Optimal Dividend Policy with Liability Constraint under a Hidden Markov Regime-Switching Model

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Abstract

This paper deals with the optimal liability and dividend strategies for an insurance company in Markov regime-switching models. The objective is to maximize the total expected discounted utility of dividend payment in the infinite time horizon in the logarithm and power utility cases, respectively. The switching process, which is interpreted by a hidden Markov chain, is not completely observable. By using the technique of the Wonham filter, the partially observed system is converted to a completely observed one and the necessary information is recovered. The upper-lower solution method is used to show the existence of classical solution of the associated second-order nonlinear Hamilton-Jacobi-Bellman equation in the two-regime case. The explicit solution of the value function is derived and the corresponding optimal dividend policies and liability ratios are obtained. In the multi-regime case, a general setting of the Wonham filter is presented, and the value function is proved to be a viscosity solution of the associated system of Hamilton-Jacobi-Bellman equations.

Key Words. Stochastic control, dividend policy, liability, reinsurance, regime-switching, hidden Markov chain.

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1 Introduction

Because of the nature of insurance companies' products, insurers tend to accumulate relatively large amounts of cash or cash equivalents, pursue capital gains, and keep sufficient capital reserves or surplus in order to pay future claims and avoid financial ruin. As public listed companies, most insurance companies undergo pressures to pay dividend to shareholders. However, the payment of dividends to shareholders may reduce an insurer's ability to survive adverse investment and underwriting experience. The study of optimal dividend payment and liability management of an insurance company has become a high priority task. The determination of optimal liability level has its short- and long-run effect on the insurance companys performance. More importantly, it is a crucial issue to evaluate the vulnerabilities of the financial status of the insurance companies by considering their leverage levels. The liability ratio, defined as the ratio of liability and surplus (L_t/X_t), measures the leverage level of the insurance companies. As the actual liability ratio exceeds the optimal liability ratio, the probability of default rises.

The dividend optimization problem is a classical problem in the literature of actuarial science and has attracted extensive attention. Initiated in the work of De Finetti (1957), there have been increasing efforts on using advanced methods of stochastic control to study the optimal dividend policy; see Gerber (1972), Asmussen and Taksar (1997), Gerber and Shiu (2004), Gerber and Shiu (2006), Kulenko and Schimidli (2008), Yao et al. (2011) and Jin et al. (2015). As a standard tool with the goal of reducing and eliminating risk to protect insurance companies against the impact of claim volatilities, reinsurance has been adopted to manage the risk of unsustainable large claims. The primary insurance carrier pays the reinsurance company a certain part of the premiums. In return, the reinsurance company is obliged to share the risk of large claims. Some recent work can be found in Asmusen et al. (2000), Bai et al. (2008), Choulli et al. (2001), Meng and Siu (2011) and references therein. A practitioner manages the liability level and dividend payment against future risks by taking into account reinsurance tools.

Moreover, empirical studies indicate in particular that traditional diffusion models are inadequate to capture the asset value movements due to the extreme economic environment. To better reflect reality, regime-switching models have been widely adopted to analyze the optimization problems from economics, finance and actuarial science. A comprehensive study of switching diffusions with "state-dependent" switching is in Yin and Zhu (2010). We incorporate a hidden Markov chain representing the discrete movements. The return rate of assets in this paper is modeled as a regime-switching diffusion process. The Markov jumps describe the economic trends and impacts that cannot be modeled as either ordinary differential equations or stochastic differential equations. A typical example is a two-state Markov chain with one state representing the soft market and the other representing the hard market. We expect that including the "state-dependent" switching process makes our proposed model outperform in depicting the actual market condition because they consider the respective uncertainties of the involved variables. However, in reality, an observation without noise is virtually impossible to detect. The state of the Markov chain is assumed observable with additive white noise. Since the states in the Markov chain are not directly observable, a nonlinear filtering technique is introduced to recover the necessary information of observations (Liptser and Shiryaev (1968)). This study adopts the well-known filter technique by Wonham (1965), according to which the partially observed system will be converted to a completely observable one. Consequentially, the feasibility of the estimation of our proposed model is assured. Related work can be referred to Yang and Yin (2004), Haussmann and Sass (2004), Elliott and Siu (2013), Elliott et al. (1995), Yu et al. (2014), Baeuerle and Rieder (2007), Korn et al. (2011), Song et al. (2011), Siu (2012), Tran and Yin (2014) and Rishel and Helmes (2006).

In this work, we focus on the financial health of the insurers that mainly issue Credit Default Swap (CDS) protections. CDS is a popular credit derivative to enhance the credit ratings of the reference assets. The CDSs are privately negotiated contracts that perform in a similar manner to insurance contacts, but their payoff function is similar to a put option. The CDS requires that the insurer put up more collateral if the market value of the securities insured falls below the predetermined level. Claims are the required payments to the insured holders of CDSs, due to either defaults of the obligor or for collateral calls when the prices of the insured securities decline. The liabilities, denoted as $L(t)$, depict how much insurance policies to offer, and are monotonic to the total notional values protected in the CDS contracts.

Unlike the classical ruin problem or the Cramér-Lundberg approach, our criterion does not focus solely upon the probability of ruin. The criterion in our problem is to maximize the expectation of the discounted value of the utility of dividend until financial ruin under optimal liabilities and dividend strategies. We find the optimal capital requirement or leverage that balances risks against expected growth and return under logarithm and power utility functions. The two control variables are liability ratio and dividend payment rate, respectively. Starting with a two-regime case, we obtain the associated Hamilton-Jacobi-Bellman (HJB) equation which is a nonlinear second-order partial differential equation (PDE) by using dynamic programming principle. The existence of the classical solution to the system of nonlinear PDEs is proved by the ordered upper-lower solution method introduced in Fleming and Pang (2004). Then we verify that the classical solution to the nonlinear PDE is indeed the value function. Further, we extend the formulation to a multi-regime Markov switching case. A general formulation of the Wonham filter and the associated system of HJB equations are obtained. The value function is shown to be the viscosity solution to the associated system of HJB equations.

The rest of the paper is organized as follows. A general formulation of asset value, surplus, insurance liabilities, liability ratio, dividend strategies, and assumptions are presented in Section 2. The Wonham filter for the two-regime case is presented. Section 3 deals with optimal liability ratio and dividend payment strategies in logarithm utilities. The upper-lower solution method are introduced, and the existence of classical solution of the HJB equation is proved in Section 3.1. The verification theorem of optimal value function is presented in Section 3.2. Section 4 deals with optimal liability ratio and dividend payment strategies in power utilities. Section 5 deals with the multi-regime case. The value function is proved to be the viscosity solution of the associated system of HJB equations. Finally, additional remarks are provided in Section 6.

2 Formulation

For an insurer, when the insurer incurs a liability at time t, he receives a premium for the amount insured. The collected premium will increase surplus at time t. Denote by β the premium rate, which represents the cost of protection per dollar of insurance liabilities. The surplus increasing from the insurance sales during the time period $[t, t+dt]$ is denoted as $\beta L(t)dt$.

To protect insurance companies against the impact of claim volatilities, reinsurance is a standard tool with the goal of reducing and eliminating risk. The primary insurance carrier pays the reinsurance company a certain part of the premiums. In return, the reinsurance company is obliged to share the risk of large claims. We assume that proportional reinsurance is adopted by the primary insurance company in our model. Within this scheme, the reinsurance company covers a fixed percentage of the liability in the CDS. Let θ be an exogenous retention level for the reinsurance policy. Note that $\theta \in (0,1]$. Denote by $g(\theta)$ be reinsurance charge rate (the cost of reinsurance protection per dollar of reinsured liability). Hence, the reinsurance charge during the time period

$[t, t + dt]$ is denoted as $g(\theta)L(t)dt$, and only $\theta L(t)dt$ will be covered by the primary insurance company. Further, from the point of view of primary insurance company, the company will choose to balance the reinsurance cost and profitability to partially hedge the risks by selecting a reasonable coverage level. It is natural that the reinsurance cost per dollar of the liability should be less than the premium collected per dollar of the liability. That is, it is intuitively that $\beta > g(\theta)$.

Remark 2.1. The reinsurance strategy plays a key role in determining the optimal liability ratios, which are shown in (3.18) and (4.10) in later sections. The adjustment of optimal liability level is determined by the reinsurance cost. A more detailed analysis of the impact of reinsurance strategies on insurance companies' liability levels is provided in the concluding remarks.

At this premium rate β and reinsurance retention level θ, there is an elastic demand for insurance contract and the insurer decides how much insurance $L(t)$ to offer at that premium rate and reinsurance retention level. One natural control variable of the insurance company is its liability, the insurance policies sold. Let $\pi(t) = L(t)/X(t)$ be the liability ratio of the insurance company. Then, the leverage, which is described as the ratio between asset values and surplus, can be written as $A(t)/X(t) = 1 + \pi(t)$. To avoid the insurance liabilities being too large, the insurers will decide the optimal liabilities to manage the sale of insurance policies.

To delineate the random environment and insurance business cycle, we use a continuous-time two-state Markov chain $\alpha(t)$ taking values in the finite space $\{1, 2\}$ to represent the soft and hard markets. In the industry's soft market, the premium rates are low, and the insurance carriers will rely on the high return of asset investment. While in the industry's hard market, the economic is in the downturn. The insurer's cannot make the high return as they once had. Premium rates will escalate to counteract the losses from the investment. $\beta(\alpha(t))$ was denoted as the premium rates varying with Markov switches. Let β_1 be the premium rate in the soft market and β_2 be the premium rate in the hard market, where $\beta_1 < \beta_2$.

The assets are invested in the financial markets. We assume that the asset value $A(t)$ in the financial market follows a geometric Brownian Motion process with regime-swtiching

$$\frac{dA(t)}{A(t)} = \mu(\alpha(t))dt + \sigma_1 dW_1(t), \tag{2.1}$$

where $\mu(\alpha(t))$ is the varying drift of the asset and σ_1 is the corresponding volatility and $W_1(t)$ is a standard Brownian motion. Hence, the surplus process in the absence of claims and dividend payment can be denoted by $\widetilde{X}(t)$ and follows

$$d\widetilde{X}(t) = [\beta(\alpha(t)) - g(\theta)]L(t)dt + A(t)[\mu(\alpha(t))dt + \sigma_1 dW_1(t)]. \tag{2.2}$$

We assume the return rate of asset $\mu(\alpha(t))$ is subjected to a two-state Markov regime-switching process, which has the generator

$$\widetilde{Q} = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix},$$

where frequencies $q_1 > 0$ and $q_2 > 0$. The return series $\mu(\alpha(t))$ takes values in the state space $\{\mu_1, \mu_2\}$, where $\mu_1 \geq \mu_2$. Specifically, when $\alpha(t) = 1$, $\mu(\alpha(t)) = \mu_1$. When $\alpha(t) = 2$, $\mu(\alpha(t)) = \mu_2$. We use μ_1 to represent the bull investment return in soft market and μ_2 to represent the bear investment return in hard market. In view of the generator of Markov chain, for small time interval

$\delta > 0$, we have the respective transition probabilities as follows.

$$\Pr\{\mu(t+\delta) = \mu_2 | \mu(t) = \mu_1, \mu(s), s \le t\} = \begin{cases} q_1\delta + o(\delta), & \text{if } \mu_2 \ne \mu_1, \\ 1 - q_1\delta + o(\delta), & \text{if } \mu_2 = \mu_1, \end{cases} \tag{2.3}$$

and

$$\Pr\{\mu(t+\delta) = \mu_1 | \mu(t) = \mu_2, \mu(s), s \le t\} = \begin{cases} q_2\delta + o(\delta), & \text{if } \mu_1 \ne \mu_2, \\ 1 - q_2\delta + o(\delta), & \text{if } \mu_1 = \mu_2. \end{cases} \tag{2.4}$$

The expected time of staying in the soft market is $1/q_1$, and the expected time of staying in the hard market is $1/q_2$.

We further consider the future claims, which are against insurer's liabilities incurred earlier. The future claims are the required payments to the insured holders. Surplus declines by the amount of future claims. $R(t)$ was denoted as the future claims up to time t. Then we assume that the claims are proportional to the amount of insurance liabilities $L(t)$. Hence, the accumulated claims up to time T is denoted as

$$R(T) = \int_0^T c(t)L(t)dt, \tag{2.5}$$

where $c(t)$ can be considered as a claim rate against liabilities.

