Optimal Consumption and Investment Strategies with Liquidity Risk and Lifetime Uncertainty for Markov Regime-Switching Jump Diffusion Models

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Abstract

In this paper, we consider the optimal consumption and investment strategies for households throughout their lifetime. Risks such as the illiquidity of assets, abrupt changes of market states, and lifetime uncertainty are considered. Taking the effects of heritage into account, investors are willing to limit their current consumption in exchange for greater wealth at their death, because they can take advantage of the higher expected returns of illiquid assets. Further, we model the liquidity risks in an illiquid market state by introducing frozen periods with uncertain lengths, during which investors cannot continuously rebalance their portfolios between different types of assets. In liquid market, investors can continuously remix their investment portfolios. In addition, a Markov regime-switching process is introduced to describe the changes in the market's states. Jumps, classified as either moderate or severe, are jointly investigated with liquidity risks. Explicit forms of the optimal consumption and investment strategies are developed using the dynamic programming principle. Markov chain approximation methods are adopted to obtain the value function. Numerical examples demonstrate that the liquidity of assets and market states have significant effects on optimal consumption and investment strategies in various scenarios.

Key Words. Stochastic control, liquidity risk, lifetime uncertainty, regime-switching, Markov chain approximation method

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1 Introduction

Recent financial markets have demonstrated the value of investing in illiquid assets, especially real estate and other immovable properties. Illiquid assets usually consist of real estate, private company interests, certain kinds of debt, and some types of art. They have some common features. First, it takes time to find a counter-party to trade with, which results in uncertain wait periods and the trading process usually requires agents with specialized abilities. Second, market volatility is closely related with to liquidity levels. For example, the real estate market became highly illiquid following the crash of the credit market during the financial crisis in 2007-2009. The liquidity level of the whole market affects the possibility of finding a suitable trading partner, and abrupt jumps or crashes in asset prices can affect the liquidity of the whole market. Third, the illiquid assets market is volatile, and in an extreme case, the market is likely to shut down during a financial crisis. Therefore, investors who own illiquid assets face a liquidity risk, which cannot be hedged because of the randomness of trading opportunities.

Recently, the influence of liquidity risks in portfolio choice has drawn increasing attention and has been studied in emerging work under various models and approaches. Amihud and Mendelson (1991) discuss the effect of liquidity risk on the yield of U.S. securities. The risk premium of liquidity risk is treated as a discount on the price of securities, which reduces investors' total wealth. However, in their settings they only consider the risks introduced by the fixed non-tradable period before the maturity date. In reality, in addition to the fixed period, market crashes can also freeze the trading activities of securities. French and Roll (1986) fix this problem by introducing two regimes representing two kinds of market states and consider different control policies in different regimes. Jang et al. (2007) improve the model by adding two independent Poisson processes to describe the regime switching process, which shows the effect of possible crashes on control policies. However, they assume that the regime-switchings only occur at the first jump of a Poisson process, which means that once the jump occurs the market will remain in that state. To capture more realistic situations, Dai et al. (2016) assume that investors can trade at any time except at some "pre-dominated markets closing nights," which means that the non-tradable times are known in advance. This setting acts as the market closing of some securities, such as stocks in major exchanges. As the closing time is fixed, it is not exactly the same as the liquidity risk that we discuss in our work. In addition, Dai et al. (2016) aim to maximize the expected utilities of the terminal total wealth of bonds and stocks, which ignores the effect of liquidity risk on consumption policies during the investors' lifetimes. Another approach to investigate liquidity risk introduced by Constantinides (1986) is to incorporate fixed or proportional transaction costs. There are extensive work adopting this method, see Davis and Norman (1990), Framstad et al. (2001), and Mei and Nogales (2018). Liu (2004) studies the consumption effect on the control policies of various classes of assets with different drifts and volatilities under transaction cost framework. Liu and Loewenstein (2013) propose a model where market crashes can trigger switching into another regime with different investment set. Transaction costs are used to represent the value that investors have to give up if they want to trade between different assets in illiquid market state.

Another essential issue that needs further discussion is whether transaction cost is a good measure for liquidity risks. Market events, such as crashes and booms, have essential influences on asset allocation strategies. First, market liquidity may change after market events. Second, parameters of investment assets may vary with different market liquidity. Third, market may close after severe crashes. Therefore, in some illiquid market, investors cannot find a counter-party to trade with, no matter how much return they can give up or how much cost to pay. Ang et al. (2014) develop an innovative approach to deal with this situation. They introduce an illiquid asset to the settings of Merton (1969). This asset cannot be consumed and can only be transferred to liquid wealth at random times when trading opportunities hit. And they utilize an independent poisson process to model the random wait time that investors require to find trading partners in the real world. The randomness is the liquidity risk that investors cannot hedge even when all assets are correlated. In addition, they suppose that investors have constant relative risk aversion and only consider consumption utilities from time zero to positive infinity.

In this paper, we investigate the effect of liquidity risks on optimal policies of an investor subject to random trading times. Specifically, the financial market contains three assets: one risk-free liquid bond, one risky liquid stock, and one risky illiquid asset with jump diffusions. Investors can only consume their liquid wealth, while illiquid wealth must be converted to liquid wealth to finance consumptions. To reflect the hybrid feature of the market liquidity states, we incorporate the widely accepted regime-switching system proposed in Hamilton (1989) and an independent poisson process representing the random trading opportunities. In liquid regime, all assets can be traded frequently; while in illiquid regime, illiquid asset can only be converted once trading opportunities hit. Additionally, we assume the switching of market regimes can be triggered by different sectors: it can occur automatically due to economic environment changes or it can be driven by severe jumps or crashes in the prices of illiquid assets. Further more, the target function consists of the consumption utilities during the investor's lifetimes as well as the heritage at the random stopping time. Under those assumption, we formulate a complex model to derive the optimal control policies of constant relative risk aversion investors with liquidity risks in a two regime-switching market.

Although the model in our paper is very complex, we obtain closed-form solutions when available in simple cases and adopt the Markov chain approximation method developed by Kushner and Dupuis (2001) to obtain the optimal values of the three control variables in general cases. The Markov chain approximation method requires little regularity of the value function and analytic properties of the associated systems of nonlinear Hamilton-Jacobi-Bellman (HJB) equations, which is common in regime-switching jump diffusion models. The Markov chain approximation method has been applied to various stochastic systems in Budhiraja and Ross (2007) and Song et al. (2006).

The Markov chain approximation method shows in the numerical examples that liquidity risk not only influences investors' investment preferences in illiquid assets, it also affects their preferences in liquid risky assets. Investors behave in a more risk-averse way and reduce allocations for both illiquid and liquid risky assets because of the possibility of no consumption before the next trading time. Further, we also demonstrate that investors use almost the same consumption policy for different expected wait periods between trading opportunities, and that they do not change consumption strategies according to whether the correlation of illiquid and liquid assets is positive or negative.

Compared with the existing literature, our work has three important innovations. First, we use more complex and realistic transition processes to approximate the market regime switchings in a real market, which is not considered in Ang et al. (2014). Incorporating a finite-state continuous-time Markov process to describe the transition between different market regimes is a more general and versatile framework for studying stochastic optimization problems that arise in financial mathematics. To be specific, we model the switching of market liquidity states as several independent poisson processes, including the regime-switching driven by a continuous-time Markov chain and severe jumps in the prices of illiquid assets. Once Poisson jumps occur, the regime switches to another state. We assume that price jumps can trigger the market regime switchings, which is more practical and realistic. For example, in the case of falling illiquid asset prices, customers prefer to hold their wealth instead of investing in the financial market, which ruins the liquidity of the total market. In contrast, investors treat increases in illiquid asset prices as a signal of market recovery. They prefer to invest more in illiquid assets and the liquidity of the market improves, as mentioned in Zheng et al. (2015).

Second, instead of using transaction costs to describe and measure the market friction when market is relatively illiquid, we follow the work of Ang et al. (2014) and include an independent poisson process to model the random trading opportunities in illiquid market state. The key reason for incorporating this poisson process to represent liquidity risks is that there are cases where investors cannot find a counter-party by just paying transaction costs in reality. Therefore, we combine the regime-switching model of Liu and Loewenstein (2013) and this independent poisson model to form a more general financial market.

Third, Ang et al. (2014) only consider the utilities from instantaneous consumption and there is no random exit time for an individual investor. Under those assumptions, it is much easier to solve the Hamilton-Jacobi-Bellman (HJB) equation, but it is not an appropriate model for an investor with uncertain lifetime in reality. We further analyze this problem by maximizing the total utilities of terminal heritage and consumption during investors' lifetimes and introducing a random lifetime variable. Therefore, we consider the market routine throughout the investors' lives. Investors try to maximize their expected utility for consumption during their lifetimes, but at the time of their death, all of their positions close, and their heritage is equal to the discounted utility of all of their assets. In addition, a linear combination of exponential distribution is adopted to describe the random death time of investors, as a linear combination of exponential distributions can provide an arbitrarily close approximation to any nonnegative distribution in the sense of weak convergence. Further, we investigate whether investors would like to reduce their consumption and increase allocation in illiquid assets, given the high return rate, to maximize their wealth.

The rest of this paper is organized as follows. In Section 2, we characterize the asset types and value function. A comprehensive description of the formulation is stated. Section 3 develops the HJB equation to derive the optimal controls. The effects of rebalancing times on optimal controls are investigated. Section 4 deals with an iterative method to approximate portfolio strategies. In Section 5, numerical examples are provided to illustrate the insights of our numerical results. Section 6 concludes the paper.

2 Formulation

We derive the optimal portfolio, consumption, and trading policies for investors in a market with liquidity risks. The financial market has three assets, liquid risk-free bonds, liquid risky stocks, and illiquid risky real estate. For the dynamics of assets, stocks follow a geometric Brownian motion and real estate follows a jump diffusion process. Investors can only use liquid wealth to finance consumption during their lives. In addition, the market has two regimes: liquid and illiquid regimes. In liquid regime, all assets can be traded with no restriction; in illiquid regime, investors can only rebalance their portfolio when trading opportunities arrive or the regime changes back to the liquid regime. And we model the switching of market liquidity states as several independent poisson processes, including the regime-switching driven by a continuous-time Markov chain and severe price jumps of illiquid assets. Finally, the value function consists of the instantaneous consumption during the investor's lifetime and the heritage utilities. We will discuss the details of the model in the following sections.

