p(K_1\cap E_u,\ldots,K_{n-1}\cap E_u) \le \bigg(\prod_{j=1}^{r}\tilde{v}_p(\underbrace{K_j\cap E_u,\ldots,K_j\cap E_u}_{r},K_{r+1}\cap E_u,\ldots,K_{n-1}\cap E_u)\bigg)^{\frac{1}{r}}, \tag{4.6}$$

with the equality holding if and only if $K_1 \cap E_u, \ldots, K_{n-1} \cap E_u$ are all dilations of each other, which is true if and only if $K_1, \ldots, K_{n-1}$ are all dilations of each other.

On the other hand, Hölder's inequality can be written as

$$\int_{S^{n-1}}\prod_{i=1}^{m} f_i(u)\,dS(u) \le \prod_{i=1}^{m}\bigg(\int_{S^{n-1}}(f_i(u))^m\,dS(u)\bigg)^{\frac{1}{m}}, \tag{4.7}$$

with the equality holding if and only if all f_i are proportional.

From (4.5), (4.6) and (4.7), we obtain

$$\begin{aligned}
&\tilde{V}_{q,i}(\mathbf{I}^p(K_1,\ldots,K_{n-1}))\\
&= \omega_n\bigg(\frac{1}{n\omega_n}\Big(\frac{2}{1-p}\Big)^{\frac{(n-i)q}{p}}\int_{S^{n-1}}\tilde{v}_p(K_1\cap E_u,\ldots,K_{n-1}\cap E_u)^{\frac{(n-i)q}{p}}\,dS(u)\bigg)^{\frac{1}{q}}\\
&\le \omega_n\bigg(\frac{1}{n\omega_n}\Big(\frac{2}{1-p}\Big)^{\frac{(n-i)q}{p}}\bigg)^{\frac{1}{q}}\bigg(\int_{S^{n-1}}\prod_{j=1}^{r}\tilde{v}_p(\underbrace{K_j\cap E_u,\ldots,K_j\cap E_u}_{r},K_{r+1}\cap E_u,\ldots,K_{n-1}\cap E_u)^{\frac{(n-i)q}{rp}}\,dS(u)\bigg)^{\frac{1}{q}}\\
&\le \omega_n\bigg(\prod_{j=1}^{r}\frac{1}{n\omega_n}\Big(\frac{2}{1-p}\Big)^{\frac{(n-i)q}{p}}\bigg)^{\frac{1}{rq}}\prod_{j=1}^{r}\bigg(\int_{S^{n-1}}\tilde{v}_p(\underbrace{K_j\cap E_u,\ldots,K_j\cap E_u}_{r},K_{r+1}\cap E_u,\ldots,K_{n-1}\cap E_u)^{\frac{(n-i)q}{p}}\,dS(u)\bigg)^{\frac{1}{rq}}\\
&= \bigg(\prod_{j=1}^{r}\tilde{V}_{q,i}(\mathbf{I}^p\,(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_{n-1}))\bigg)^{\frac{1}{r}}.
\end{aligned}$$

In view of the equality conditions in (4.6) and (4.7), it follows that the equality holds if and only if $K_1, \ldots, K_{n-1}$ are all dilations of each other.

In view of that D_i $(i = 1, 2, \ldots, n)$ are dilated copies of each other, we have

$$\tilde{V}_{q,i}(\mathbf{I}^p(D_1, \ldots, D_{n-1}))^r = \prod_{j=1}^{r} \tilde{V}_{q,i}(\mathbf{I}^p\,(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1})).$$

Hence,

$$S_{V_{q,i}}(\mathbf{I}^p(K_1, \ldots, K_{n-1}), \mathbf{I}^p(D_1, \ldots, D_{n-1})) \leq \Big(\prod_{j=1}^{r} \tilde{V}_{q,i}(\mathbf{I}^p\,(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1}))\Big)^{\frac{1}{r}} + \Big(\prod_{j=1}^{r} \tilde{V}_{q,i}(\mathbf{I}^p\,(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1}))\Big)^{\frac{1}{r}}, \tag{4.8}$$

with the equality holding if and only if $K_1, \ldots, K_{n-1}$ are all dilations of each other.

By using inequality (3.2) in Lemma 3.3 on the right-hand side of inequality (4.8), we obtain

$$\begin{aligned} &S_{V_{q,i}}(\mathbf{I}^p(K_1, \ldots, K_{n-1}), \mathbf{I}^p(D_1, \ldots, D_{n-1})) \\ &\quad\leq \prod_{j=1}^{r}\Big(\tilde{V}_{q,i}(\mathbf{I}^p\,(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1})) + \tilde{V}_{q,i}(\mathbf{I}^p\,(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1}))\Big)^{\frac{1}{r}} \\ &\quad\leq \Big(\prod_{j=1}^{r} S_{V_{q,i}}(\mathbf{I}^p(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1}), \mathbf{I}^p(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1}))^{\frac{1}{r}}\Big)^{r}. \end{aligned}$$

In view of the equality conditions in inequality (3.2), it follows that the equality holds if and only if $K_1, \ldots, K_{n-1}$ are all dilations of each other. □

Remark 4.4. Taking $q = 1$ in (4.4), we obtain

$$\begin{aligned} &S_{\tilde{W}_i}(\mathbf{I}^p(K_1, \ldots, K_{n-1}), \mathbf{I}^p(D_1, \ldots, D_{n-1})) \\ &\quad\leq \prod_{j=1}^{r} S_{\tilde{W}_i}(\mathbf{I}^p(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1}), \mathbf{I}^p(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1})), \end{aligned}$$

with the equality holding if and only if $K_1, \ldots, K_{n-1}$ are all dilations of each other. Here, $S_{\tilde{W}_i}$ denotes the well known i-dual quermassintegral sum function.

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