We are working on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, where $\mathcal{F}_t$ is the σ-algebra generated by $\{W_1(s), \alpha(s) : 0 \le s \le t\}$ and $\{\mathcal{F}_t\}$ is the filtration satisfying the usual conditions. A dividend strategy $D(\cdot)$ is an $\mathcal{F}_t$-adapted process $\{Z(t) : t \ge 0\}$ corresponding to the accumulated amount of dividends paid up to time t such that $Z(t)$ is a nonnegative and nondecreasing stochastic process that is right continuous and have left limits with $D(0^-) = 0$. In this paper, we consider the optimal dividend strategy where the dividend payments are proportional to the surplus with a dividend payment rate $z(t)$. For dividend payment rate $z(t)$, we assume $z(t)$ is non-negative and subject to an upper bound. As a result, we write $Z(t)$ as

$$dZ(t) = z(t)X(t)dt, \tag{2.6}$$

where $z(t)$ is an $\mathcal{F}_t$-adapted process. Thus, taking into account the impact of reinsurance, the insurer's surplus process in the presence of claims and dividend payments is given by

$$dX(t) = d\widetilde{X}(t) - \theta dR(t) - dZ(t). \tag{2.7}$$

Together with the initial condition, (2.7) follows

$$\begin{cases} dX(t) = [(\beta(\alpha(t)) - g(\theta) - \theta c(t))L(t) + \mu(\alpha(t))A(t) - z(t)X(t)]dt + A(t)\sigma_1 dW_1(t), \\ X(0) = x \ge 0 \end{cases} \tag{2.8}$$

for all $t < \tau$ and we impose $X(t) = 0$ for all $t > \tau$, where $\tau = \inf\{t \ge 0 : X(t) < 0\}$ represents the time of financial ruin. In view of (2.8), the surplus increment consists of two parts: one is from the asset appreciation with growth rate μ, the other is from the net gain of insurance contract sales (net collected premium minus claims).

Recall that $\pi(t)$ represents the liability ratio, the decision maker manages the underwriting process and determines the optimal liability ratio. Thus, $\pi(t)$ is a control variable. Denote $\Gamma = [0, \infty)$. Assume that $\pi \in \Gamma$. (2.8) can be written as

$$\begin{cases} \dfrac{dX(t)}{X(t)} = [\pi(t)(\beta(\alpha(t)) - g(\theta) - \theta c(t) + \mu(\alpha(t))) + \mu(\alpha(t)) - z(t)]dt + (\pi(t)+1)\sigma_1 dW_1(t), \\ X(0) = x. \end{cases} \tag{2.9}$$

For liability ratio $\pi(t)$, we further assume that $\forall\ T \in (0, \infty)$,

$$\mathbb{E}\int_0^T \pi^2(t)dt < \infty. \tag{2.10}$$

The representative financial institute is risk averse and the objective is to maximize the expectation of the discounted value of the utility of dividend until financial ruin. Denote by $\rho > 0$ the discount factor. For an arbitrary admissible pair $u = (\pi, z)$, the performance function is the expected discounted dividend until ruin, and is given by

$$J(x, u(\cdot)) = \mathbb{E}_x\Big[\int_0^\tau e^{-\rho t}U(z(t)X(t))dt\Big], \tag{2.11}$$

where $\mathbb{E}_x$ denotes the expectation conditioned on $X(0) = x$, U denotes the utility functions of the dividend payment.

We are interested in finding the optimal dividend payment rate, investment strategy and liability ratio to maximize the performance function $J(x, u(\cdot))$. Define $V(x)$ as the optimal value of the corresponding problem. That is,

$$V(x) = \sup_u J(x, u(\cdot)). \tag{2.12}$$

Remark 2.2. In view of (2.6), the divided payment amount is assumed to be proportional to the surplus level. The divided payment amount is limited when surplus. Intuitively, the risk of financial ruin due to a large lump sum of dividend payments can be mitigated. From (2.9), we can the financial ruin can be avoided almost surely due to the proportional type of dividend payment strategies. That is, $\mathbb{P}(\tau < \infty) = 0$. Therefore, (2.11) can be rewritten as

$$J(x, u(\cdot)) = \mathbb{E}_x\Big[\int_0^\infty e^{-\rho t}U(z(t)X(t))dt\Big]. \tag{2.13}$$

In practice, we do not have direct information of the Markov regime-switching process, but can only observe $\mu(\alpha(t))$ with its assumption of white noise innovation. By using the technique in Yang et al. (2015), we observe $\phi(t)$, whose dynamics is given by

$$d\phi(t) = \mu(\alpha(t))dt + \sigma_2 dW_2(t), \tag{2.14}$$

where σ_2 is a constant. $W_2(t)$ is used to model the observation noises. $W_2(t)$ is a standard scalar Brownian motion, and is independent with $W_1(t)$ and $\alpha(t)$. We employ the nonlinear filtration technique introduced by Wonham (1965) to estimate the dynamic state of the Markov regime-switching process on the basis of the data perturbed by white noise. This technique reduces the partially observed problem to a completely observed problem.

The first key issue for the estimation of a Markov process is to confirm if we have the conditional probability of the states of the Markov process based on the available data up to time. Let $p(t)$ be the conditional probability of state of the Markov jump process that represents the rate of return performs in a soft market. That is,

$$p(t) = \Pr[\mu(t) = \mu_1 | \mathcal{F}_s, 0 \leq s \leq t], \tag{2.15}$$

with $p_0 = \mathbb{P}[\mu(0) = \mu_1]$. p_0 is the initial point of conditional probability. Let $\Lambda = (0, 1)$. Then $p(t) \in \Lambda$. In light of the nonlinear filtering results in Wonham (1965), the Wonham filter for the two-regime case is given by

$$dp(t) = [-q_1 p(t) + q_2(1 - p(t))]dt - \frac{\widehat{\mu}(p(t))}{\sigma_2^2} \left(d\phi(t) - \bar{\mu}(p(t))dt\right), \tag{2.16}$$

where

$$\bar{\mu}(p(t)) = \mu_1 p(t) + \mu_2(1 - p(t)),$$

$$\widehat{\mu}(p(t)) = (\mu_1 - \mu_2)p(t)(1 - p(t)).$$

Let

$$dW_3(t) = \frac{d\phi(t) - \bar{\mu}(p(t))dt}{\sigma_2}. \tag{2.17}$$

In view of (2.16), the conditional probability $p(t)$ follows

$$dp(t) = [-q_1 p(t) + q_2(1 - p(t))]dt + \frac{\widehat{\mu}(p(t))}{\sigma_2} dW_3(t), \tag{2.18}$$

where $p(0) = p_0$, and $W_3(t)$ is a Brownian motion which is also called the innovations process. The innovations process $W_3(t)$ is independent with $W_1(t)$. Similar to the work in Yang et al. (2015), we find the best estimate for the asset price and surplus process in the sense of least mean square prediction error. Then we can transfer the partially observable system to a completely observable system as follows. Thus, in view of the conditional probability defined in (2.15), the asset value process in (2.1) can be rewritten in terms of the conditional probability. That is,

$$\frac{dA(t)}{A(t)} = \bar{\mu}(p(t))ds + \sigma_1 dW_1(t). \tag{2.19}$$

Let $\bar{\beta}(p(t)) = \beta_1 p(t) + \beta_2(1 - p(t))$. Now the completely observable dynamics of surplus process (2.9) can be written as

$$\begin{cases} \dfrac{dX(t)}{X(t)} = \{\pi(t)[\bar{\beta}(p(t)) - g(\theta) - \theta c(t) + \bar{\mu}(p(t))] + \bar{\mu}(p(t)) - z(t)\}dt + (\pi(t) + 1)\sigma_1 dW_1(t), \\ X(0) = x. \end{cases} \tag{2.20}$$

Our original problem of maximizing the objective function $J(x, u(\cdot))$ in (2.13) is equivalent to maximizing $J(x, p, u(\cdot))$ based on nonlinear filtering technique.

$$J(x, p, u(\cdot)) = \mathbb{E}_{x,p}\Big[\int_0^\infty e^{-\rho t} U(z(t)X(t))dt\Big], \tag{2.21}$$

where $\mathbb{E}_{x,p}$ is the conditional expectation given $X(0) = x$, and $p(0) = p$. The optimization problem can be rewritten as follows

$$
\begin{aligned}
&\text{Maximize} \quad J(x,p,u(\cdot)) \\
&\text{s.t.} \quad \begin{cases} \dfrac{dX(t)}{X(t)} = \{\pi(t)[\bar{\beta}(p(t)) - g(\theta) - \theta c(t) + \bar{\mu}(p(t))] + \bar{\mu}(p(t)) - z(t)\}dt + (\pi(t)+1)\sigma_1 dW_1(t), \\ X(0) = x, \\ dp(t) = [-q_1 p(t) + q_2(1-p(t))]dt + \dfrac{\widehat{\mu}(p(t))}{\sigma_2} dW_3(t), \\ p(0) = p. \end{cases}
\end{aligned}
\tag{2.22}
$$

We are working on a filtered probability space $(\Omega, \widetilde{\mathcal{F}}, \{\widetilde{\mathcal{F}}_t\}, P)$, where $\widetilde{\mathcal{F}}_t$ is the σ-algebra generated by $\{W_1(s), W_3(s), \alpha(s) : 0 \le s \le t\}$ and $\{\widetilde{\mathcal{F}}_t\}$ is the filtration satisfying the usual conditions. A strategy $u(\cdot) = \{(\pi(t), z(t)) : t \ge 0\}$ being progressively measurable with respect to $\{W_1(s), W_3(s), \alpha(s) : 0 \le s \le t\}$, augmented by the $\mathbb{P}$-null sets, is called an admissible strategy. Denote the collection of all admissible strategies or admissible controls by $\mathcal{A}$. Let N be a sufficiently large positive constant to guarantee the feasibility of the optimal dividend strategy. Then the admissible strategy set $\mathcal{A}$ can be defined as

$$
\mathcal{A} = \Big\{u(t) = (\pi(t), z(t)) \in \Gamma \times \mathbb{R} : \mathbb{E}\int_0^T \pi^2(t)dt < \infty;\ 0 \le z(t) \le N < \infty\Big\}. \tag{2.23}
$$

The value function follows

$$
V(x,p) = \sup_{u(\cdot)\in\mathcal{A}} J(x,p,u(\cdot)). \tag{2.24}
$$

To solve a stochastic control problem, one usually uses a dynamic programming approach. This in turn requires considering the generator (an operator) of the controlled process involved and use it to derive a partial differential equation, known as HJB equation, satisfied by the value function. The solution of the HJB equation then yields the optimal control and optimal value function. Assuming the existence of optimal control, for an arbitrary $V(\cdot,\cdot) \in C^2(\mathbb{R} \times \Lambda)$, define an operator $\mathcal{L}^u$ by

$$
\begin{aligned}
\mathcal{L}^u V(x,p) = {} & \frac{1}{2}V_{xx}\sigma_1^2(\pi+1)^2x^2 + \frac{\widehat{\mu}^2(p)}{2\sigma_2^2}V_{pp} + V_x x\{\pi[\bar{\beta}(p) - g(\theta) - \theta c + \bar{\mu}(p)] + \bar{\mu}(p) - z\} \\
& + V_p[-q_1 p + q_2(1-p)].
\end{aligned}
\tag{2.25}
$$

where V_x, V_p, V_{xx}, and V_{pp} denote the first-order and the second-order partial derivatives with respect to x and p, respectively. Formally, the value function (2.24) satisfies the Hamilton-Jacobi-Bellman equation

$$
\max_u\{\mathcal{L}^u V(x,p) - \rho V(x,p) + U(zx)\} = 0. \tag{2.26}
$$

Using π and z to represent the controls, (2.26) can be rewritten as

$$
\begin{aligned}
&\max_\pi[\frac{1}{2}V_{xx}\sigma_1^2(\pi+1)^2x^2 + V_x x\pi(\bar{\beta}(p) - g(\theta) - \theta c + \bar{\mu}(p))] + \max_z[-zxV_x + U(zx)] \\
&\qquad + \frac{\widehat{\mu}^2(p)}{2\sigma_2^2}V_{pp} + \bar{\mu}(p)xV_x + V_p[-q_1 p + q_2(1-p)] - \rho V(x,p) = 0.
\end{aligned}
\tag{2.27}
$$

3 Logarithm Utility Function

We will consider two major types of utility functions: the logarithm utility and power utility in the two-regime case. Each type of the utility function is adopted by the practitioners based on their specific return and risk objectives. The logarithm utility function put much weight on penalizing liabilities that would lead to zero or low dividend payments. The power utility function is a more versatile formulation with constant relative risk aversion. The power utility function case will be analyzed in Section 4.