2.1 Assumptions

Before we characterize the process of different assets, we first state the main assumptions of this paper.

1. There are only three types of assets in the market: bonds, stocks, and real estate. Bonds and stocks are considered liquid assets. Real estate is considered illiquid asset, and investors must wait for a trading opportunity when the market state is illiquid.

2. There are only two market regimes: liquid, $l_t = 0$; and illiquid, $l_t = 1$. We model the transition process using couples of independent Poisson processes, which we describe in more detail in the next section.

3. Bonds and stocks are liquid assets that can be traded continuously in any market state. However, real estate can only be traded frequently in a liquid market. In an illiquid market, investors must wait for a trading opportunity that occurs at τ_1, where τ_1 satisfies:

$$\Pr(\tau_1 \leq t) = 1 - e^{-\lambda_1 t},$$

with $\tau_1 > 0$.

2.2 Regime switchings

To describe the impact of asset price jumps on market liquidity, we model the switching process of market regimes as in Liu and Loewenstein (2013). We define the jumps correlated with regime switchings as N_{it}^q with $q \in \{U, D\}$, where U stands for a price up jump and D stands for a price down jump, and $i \in \{1, 2\}$, $i.e$, when N_{1t}^q happens, the market regime transfers to another regime; when N_{2t}^q happens, the regime does not change. In addition, there is another independent Poisson process that represents the market switchings caused by other general economic factors, which is denoted by N_t^R. Therefore, we have the following equations:

$$\begin{cases} N_t^U = N_{1t}^U + N_{2t}^U, \\ N_t^D = N_{1t}^D + N_{2t}^D, \quad \text{and} \end{cases} \tag{2.1}$$

$$dl_t = \begin{cases} dN_{1t}^D + dN_t^R, & \text{if } l_{t^-} = 0; \\ -(dN_{1t}^U + dN_t^R), & \text{if } l_{t^-} = 1. \end{cases} \tag{2.2}$$

To illustrate this process, we assume that the model is in a liquid state with $l_t = 0$. If N_{1t}^D or N_t^R happens, there is a crash in illiquid asset prices or a general market crash (with no change in illiquid asset prices). Then the regime transfers to an illiquid state ($l_t = 1$). If N_{2t}^D happens, we have a downward jump in illiquid asset prices, but the regime remains liquid. In contrast, when $l_t = 1$, if N_{1t}^U or N_t^R happens, we have an upward jump in illiquid asset prices or the market transfers by itself (with no change in illiquid asset prices). Then the regime transfers to the liquid state ($l_t = 0$). If N_{2t}^U occurs, we have an upward jump in illiquid asset prices, but the regime remains illiquid. To capture the differences in the transition frequencies, we use different values for parameters in different regimes, with $\eta_1^U(i), \eta_2^U(i), \eta_1^D(i)$, and $\eta_2^D(i)$ corresponding to the Poisson processes N_1^U, N_2^U, N_1^D and N_2^D in the market liquidity regime $l_t = i$, where $i \in \{0, 1\}$, and with $\xi(0)$ and $\xi(1)$ corresponding to the general market regime switching process N_t^R in regime $l_t = 0$ and $l_t = 1$, respectively.

2.3 Assets

1. The riskless liquid bond A_t follows a rate of return $r(l_t)$ depending on the market regime with

$$dA_t = r(l_t) A_t dt. \tag{2.3}$$

2. The risky liquid stock S_t follows a geometric Brownian motion process with drift $\mu(l_t)$ and volatility $\sigma(l_t)$ as

$$dS_t = \mu(l_t)S_t dt + \sigma(l_t)S_t dB_t^1. \tag{2.4}$$

3. Real estate is an illiquid risky asset with price P_t, which follows a geometric Brownian motion with drift $k(l_t)$, volatility $\psi(l_t)$, and the correlation coefficient $\rho(l_t)$ such that

$$\begin{aligned} dP_t =& (k(l_t) - v(l_t))P_{t^-}dt + \psi(l_t)\rho(l_t)P_{t^-}dB_t^1 \\ &+ \psi(l_t)\sqrt{1-\rho(l_t)^2}P_{t^-}dB_t^2 + (J_t^U - 1)P_{t^-}dN_t^U + (J_t^D - 1)P_{t^-}dN_t^D, \end{aligned} \tag{2.5}$$

with

$$v(l_t) = \eta^U(l_t)\mathbb{E}(J_t^U - 1) + \eta^D(l_t)\mathbb{E}(J_t^D - 1),$$

where B_t^1 and B_t^2 are two independent Brownian motions and J_t^U and J_t^D capture the up jumps and down jumps in prices of illiquid assets. N_t^U and N_t^D are independent Poisson processes. In particular, N_{it}^q has a transition intensity $\eta_i^q(l_t)$ with $\eta^q(l_t) = \eta_1^q(l_t)+\eta_2^q(l_t)$ for $i = 1, 2$ with $l_t \in \{0, 1\}$ and $q \in \{U, D\}$. $v(l_t)$ represents the expected return compensation of jumps with $k(l_t) > v(l_t)$.

Let us work with a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, where $\{\mathcal{F}_t\}$ is a filtration satisfying the usual condition. That is, $\mathcal{F}_t$ is a family of σ-algebras such that $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$ and $\mathcal{F}_0$ contains all of the null sets. Now we have a vector of two independent Brownian motions $B_t = (B_t^1, B_t^2)$, and a vector of five independent Poisson processes, namely $N_t = (N_t^R, N_{1t}^D, N_{2t}^D, N_{1t}^U, N_{2t}^U)$. $\mathbb{P}$ is the corresponding measure and $\mathcal{F}_t$ is the σ-algebra generated by $\{N_s, X_s : 0 \leq s \leq t\}$.

2.4 Value function

We model the investor's objectives as two components: one is the utility of continuous consumption during the investor's lifetime and the other is the ultimate wealth at a random exit time τ_2, which can capture uncertain events such as accidents, illness and retirement. It is well-known that a linear combination of exponential distribution can provide an arbitrarily close approximation to any nonnegative distributions, see, for example, Dufresne (2007). Due to this fact, we assume that τ_2 follows a linear combination of exponential distributions. Then, the survival probability is

$$\Pr(\tau_2 \leq t) = 1 - \sum_{i=1}^{n} p_i e^{-\widehat{\lambda}_i t}, \tag{2.6}$$

with $\sum_{i=1}^{n} p_i = 1$, $\widehat{\lambda}_i \geq 0$, $t \geq 0$, $i = 1, \ldots, n$, $1 \leq n$.

We divide the wealth of an investor into liquid assets and illiquid assets, where only liquid assets can be consumed immediately and illiquid assets must be converted into liquid assets before being used as a source of consumption. This means that in an illiquid state ($l_t = 1$), investors will always choose a positive proportion of liquid wealth ($W_t > 0$) to meet future consumption, because they must wait for an uncertain period before illiquid assets are available for consuming. This is similar to bankruptcy events in real world. Many companies declare bankruptcy not because they have negative total wealth, but because they cannot transfer their illiquid assets immediately to liquid assets to meet their current obligations.

After combining bonds and stocks, we have the joint evolution process of liquid wealth W_t and illiquid wealth Z_t as

$$dW_t = (r(l_t) + \theta_{t^-}(\mu(l_t) - r(l_t)) - c_t)W_{t^-}dt + \theta_{t^-}\sigma(l_t)W_{t^-}dB_t^1 - dI_t, \tag{2.7}$$

and

$$\begin{aligned} dZ_t =& (k(l_t) - v(l_t))Z_{t^-}dt + \psi(l_t)\rho(l_t)Z_{t^-}dB_t^1 + \psi(l_t)\sqrt{1-\rho(l_t)^2}Z_{t^-}dB_t^2 \\ &+ (J_t^U - 1)Z_{t^-}dN_t^U + (J_t^D - 1)Z_{t^-}dN_t^D + dI_t. \end{aligned} \tag{2.8}$$

Note that θ denotes the proportion of total liquid wealth allocated to liquid risky assets, stocks denoted by S_t; then $(1-\theta)$ is the proportion of the portfolio in liquid riskless bonds. The liquid wealth decreases at the consumption rate $c_t = C_t/W_t$, where C_t is the amount of consumption. In addition, when a trading opportunity arrives in state $l_t = 1$, the investor can convert dI_t amount of liquid assets to illiquid assets at no costs.

Following the work of Ang et al. (2014), we assume that an investor has a Constant Relative Risk Aversion (CRRA) utility function, and we model the value function $F(W, Z, l)$ as follows:

$$F(W,Z,l) = \sup_{\{\theta,I,c\}} \mathbb{E}_{\{W,Z,l\}}\left[\int_0^{\tau_2} e^{-\beta t}\frac{C_t^{1-\gamma}}{1-\gamma}dt + e^{-\beta\tau_2}\frac{(W_{\tau_2}+Z_{\tau_2})^{1-\gamma}}{1-\gamma}\right], \tag{2.9}$$

where $\mathbb{E}_{\{W,X,l\}}$ is the expectation conditioned on initial values $W(0) = W$, $Z(0) = Z$, and $l(0) = l$ with W stands for the value of liquid assets, consisting of bonds B and stocks S; Z stands for the value of illiquid assets only including P; l is the regime state. β is the discount factor ($\beta > 0$), and γ is the constant risk aversion ($\gamma > 1$). The first part in the value function (2.9) is the instantaneous consumption; the second part is the heritage when investors reach the stopping time τ_2 due to some reasons.

Finally, we assume that the discount rate β satisfies

$$\beta > (1-\gamma)r + \frac{1-\gamma}{2\gamma(1-\rho^2)}\left(\left(\frac{\mu-r}{\sigma}\right)^2 - 2\rho\left(\frac{\mu-r}{\sigma}\right)\left(\frac{k-r}{\psi}\right) + \left(\frac{k-r}{\psi}\right)^2\right), \tag{2.10}$$

as in the Merton two-risky-asset model. Further, we assume the following relationship between Sharpe ratios of illiquid and liquid assets:

$$\frac{k-r}{\psi} > \frac{\mu-r}{\sigma}. \tag{2.11}$$

3 Solution to the model

Our first step is to show that investors do not short any assets for leverage, *i.e*, even though the return on illiquid assets is higher than that on liquid assets, investors do not short liquid wealth to fund illiquid assets when facing the liquidity risks.

Theorem 3.1. *Any optimal solution to this model has* $W_{\tau_i} > 0$ *and* $Z_{\tau_i} \geq 0$*, where* τ_i *is the time when investors can rebalance their portfolio.*

Lemma 3.2. *Assume the jump distribution in* log *coordinates is exponentially distributed with the parameter* $\lambda > 1$ *and for transition intensities we have*

$$\eta_2^U(1)\frac{1}{\lambda-1} - \eta_2^D(1)\frac{1}{\lambda+1} > 0.$$

Then investors will always have positive illiquid wealth, i.e, $Z > 0$*, in* $l_t = 1$*, if and only if*

$$\frac{k-v-r}{\psi} > \rho\frac{\mu-r}{\sigma}.$$

The proofs of Lemma 3.1 and Lemma 3.2 are provided in the appendix. From Theorem 3.1 and Lemma 3.2, we know that under special conditions, investors will not short any assets during rebalancing times. And we intend to give more details for the controls by deriving the HJB equation first.

3.1 HJB equations

Note that in value function (2.9), the stopping time τ_2 is random and is modeled as a linear combination of an exponential random variable. We use similar techniques as in Liu and Loewenstein (2013) to simplify (2.9) as

$$\begin{aligned} F(W,Z,l) = \sup_{\{\theta,I,c\}} \mathbb{E}\Bigg[\int_0^\infty & e^{-(\pi(l_t)+\beta)t}\Bigg(\sum_{i=1}^n p_i e^{-\widehat{\lambda}_i t}\Big(\eta_1^D(l_t)F(W_t, Z_t J_t^D, 1-l) \\ & +\eta_2^D(l_t)F(W_t, Z_t J_t^D, l) + \eta_1^U(l_t)F(W_t, Z_t J_t^U, 1-l) + \eta_2^U(l_t)F(W_t, Z_t J_t^U, l) \\ & +\xi(l_t)F(W_t, Z_t, 1-l)\Big) + \sum_{i=1}^n p_i\widehat{\lambda}_i e^{-\widehat{\lambda}_i t}\frac{(W_t+Z_t)^{1-\gamma}}{1-\gamma} + \sum_{i=1}^n p_i e^{-\widehat{\lambda}_i t}\frac{C_t^{1-\gamma}}{1-\gamma}\Bigg)dt\Bigg], \\ \pi(l_t) :=& \xi(l_t) + \eta_1^U(l_t) + \eta_2^U(l_t) + \eta_1^D(l_t) + \eta_2^D(l_t), \end{aligned} \tag{3.1}$$

where J_t^U and J_t^D are jump sizes of the illiquid asset price, for which we will give specific distributions in the numerical analysis section. And we include the derivation of (3.1) in the Appendix.

Note that the utility function is homothetic and the return process has constant moments; then, the value function F must be homogeneous with the degree $1-\gamma$ and with the form

$$F(W,Z,l) = W^{1-\gamma}g(x,l) \text{ with } x = \frac{Z}{W}. \tag{3.2}$$

As in our work the initial total assets are fixed, $i.e$, $W+Z$ is a constant, we have

$$F(W,Z,l) =(W+Z)^{1-\gamma}\frac{g(x,l)}{(1+x)^{1-\gamma}}. \tag{3.3}$$

Therefore, it is more meaningful in the following sections to find the value and depict the graph for $\frac{g(x,l)}{(1+x)^{1-\gamma}}$, denoted as $H(x,l)$.

3.1.1 At rebalancing times

Now we consider the situation when investors can rebalance their wealth, *i.e*, when trading opportunities arrive. At that time, investors can convert their assets at no cost. If we denote the new value function immediately after rebalancing assets as $\widetilde{F}$, we have

$$\widetilde{F}(W_t, Z_t, l_t) = \max_{I_t \in [-Z_t, W_t)} F(W_t - I_t, Z_t + I_t, l). \tag{3.4}$$

As $\widetilde{F}$ must also be homogeneous with the degree $1-\gamma$, there exists $\tilde{g}$ *s.t.* $\widetilde{F}(W, Z, l) = W^{1-\gamma}\tilde{g}(x, l)$ with $x = \frac{Z}{W}$. Now we characterize the behavior of $\widetilde{F}$ as follows.

As there is no costs to rebalance, we should have

$$(W-\omega)^{1-\gamma}\tilde{g}\left(\frac{Z+\omega}{W-\omega}, l\right) = W^{1-\gamma}\tilde{g}\left(\frac{Z}{W}, l\right) \text{ with } -Z \le \omega < W.$$

Then, differentiating both sides with respect to ω, we get

$$-(1-\gamma)\tilde{g}\left(\frac{Z+\omega}{W-\omega}, l\right)(W-\omega)^{-\gamma} + (W-\omega)^{1-\gamma}\tilde{g}_\omega\left(\frac{Z+\omega}{W-\omega}, l\right),$$

which implies

$$-(1-\gamma)\tilde{g}\left(\frac{Z+\omega}{W-\omega}, l\right)(W-\omega)^{-\gamma} + (W-\omega)^{1-\gamma}\frac{\partial\tilde{g}\left(\frac{Z+\omega}{W-\omega}, l\right)}{\partial\frac{Z+\omega}{W-\omega}}\frac{W+Z}{(W-\omega)^2} = 0.$$

Then,

$$(1-\gamma)\tilde{g}\left(\frac{Z+\omega}{W-\omega}, l\right) = \frac{\partial\tilde{g}\left(\frac{Z+\omega}{W-\omega}, l\right)}{\partial\frac{Z+\omega}{W-\omega}}\frac{W+Z}{W-\omega}.$$

Let $\omega = 0$. We have $(1-\gamma)\tilde{g}(x,l) = \tilde{g}_x(x,l)(1+x)$. Integrating both sides, we get $\tilde{g}(x,l) = T(1+x)^{1-\gamma}$, with T being a constant. Finally, we have $\widetilde{F}(W,Z,l) = TW^{1-\gamma}\left(1+\frac{Z}{W}\right)^{1-\gamma}$.

3.1.2 Between rebalancing times

Now we characterize the behavior between rebalancing times. We discuss the cases when the initial state is liquid and when it is illiquid.

For the illiquid state. For $l = 1$. Using Itô's lemma *w.r.t.* (3.1), we get the following HJB equation:

$$\begin{aligned} 0 = \max_{\{c,\theta\}} \Bigg[& \sum_{i=1}^{n} p_i e^{-\widehat{\lambda}_i}\frac{(cW)^{1-\gamma}}{1-\gamma} + f(W,Z,1) + \sum_{i=1}^{n} p_i\widehat{\lambda}_i e^{-\widehat{\lambda}_i}\frac{(W+Z)^{1-\gamma}}{1-\gamma} - (\pi(1)+\beta)F \\ & + F_W W(r+\theta(\mu-r)-c) + F_Z Z(k-v) + \lambda_1(\widetilde{F}-F) \\ & + 0.5F_{WW}W^2\theta^2\sigma^2 + 0.5F_{ZZ}Z^2\psi^2 + F_{WZ}WZ\psi\sigma\rho\theta \Bigg], \end{aligned} \tag{3.5}$$

where

$$f(W,Z,1)=\sum_{i=1}^{n}p_ie^{-\widehat{\lambda}_i}\Big(\eta_1^U(1)\mathbb{E}[F(W,ZJ^U,0)]+\eta_2^U(1)\mathbb{E}[F(W,ZJ^U,1)]+\eta_1^D(1)\mathbb{E}[F(W,ZJ^D,0)]$$
$$+\eta_2^D(1)\mathbb{E}[F(W,ZJ^D,1)]+\xi(1)\mathbb{E}[F(W,Z,0)]\Big). \tag{3.6}$$

$\widetilde{F}$ is the value function after rebalancing. In the illiquid state $l=1$, the market can only jump to a liquid state when N_t^R or N_{1t}^U hits. Therefore, there is no possibility that the price of illiquid assets will fall, but the regime becomes liquid ($l=0$). Then, we know $\eta_1^D(1)$ always equals 0.

Plugging in

$$F(W,Z,l)=W^{1-\gamma}g(x,l) \text{ with } x=\frac{Z}{W}$$

into (3.5), we have

$$\begin{aligned}
F_W &= (1-\gamma)W^{-\gamma}g(x,l)+W^{1-\gamma}g_x(x,l)\frac{-Z}{W^2},\\
F_Z &= W^{1-\gamma}g_x(x,l)\frac{1}{W},\\
F_{WW} &= (1-\gamma)(-\gamma)W^{-\gamma-1}g(x,l)+2\gamma ZW^{-\gamma-2}g_x(x,l)+Z^2W^{-\gamma-3}g_{xx}(x,l),\\
F_{ZZ} &= W^{-\gamma-1}g_{xx}(x,l),\\
F_{ZW} &= -\gamma W^{-\gamma-1}g_x(x,l)-W^{-\gamma-2}Zg_{xx}(x,l).
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
g_x(x,l) &= \frac{\partial g(x,l)}{\partial x},\\
g_{xx}(x,l) &= \frac{\partial^2 g(x,l)}{\partial x^2}.
\end{aligned} \tag{3.8}$$

Then, we can get a simplified version of the HJB equation for $g(x,l)$ when $l=1$.