3.1 Optimal Controls and Value Function

We construct a solution of (2.27) with the form

$$V(x,p) = k\ln x + H(p). \tag{3.1}$$

(3.1) will be verified to be the solution of (2.27) in Section 3.2 with appropriate constant k and function $H(p)$. To determine k and $H(p)$, we plug (3.1) into (2.27). Then we have

$$\begin{aligned}\max_{\pi}[-\frac{k}{2}\sigma_1^2\pi^2 + \pi k(\bar{\beta}(p) - g(\theta) - \theta c + \bar{\mu}(p) - \sigma_1^2)] + \max_{z}[-kz + \ln z] + (1-k\rho)\ln x \\ + \frac{\widehat{\mu}^2(p)}{2\sigma_2^2}H_{pp}(p) + (-q_1 p + q_2(1-p))H_p(p) - \rho H(p) - \frac{k}{2}\sigma_1^2 + k\bar{\mu}(p) = 0.\end{aligned} \tag{3.2}$$

Since (3.2) holds for all x, we have

$$k = \frac{1}{\rho}, \tag{3.3}$$

and

$$V(x,p) = \frac{1}{\rho}\ln x + H(p). \tag{3.4}$$

In view of (3.2), the optimal liability ratio π^* is obtained as

$$\pi^* = \max\Big\{\frac{\bar{\beta}(p) - g(\theta) - \theta c + \bar{\mu}(p) - \sigma_1^2}{\sigma_1^2}, 0\Big\}. \tag{3.5}$$

The optimal dividend payment rate follows

$$z^* = \rho. \tag{3.6}$$

Substituting the optimal controls into (3.2), it yields that

$$\frac{\widehat{\mu}^2(p)}{2\sigma_2^2}H_{pp}(p) + (-q_1 p + q_2(1-p))H_p(p) - \rho H(p) + Y(p) = 0, \tag{3.7}$$

where

$$Y(p) = \frac{(\bar{\beta}(p) - g(\theta) - \theta c + \bar{\mu}(p) - \sigma_1^2)^2}{2\rho\sigma_1^2} I_{\{\bar{\beta}(p) - g(\theta) - \theta c + \bar{\mu}(p) \geq \sigma_1^2\}} + \ln\rho - 1 + \frac{2\bar{\mu}(p) - \sigma_1^2}{2\rho}.$$

$I_{\{\cdot\}}$ is the indicator function. Let $\widetilde{H}(p)$ be a classical solution of (3.7), then we will verify that the proposed value function

$$\widetilde{V}(x,p) = \frac{1}{\rho}\ln x + \widetilde{H}(p) \tag{3.8}$$

equals the value function $V(x,p)$ defined in (2.24).

Let

$$f(H,p) = -\rho H(p) + Y(p),$$

and

$$\widetilde{\mathcal{L}}H(p) = \frac{\widehat{\mu}^2(p)}{2\sigma_2^2}H_{pp}(p) + (-q_1 p + q_2(1-p))H_p(p).$$

To obtain the classical solution of (3.7), we use the upper-lower solution method in Fleming and Pang (2004) and Pao (1992). The lower solution and upper solution will be defined as follows.

Definition 3.1. A solution $H_1(p)$ is said to be a lower solution of (3.7) iff $\forall p \in \Lambda$, $H_1(p) \in C^2(\Lambda)$ and $H_1(p)$ satisfies

$$\widetilde{\mathcal{L}}H(p) + f(H,p) \geq 0. \tag{3.9}$$

A solution $H_2(p)$ is said to be an upper solution of (3.7) iff $\forall p \in \Lambda$, $H_2(p) \in C^2(\Lambda)$ and $H_2(p)$ satisfies

$$\widetilde{\mathcal{L}}H(p) + f(H,p) \leq 0. \tag{3.10}$$

Moreover, if $\forall p \in \Lambda$,

$$H_1(p) \leq H_2(p),$$

we say $H_1(p)$ and $H_2(p)$ are an ordered pair of lower solution and upper solution. To proceed, we will first find an ordered pair of lower and upper solution $(H_1(p), H_2(p))$. Then we can prove the existence of a classical solution $\widetilde{H}(p)$ of (3.7).

3.2 Verification Theorem

Lemma 3.2. *Let*

$$\widehat{h} = \frac{1}{\rho}\left(\ln \rho - 1 - \frac{1}{2\rho}\sigma_1^2 + \frac{\mu_2}{\rho}\right). \tag{3.11}$$

Then $\widehat{h}$ is a lower solution of (3.7). *Moreover, $\forall p \in \Lambda$, $f(\widehat{h},p) \geq 0$.*

Proof. Since $\widehat{h}$ is constant,

$$\frac{\widehat{\mu}^2(p)}{2\sigma_2^2}\widehat{h}_{pp} + (-q_1 p + q_2(1-p))\widehat{h}_p = 0.$$

To verify that $\widehat{h}$ is a lower solution of (3.7), it is sufficient to verify $f(\widehat{h},p) > 0$, $\forall p \in \Lambda$. In view of (3.7), we have

$$\begin{aligned} f(\widehat{h},p) &= \frac{(\bar{\beta}(p) - g(\theta) - \theta c + \bar{\mu}(p) - \sigma_1^2)^2}{2\rho\sigma_1^2} I_{\{\bar{\beta}(p) - g(\theta) - \theta c + \bar{\mu}(p) \geq \sigma_1^2\}} + \frac{1}{\rho}(\bar{\mu}(p) - \mu_2) \\ &= \frac{(\bar{\beta}(p) - g(\theta) - \theta c + \bar{\mu}(p) - \sigma_1^2)^2}{2\rho\sigma_1^2} I_{\{\bar{\beta}(p) - g(\theta) - \theta c + \bar{\mu}(p) \geq \sigma_1^2\}} + \frac{1}{\rho}(\mu_1 - \mu_2)(1-p) \\ &\geq 0. \end{aligned}$$

Then,

$$\widetilde{\mathcal{L}}\widehat{h} + f(\widehat{h},p) \geq 0.$$

Hence, $\widehat{h}$ is a lower solution of (3.7), and $f(\widehat{h},p) \geq 0$, $\forall p \in \Lambda$. □

Lemma 3.3. *Let* $\varphi = \beta_2 + \mu_2 - g(\theta) - \theta c - \sigma_1^2$, *and*

$$\tilde{\lambda} = \max_{p\in(0,1)} (q_2 - (q_1+q_2)p)((\mu_1-\mu_2+\beta_1-\beta_2)p+\varphi)(\mu_1-\mu_2+\beta_1-\beta_2),$$

and

$$\bar{h}(p) = \frac{((\mu_1-\mu_2+\beta_1-\beta_2)p+\varphi)^2}{\rho^2\sigma_1^2} + \frac{1}{\rho}\left(\frac{(\mu_1-\mu_2)^2(\mu_1-\mu_2+\beta_1-\beta_2)^2}{\rho^2\sigma_1^2\sigma_2^2} + \ln\rho - \frac{1}{2\rho}\sigma_1^2 + \frac{\mu_1}{\rho} + \widetilde{N}\right), \tag{3.12}$$

where $\widetilde{N}$ *is a sufficiently large positive constant such that*

$$\widetilde{N} > \max\left\{0, \frac{2\tilde{\lambda}}{\rho^2\sigma_1^2} - \frac{(\mu_1-\mu_2)(1-p)}{\rho} + \frac{((\mu_1-\mu_2+\beta_1-\beta_2)p+\varphi)^2}{2\rho\sigma_1^2}\right\}.$$

Then $\bar{h}(p) > \widehat{h}$, *and* $\bar{h}(p)$ *is an upper solution of* (3.7).

Proof. In view of the definition of $\widehat{h}$ and $\bar{h}(p)$, we have

$$\bar{h}(p) - \widehat{h} > \frac{((\mu_1-\mu_2+\beta_1-\beta_2)p+\varphi)^2}{\rho^2\sigma_1^2} + \frac{(\mu_1-\mu_2)^2(\mu_1-\mu_2+\beta_1-\beta_2)^2}{\rho^3\sigma_1^2\sigma_2^2} + \frac{\mu_1-\mu_2}{\rho^2}$$

Hence, $\bar{h}(p) > \widehat{h}$ as $\rho > 0$ and $\mu_1 > \mu_2$. On the other hand, $\bar{h}(p)$ is a quadratic function of q, we have

$$\bar{h}_{pp}(p) = \frac{2(\mu_1-\mu_2+\beta_1-\beta_2)^2}{\rho^2\sigma_1^2} \quad \text{and} \quad \bar{h}_p(p) = \frac{2(\mu_1-\mu_2+\beta_1-\beta_2)}{\rho^2\sigma_1^2}((\mu_1-\mu_2+\beta_1-\beta_2)p+\varphi).$$

Then,

$$\begin{aligned}\widetilde{\mathcal{L}}\bar{h}(p) &= \frac{\widehat{\mu}^2(p)}{2\sigma_2^2}\bar{h}_{pp} + (-q_1p + q_2(1-p))\bar{h}_q \\ &\le \frac{(\mu_1-\mu_2)^2(\mu_1-\mu_2+\beta_1-\beta_2)^2}{\rho^2\sigma_1^2\sigma_2^2} + \frac{2\tilde{\lambda}}{\rho^2\sigma_1^2}.\end{aligned} \tag{3.13}$$

Further,

$$\begin{aligned}& f(\bar{h},p) \\ &= -\rho\bar{h}(p) + Y(p) \\ &\le -\frac{((\mu_1-\mu_2+\beta_1-\beta_2)p+\varphi)^2}{\rho\sigma_1^2} - \left(\frac{(\mu_1-\mu_2)^2(\mu_1-\mu_2+\beta_1-\beta_2)^2}{\rho^2\sigma_1^2\sigma_2^2} + \ln\rho - \frac{1}{2\rho}\sigma_1^2 + \frac{\mu_1}{\rho} + \widetilde{N}\right) \\ &\quad + \frac{((\mu_1-\mu_2+\beta_1-\beta_2)p+\varphi)^2}{2\rho\sigma_1^2} + \ln\rho - 1 + \frac{2\bar{\mu}(p)-\sigma_1^2}{2\rho} \\ &\le -\frac{((\mu_1-\mu_2+\beta_1-\beta_2)p+\varphi)^2}{2\rho\sigma_1^2} - \frac{(\mu_1-\mu_2)^2(\mu_1-\mu_2+\beta_1-\beta_2)^2}{\rho^2\sigma_1^2\sigma_2^2} - \frac{(\mu_1-\mu_2)(1-p)}{\rho} - \widetilde{N}.\end{aligned} \tag{3.14}$$

Hence, combining the equations (3.13) and (3.14), the left side of (3.10) follows

$$\begin{aligned}\widetilde{\mathcal{L}}\bar{h}(p) + f(\bar{h},p) &\le -\frac{((\mu_1-\mu_2+\beta_1-\beta_2)p+\varphi)^2}{2\rho\sigma_1^2} + \frac{2\tilde{\lambda}}{\rho^2\sigma_1^2} - \frac{(\mu_1-\mu_2)(1-p)}{\rho} - \widetilde{N} \\ &\le 0.\end{aligned}$$

Therefore, $\bar{h}(p)$ is an upper solution of (3.7). □

Theorem 3.4. *There exists a classical solution of equation* (3.7) *denoted by* $\widetilde{H}(p)$ *such that*

$$\widehat{h} \leq \widetilde{H}(p) \leq \bar{h}(p), \tag{3.15}$$

where $\widehat{h}$ *and* $\bar{h}(p)$ *are defined in* (3.11) *and* (3.12), *respectively.*

Proof. In accordance with the Lemma 3.2 and Lemma 3.3, an ordered pair of lower solution and upper solution of equation (3.7) are obtained. The existence of a classical solution can be proved by Theorem 5.2 in Chapter 7 in Pao (1992). □

Theorem 3.5. *Suppose there exists a function* $\widetilde{H}(p)$ *such that* (3.15) *holds, and that* $\widetilde{H}(p)$ *solves* (3.7). *Let*

$$\widetilde{V}(x,p) = \frac{1}{\rho}\ln x + \widetilde{H}(p). \tag{3.16}$$

Then,

(a) *For all admissible pairs of control policies* $u = (\pi, z) \in \mathcal{A}$,

$$\widetilde{V}(x,p) \geq J(x,p,u) = \mathbb{E}_{x,p}\int_0^\infty e^{-\rho t}\ln(z(t)X(t))dt. \tag{3.17}$$

(b) *If* $u^* = (\pi^*, z^*)$ *satisfies the following:*

$$\begin{aligned} \pi^* &= \frac{\bar{\beta}(p) - g(\theta) - \theta c + \bar{\mu}(p) - \sigma_1^2}{\sigma_1^2} I_{\{\bar{\beta}(p)-g(\theta)-\theta c+\bar{\mu}(p)\geq\sigma_1^2\}}, \\ z^* &= \rho. \end{aligned} \tag{3.18}$$

Then $u^* \in \mathcal{A}$. *We have*

$$\widetilde{V}(x,p) \geq J(x,p,u^*) = \mathbb{E}_{x,p}\int_0^\infty e^{-\rho t}\ln(z^*(t)X(t))dt. \tag{3.19}$$

Moreover, $\widetilde{V}(x,p)$ *is the value function defined in* (2.24). *That is,* $\widetilde{V}(x,p) = V(x,p)$.