$$0=\max_{\{c,\theta\}}\Bigg\{\sum_{i=1}^{n}p_ie^{-\widehat{\lambda}_i}\frac{c^{1-\gamma}}{1-\gamma}+\sum_{i=1}^{n}p_i\widehat{\lambda}_ie^{-\widehat{\lambda}_i}\frac{(1+x)^{1-\gamma}}{1-\gamma}+G(x,1)-(\pi(1)+\beta)g(x,1)$$
$$+\lambda_1(\tilde{g}(x,1)-g(x,1))+A(x,c,\theta)g(x,1)+B(x,c,\theta)\frac{\partial g(x,1)}{\partial x}+C(x,c,\theta)\frac{\partial^2 g(x,1)}{\partial x^2}\Bigg\}, \tag{3.9}$$

where

$$\begin{aligned}
G(x,1) &= \sum_{i=1}^{n}p_ie^{-\widehat{\lambda}_i}\Big(\eta_1^U(1)\mathbb{E}\left[\frac{(1+xJ^U)^{1-\gamma}}{(1+x_L^*)^{1-\gamma}}\right]g(x_L^*,0)+\eta_2^U(1)\mathbb{E}[g(xJ^U,1)]\\
&\quad+\eta_1^D(1)\mathbb{E}\left[\frac{(1+xJ^D)^{1-\gamma}}{(1+x_L^*)^{1-\gamma}}\right]g(x_L^*,0)+\eta_2^D(1)\mathbb{E}[g(xJ^D,1)]+\xi(1)\frac{(1+x)^{1-\gamma}}{(1+x_L^*)^{1-\gamma}}g(x_L^*,0)\Big),\\
A(x,c,\theta) &= (1-\gamma)(r+\theta(\mu-r)-c)-0.5\gamma(1-\gamma)\sigma^2\theta^2,\\
B(x,c,\theta) &= [k-v-(r+\theta(\mu-r)-c)+\gamma\theta^2\sigma^2-\gamma\psi\theta\rho\sigma]x, \text{ and}\\
C(x,c,\theta) &= [0.5\theta^2\sigma^2+0.5\psi^2-\psi\theta\rho\sigma]x^2.
\end{aligned} \tag{3.10}$$

For the derivation of $G(x,1)$ in (3.10), readers can refer to the Appendix.
In addition, the solution $g(x,1)$ satisfies the following two equations, which will be proved in the Appendix:

$$\lim_{x\to\infty} g_x(x,1)x = 0,$$
$$\lim_{x\to\infty} g_{xx}(x,1)x^2 = 0. \tag{3.11}$$

To solve this HJB equation, we have to better understand the behavior of $\tilde{g}$ and $g(x,0)$. $\tilde{g}$ is the value function right after rebalancing when trading opportunities arrive, which is described by $\tilde{g}(x) = T(1+x)^{1-\gamma}$. In addition, upon rebalancing, investors always choose dI_t $s.t.$ $\frac{Z}{W} = x_I^*$ with $x_I^* = \arg\max \frac{g(x,1)}{(1+x)^{1-\gamma}}$ (see Lemma 3.3 below). Hence, $\frac{\tilde{g}(x_I^*)}{(1+x^*)^{1-\gamma}} = \max \frac{g(x,1)}{(1+x)^{1-\gamma}} = T$, and it is a constant, where x_I^* is the optimal allocation in the illiquid state $l = 1$.

Then, let us consider $g(x,0)$. From Equation (3.3) and the fact that investors can rebalance their portfolio at anytime without any costs and that $W + Z$ is fixed initially, we know that for any $x \in [0,1)$ investors can immediately change to the optimal allocation of assets in liquid states. Therefore, $\frac{g(x,0)}{(1+x)^{1-\gamma}}$ is a constant and is denoted by K. Then $g(x,0) = K(1+x)^{1-\gamma}$. The optimal asset allocation in liquid state x_L^* is discussed in detail in (3.14), which we characterize later.

Lemma 3.3. *Let* $H(x,1) = \frac{g(x,1)}{(1+x)^{1-\gamma}}$. *Then* $H(x,1)$ *is concave for* $x \in [0,\infty)$*, and investors always return to* x_I^* *with* $x_I^* = \arg\max H(x,1)$ *when they can rebalance their portfolios.*

The proof of Lemma 3.3 is provided in the appendix. Then, let us consider the general control formulas for $\theta(x)$ and $c(x)$ in $l = 1$. The proofs of Theorem 3.4 and Theorem 3.5 are provided in the appendix as well.

Theorem 3.4. *The optimal portfolio policy in* $l = 1$ *is*

$$\theta(x) = \frac{\frac{\mu-r}{\sigma^2}(\gamma-1)g(x,1) + \frac{\mu-r+\rho\psi\sigma\gamma}{\sigma^2}xg_x(x,1) + \frac{\rho\psi}{\sigma}x^2g_{xx}(x,1)}{\gamma(\gamma-1)g(x,1) + 2\gamma xg_x(x,1) + x^2g_{xx}(x,1)}.$$

In particular, for $\rho = 0$*, we have*

$$\theta(x_I^*) = \frac{\mu-r}{\sigma^2}\frac{1+x_I^*}{\gamma(1+x_I^{*2}(\kappa-1))},$$

with $\kappa \geq 1$.

Theorem 3.5. *The optimal consumption policy in* $l = 1$ *satisfies the following equation:*

$$c(x) = \left[\frac{1}{\sum_{i=1}^n p_i e^{-\widehat{\lambda}_i}}\Big((1-\gamma)g(x,1) - xg_x(x,1)\Big)\right]^{-\frac{1}{\gamma}}.$$

In particular, we have a simplified version for consumption at x_I^**:*

$$c(x_I^*) = (1+x_I^*)\left(\frac{1}{\sum_{i=1}^n p_i e^{-\widehat{\lambda}_i}}(1-\gamma)T\right)^{-\frac{1}{\gamma}}.$$

Compared to the consumption value in Merton (1969), which is a constant, our consumption at x_I^* is much lower because the third asset is illiquid, and it has to wait a random amount of time before it can be consumed. However, the investors with more illiquid assets are expected to be wealthier in the future. They prefer to consume more of their total liquid assets during their lives, smoothing their total consumption, which means $c(x)$ is not strictly decreasing $w.r.t$ x. In view of the optimal consumption strategies obtained in Theorem 3.5, we consider the derivative of $c(x)$. As

$$\frac{dc(x)}{dx} = -\frac{1}{\gamma}\left[\frac{1}{\sum_{i=1}^n p_i e^{-\widehat{\lambda}_i}}\Big((1-\gamma)g(x,1) - xg_x(x,1)\Big)\right]^{-\frac{1}{\gamma}-1}\frac{1}{\sum_{i=1}^n p_i e^{-\widehat{\lambda}_i}}\Big(-\gamma g_x(x,1) - xg_{xx}(x,1)\Big),$$

we can see that $\frac{dc(x)}{dx} \geq 0$ if and only if $\gamma g_x(x,1) + xg_{xx}(x,1) \geq 0$.

For the liquid state. Now let us consider the case of $l = 0$.

For $l = 0$, we can use similar techniques as in Section 3.1.1 to get $g(x,0) = K(1+x)^{1-\gamma}$ for some constant K. Then we can get

$$\begin{aligned}0 = \max_{\{\theta,c,x\}}\Bigg\{&\sum_{i=1}^n p_i e^{-\widehat{\lambda}_i}\frac{c^{1-\gamma}}{1-\gamma} + \bar{G}(x,0) + \sum_{i=1}^n p_i\widehat{\lambda}_i e^{-\widehat{\lambda}_i}\frac{(1+x)^{1-\gamma}}{1-\gamma} - (\pi(0)+\beta)K(1+x)^{1-\gamma}\\ &+ \bar{A}(x,c,\theta)K(1+x)^{1-\gamma} + \bar{B}(x,c,\theta)(1-\gamma)K(1+x)^{-\gamma} + \bar{C}(x,c\theta)\gamma(\gamma-1)K(1+x)^{-\gamma-1}\Bigg\}.\end{aligned} \tag{3.12}$$

Using the same techniques as in Equations (**??**) and (**??**), we get

$$\begin{aligned}\bar{G}(x,0) =&\sum_{i=1}^n p_i e^{-\widehat{\lambda}_i}\Big(\eta_1^U(0)\mathbb{E}[g(xJ^U,1)] + \eta_2^U(0)\mathbb{E}[K(1+xJ^U)^{1-\gamma}] + \eta_1^D(0)\mathbb{E}[g(xJ^D,1)]\\ &+ \eta_2^D(0)\mathbb{E}[K(1+xJ^D)^{1-\gamma}] + \xi(0)g(x,1)\Big),\\ \bar{A}(x,c,\theta) =&(1-\gamma)(\bar{r}+\theta(\bar{\mu}-\bar{r})-c) - 0.5\gamma(1-\gamma)\bar{\sigma}^2\theta^2,\\ \bar{B}(x,c,\theta) =&[\bar{k} - v - (\bar{r}+\theta(\bar{\mu}-\bar{r})-c) + \gamma\theta^2\bar{\sigma}^2 - \gamma\bar{\psi}\theta\rho\bar{\sigma}]x, \text{ and}\\ \bar{C}(x,c,\theta) =&[0.5\theta^2\bar{\sigma}^2 + 0.5\bar{\psi}^2 - \bar{\psi}\theta\rho\bar{\sigma}]x^2.\end{aligned}$$

Similarly, we always have $\eta_1^U(0) = 0$ to reflect that there is no possibility of a price up jump of illiquid assets, but the regime changes from liquid to illiquid. Note that $G(x,0)$ is a function of x; therefore, it becomes straightforward for us to find the optimal control in $l = 0$ under the first-order condition. We have

$$\begin{aligned}c_L^*(x) =&\left[\frac{(1-\gamma)K(1+x)^{1-\gamma}}{\sum_{i=1}^n p_i e^{-\widehat{\lambda}_i}}\right]^{-\frac{1}{\gamma}},\\ \theta_L^*(x) =&\frac{(1+x)(\bar{\mu}-\bar{r}) - x\gamma\bar{\psi}\rho\bar{\sigma}}{\bar{\sigma}^2\gamma},\end{aligned} \tag{3.13}$$

where $c_L^*(x)$ and $\theta_L^*(x)$ are the optimal policies in the liquid state, which maximizes (3.12).