Proof. Applying Itô's lemma to $e^{-\rho t}\widetilde{V}(x,p)$, we have

$$\begin{aligned} \mathbb{E}e^{-\rho T}\widetilde{V}(X(T),p(T)) - \widetilde{V}(x,p) &= \mathbb{E}\int_0^T e^{-\rho t}[\mathcal{L}^u\widetilde{V}(X(t),p(t)) - \rho\widetilde{V}(X(t),p(t))]dt \\ &\leq -\mathbb{E}\int_0^T e^{-\rho t}\ln(z(t)X(t))dt. \end{aligned} \tag{3.20}$$

Hence,

$$\begin{aligned} \widetilde{V}(x,p) &\geq \mathbb{E}\int_0^T e^{-\rho t}\ln(z(t)X(t))dt + \mathbb{E}e^{-\rho T}\widetilde{V}(X(T),p(T)) \\ &= \mathbb{E}\int_0^T e^{-\rho t}\ln(z(t)X(t))dt + \mathbb{E}e^{-\rho T}[\frac{1}{\rho}\ln X(T) + \widetilde{H}(p(T))]. \end{aligned} \tag{3.21}$$

To verify (3.19), we need to show that

$$\limsup_{T\to\infty}\mathbb{E}e^{-\rho T}\widetilde{V}(X(T),p(T)) \geq 0. \tag{3.22}$$

Since $\widetilde{H}(p)$ is bounded with lower solution $\widehat{h}$ and upper solution $\bar{h}(p)$ in (3.15), where $\widetilde{H}(p)$ is independent of T. We have

$$\limsup_{T\to\infty} \mathbb{E}e^{-\rho T}\widetilde{H}(p(T)) \geq 0. \tag{3.23}$$

Hence, it is sufficient to show that

$$\limsup_{T\to\infty} \mathbb{E}e^{-\rho T}\ln X(T) \geq 0. \tag{3.24}$$

Applying Itô's lemma to $\ln X(t)$, we have

$$d\ln X(t) = [\pi(t)(\bar{\beta}(p(t))-g(\theta)-\theta c(t)+\bar{\mu}(p(t)))+\bar{\mu}(p(t))-z(t)-\frac{1}{2}\sigma_1^2(\pi(t)+1)^2]dt+(\pi(t)+1)\sigma_1 dW_1. \tag{3.25}$$

Hence,

$$\begin{aligned} &\mathbb{E}e^{-\rho T}\ln X(T) \\ &= e^{-\rho T}\mathbb{E}\int_0^T [\pi(t)(\bar{\beta}(p(t)) - g(\theta) - \theta c(t) + \bar{\mu}(p(t))) + \bar{\mu}(p(t)) - z(t) - \frac{1}{2}\sigma_1^2(\pi(t)+1)^2]dt \\ &\quad +e^{-\rho T}\ln x. \end{aligned} \tag{3.26}$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}\sigma_1^2\pi^2(t), \\ A_2 &= \pi(t)p(t)(\beta_1 - \beta_2 + \mu_1 - \mu_2), \\ A_3 &= \pi(t)(\beta_2 + \mu_2 - g(\theta) - \theta c(t) - \sigma_1^2), \\ A_4 &= -z(t), \\ A_5 &= \bar{\mu}(p) - \frac{1}{2}\sigma_1^2. \end{aligned}$$

To proceed, we will prove

$$\limsup_{T\to\infty} e^{-\rho T}\mathbb{E}\int_0^T A_i dt \geq 0, \quad \forall i = 1,\dots,5.$$

The definition of admissible strategies yields that

$$\begin{aligned} \limsup_{T\to\infty} e^{-\rho T}\mathbb{E}\int_0^T A_1 dt &\geq \lim_{T\to\infty} -\frac{\sigma_1^2}{2}e^{-\rho T}\mathbb{E}\int_0^T \pi^2(t)dt \\ &= 0. \end{aligned} \tag{3.27}$$

The Cauchy-Schwarz inequality then leads to

$$\begin{aligned} \limsup_{T\to\infty} e^{-\rho T}\mathbb{E}\int_0^T A_2 dt &\geq -\lim_{T\to\infty} \frac{1}{2}e^{-\rho T}(\beta_1 - \beta_2 + \mu_1 - \mu_2)\mathbb{E}\int_0^T [p^2(t) + \pi^2(t)]dt \\ &\geq -\lim_{T\to\infty} \frac{1}{2}e^{-\rho T}(\beta_1 - \beta_2 + \mu_1 - \mu_2)\mathbb{E}\int_0^T [1 + \pi^2(t)]dt \\ &= 0. \end{aligned} \tag{3.28}$$

Referring to (3.27), we have

$$\begin{aligned}\limsup_{T\to\infty} e^{-\rho T}\mathbb{E}\int_0^T A_3 dt &\geq -\lim_{T\to\infty}\frac{1}{2}e^{-\rho T}\mathbb{E}\int_0^T [(\beta_2+\mu_2-g(\theta)-\theta c(t)-\sigma_1^2)^2+\pi^2(t)]dt \\ &= 0.\end{aligned} \tag{3.29}$$

Moreover,

$$\begin{aligned}\limsup_{T\to\infty} e^{-\rho T}\mathbb{E}\int_0^T A_4 dt &\geq -\lim_{T\to\infty} e^{-\rho T} NT \\ &= 0.\end{aligned} \tag{3.30}$$

Due to the boundness of q, we have

$$\begin{aligned}\limsup_{T\to\infty} e^{-\rho T}\mathbb{E}\int_0^T A_5 dt &\geq -\lim_{T\to\infty}\frac{1}{2}e^{-\rho T}\mathbb{E}\int_0^T [-\frac{1}{2}\sigma_1^2+\mu_1]dt \\ &= 0.\end{aligned} \tag{3.31}$$

Hence, combining (3.27) to (3.31), we obtain

$$\limsup_{T\to\infty}\mathbb{E}e^{-\rho T}\ln X(T)\geq 0.$$

Therefore, (3.22) is satisfied so (3.19) is verified.

Consider the liability ratio and dividend payment rate strategies $u^*=(\pi^*,z^*)$,

$$\begin{aligned}\pi^* &= \frac{\bar{\beta}(p)-g(\theta)-\theta c+\bar{\mu}(p)-\sigma_1^2}{\sigma_1^2} I_{\{\bar{\beta}(p)-g(\theta)-\theta c+\bar{\mu}(p)\geq\sigma_1^2\}}, \\ z^* &= \rho.\end{aligned}$$

It is not hard to show that

$$\begin{aligned}\pi^* &\in \arg\max_{\pi}[-\frac{k}{2}\sigma_1^2\pi^2+\pi k(\bar{\beta}(p)-g(\theta)-\theta c+\bar{\mu}(p)-\sigma_1^2)], \\ z^* &\in \arg\max_{z}[-kz+\ln z].\end{aligned}$$

Then $u^*\in\mathcal{A}$. Let $(X^*(t),p^*(t))$ be the corresponding trajectories of u^*. We have

$$\mathcal{L}^{u^*}\widetilde{V}(X^*(t),p^*(t))-\rho\widetilde{V}(X^*(t),p^*(t))=-\ln(z^*(t)X^*(t)).$$

Hence,

$$\widetilde{V}(x,p)=\mathbb{E}\int_0^T e^{-\rho t}\ln(z^*(t)X^*(t))dt+\mathbb{E}e^{-\rho T}\widetilde{V}(X^*(T),p^*(T)).$$

To prove (3.18), we need only verify

$$\widetilde{V}(x,p)\leq\mathbb{E}\int_0^\infty e^{-\rho t}\ln(z^*(t)X^*(t))dt. \tag{3.32}$$

That is, it is sufficient to show that

$$\liminf_{T\to\infty}\mathbb{E}e^{-\rho T}\widetilde{V}(X^*(T),p^*(T))\leq 0. \tag{3.33}$$

Similar to (3.26),

$$
\begin{aligned}
&\mathbb{E}e^{-\rho T}\ln X^*(T)\\
&= e^{-\rho T}\mathbb{E}\int_0^T [\pi^*(t)(\bar{\beta}(p(t)) - g(\theta) - \theta c(t) + \bar{\mu}(p(t))) + \bar{\mu}(p(t)) - z^*(t) - \frac{1}{2}\sigma_1^2(\pi^*(t)+1)^2]dt\\
&\quad + e^{-\rho T}\ln x.\\
&\le e^{-\rho T}\Big\{\mathbb{E}\int_0^T \left(\frac{(\bar{\beta}(p(t)) - g(\theta) - \theta c + \bar{\mu}(p(t)) - \sigma_1^2)^2}{\sigma_1^2}\right)dt + (\mu_1 - \rho)T\Big\} + e^{-\rho T}\ln x.
\end{aligned}
\tag{3.34}
$$

By virtue of the techniques in (3.27) to (3.31), we have

$$
\liminf_{T\to\infty} \mathbb{E}e^{-\rho T}\ln(X^*(T)) \le 0. \tag{3.35}
$$

Considering the fact that $\widetilde{H}(p)$ is bounded, we have

$$
\liminf_{T\to\infty} \mathbb{E}e^{-\rho T}\widetilde{H}(p^*) \le 0. \tag{3.36}
$$

Thus, (3.32) is satisfied. Combining with (3.19) and (3.32), we have

$$
V(x,p) = \mathbb{E}\int_0^\infty e^{-\rho t}\ln(z^*(t)X^*(t))dt.
$$

Then (b) is proved. □

4 Power Utility Function

In this section, we consider a power utility function of the following form

$$
U(x) = \frac{1}{\gamma}x^\gamma, \quad x > 0,
$$

where $0 < \gamma < 1$.

4.1 Optimal Controls and Value Function

We construct a solution of (2.27) with the form

$$
V(x,p) = \frac{x^\gamma}{\gamma}F(p). \tag{4.1}
$$

With appropriate values of γ and $F(p)$, (4.1) will be verified to be the solution of (2.27). To determine γ and $F(p)$, we plug 4.1 into (2.27). Then we have

$$
\begin{aligned}
&\max_\pi\left[\frac{1}{2}\sigma_1^2(\gamma-1)F(p)(\pi+1)^2x^\gamma + F(p)\left(\bar{\beta}(p) - g(\theta) - \theta c + \bar{\mu}(p)\right)\pi x^\gamma\right] + \max_z\left[-zF(p) + \frac{z^\gamma}{\gamma}\right]x^\gamma\\
&+\frac{\widehat{\mu}^2(p)}{2\sigma_2^2\gamma}F_{pp}(p)x^\gamma + \bar{\mu}(p)F(p)x^\gamma + \frac{1}{\gamma}F_p(p)\left[-\lambda_1 p + \lambda_2(1-p)\right]x^\gamma - \frac{\rho}{\gamma}F(p)x^\gamma = 0.
\end{aligned}
\tag{4.2}
$$

Since (4.2) holds for all x, we have

$$\max_{\pi}\left[\frac{1}{2}\sigma_1^2(\gamma-1)(\pi+1)^2+\left(\bar\beta(p)-g(\theta)-\theta c+\bar\mu(p)\right)\pi\right]F(p)+\max_{z}\left[-zF(p)+\frac{z^\gamma}{\gamma}\right] \\ +\frac{\widehat{\mu}^2(p)}{2\sigma_2^2\gamma}F_{pp}(p)+\bar\mu(p)F(p)+\frac{1}{\gamma}F_p(p)\left[-\lambda_1 p+\lambda_2(1-p)\right]-\frac{\rho}{\gamma}F(p)=0. \tag{4.3}$$

In view of (4.3), the optimal liability ratio π^* is obtained as

$$\pi^*=\frac{\bar\beta(p)-g(\theta)-\theta c+\bar\mu(p)-\sigma_1^2(1-\gamma)}{\sigma_1^2(1-\gamma)}I_{\{\bar\beta(p)-g(\theta)-\theta c+\bar\mu(p)\geq\sigma_1^2(1-\gamma)\}}.$$

The optimal dividend payment rate follows

$$z^*=(F(p))^{\frac{1}{\gamma-1}}.$$

Substituting the optimal controls into (4.3), it yields that

$$\frac{\widehat{\mu}^2(p)}{2\sigma_2^2}F_{pp}(p)+\left[-\lambda_1 p+\lambda_2(1-p)\right]F_p(p)+(1-\gamma)\left(F(p)\right)^{\frac{\gamma}{\gamma-1}}+Y(p)F(p)=0, \tag{4.4}$$

where

$$\begin{aligned} Y(p) &= \frac{\gamma}{2\sigma_1^2(1-\gamma)}\left(\bar\beta(p)-g(\theta)-\theta c+\bar\mu(p)\right)^2-\gamma\left(\bar\beta(p)-g(\theta)-\theta c+\frac{\rho}{\gamma}\right) \\ &\quad -\frac{\gamma}{2\sigma_1^2(1-\gamma)}\left(\bar\beta(p)-g(\theta)-\theta c+\bar\mu(p)-\sigma_1^2(1-\gamma)\right)^2 I_{\{\bar\beta(p)-g(\theta)-\theta c+\bar\mu(p)<\sigma_1^2(1-\gamma)\}}. \end{aligned}$$

Let

$$R(p)=\ln F(p).$$

Then (4.4) is equivalent to

$$\frac{\widehat{\mu}^2(p)}{2\sigma_2^2}\left[R_{pp}(p)+R_p^2(p)\right]+\left[-\lambda_1 p+\lambda_2(1-p)\right]R_p(p)+L(p)=0, \tag{4.5}$$

where

$$L(p)=(1-\gamma)e^{\frac{1}{\gamma-1}R(p)}+Y(p).$$

Let $\widetilde{R}(p)$ be a classical solution of (4.5), then we will verify that the proposed value function

$$\widetilde{V}(x,p)=\frac{x^\gamma}{\gamma}e^{\widetilde{R}(p)}$$

equals the value function $V(x,p)$ defined in (2.24).

Similar to the method in Section 3, we try to find the ordered lower solution and upper solution of (4.5), and then prove the existence of a classical solution $\widetilde{R}(p)$ of (4.5).

Lemma 4.1. *Let R_0 be a constant defined by*

$$e^{\frac{1}{\gamma-1}R_0}=\frac{1}{2}\gamma\sigma_1^2+\frac{\rho}{1-\gamma}.$$

and $\overline{R}(p)\equiv\overline{R}$ with $\overline{R}<R_0$ being a constant. Then $\overline{R}$ is a lower solution of (4.5).

Proof. Since $\overline{R}(p)$ is a constant,

$$\frac{\widehat{\mu}^2(p)}{2\sigma_2^2}\left[\overline{R}_{pp}(p)+\overline{R}_p^2(p)\right]+\left[-\lambda_1 p+\lambda_2(1-p)\right]\overline{R}_p(p)=0.$$

To verify that $\overline{R}(p)$ is a lower solution of (4.5), it is sufficient to show that $L(p)>0$. It is sufficient to show

$$\begin{aligned}
&(1-\gamma)e^{\frac{1}{\gamma-1}\overline{R}}+\frac{\gamma}{2\sigma_1^2(1-\gamma)}\left(\bar{\beta}(p)-g(\theta)-\theta c+\bar{\mu}(p)\right)^2\\
&\quad-\gamma\left(\bar{\beta}(p)-g(\theta)-\theta c\right)-\rho\\
&\quad-\frac{\gamma}{2\sigma_1^2(1-\gamma)}\left(\bar{\beta}(p)-g(\theta)-\theta c+\bar{\mu}(p)-\sigma_1^2(1-\gamma)\right)^2\\
&=(1-\gamma)e^{\frac{1}{\gamma-1}\overline{R}}+\gamma\bar{\mu}(p)-\rho-\frac{\gamma}{2}\sigma_1^2(1-\gamma)\\
&\geq(1-\gamma)e^{\frac{1}{\gamma-1}\overline{R}}-\frac{1}{2}\gamma(1-\gamma)\sigma_1^2-\rho\\
&\geq 0,
\end{aligned}$$

which is guaranteed by the choice of $\overline{R}$. □

Lemma 4.2. *Suppose that $Y(p)<-\bar{y}$ for some constant $\bar{y}>0$. Let*

$$\widehat{R}(p)=B\left[\bar{\beta}(p)-g(\theta)-\theta c+\bar{\mu}(p)\right]^2+C,$$

where $B>0$ is a sufficient small constant and C is a sufficient large positive constant. Then $\widehat{R}(p)$ is a upper solution of (4.5).

Proof. Since

$$\widehat{R}(p)=B\left[\left(\mu_1-\mu_2+\beta_1-\beta_2\right)p+\psi\right]^2+C,$$

$$\begin{aligned}
\widehat{R}_p(p)&=2B\left(\mu_1-\mu_2+\beta_1-\beta_2\right)\left(\bar{\beta}(p)-g(\theta)-\theta c+\bar{\mu}(p)\right),\\
\widehat{R}_{pp}(p)&=2B\left(\mu_1-\mu_2+\beta_1-\beta_2\right)^2,
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{\widehat{\mu}^2(p)}{2\sigma_2^2}\left[\widehat{R}_{pp}(p)+\widehat{R}_p^2(p)\right]+\left[-\lambda_1 p+\lambda_2(1-p)\right]\widehat{R}_p(p)\\
&=\frac{\widehat{\mu}^2(p)}{2\sigma_2^2}\left\{2B\left(\mu_1-\mu_2+\beta_1-\beta_2\right)^2+\left[2B\left(\mu_1-\mu_2+\beta_1-\beta_2\right)\left(\bar{\beta}(p)-g(\theta)-\theta c+\bar{\mu}(p)\right)\right]^2\right\}\\
&\quad+2\left[-\lambda_1 p+\lambda_2(1-p)\right]B\left(\mu_1-\mu_2+\beta_1-\beta_2\right)\left(\bar{\beta}(p)-g(\theta)-\theta c+\bar{\mu}(p)\right)\\
&=B\left\{\frac{\widehat{\mu}^2(p)}{2\sigma_2^2}\left[2\left(\mu_1-\mu_2+\beta_1-\beta_2\right)^2+\left[2\left(\mu_1-\mu_2+\beta_1-\beta_2\right)\left(\bar{\beta}(p)-g(\theta)-\theta c+\bar{\mu}(p)\right)\right]^2\right]\right.\\
&\quad\left.+2\left[-\lambda_1 p+\lambda_2(1-p)\right]\left(\mu_1-\mu_2+\beta_1-\beta_2\right)\left(\bar{\beta}(p)-g(\theta)-\theta c+\bar{\mu}(p)\right)\right\}\\
&\leq BK,
\end{aligned}$$

where

$$K = \max_{p\in[0,1]} \left\{ \frac{\widehat{\mu}^2(p)}{2\sigma_2^2} \left[2(\mu_1-\mu_2+\beta_1-\beta_2)^2 + \left[2(\mu_1-\mu_2+\beta_1-\beta_2)\left(\bar{\beta}(p)-g(\theta)-\theta c+\bar{\mu}(p)\right)\right]^2 \right] \right.$$
$$\left. +2\left[-\lambda_1 p+\lambda_2(1-p)\right](\mu_1-\mu_2+\beta_1-\beta_2)\left(\bar{\beta}(p)-g(\theta)-\theta c+\bar{\mu}(p)\right)\right\}.$$

Since $(1-\gamma)e^{\frac{1}{\gamma-1}\widehat{R}(p)} \le (1-\gamma)e^{\frac{1}{\gamma-1}C}$, we have

$$\begin{aligned}&\frac{\widehat{\mu}^2(p)}{2\sigma_2^2}\left[\widehat{R}_{pp}(p)+\widehat{R}_p^2(p)\right]+\left[-\lambda_1 p+\lambda_2(1-p)\right]\widehat{R}_p(p)+(1-\gamma)e^{\frac{1}{\gamma-1}\widehat{R}(p)}+Y(p)\\ &\le BK+Y(p)+(1-\gamma)e^{\frac{1}{\gamma-1}C}.\end{aligned} \tag{4.6}$$

Since $Y(p) < -\bar{y}$, taking $B > 0$ sufficient small and $C > 0$ sufficient large, the right-hand-side of (4.6) is less than zero, which shows $\widehat{R}(p)$ is an upper solution. □

The following theorem follows from Theorem 5.2 in Chapter 7 in Pao (1992), Lemmas 4.1 and 4.2.

Theorem 4.3. *Let the conditions in Lemmas 4.1 and 4.2 hold. Then there exists a classical solution of equation (4.5) denoted by $\tilde{R}(p)$ such that*

$$\bar{R}(p) \le \tilde{R}(p) \le \widehat{R}(p). \tag{4.7}$$

Theorem 4.4. *Suppose $Y(p) < -\max_{p\in(0,1)}\left\{\frac{\gamma(\bar{\beta}(p)-g(\theta)-\theta c+\bar{\mu}(p))^2}{2\sigma_1^2(1-\gamma)^2}, \frac{\gamma}{2}\sigma_1^2\right\}$, and there exists a function $\tilde{R}(q)$ such that (4.7) holds and that $\tilde{R}(q)$ solves (4.5). Let*

$$\tilde{V}(x,p) = \frac{x^\gamma}{\gamma}e^{R(p)}. \tag{4.8}$$

Then,

(a) *For all admissible pairs of control policies $u = (\pi, z) \in \mathcal{A}$,*

$$\tilde{V}(x,p) \ge J(x,p,u) = \mathbb{E}_{x,p}\int_0^\infty e^{-\rho t}\frac{(z(t)X(t))^\gamma}{\gamma}\mathrm{d}t. \tag{4.9}$$

(b) *If $u^* = (\pi^*, z^*)$ satisfies the following:*

$$\begin{aligned}\pi^* &= \frac{\bar{\beta}(p)-g(\theta)-\theta c+\bar{\mu}(p)-\sigma_1^2(1-\gamma)}{\sigma_1^2(1-\gamma)}I_{\{\bar{\beta}(p)-g(\theta)-\theta c+\bar{\mu}(p)\ge\sigma_1^2(1-\gamma)\}},\\ z^* &= e^{\frac{1}{\gamma-1}R(p)}.\end{aligned} \tag{4.10}$$

Then $u^* \in \mathcal{A}$, and we have

$$\tilde{V}(x,p) \ge J(x,p,u^*) = \mathbb{E}_{x,p}\int_0^\infty e^{-\rho t}\frac{(z^*(t)X(t))^\gamma}{\gamma}\mathrm{d}t.$$

Moreover, $\tilde{V}(x,p)$ is the value function define in (2.24). That is $\tilde{V}(x,p) = V(x,p)$.

Proof. Applying Itô's lemma to $e^{-\rho t}\tilde{V}(x,p)$, we have

$$\begin{aligned}\mathbb{E}e^{-\rho T}\tilde{V}(X(T),p(T)) - \tilde{V}(x,p) &= \mathbb{E}\int_0^T e^{-\rho t}\mathcal{L}^u\tilde{V}(X(t),p(t)) - \rho\tilde{V}(X(t),p(t))]dt \\ &\leq -\mathbb{E}\int_0^T e^{-\rho t}\frac{(z(t)X(t))^\gamma}{\gamma}dt.\end{aligned}$$

Hence,

$$\begin{aligned}\tilde{V}(x,p) &\geq \mathbb{E}\int_0^T e^{-\rho t}\frac{(z(t)X(t))^\gamma}{\gamma}dt + \mathbb{E}e^{-\rho T}\tilde{V}(X(T),p(T)) \\ &= \mathbb{E}\int_0^T e^{-\rho t}\frac{(z(t)X(t))^\gamma}{\gamma}dt + \mathbb{E}e^{-\rho T}\frac{X^\gamma(T)}{\gamma}e^{R(p)} \\ &\geq \mathbb{E}\int_0^T e^{-\rho t}\frac{(z(t)X(t))^\gamma}{\gamma}dt.\end{aligned}$$

Now, let us consider the liability ratio and dividend payment rate strategies $u^* = (\pi^*, z^*)$. Let $X^*(t)$ be the corresponding trajectories of u^*. We have

$$\mathcal{L}^{u^*}\tilde{V}(X^*(t),p(t)) - \rho\tilde{V}(X^*(t),p(t)) = -\frac{(z^*(t)X^*(t))^\gamma}{\gamma}.$$

Hence,

$$\tilde{V}(x,p) = \mathbb{E}\int_0^T e^{-\rho t}\frac{(z^*(t)X^*(t))^\gamma}{\gamma}dt + \mathbb{E}e^{-\rho T}\tilde{V}(X^*(T),p(T)).$$

To prove $\tilde{V}(x,p) = V(x,p)$, we only need to show

$$\tilde{V}(x,p) \leq \mathbb{E}\int_0^\infty e^{-\rho t}\frac{(z^*(t)X^*(t))^\gamma}{\gamma}dt.$$

That is to show

$$\liminf_{T\to\infty}\mathbb{E}e^{-\rho T}\tilde{V}(X^*(T),p(T)) = \liminf_{T\to\infty}\mathbb{E}e^{-\rho T}\frac{(X^*(T))^\gamma}{\gamma}e^{R(p)} = 0. \tag{4.11}$$

Since $R(p)$ is bounded, it is sufficient to show

$$\liminf_{T\to\infty}\mathbb{E}e^{-\rho T}\left(X^*(T)\right)^\gamma = 0.$$

Applying Itô's formula to $(X^*(t))^\gamma$, it yields that

$$\begin{aligned}(X^*(T))^\gamma &= x^\gamma\exp\left\{\gamma\int_0^T\left[\pi^*(t)\left(\bar{\beta}(p(t)) - g(\theta) - \theta c(t) + \bar{\mu}(p(t))\right) + \bar{\mu}(p(t)) - z^*(t)\right.\right. \\ &\quad\left.\left. -\frac{1}{2}(\pi^*(t)+1)^2\sigma_1^2\right]dt + \gamma\int_0^T(\pi^*(t)+1)\sigma_1 dW_1(t)\right\}.\end{aligned}$$