Then, by reconsidering the HJB function after plugging in the optimal control $c_L^*(x)$ and $\theta_L^*(x)$, we have

$$0 = \max_x \Bigg(\sum_{i=1}^{n} p_i e^{-\widehat{\lambda}_i} \frac{c_L^*(x)^{1-\gamma}}{1-\gamma} + \sum_{i=1}^{n} p_i \widehat{\lambda}_i e^{-\widehat{\lambda}_i} \frac{(1+x)^{1-\gamma}}{1-\gamma} + \Bigg(\bar{A}(x, c_L^*(x), \theta_L^*(x)) - \pi(0) - \beta \Bigg) K \\ \times (1+x)^{1-\gamma} + \bar{B}(x, c_L^*(x), \theta_L^*(x))(1-\gamma) K (1+x)^{-\gamma} + \bar{C}(x, c_L^*(x), \theta_L^*(x)) \gamma(\gamma-1) K (1+x)^{-\gamma-1} \\ + \sum_{i=1}^{n} p_i e^{-\widehat{\lambda}_i} \Big(\eta_1^U(0) \mathbb{E}[g(xJ^U, 1)] + \eta_2^U(0) \mathbb{E}[K(1+xJ^U)^{1-\gamma}] + \eta_1^D(0) \mathbb{E}[g(xJ^D, 1)] \\ + \eta_2^D(0) \mathbb{E}[K(1+xJ^D)^{1-\gamma}] + \xi(0) g(x,1) \Big) \Bigg). \tag{3.14}$$

Let x_L^* denote the optimal asset allocation between liquid and illiquid assets in $l = 0$, which maximizes Equation (3.14). We need to find K to satisfy Equation (3.14) and the value function in $l = 0$, which is $g(x, 0) = K(1+x)^{1-\gamma}$. Note that as we cannot solve K explicitly, we use a numerical method.

Remark 3.6. We briefly summarize the steps for readers to follow the main idea. Our ultimate objective is to derive formulas for the optimal control policies c, θ and the optimal asset allocation x for each regime. Our first step is to use Theorem 3.1 and Lemma 3.2 to prove that investors does not use leverage. During trading opportunities, no matter how investors remix their portfolio, they always have positive position in liquid and illiquid wealth. Therefore, the optimal asset allocation $x \in (0, 1)$. Second, we try to simplify the value function $F(W, Z, l)$ by reducing unknown variables. Under the properties of the utility function and the return process, the value function $F(W, Z, l)$ can be represented as $(W + Z)^{1-\gamma} g(x, l)$. Third, we show that the value function at the instant of trading $\widetilde{F}(W, Z, l)$ is equal to a constant T times $(1 + x)^{1-\gamma}$, which means investors will always return to the optimal x_L^* with the largest T at trading opportunities. Fourth, we apply Lemma 3.3 to prove that such optimal x_L^* indeed exists as $H(x, 1)$ is concave. Then, we characterize the HJB equations of $g(x, 1)$ and $g(x, 0)$ in Eq.(3.9) and Eq.(3.12). Finally, we present the formulas for the optimal control policies in terms of $g(x, l)$, and provide numerical solutions in the next section.

4 Numerical method

In this section, we solve this HJB equation by an iteration method based on the Markov chain approximation method in Kushner and Dupuis (2001). This method applies to continuous time stochastic problems with jump diffusions and the most important thing is that this method facilitates convergence under weak conditions without too many state variables. Now we recall the HJB equation for both $l = 0$ and $l = 1$.

For $l = 0$, we know that K is a constant, and we try to find the three controls to maximize it.

Let

$$N(c,\theta,x) = \sum_{i=1}^{n} p_i e^{-\widehat{\lambda}_i}\frac{c^{1-\gamma}}{1-\gamma} + \sum_{i=1}^{n} p_i\widehat{\lambda}_i e^{-\widehat{\lambda}_i}\frac{(1+x)^{1-\gamma}}{1-\gamma}$$
$$+ \sum_{i=1}^{n} p_i e^{-\widehat{\lambda}_i}\big(\eta_1^U(0)\mathbb{E}[g(xJ^U,1)] + \eta_1^D(0)\mathbb{E}[g(xJ^D,1)] + \xi(0)g(x,1)\big),$$

$$D(c,\theta,x) = \big(\bar{A}(x,c,\theta) - \pi(0) - \beta\big)(1+x)^{1-\gamma} + \bar{B}(x,c,\theta)(1-\gamma)(1+x)^{-\gamma} \tag{4.1}$$
$$+ \bar{C}(x,c,\theta)\gamma(\gamma-1)(1+x)^{-\gamma-1} + \sum_{i=1}^{n} p_i e^{-\widehat{\lambda}_i}\big(\eta_2^U(0)\mathbb{E}[(1+xJ^U)^{1-\gamma}]\big)$$
$$+ \sum_{i=1}^{n} p_i e^{-\widehat{\lambda}_i}\big(\eta_2^D(0)\mathbb{E}[(1+xJ^D)^{1-\gamma}]\big).$$

Then we have

$$K = \max_{\{c,\theta,x\}} \frac{-N(c,\theta,x)}{D(c,\theta,x)}, \tag{4.2}$$

with the optimal control values given by

$$(c_L^*, \theta_L^*, x_L^*) = \arg\max_{\{c,\theta,x\}} \frac{-N(c,\theta,x)}{D(c,\theta,x)}. \tag{4.3}$$

We can get $K = \frac{-N(c_L^*,\theta_L^*,x_L^*)}{D(c_L^*,\theta_L^*,x_L^*)}$ and $g(x_L^*,0) = K(1+x_L^*)^{1-\gamma}$, which is denoted as g_L^* from now on. Then the other two controls can be derived from $g(x_L^*,0)$ using Equation (3.13).

For $l=1$, $g(x,1)$ is a function that depends on the value of x, and the HJB function is as follows:

$$0 = \max_{\{c,\theta\}}\Bigg\{\sum_{i=1}^{n} p_i e^{-\widehat{\lambda}_i}\frac{c^{1-\gamma}}{1-\gamma} + \sum_{i=1}^{n} p_i\widehat{\lambda}_i e^{-\widehat{\lambda}_i}\frac{(1+x)^{1-\gamma}}{1-\gamma} + \widehat{G}(x,1) - (\pi(1)+\beta+\lambda_1 - A(x,c,\theta))g(x,1)$$
$$+ B(x,c,\theta)\frac{\partial g(x,1)}{\partial x} + C(x,c,\theta)\frac{\partial^2 g(x,1)}{\partial x^2}\Bigg\}, \tag{4.4}$$

where

$$\widehat{G}(x,1) = \sum_{i=1}^{n} p_i e^{-\widehat{\lambda}_i}\Bigg(\eta_1^U(1)\mathbb{E}\left[\frac{(1+xJ^U)^{1-\gamma}}{(1+x_L^*)^{1-\gamma}}\right]g_L^* + \eta_2^U(1)\mathbb{E}[g(xJ^U,1)] + \eta_1^D(1)\mathbb{E}\left[\frac{(1+xJ^D)^{1-\gamma}}{(1+x_L^*)^{1-\gamma}}\right]g_L^*$$
$$+ \eta_2^D(1)\mathbb{E}[g(xJ^D,1)] + \xi(1)\frac{(1+x)^{1-\gamma}}{(1+x_L^*)^{1-\gamma}}g_L^* + \lambda_1 T(1+x)^{1-\gamma}\Bigg),$$

$$A(x,c,\theta) = (1-\gamma)(r+\theta(\mu-r)-c) - 0.5\gamma(1-\gamma)\sigma^2\theta^2,$$
$$B(x,c,\theta) = [k - v - (r+\theta(\mu-r)-c) + \gamma\theta^2\sigma^2 - \gamma\psi\theta\rho\sigma]x, \text{ and}$$
$$C(x,c,\theta) = [0.5\theta^2\sigma^2 + 0.5\psi^2 - \psi\theta\rho\sigma]x^2.$$

First, we use a change of variables and let $z = \ln x$. Then we can easily simplify Equation (4.4) as

$$0 = \max_{\{c,\theta\}} \Bigg\{ \sum_{i=1}^{n} p_i e^{-\widehat{\lambda}_i} \frac{c^{1-\gamma}}{1-\gamma} + \sum_{i=1}^{n} p_i \widehat{\lambda}_i e^{-\widehat{\lambda}_i} \frac{(1+e^z)^{1-\gamma}}{1-\gamma} + \widehat{G}(z,1) - (\pi(1) + \beta + \lambda_1 - A(e^z, c, \theta)) g(z,1)$$
$$+ g_z(z,1)\Bigg(k - v - (r + \theta(\mu - r) - c) + (\gamma - 0.5)\theta^2\sigma^2 - (\gamma - 1)\psi\theta\rho\sigma - 0.5\psi^2 \Bigg)$$
$$+ g_{zz}(z,1)\Bigg(0.5\theta^2\sigma^2 + 0.5\psi^2 - \psi\theta\rho\sigma \Bigg) \Bigg\}, \tag{4.5}$$

where

$$\widehat{G}(z,1) = \sum_{i=1}^{n} p_i e^{-\widehat{\lambda}_i} \Big(\eta_1^U(1) \mathbb{E}\left[\frac{(1+e^{z+\bar{J}^U})^{1-\gamma}}{(1+e^{z_L^*})^{1-\gamma}} \right] g_L^* + \eta_2^U(1) \mathbb{E}[g(z+\bar{J}^U, 1)]$$
$$+ \eta_1^D(1) \mathbb{E}\left[\frac{(1+e^{z+\bar{J}^D})^{1-\gamma}}{(1+e^{z_L^*})^{1-\gamma}} \right] g_L^* + \eta_2^D(1)\mathbb{E}[g(z+\bar{J}^D, 1)] + \xi(1) \frac{(1+e^z)^{1-\gamma}}{(1+e^{z_L^*})^{1-\gamma}} g_L^* \Big)$$
$$+ \lambda_1 T (1+e^z)^{1-\gamma},$$
$$A(e^z, c, \theta) = (1-\gamma)(r + \theta(\mu - r) - c) - 0.5\gamma(1-\gamma)\sigma^2\theta^2,$$

and $\bar{J}^U$ and $\bar{J}^D$ are jump distributions in log coordinates.

We discretize $g(z,1)$ by considering $g(z_n, 1)$ with $z_n = nh$, where h is a relatively small step size, and approximate the derivative of $g(z_n, 1)$ using

$$g_{z+}(z_n, 1) = \frac{g_{n+1} - g_n}{h},$$
$$g_{z-}(z_n, 1) = \frac{g_n - g_{n-1}}{h}, \text{ and} \tag{4.6}$$
$$g_{zz}(z_n, 1) = \frac{g_{n+1} + g_{n-1} - 2g_n}{h^2},$$

where g_{z-} represents the derivatives of g with negative coefficients and g_{z+} with positive coefficients.

Then we can simplify Equation (4.4) as

$$g_n = \max_{\{c,\theta\}} \Bigg[p_n^d(c,\theta) g_{n-1} + p_n^u(c,\theta) g_{n+1} + \Big(\sum_{i=1}^{n} p_i e^{-\widehat{\lambda}_i} \frac{c^{1-\gamma}}{1-\gamma} + \sum_{i=1}^{n} p_i \widehat{\lambda}_i e^{-\widehat{\lambda}_i} \frac{(1+e^{z_n})^{1-\gamma}}{1-\gamma} + \widehat{G}(z_n, 1) \Big)$$
$$\Delta t_n(c,\theta) \Bigg], \tag{4.7}$$

where

$$\Delta t_n(c,\theta) = \frac{h^2}{D_1 h^2 + D_2 h + \theta^2\sigma^2 + \psi^2 - 2\psi\theta\rho\sigma},$$
$$p_n^d(c,\theta) = \frac{[v + r + \theta(\mu - r) + (\gamma - 1)\psi\theta\rho\sigma + 0.5\psi^2]h + 0.5\theta^2\sigma^2 + 0.5\psi^2 - \psi\theta\rho\sigma}{h^2} \Delta t_n(c,\theta), \text{ and}$$
$$p_n^u(c,\theta) = \frac{[k + c + (\gamma - 0.5)\theta^2\sigma^2]h + 0.5\theta^2\sigma^2 + 0.5\psi^2 - \psi\theta\rho\sigma}{h^2} \Delta t_n(c,\theta),$$

and

$$D_1 = (\gamma-1)(r+\theta(\mu-r)-c) - 0.5\gamma(\gamma-1)\sigma^2\theta^2 + \pi(1) + \beta + \lambda_1 \text{ and}$$
$$D_2 = v + r + \theta(\mu-r) + (\gamma-1)\psi\theta\rho\sigma + 0.5\psi^2 + k + c + (\gamma-0.5)\theta^2\sigma^2.$$

Then, we start our iterative methods to solve g_L^* and $g(x,1)$ simultaneously in log coordinates. We denote $g^i(z_n,1)$ by g_n^i, and use the value function of the standard Merton problem as our initial guess for g_n^0,

$$g_n^0 = \frac{1}{1-\gamma}\left[\frac{\beta-\mu(1-\gamma)+0.5(1-\gamma)\gamma\sigma^2}{\gamma}\right]^{-\gamma}, \tag{4.8}$$

as suggested in Fleming and Soner (2006). We set $z_I^* = z_I^k$ as our initial guess of the optimal asset allocation in $l=1$. For $l=0$, we denote g_L^* and K in each iteration as g_L^i and K^i. The whole process is as follows.

1. Given each iteration g_n^i, we compute the optimal control for the ith iteration in $l=0$.

$$\begin{aligned}
N^i(c,\theta,e^{z_n}) &= \sum_{i=1}^{n} p_i e^{-\widehat{\lambda}_i}\frac{c^{1-\gamma}}{1-\gamma} + \sum_{i=1}^{n} p_i\lambda_1 e^{-\widehat{\lambda}_i}\frac{(1+e^{z_n})^{1-\gamma}}{1-\gamma}\\
&\quad + \sum_{i=1}^{n} p_i e^{-\widehat{\lambda}_i}\big(\eta_1^U(0)\mathbb{E}[g^i_{z+\bar{J}^U}] + \eta_1^D(0)\mathbb{E}[g^i_{z+\bar{J}^D}] + \xi(0)g_z^i\big),\\
D^i(c,\theta,e^{z_n}) &= \big(\bar{A}(e^{z_n},c,\theta)-\pi(0)-\beta\big)(1+e^{z_n})^{1-\gamma} + \bar{B}(e^{z_n},c,\theta)(1-\gamma)(1+e^{z_n})^{-\gamma}\\
&\quad + \bar{C}(e^{z_n},c,\theta)\gamma(\gamma-1)(1+e^{z_n})^{-\gamma-1} + \sum_{i=1}^{n} p_i e^{-\widehat{\lambda}_i}\big(\eta_2^U(0)\mathbb{E}[(1+e^{z_n+\bar{J}^U})^{1-\gamma}]\\
&\quad + \eta_2^D(0)\mathbb{E}[(1+e^{z_n+\bar{J}^D})^{1-\gamma}]\big),\\
(c_L^i,\theta_L^i,z_L^i) &= \arg\max_{\{c,\theta,z_n\}} \frac{-N^i(c,\theta,e^{z_n})}{D^i(c,\theta,e^{z_n})},
\end{aligned} \tag{4.9}$$

2. With these optimal policies, we can get K^i and the value function g_L^i in the ith iteration.

$$\begin{aligned}
K^i &= \frac{-N^i\left(c_L^i,\theta_L^i,e^{z_L^i}\right)}{D^i\left(c_L^i,\theta_L^i,e^{z_L^i}\right)},\\
g_L^i &= K^i(1+e^{z_L^i})^{1-\gamma}.
\end{aligned} \tag{4.10}$$

3. Given z_I^k and g_n^i, we compute the constant T^{i+1} as $g_n^i(z^*=z_I^k) = T^{i+1}(1+e^{z_I^k})^{1-\gamma}$.

4. Given each iteration g_n^i, we compute the $i+1$ step optimal control policies in illiquid regime

$l = 1$ for each point z_n as

$$
\begin{aligned}
c_n^{i+1} = &\arg\max_c \Bigg[p_n^d(c,\theta) g_{n-1}^i + p_n^u(c,\theta) g_{n+1}^i + \Big(\sum_{i=1}^n p_i e^{-\widehat{\lambda}_i} \frac{c^{1-\gamma}}{1-\gamma} + \sum_{i=1}^n p_i \lambda_1 e^{-\widehat{\lambda}_i} \frac{(1+e^{z_n})^{1-\gamma}}{1-\gamma} \\
&+ \widehat{G}(z_n, 1) \Big) \Delta t_n(c,\theta) \Bigg] \text{ and} \\
\theta_n^{i+1} = &\arg\max_\theta \Bigg[p_n^d(c,\theta) g_{n-1}^i + p_n^u(c,\theta) g_{n+1}^i + \Big(\sum_{i=1}^n p_i e^{-\widehat{\lambda}_i} \frac{c^{1-\gamma}}{1-\gamma} + \sum_{i=1}^n p_i \widehat{\lambda}_i e^{-\widehat{\lambda}_i} \frac{(1+e^{z_n})^{1-\gamma}}{1-\gamma} \\
&+ \widehat{G}(z_n, 1) \Big) \Delta t_n(c,\theta) \Bigg].
\end{aligned}
\tag{4.11}
$$

5. Using the results of Step 4, we can compute the value for the next iteration.

$$
\begin{aligned}
g_n^{i+1} = &p_n^d(c_n^{i+1}, \theta_n^{i+1}) g_{n-1}^i + p_n^u(c_n^{i+1}, \theta_n^{i+1}) g_{n+1}^i \\
&+ \Big(\sum_{i=1}^n p_i e^{-\widehat{\lambda}_i} \frac{{c_n^{i+1}}^{1-\gamma}}{1-\gamma} + \sum_{i=1}^n p_i \widehat{\lambda}_i e^{-\widehat{\lambda}_i} \frac{(1+e^{z_n})^{1-\gamma}}{1-\gamma} + \widehat{G}(z_n, 1) \Big) \Delta t_n(c_n^{i+1}, \theta_n^{i+1}).
\end{aligned}
\tag{4.12}
$$

6. Repeat Steps $1-5$ until the convergence of g_n^i, and store the value of the constant T in $T(z_I^k)$, which means the value of T *w.r.t.* our guess of the optimal asset allocation $z_I^* = z_I^k$.

7. Repeat Steps $1-6$ for each $z_I^k + h$ and $z_I^k - h$. If $T(z_I^k) > T(z_I^k - h)$ and $T(z_I^k) > T(z_I^k + h)$, then the iterations stop and the optimal asset allocation between liquid and illiquid assets is $\frac{e^{z^k}}{e^{z^k}+1}$. Otherwise, if $T(z_I^k) < T(z_I^k - h)$, then let $k = k-1$ and repeat Steps $1-6$. If $T(z_I^k) < T(z_I^k + h)$, then let $k = k+1$ and repeat Steps $1-6$.

5 Numerical examples

To gain a better understanding of how investors balance liquid and illiquid assets, we now provide a numerical result to see how they behave under different parameters. For the parameters of investment assets, we adopt parameter values in Ang et al. (2014) in the illiquid regime, and use modified the value slightly in the liquid regime. For the parameters of price jumps, we adopt the parameter values in Liu and Loewenstein (2013) and modified a little to enhance the convergence of our numerical method.

First,we assume that the jump size satisfies an exponential distribution in log-scale with the parameter $\lambda = 10$. Let the magnitudes of the up and down jump size be u and d, respectively. u and d are positive. The size of the jumps are then denoted as u when jumping up and $-d$ when jumping down. That is, $\mathbb{P}\left\{ \bar{J}^q = u\mathbb{I}_{\{q=U\}} + d\mathbb{I}_{\{q=D\}} \right\} = \lambda e^{-\lambda(u\mathbb{I}_{\{q=U\}} + d\mathbb{I}_{\{q=D\}})}$ for $u, d \in [0, \infty)$, where $\mathbb{I}_{\{\cdot\}}$ is an indicator function. Then we have

$$\begin{aligned}&\mathbb{P}(e^{\bar{J}^U} \leq x) = \mathbb{P}(\bar{J}^U \leq \ln x) = 1 - e^{-\lambda \ln x} = 1 - x^{-\lambda} \text{ for } x \in [1, \infty) \text{ and}\\&\mathbb{P}(e^{-\bar{J}^D} \leq y) = \mathbb{P}(\bar{J}^U \geq -\ln y) = e^{\lambda \ln y} = y^{\lambda} \text{ for } y \in (0, 1].\end{aligned} \tag{5.1}$$

We can get the expectation of a jump size as $\mathbb{E}[e^{\bar{J}^U}] = 1.1111$ and $\mathbb{E}[e^{-\bar{J}^D}] = 0.9091$ for $\lambda = 10$, which means that the average size of an up jump is 11.11% and of a down jump is 9.09%.