Thus, by Hölder's inequality, we have

$$
\begin{aligned}
\mathbb{E}e^{-\rho T}\left(X^*(T)\right)^{\gamma} &= x^{\gamma}\mathbb{E}e^{-\rho T}\exp\left\{\gamma\int_0^T\left[\pi^*(t)\left(\bar{\beta}(p(t))-g(\theta)-\theta c(t)+\bar{\mu}(p(t))\right)\right.\right.\\
&\quad\left.\left.+\bar{\mu}(p(t))-z^*(t)-\frac{1}{2}(\pi^*(t)+1)^2\sigma_1^2\right]dt+\gamma\int_0^T(\pi^*(t)+1)\sigma_1 dW_1(t)\right\}\\
&= x^{\gamma}e^{-\rho T}\mathbb{E}\left\{\exp\left\{\gamma\int_0^T\left[\pi^*(t)\left(\bar{\beta}(p(t))-g(\theta)-\theta c(t)+\bar{\mu}(p(t))\right)\right.\right.\right.\\
&\quad\left.\left.+\bar{\mu}(p(t))-z^*(t)-\frac{1}{2}(\pi^*(t)+1)^2\sigma_1^2+\gamma(\pi^*(t)+1)^2\sigma_1^2\right]dt\right\}\\
&\quad\left.\times\exp\left\{-\gamma^2\int_0^T(\pi^*(t)+1)^2\sigma_1^2dt+\gamma\int_0^T(\pi^*(t)+1)\sigma_1 dW_1(t)\right\}\right\}\\
&\leq x^{\gamma}e^{-\rho T}\left\{\mathbb{E}\exp\left\{2\gamma\int_0^T\left[\pi^*(t)\left(\bar{\beta}(p(t))-g(\theta)-\theta c(t)+\bar{\mu}(p(t))\right)\right.\right.\right.\\
&\quad\left.\left.\left.+\bar{\mu}(p(t))-z^*(t)-\frac{1}{2}(\pi^*(t)+1)^2\sigma_1^2+\gamma(\pi^*(t)+1)^2\sigma_1^2\right]dt\right\}\right\}^{\frac{1}{2}}\\
&\quad\times\left\{\mathbb{E}\exp\left\{-2\gamma^2\int_0^T(\pi^*(t)+1)^2\sigma_1^2dt+2\gamma\int_0^T(\pi^*(t)+1)\sigma_1 dW_1(t)\right\}\right\}^{\frac{1}{2}}.
\end{aligned}
$$

Since

$$
\exp\left\{-2\gamma^2\int_0^T(\pi^*(t)+1)^2\sigma_1^2dt+2\gamma\int_0^T(\pi^*(t)+1)\sigma_1 dW_1(t)\right\}
$$

is a positive supermartingale, we have

$$
\mathbb{E}\exp\left\{-2\gamma^2\int_0^T(\pi^*(t)+1)^2\sigma_1^2dt+2\gamma\int_0^T(\pi^*(t)+1)\sigma_1 dW_1(t)\right\}\leq 1.
$$

Thus,

$$
\begin{aligned}
\mathbb{E}e^{-\rho T}\left(X^*(T)\right)^{\gamma} &\leq x^{\gamma}e^{-\rho T}\left\{\mathbb{E}\exp\left\{2\gamma\int_0^T\left[\pi^*(t)\left(\bar{\beta}(p(t))-g(\theta)-\theta c(t)+\bar{\mu}(p(t))\right)\right.\right.\right.\\
&\quad\left.\left.\left.+\bar{\mu}(p(t))-z^*(t)-\frac{1}{2}(\pi^*(t)+1)^2\sigma_1^2+\gamma(\pi^*(t)+1)^2\sigma_1^2\right]dt\right\}\right\}^{\frac{1}{2}}\\
&= x^{\gamma}\left\{\mathbb{E}\exp\left\{2\int_0^T\left[Y(p(t))+\frac{1}{2}\gamma(\pi^*(t)+1)^2\sigma_1^2-\gamma z^*(t)\right]\mathrm{d}t\right\}\right\}^{\frac{1}{2}}.
\end{aligned}
$$

Since $Y(p) < -\max_{p\in(0,1)}\left\{\frac{\gamma(\bar{\beta}(p)-g(\theta)-\theta c+\bar{\mu}(p))^2}{2\sigma_1^2(1-\gamma)^2},\frac{\gamma}{2}\sigma_1^2\right\}$ and $z^*(t)>0$, we have

$$
\liminf_{T\to\infty}\mathbb{E}e^{-\rho T}\left(X^*(T)\right)^{\gamma}\leq\liminf_{T\to\infty}x^{\gamma}\mathbb{E}\exp\left\{2\int_0^T\left[Y(p(t))+\frac{1}{2}\gamma(\pi^*(t)+1)^2\sigma_1^2-\gamma z^*(t)\right]\mathrm{d}t\right\}=0.
$$

The proof is completed. □

5 Multi-Regime Case

To delineate a general random environment and other random factors, a continuous-time Markov chain $\alpha(t)$ taking values in the finite space $\mathcal{M} = \{1, \ldots, m\}$ is adopted, where $m \geq 3$. The market switching process is represented by the Markov chain $\alpha(t)$. Let the continuous-time Markov chain $\alpha(t)$ be generated by $Q = (q_{ij}) \in \mathbb{R}^{m \times m}$. That is,

$$\Pr\{\alpha(t+\delta) = j | \alpha(t) = i, \alpha(s), s \leq t\} = \begin{cases} q_{ij}\delta + o(\delta), & \text{if } j \neq i, \\ 1 + q_{ii}\delta + o(\delta), & \text{if } j = i, \end{cases}$$

where $q_{ij} \geq 0$ for $i, j = 1, 2, \ldots, m$ with $j \neq i$ and $q_{ii} = -\sum_{j \neq i} q_{ij} < 0$ for each $i = 1, 2, \ldots, m$. Further, for each $i \in \mathcal{M}$, the drift $\mu(i)$ is denoted as μ_i.

Let $p_i(t)$ be the conditional probability of state of the Markov jump process that represents the rate of return performs in market mode $\alpha(t) = i$. That is,

$$p_i(t) = \Pr[\mu(t) = \mu_i | \mathcal{F}_s, 0 \leq s \leq t], \tag{5.1}$$

for $i \in \mathcal{M}$. Let $p(t) = (p_1(t), p_2(t), \ldots, p_m(t))' \in \mathbb{R}^{m \times 1}$, where $p(0) = p = (p_1, p_2, \ldots, p_m)' \in \mathbb{R}^{m \times 1}$ is the initial distribution of $\alpha(t)$. The Wonham filter for the multi-regime case is then given by the following system of equations

$$dp_i(t) = \sum_{k=1}^{m} q_{ki} p_i dt - \frac{\widehat{\mu}(p_i(t))}{\sigma_2^2} \bar{\mu}(t) dt + \frac{\widehat{\mu}(p_i(t))}{\sigma_2^2} d\psi(t), \tag{5.2}$$

for $i = 1, 2, \ldots, m$, where

$$\bar{\mu}(p(t)) = \sum_{i=1}^{m} p_i(t)\mu_i,$$

$$\widehat{\mu}(p_i(t)) = \mu_i - \bar{\mu}(p(t)).$$

Let

$$dW_4(t) = \frac{d\psi(t) - \bar{\mu}(p(t))dt}{\sigma_2}. \tag{5.3}$$

In view of (5.2), the conditional probability $p_i(t)$ follows

$$dp_i(t) = \sum_{k=1}^{m} q_{ki} p_i dt + \frac{\widehat{\mu}(p_i(t))}{\sigma_2} dW_4(t), \tag{5.4}$$

where $p_i(0) = p_i$, and $W_4(t)$ is a Brownian motion as the innovations process. The innovations process $W_4(t)$ is independent with $W_1(t)$. Thus, in view of the conditional probability defined in (5.1), the asset value process in (2.1) can be rewritten in terms of the conditional probability. That is,

$$\frac{dA(t)}{A(t)} = \bar{\mu}(p(t))ds + \sigma_1 dW_1(t). \tag{5.5}$$

Let $\bar{\beta}(p(t)) = \sum_{k=1}^{m} \beta_i p_i(t)$. Now the dynamics of surplus process (2.8) can be written as

$$\begin{cases} \dfrac{dX(t)}{X(t)} = \{\pi(t)[\bar{\beta}(p(t)) - g(\theta) - \theta c(t) + \bar{\mu}(p(t))] + \bar{\mu}(p(t)) - z(t)\}dt + (\pi(t)+1)\sigma_1 dW_1(t), \\ X(0) = x. \end{cases} \tag{5.6}$$

Our original problem of maximizing the objective function $J(x, u(\cdot))$ in (2.11) is equivalent to maximizing $J(x, p, u(\cdot))$ based on nonlinear filtering technique.

$$J(x,p,u(\cdot)) = \mathbb{E}_{x,p}\Big[\int_0^{\tau} e^{-\rho t} U(z(t)X(t))dt\Big], \tag{5.7}$$

where $\mathbb{E}_{x,p}$ is the conditional expectation given $X(0) = x$, and $p(0) = p$.

Let $O(t) = \text{diag}(u_1, u_2, \dots, u_m) - \bar{\mu}(p(t))I$, where $I \in \mathbb{R}^{m\times m}$ is the identity matrix. (5.4) can be rewritten as

$$dp(t) = Q'p(t)dt + \phi(t)dW_4(t), \tag{5.8}$$

where

$$\phi(t) = \frac{O(t)p(t)}{\sigma_2},$$

and Q' is the transpose of Q. Let $\eta(t) = \pi(t)[\bar{\beta}(p(t)) - g(\theta) - \theta c(t) + \bar{\mu}(p(t))] + \bar{\mu}(p(t)) - z(t)$,

$$\Phi(t) = \begin{pmatrix} \eta(t)X(t) \\ Q'p(t) \end{pmatrix}, \; B(t) = \begin{pmatrix} X(t) \\ p(t) \end{pmatrix}, \; b = \begin{pmatrix} x \\ p \end{pmatrix}.$$

Define

$$\omega(t) = \begin{pmatrix} W_1(t) \\ W_4(t)e_m \end{pmatrix},$$

and $\Sigma(t) = \text{diag}((\pi + 1)\sigma_1 X(t), \phi(t))$, where $e_m \in \mathbb{R}^m$ is the column vector with all components being one. The optimization problem can be rewritten as follows

$$\begin{aligned} &\text{Maximize } \; J(x,p,u(\cdot)) \\ &\text{s.t.} \quad \begin{cases} dB(t) = \Phi(t)dt + \Sigma(t)d\omega(t), \\ B(0) = b. \end{cases} \end{aligned} \tag{5.9}$$

The value function is given by

$$V(x,p) = \sup_{u(\cdot)\in\mathcal{A}} J(x,p,u(\cdot)). \tag{5.10}$$

Then the associated HJB equation follows

$$\sup_{u\in\mathcal{A}} D_b V \cdot \Phi + \frac{1}{2}\text{tr}\{(\Sigma\Sigma')D_b^2 V\} = \rho V, \tag{5.11}$$

and D_b and D_b^2 denote the gradient vector and matrix of second order partial derivatives of V. That is, $D_bV = (V_{b_1}, \dots, V_{b_{m+1}}) = (V_x, V_{p_1} \dots, V_{p_m})$, and $D_b^2 = (V_{b_i b_j})$, $i, j = 1, \dots, m+1$. $\text{tr}\{\cdot\}$ is defined such that $\text{tr}\{\widetilde{A}\widetilde{B}\} = \sum_{i,j=1}^n \widetilde{A}_{ij}\widetilde{B}_{ij}$ if $\widetilde{A}$ and $\widetilde{B}$ are $n \times n$ matrices.

5.1 Viscosity Solutions

In multi-regime cases, where the classical solutions are not available. we refer to the viscosity solutions. The notion of viscosity solution was introduced by Crandell and Lions (1983) for first-order Hamilton-Jacobi equations. Following the standard definition of viscosity solutions by Crandell et al. (1992) and Fleming and Soner (2006). Let Ξ be an open set of $\mathbb{R}^{m+1}$.