In addition, we assume the parameters for τ_2 are $p_i = 0.25$, for $i = 1, 2, 3, 4$, $\widehat{\lambda}_1 = 0.05, \widehat{\lambda}_2 = 0.025, \widehat{\lambda}_3 = 0.02, \widehat{\lambda}_4 = 0.0125$. The average trading opportunity in the illiquid state $l = 1$ is assumed to be once per year, which means $\lambda_1 = 1$. In terms of regime switchings, we assume that automatic market switches or price crashes occur twice a year on average; then, $\eta_1^D(0) + \xi(0) = 2$. In addition, as the probability that the regime switching resulting from a drop in illiquid asset prices is larger than the one from self-adjustments by the market, we set $\eta_1^D(0) > \xi(0)$. In addition, we assume that the average duration of an illiquid state is 1.25 years, *i.e*, $\eta_1^U(1) + \xi(1) = 0.8$. Similarly, we assume that there is a greater likelihood of price jumps in illiquid assets; this means $\eta_1^U(1) > \xi(1)$. We set the parameters of the jumps or crashes that do not result in regime switching to be 0.2, reflecting their small possibility. The values of all of the parameters are presented in Table 5.1.

γ	β	r	$\bar{r}$	μ	$\bar{\mu}$	k	$\bar{k}$	ρ	ψ	$\bar{\psi}$
6	0.1	0.04	0.03	0.12	0.10	0.18	0.15	0.15	0.2	0.18
λ_1	p_1	$\widehat{\lambda}_1$	p_2	$\widehat{\lambda}_2$	p_3	$\widehat{\lambda}_3$	p_4	$\widehat{\lambda}_4$	σ	$\bar{\sigma}$
1	0.25	0.05	0.25	0.025	0.25	0.02	0.25	0.0125	0.18	0.15
$\eta_1^U(0)$	$\eta_2^U(0)$	$\eta_1^D(0)$	$\eta_2^D(0)$	$\eta_1^U(1)$	$\eta_2^U(1)$	$\eta_1^D(1)$	$\eta_2^D(1)$	λ	$\xi(0)$	$\xi(1)$
0	0.2	1.6	0.04	0.6	0.2	0	0.2	10	0.4	0.2

Table 5.1: Parameters and Values

Note that in our setting $\eta_2^D(0) = 0.04$, which is very small compared to other parameters, because if there is a crisis in illiquid assets, the market state is unlikely to change from an illiquid to liquid state. In reality, when investors face a relatively large drop in asset prices, they commonly use options to hedge their positions. These transactions increase the supply side of this asset and make the price drop more, which destroys the market liquidity further. Then liquidity gets worse and it is even more difficult to find trade partners to clear the positions. On the other hand, in an illiquid state, $l = 1$, it is hard to restore the confidence of investors; therefore, we usually need to wait for a longer time to see a market recovery than to see a market falling. Hence, we set $\eta_1^U(1) < \eta_1^D(0)$ and $\xi(1) < \xi(0)$.

5.1 Numerical results of the control variables

Under our assumptions and the set of parameter values given in Table 5.1, we find that the optimal asset allocation in different regimes is $x_I^* = 11.92\%$ and $x_L^* = 26.89\%$. The optimal controls in each regime are presented in the following figures.

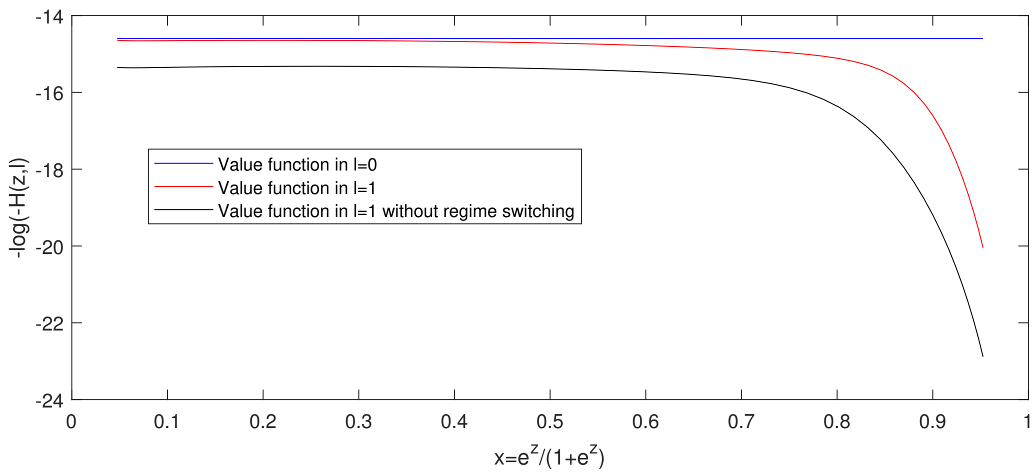

Figure 5.1: Value Function

Figure 5.1 describes the value function in the liquid and illiquid regimes and the case without price jumps and regime switchings under the same set of parameters. The x-axis is the ratio of illiquid assets to total liquid assets, calculated as $\frac{e^z}{1+e^z}$, where z is the same variable as in Section 4. The y-axis is the negative logarithm of the expected discounted utility, which is donated by $-\log(-H(z,l))$ with $H(z,l) = \frac{g(z,l)}{(1+e^z)^{1-\gamma}}$. Now, we would like to explain the visibility of this scale change first. Note that from (2.9), (3.3), and the assumption $\gamma > 1$, we know the expected discounted utility $F(W,Z,L) < 0$. Hence, the value functions $g(z,l)$ and $H(z,l)$ are also less than 0. As the absolute value of $H(z,l)$ is very large, we take a negative logarithm to reduce the scale.

We want to compare our model with the one without any regime switching. In the case without jumps and trading opportunities, the market is always in the illiquid state as in our model, and investors can only wait for a random trading opportunity to rebalance their portfolios. Hence, the model becomes very similar to the model in Ang et al. (2014), except that our model considers discounted heritage wealth at the time of investors' death, which is determined by τ_2. The only difference between this non-regime-switching model and our two states model is that no regime switching means that investors always stay in $l = 1$. Investors cannot jump back to the liquid state $l = 0$ and have fewer opportunities to rebalance. Therefore, it is sensible for investors to put more wealth in liquid assets, so that they do not consume all of their liquid asset before the next trading opportunity arrives. Note that the difference between the red curve and the black curve can be treated as the value of the opportunities associated with regime switching.

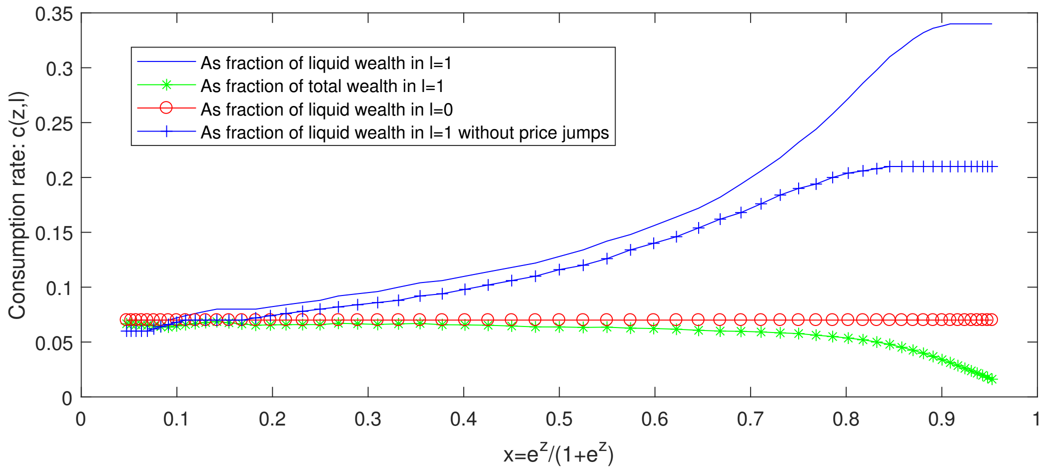

Figure 5.2: Consumption Policy

Figure 5.2 presents the optimal consumption policies in each state. The blue curve plots the optimal consumption policy in state 1, as a fraction of liquid wealth. As shown in Theorem 3.5, it is a function of the current ratio of illiquid assets to liquid assets. As the proportion of illiquid assets rise, investors consume a larger proportion of their liquid wealth, which smooths lifetime consumption. The green-star curve denotes the consumption policy as a proportion of the total wealth, $\frac{c}{1+x}$. In a relatively large range, the consumption as a fraction of the total wealth is rather flat and less than that in the state $l = 0$. As the proportion of illiquid assets becomes larger, the cost of maintaining a smooth consumption becomes larger, leading to less consumption as a proportion of total wealth. Then, we consider the blue curve with sign "+", which is the policy in a case with no jumps or regime switchings. Compared to the case with regime switchings, the optimal policies are almost the same, except when x tends to 1. Therefore, although investors have opportunities to see their illiquid assets jump up to much higher values, they do not lower their consumption and are reluctant to invest more wealth in illiquid assets. As investors' illiquid wealth tends to 1, they consume even larger proportion. The benefit of possible lucky up jumps is not enough to offset the effect of consumption smoothing for a risk-averse investor.