Definition 5.1. A continuous function $V : \Xi \to \mathbb{R}$ is said to be a viscosity subsolution of (5.11) for each $b \in \Xi$, if any twice continuously differentiable function $\xi \in C^2(\Xi)$ with $v(b_0) = \xi(b_0)$ such that $V - \xi$ reaches its maximum at b_0 satisfies

$$\sup_u D_b\xi \cdot \Phi + \frac{1}{2}\mathrm{tr}\{(\Sigma\Sigma')D_b^2\xi\} - \rho\xi \geq 0, \tag{5.12}$$

A continuous function $V : \Xi \to \mathbb{R}$ is said to be a viscosity supersolution of (5.11) for each $b \in \Xi$, if any twice continuously differentiable function $\xi \in C^2(\Xi)$ with $v(b_0) = \xi(b_0)$ such that $V - \xi$ reaches its minimum at b_0 satisfies

$$\sup_u D_b\xi \cdot \Phi + \frac{1}{2}\mathrm{tr}\{(\Sigma\Sigma')D_b^2\xi\} - \rho\xi \leq 0, \tag{5.13}$$

Finally, a continuous function $V : \Xi \to \mathbb{R}$ is said to be a viscosity solution of (5.11) if it is both a viscosity subsolution and a viscosity supersolution for each $b \in \Xi$.

Lemma 5.2. *Let*

$$\begin{aligned} S(t) = \int_0^t \{\tilde{\pi}(s)[\bar{\beta}(p(s)) - g(\theta) - \theta c(s) + \bar{\mu}(p(s))] + \bar{\mu}(p(s)) - \tilde{z}(s) - \frac{1}{2}(\tilde{\pi}(s)+1)^2\sigma_1^2\}ds \\ + \int_0^t (\tilde{\pi}(s)+1)\sigma_1 dW_1(s). \end{aligned}$$

Assume that

$$\rho > \max_\kappa \Big\{ -\frac{\gamma(1-\gamma)\sigma_1^2}{2}\Big(\frac{4(\eta - \bar{\mu}(p)+\kappa)}{\sigma_1^2(1-\gamma)} - \frac{4(\eta-\bar{\mu}(p)+\kappa)^2}{\sigma_1^4(1-\gamma)^2}\Big)\Big\} + \varepsilon,$$

where $0 < \kappa < N$, $0 < \gamma < 1$, ε is a small positive constant. Then,

$$\mathbb{E}e^{\gamma S(t)} \leq e^{(\rho-\varepsilon)t}.$$

Proof. By using Itô's lemma,

$$\begin{aligned} \frac{de^{\gamma S(t)}}{e^{\gamma S(t)}} &= \Big\{\gamma(\tilde{\pi}(t)[\bar{\beta}(p(t)) - g(\theta) - \theta c(t) + \bar{\mu}(p(t))] + \bar{\mu}(p(t)) - \tilde{z}(t)) - \frac{\gamma(1-\gamma)}{2}(\tilde{\pi}(t)+1)^2\sigma_1^2\Big\}dt \\ &\quad + \gamma(\tilde{\pi}(t)+1)\sigma_1 dW_1(t) \\ &= \Big\{ -\frac{\gamma(1-\gamma)\sigma_1^2}{2}\Big(\frac{4(\eta(t) - \bar{\mu}(p(t)) + \tilde{z}(t))}{\sigma_1^2(1-\gamma)} - \frac{4(\eta(t)-\bar{\mu}(p(t))+\tilde{z}(t))^2}{\sigma_1^4(1-\gamma)^2}\Big) \\ &\quad - \frac{\gamma(1-\gamma)\sigma_1^2}{2}\Big(\tilde{\pi}(t) + 1 - \frac{2(\eta(t)-\bar{\mu}(p(t))+\tilde{z}(t))}{\sigma_1^2(1-\gamma)}\Big)^2\Big\}dt + \gamma(\tilde{\pi}(t)+1)\sigma_1 dW_1(t) \\ &\leq \Big\{\rho - \varepsilon - \frac{\gamma(1-\gamma)\sigma_1^2}{2}\Big(\tilde{\pi}(t)+1-\frac{2(\eta(t)-\bar{\mu}(p(t))+\tilde{z}(t))}{\sigma_1^2(1-\gamma)}\Big)^2\Big\}dt + \gamma(\tilde{\pi}(t)+1)\sigma_1 dW_1(t) \\ &\leq (\rho-\varepsilon)dt + \gamma(\tilde{\pi}(t)+1)\sigma_1 dW_1(t). \end{aligned}$$

Then, it follows

$$\mathbb{E}e^{\gamma S(t)} \le 1 + (\rho - \varepsilon)\int_0^t \mathbb{E}e^{\gamma S(r)} dr.$$

Hence, by using the Gronwall's inequality, we have

$$\mathbb{E}e^{\gamma S(t)} \le e^{(\rho-\varepsilon)t}.$$

□

Theorem 5.3. *Assume that the conditions in Lemma* 5.2 *are satisfied.* $V(x,p)$ *is continuous with respect to* x *and* p.

Proof. Given the optimal control $\tilde{u} = (\tilde{\pi}, \tilde{z})$, $X(t)$ follows

$$\frac{dX(t)}{X(t)} = \{\tilde{\pi}(t)[\bar{\beta}(p(t)) - g(\theta) - \theta c(t) + \bar{\mu}(p(t))] + \bar{\mu}(p(t)) - \tilde{z}(t)\}dt + (\tilde{\pi}(t)+1)\sigma_1 dW_1(t).$$

Then, $X(t) = xe^{S(t)}$.

Case 1. We consider the logarithm utility function. Fix p. For x_1 and $x_2 \in \mathbb{R}$,

$$\begin{aligned} V(x_1,p) - V(x_2,p) &= \mathbb{E}\Big[\int_0^\tau e^{-\rho t}(\ln x_1 - \ln x_2)dt\Big] \\ &\le K_1 \int_0^\infty e^{-\rho t}|\ln x_1 - \ln x_2| t dt \\ &\le \frac{K_1}{\rho^2}|\ln x_1 - \ln x_2| \end{aligned}$$

for sufficiently large constant K_1. Then $V(x,p)$ is continuous with respect to x.

To proceed, we prove $V(x,p)$ is continuous with respect to p. Fix x. For p_1 and $p_2 \in \mathbb{R}^m$, let

$$\begin{aligned} S(p_1) = &\int_0^t \{\tilde{\pi}(s)[\bar{\beta}(p_1(s)) - g(\theta) - \theta c(s) + \bar{\mu}(p_1(s))] + \bar{\mu}(p_1(s)) - \tilde{z}(s) - \frac{1}{2}(\tilde{\pi}(s)+1)^2\sigma_1^2\}ds \\ &+ \int_0^t (\tilde{\pi}(s)+1)\sigma_1 dW_1(s), \\ S(p_2) = &\int_0^t \{\tilde{\pi}(s)[\bar{\beta}(p_2(s)) - g(\theta) - \theta c(s) + \bar{\mu}(p_2(s))] + \bar{\mu}(p_2(s)) - \tilde{z}(s) - \frac{1}{2}(\tilde{\pi}(s)+1)^2\sigma_1^2\}ds \\ &+ \int_0^t (\tilde{\pi}(s)+1)\sigma_1 dW_1(s). \end{aligned}$$

We have

$$\begin{aligned} V(x,p_1) - V(x,p_2) &= \mathbb{E}\Big[\int_0^\tau e^{-\rho t}\ln x(S(p_1) - S(p_2))dt\Big] \\ &\le K_2 \ln x \int_0^\infty e^{-\rho t}\|p_1 - p_2\| t dt \\ &\le \frac{K_2 \ln x}{\rho^2}\|p_1 - p_2\| \end{aligned}$$

for sufficiently large constant K_2, where $\|\cdot\|$ is the norm in $\mathbb{R}^m$. Then $V(x,p)$ is continuous with respect to p.

Case 2. We consider the power utility function. Recall that $U(zx) = \frac{(zx)^\gamma}{\gamma}$, where $0 < \gamma < 1$. Fix p. For x_1 and $x_2 \in \mathbb{R}$,

$$\begin{aligned} V(x_1,p) - V(x_2,p) &= \mathbb{E}\Big[\int_0^\tau e^{-\rho t}\frac{\tilde{z}^\gamma}{\gamma}(x_1^\gamma - x_2^\gamma)e^{\gamma S(t)}dt\Big] \\ &\le K_3\int_0^\infty e^{-\rho t}|x_1^\gamma - x_2^\gamma|\mathbb{E}e^{\gamma S(t)}dt \end{aligned}$$

for sufficiently large constant K_3. In view of Lemma 5.2, we have

$$V(x_1,p) - V(x_2,p) \le K_3\int_0^\infty |x_1^\gamma - x_2^\gamma|e^{-\varepsilon t}dt \le \frac{K_3}{\varepsilon}|x_1^\gamma - x_2^\gamma|.$$

Then $V(x,p)$ is continuous with respect to x.

To proceed, we prove $V(x,p)$ is continuous with respect to p. Fix x. For p_1 and $p_2 \in \mathbb{R}^m$, we have

$$\begin{aligned} V(x,p_1) - V(x,p_2) &= \mathbb{E}\Big[\int_0^\tau e^{-\rho t}\frac{\tilde{z}^\gamma}{\gamma}(e^{\gamma S(p_1)} - e^{\gamma S(p_2)})dt\Big] \\ &\le \frac{N^\gamma}{\gamma}\int_0^\infty e^{-\rho t}\|e^{\gamma S(p_1)} - e^{\gamma S(p_2)}\|dt \end{aligned}$$

Similarly to Case 1,

$$V(x,p_1) - V(x,p_2) \le \frac{K_4}{\rho}\|p_1 - p_2\|$$

for sufficiently large constant K_4. Then $V(x,p)$ is continuous with respect to p. □

Theorem 5.4. *The value function $V(x,p)$ is a viscosity solution of* (5.11).

Proof. Suppose $V - \xi$ has a local maximum at b_0 in the neighbourhood $N(b_0)$, where $b_0 = (X(t_0), p(t_0))$. Let $b_\varepsilon := (X(t_0 + \varepsilon\wedge\tau), p(t_0 + \varepsilon\wedge\tau)) \in N(b_0)$. Then

$$V(b_0) - \xi(b_0) \ge V(b_\varepsilon) - \xi(b_\varepsilon).$$

Let $\chi(b) = \xi(b) + V(b_0) - \xi(b_0)$. Then $\chi(b_0) = V(b_0)$, and $\chi(b_\varepsilon) \ge V(b_\varepsilon)$. By using Dynkin's formula, we have

$$\begin{aligned} \mathbb{E}e^{-\rho\varepsilon\wedge\tau}\chi(b_\varepsilon) - \chi(b_0) &= \mathbb{E}\int_0^{\varepsilon\wedge\tau} e^{-\rho t}[D_b\chi\cdot\Phi + \frac{1}{2}\mathrm{tr}\{(\Sigma\Sigma')D_b^2\chi\} - \rho\chi]dt \\ &= \mathbb{E}\int_0^{\varepsilon\wedge\tau} e^{-\rho t}[D_b\xi\cdot\Phi + \frac{1}{2}\mathrm{tr}\{(\Sigma\Sigma')D_b^2\xi\} - \rho\xi]dt. \end{aligned}$$

In view of dynamic programming principle, $\forall\theta > 0$, there exists an admissible control u^θ such that

$$\begin{aligned} -\theta &\le \mathbb{E}e^{-\rho\varepsilon\wedge\tau}V(b_\varepsilon) - V(b_0) \\ &\le \mathbb{E}e^{-\rho\varepsilon\wedge\tau}\chi(b_\varepsilon) - \chi(b_0) \\ &\le \mathbb{E}\int_0^{\varepsilon\wedge\tau} e^{-\rho t}[D_b\xi\cdot\Phi + \frac{1}{2}\mathrm{tr}\{(\Sigma\Sigma')D_b^2\xi\} - \rho\xi]dt. \end{aligned}$$

Let $\theta \to 0$ It follows that

$$\mathbb{E}\int_0^{\varepsilon\wedge\tau} e^{-\rho t}[D_b\xi\cdot\Phi + \frac{1}{2}\mathrm{tr}\{(\Sigma\Sigma')D_b^2\xi\} - \rho\xi]dt \ge 0.$$

Divided by $\varepsilon \wedge \tau$ on both sides, and let $\varepsilon \to 0$, we have

$$D_b\xi \cdot \Phi + \frac{1}{2}\text{tr}\{(\Sigma\Sigma')D_b^2\xi\} - \rho\xi \geq 0.$$

Thus, $V(x, p)$ is a viscosity supersolution of (5.11).

To proceed, it is sufficient to prove $V(x, p)$ is a viscosity susolution of (5.11). Similarly, suppose $V - \xi$ has a local minimum at b_0 in the neighbourhood $N(b_0)$, where $b_0 = (X(t_0), p(t_0))$. Let $b_\varepsilon := (X(t_0 + \varepsilon \wedge \tau), p(t_0 + \varepsilon \wedge \tau)) \in N(b_0)$. Then

$$V(b_0) - \xi(b_0) \leq V(b_\varepsilon) - \xi(b_\varepsilon).$$

Let $\chi(b) = \xi(b) + V(b_0) - \xi(b_0)$. Then $\chi(b_0) = V(b_0)$, and $\chi(b_\varepsilon) \leq V(b_\varepsilon)$. By using Dynkin's formula, we have

$$\begin{aligned}
\mathbb{E}e^{-\rho\varepsilon\wedge\tau}\chi(b_\varepsilon) - \chi(b_0) &= \mathbb{E}\int_0^{\varepsilon\wedge\tau} e^{-\rho t}[D_b\chi \cdot \Phi + \frac{1}{2}\text{tr}\{(\Sigma\Sigma')D_b^2\chi\} - \rho\chi]dt \\
&= \mathbb{E}\int_0^{\varepsilon\wedge\tau} e^{-\rho t}[D_b\xi \cdot \Phi + \frac{1}{2}\text{tr}\{(\Sigma\Sigma')D_b^2\xi\} - \rho\xi]dt.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathbb{E}e^{-\rho\varepsilon\wedge\tau}V(b_\varepsilon) - V(b_0) &\geq \mathbb{E}e^{-\rho\varepsilon\wedge\tau}\chi(b_\varepsilon) - \chi(b_0) \\
&= \mathbb{E}\int_0^{\varepsilon\wedge\tau} e^{-\rho t}[D_b\xi \cdot \Phi + \frac{1}{2}\text{tr}\{(\Sigma\Sigma')D_b^2\xi\} - \rho\xi]dt.
\end{aligned}$$

In view of principle of optimality, $V(b_0) \geq \mathbb{E}e^{-\rho\varepsilon\wedge\tau}V(b_\varepsilon)$. Then

$$\mathbb{E}\int_0^{\varepsilon\wedge\tau} e^{-\rho t}[D_b\xi \cdot \Phi + \frac{1}{2}\text{tr}\{(\Sigma\Sigma')D_b^2\xi\} - \rho\xi]dt \leq 0.$$

Divided by $\varepsilon \wedge \tau$ on both sides, and let $\varepsilon \to 0$, we have

$$D_b\xi \cdot \Phi + \frac{1}{2}\text{tr}\{(\Sigma\Sigma')D_b^2\xi\} - \rho\xi \leq 0.$$

Thus, $V(x, p)$ is a viscosity subsolution of (5.11). Hence $V(x, p)$ is a viscosity solution of (5.11). □

5.2 An Example

We will consider a special power utility example where the value function is of the form

$$V(x, p) = \frac{x^\gamma}{\gamma}F(p) \tag{5.14}$$

for certain function $F(p)$. Recall that $\eta = \pi[\bar{\beta}(p) - g(\theta) - \theta c + \bar{\mu}(p)] + \bar{\mu}(p) - z$. Plugging (5.14) into (5.11), (5.11) can be rewritten as

$$\sup_{u\in\mathcal{A}}\Big\{x^\gamma F(p)\eta + \frac{x^\gamma}{\gamma}\sum_{k=1}^m\sum_{i=1}^m q_{ik}p_i\frac{\partial F}{\partial p_k} + \frac{1}{2}(\pi+1)^2\sigma_1^2(\gamma-1)x^\gamma F(p) + \frac{1}{2}\frac{x^\gamma}{\gamma}\sum_{i=1}^m \phi_i^2\frac{\partial^2 F}{\partial p_i^2}\Big\} = \frac{x^\gamma}{\gamma}\rho F(p).$$

Hence, given the form of the value function as in (5.14), if follows that

$$\sup_{u\in\mathcal{A}}\Big\{F(p)\eta + \frac{1}{\gamma}\sum_{k=1}^m\sum_{i=1}^m q_{ik}p_i\frac{\partial F}{\partial p_k} + \frac{1}{2}(\pi+1)^2\sigma_1^2(\gamma-1)F(p) + \frac{1}{2\gamma}\sum_{i=1}^m \phi_i^2\frac{\partial^2 F}{\partial p_i^2}\Big\} - \frac{1}{\gamma}\rho F(p) = 0. \tag{5.15}$$

Theorem 5.5. *If $F(p)$ is the viscosity solution of* (5.15)*, the value function $V(x,p)$ in* (5.14) *is the viscosity solution of* (5.11)*.*

Proof. We will prove $V(x,p) = \frac{x^\gamma}{\gamma}F(p)$ is a viscosity supersolution. The part of viscosity subsolution can be proved similarly.

Suppose $V-\xi$ has a local maximum at b_0 in the neighbourhood $N(b_0)$, where $b_0 = (X(t_0), p(t_0))$. Let $b_\varepsilon := (X(t_0+\varepsilon\wedge\tau), p(t_0+\varepsilon\wedge\tau)) \in N(b_0)$. Then $V(b_0)-\xi(b_0) \geq V(b_\varepsilon)-\xi(b_\varepsilon)$. Thus,

$$\frac{X^\gamma(t_0)}{\gamma}F(p(t_0))-\xi(X(t_0),p(t_0)) \geq \frac{X^\gamma(t_0+\varepsilon\wedge\tau)}{\gamma}F(p(t_0+\varepsilon\wedge\tau))-\xi(X(t_0+\varepsilon\wedge\tau),p(t_0+\varepsilon\wedge\tau)). \tag{5.16}$$

On the other hand, suppose $F(p)$ is the viscosity solution of (5.15). Then for all $\xi \in C^2$ such that $F-\xi$ has local maximum, it follows that

$$\sup_{u\in\mathcal{A}}\Big\{F(p)\eta+\frac{1}{\gamma}\sum_{k=1}^{m}\sum_{i=1}^{m}q_{ik}p_i\frac{\partial F}{\partial p_k}+\frac{1}{2}(\pi+1)^2\sigma_1^2(\gamma-1)F(p)+\frac{1}{2\gamma}\sum_{i=1}^{m}\phi_i^2\frac{\partial^2 F}{\partial p_i^2}\Big\}-\frac{1}{\gamma}\rho F(p) \geq 0. \tag{5.17}$$

Then

$$F(p(t_0+\varepsilon\wedge\tau))-\xi(p(t_0+\varepsilon\wedge\tau)) \geq F(p(t_0))-\xi(p(t_0)). \tag{5.18}$$

Let $p(t_0) = p(t_0+\varepsilon\wedge\tau)$, and $X(t_0+\varepsilon\wedge\tau) = X(t_0)+\Delta x$. In view of (5.16), we have

$$\xi(X(t_0)+\Delta x, p(t_0))-\xi(X(t_0),p(t_0)) \geq \frac{F(p(t_0))}{\gamma}((X(t_0)+\Delta x)^\gamma - X^\gamma(t_0))$$

Multiplying $\frac{1}{\Delta x}$ on both sides and letting $\Delta x \to 0$, we obtain

$$\frac{\partial\xi}{\partial x} \geq \frac{\partial V}{\partial x} \tag{5.19}$$

at b_0. Similarly, we have

$$\xi(X(t_0)-\Delta x, p(t_0))-\xi(X(t_0),p(t_0)) \geq \frac{F(p(t_0))}{\gamma}((X(t_0)-\Delta x)^\gamma - X^\gamma(t_0)) \tag{5.20}$$

Combining (5.18) and (5.20), we have

$$\frac{\partial^2\xi}{\partial x^2} \geq \frac{\partial^2 V}{\partial x^2} \tag{5.21}$$

at b_0.

To proceed, let $p(t_0+\varepsilon\wedge\tau) = p(t_0)+\Delta p$, and $X(t_0+\varepsilon\wedge\tau) = X(t_0)$. We have

$$\xi(X(t_0), p(t_0)+\Delta p)-\xi(X(t_0),p(t_0)) \geq \frac{X^\gamma(t_0)}{\gamma}(F(p(t_0)+\Delta p)-F(p(t_0))).$$

Therefore, for small Δp,

$$F(p(t_0))-\xi(X(t_0),p(t_0))\frac{\gamma}{X^\gamma(t_0)} \geq F(p(t_0)+\Delta p)-\xi(X(t_0),p(t_0)+\Delta p). \tag{5.22}$$

Note that (5.22) is satisfied in the neighborhood of $p(t_0)$.

On the other hand, recall that $F(p)$ is the viscosity solution of (5.15). Then, for $\xi \in C^2$ such that $F - \xi \frac{\gamma}{X^\gamma}$ has local maximum at b_0, it follows that

$$\sup_{u\in\mathcal{A}} \Big\{ F(p(t_0))\eta + \frac{1}{\gamma}\sum_{k=1}^{m}\sum_{i=1}^{m} q_{ik}p_i \frac{\partial \xi(b_0)}{\partial p_k}\frac{\gamma}{X^\gamma(t_0)} + \frac{1}{2}(\pi+1)^2\sigma_1^2(\gamma-1)F(p(t_0)) + \frac{1}{2\gamma}\sum_{i=1}^{m}\phi_i^2 \frac{\partial^2\xi(b_0)}{\partial p_i^2}\frac{\gamma}{X^\gamma(t_0)}\Big\} - \rho\frac{1}{\gamma}F(p(t_0)) \geq 0.$$

Hence,

$$\sup_{u\in\mathcal{A}} \Big\{ X^\gamma(t_0)F(p(t_0))\eta + \sum_{k=1}^{m}\sum_{i=1}^{m} q_{ik}p_i \frac{\partial \xi(b_0)}{\partial p_k} + \frac{1}{2}(\pi+1)^2\sigma_1^2(\gamma-1)X^\gamma(t_0)F(p(t_0)) + \frac{1}{2}\sum_{i=1}^{m}\phi_i^2 \frac{\partial^2\xi(b_0)}{\partial p_i^2}\Big\} - \rho\frac{1}{\gamma}X^\gamma(t_0)F(p(t_0)) \geq 0.$$

Then,

$$\sup_{u\in\mathcal{A}} \Big\{ \frac{\partial V(b_0)}{\partial x}\eta X(t_0) + \sum_{k=1}^{m}\sum_{i=1}^{m} q_{ik}p_i \frac{\partial \xi(b_0)}{\partial p_k} + \frac{1}{2}(\pi+1)^2\sigma_1^2 X^2(t_0)\frac{\partial^2 V(b_0)}{\partial x^2} + \frac{1}{2}\sum_{i=1}^{m}\phi_i^2 \frac{\partial^2\xi(b_0)}{\partial p_i^2}\Big\} - \rho\frac{1}{\gamma}X^\gamma(t_0)F(p(t_0)) \geq 0.$$

Together with (5.19) and (5.21), we have

$$\sup_{u\in\mathcal{A}} \Big\{ \frac{\partial \xi(b_0)}{\partial x}\eta X(t_0) + \sum_{k=1}^{m}\sum_{i=1}^{m} q_{ik}p_i \frac{\partial \xi(b_0)}{\partial p_k} + \frac{1}{2}(\pi+1)^2\sigma_1^2 X^2(t_0)\frac{\partial^2 \xi(b_0)}{\partial x^2} + \frac{1}{2}\sum_{i=1}^{m}\phi_i^2 \frac{\partial^2\xi(b_0)}{\partial p_i^2}\Big\} - \rho\xi(b_0) \geq 0. \tag{5.23}$$

Note that (5.23) holds for any twice continuously differentiable function $\xi \in C^2(\Xi)$ with $V(b_0) = \xi(b_0)$ such that $V - \xi$ reaches its maximum at b_0. Thus $V(x,p)$ in (5.14) is the viscosity supersolution of (5.11). By using similar technique, the proof of viscosity subsolution can be obtained. Hence, the value function $V(x,p)$ in (5.14) is the viscosity solution of (5.11). □

6 Concluding Remarks

In this paper, we derive the optimal liability ratio and dividend optimization of an insurance company in a regime-switching model. The switching process, described as a continuous-time Markov chain, is partially observable. We aim to maximize the total expected discounted utility of dividend in the infinite time horizon in the logarithm and power utility cases, respectively. By using the technique of Wonham filters, the partially observed information is converted to a completely observed one. The associated HJB equation is derived following the dynamic programming approach. Furthermore, we adopt the upper-lower method to solve for the stochastic control problem in the two-regime case and obtain the explicit classical solution of value function and corresponding optimal liability ratio and dividend strategies under simple condition. A general multiple-regime case is studied and the general setting of the Wonham filter is provided. The value function is proved to be the viscosity solution of the associated system of HJB equations.

Note that the volatility is not regime-dependent in current formulation, which reduce the difficulties to construct the Wonham filter. In future studies, we will study the cases where the volatilities depend on the Markov switching process. Further, we can consider investment strategies in the decision making process, where the insurance company will determine the optimal investment strategies for their asset portfolios. Currently we only focus on the asset and liability management of an insurance company where asset are all invested in the risky asset. Thus, taking into account the extensions, the model becomes more versatile but more complicated. Solving the coupled system of HJB equations analytically is very difficult. Nevertheless, numerical approximation method can provide a viable alternative.

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