In Figure 5.3, we plot the optimal policies for liquid risky assets in each case. In the Merton's two-assets model, the optimal policy is $\frac{\bar{\mu}-\bar{r}}{\gamma\bar{\sigma}^2} = 0.36$, which is equal to the optimal policy in the state $l = 1$, when $x = 0$. In the state $l = 1$, shown by the red curve, $\theta(z, l)$ increases a bit as z goes up. Investors increase their ratios of liquid risky assets to total liquid wealth to take advantage of the higher expected returns of liquid risky assets. As at that stage the wealth locked in illiquid assets is not significant, to smooth future consumptions, investors take risks and invest in liquid risky assets to have more to consume in the future. After x increases to some value around 0.3, the red

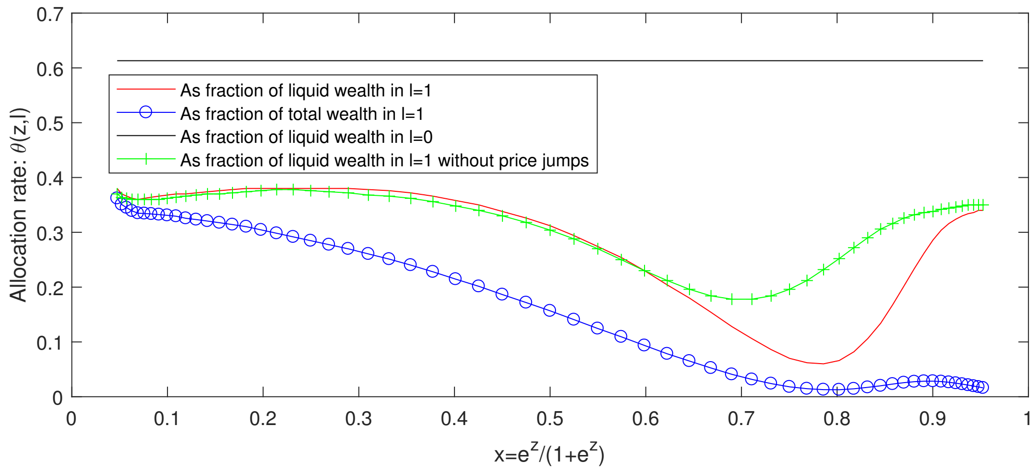

Figure 5.3: Asset Allocation Policy

curve begins to decrease, which means investors reduce the ratio of liquid risky assets to total liquid wealth. In this situation, the lack of liquidity becomes dominant. Then, investors decrease their positions in liquid risky assets to avoid extra risks and choose to have a more predictable liquid wealth. After x increases to about 0.8, in Figure 5.3, the red curve begins to increase again. In Figure 5.2, consumption as a fraction of total wealth begins to decrease significantly, as shown by the green curve. In this case, when the proportion of illiquid assets is sufficiently large, investors find it is hard to maintain their usual consumption. Instead, they lower the consumption level and put more liquid wealth into liquid risky assets to take advantage of the high expected returns of stocks to increase consumption in the future. This is also reflected in the value function in Figure 5.1. When x is relatively small, for example in the range of $x \in [0, 0.7]$, $g(x, 1)$ is almost flat with only very small differences. But when x becomes large enough, consumption decreases rapidly, as shown by the green curve in Figure 5.2. The value function $g(x, 1)$ also begins to fall. Therefore, the decrease in consumption is directly reflected in the value function. The discounted value of heritage only contributes a small amount to our value function.

Another aspect is the difference between the red curve and the green curve with sign "+". In fact, when x is not large enough, they are almost the same. The slight difference in Figure 5.3 is due to some computational errors When x becomes larger, investors have more wealth locked in illiquid wealth, and the results change. In the model with regime switchings, investors have opportunities to move into a liquid state in which they can convert their wealth freely. Therefore, illiquid assets and liquid risky assets are more substitutable goods in regime switching model than in a pure illiquid market without any jumps. Investors prefer smooth consumption, even though they have a large amount of wealth locked in illiquid assets when x is large. In the case without regime switchings, investors take extra risks to allocate more liquid wealth to liquid risky assets to increase future liquid wealth. However, in the case with regime switchings, investors know that they can convert asset types freely without long wait periods. Therefore, they do not take extra risks and invest more in bonds. Hence, we can see that the green cure is above the red one when x is large.

The black straight line stands for the optimal control in liquid regime, which is a constant and obviously larger than that in illiquid regime since investors can frequently trade their assets. Then, the blue curve reflects the proportion of the total wealth allocated to liquid risky stocks in illiquid regime. It denotes the same control policy as the red curves does but under different scales. When the proportion of illiquid wealth x goes up, the proportion of liquid risky stocks in total wealth decreases first and then increases a little bit after x becomes very large. With extreme large wealth locked in illiquid assets, it is very hard to maintain previous consumption. Therefore, investors begin to decrease consumption and increase investments in stocks to take advantage of the higher return of stocks.

5.2 Control policies with different trading intensities

In addition, we consider the reaction of investors to different intensities of jumps and trading opportunities, and summarize the results. When we change the parameter of the trading opportunities, we get the behaviors of $c(x,1)$ and $\theta(x,1)$. In Figure 5.4, we use four different values for λ_1, ranging

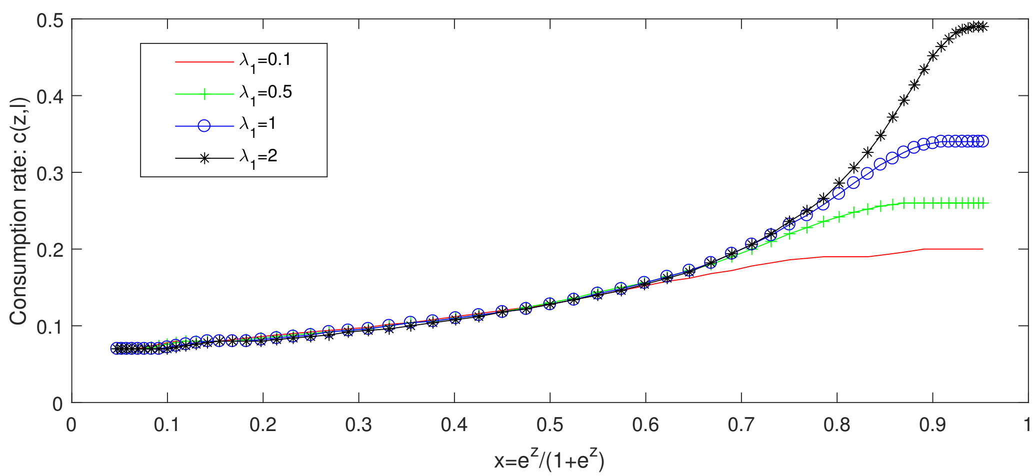

Figure 5.4: Consumption Policies $w.r.t$ Different λ_1

from 0.1 to 4, which means that the average wait time between trading opportunities changes from 10 years to 0.25 years. For $\lambda_1 = 0.1$, the consumption policies are relatively flat and can be regarded as a constant, because the average wait between trading opportunities is relatively large, and investors have to wait a relatively long time to rebalance their portfolio. Therefore, with fixed total wealth and a large x where investors have a large proportion of their wealth locked in illiquid assets, they cannot increase their consumption rate by totally consuming their liquid assets, because they fear that they may reach financial ruin before the arrival of the next trading opportunity. However, when λ_1 is relatively large, which means trading opportunities are relatively frequent, we can see an increasing trend in the optimal consumption policies $w.r.t.$ x. Faced with shorter wait time, investors are more concerned about consumption smoothing. They would like to increase their consumption rate to maintain a total consumption cW when x increases, except when x is extremely large. Further, when $x > 0.8$, we can see another flat trend, as investors cannot continuously increase consumption without destroying their liquid wealth.

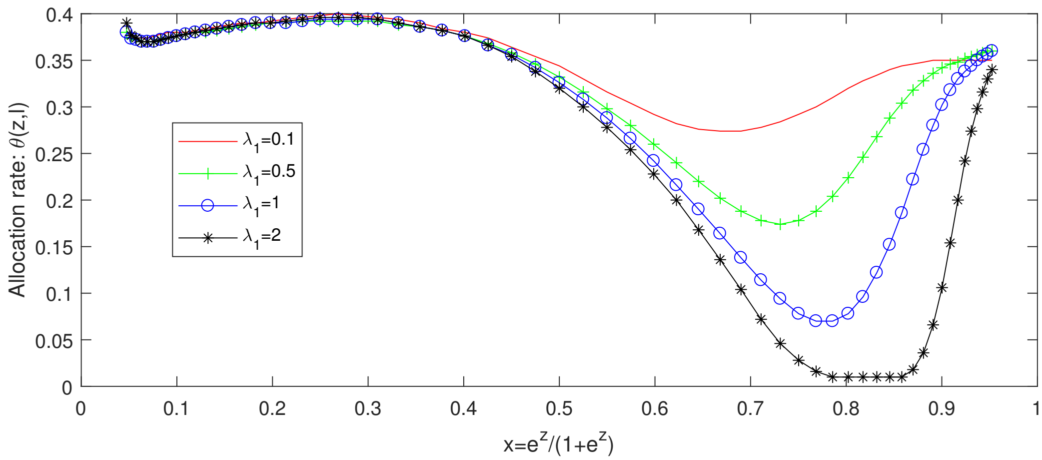

Figure 5.5: Asset Allocation $w.r.t$ Different λ_1

In Figure 5.5, we compare another control $\theta(x, 1)$ for different values of λ_1. For all of the different values of λ_1, we can find that the control $\theta(x, 1)$ has an initial increasing trend followed by a decreasing trend, and finally an increasing trend, because for a very small x, investors can take advantage of the high return of liquid risky assets to maximize their expected utility. With more wealth locked in illiquid assets, investors choose to decrease their total risks by reducing their positions in liquid risky assets. Finally, when x is very large, investors have nearly all of their wealth in illiquid assets, and they choose to be more risk-seeking, focusing on the higher returns of both risky assets by increasing $\theta(x, 1)$. One explanation is that when investors already have high levels of risk in illiquid assets, even if they reduce their liquid risky assets to 0, they cannot reduce their total risk efficiently.

We also compare the different graphs with different values of λ_1. For $\lambda_1 = 0.5$, we can treat the whole market as relatively illiquid. Then θ is relatively flat. Investors do not decease their position in liquid risky assets to decrease total risks when their wealth is locked in illiquid assets, because with large wait times between trading opportunities, illiquid assets and liquid risky assets are not close substitute assets. However, for a larger λ_1, we can see a large drop in θ when $x > 0.5$. With most of their wealth locked in illiquid assets, investors would like to reduce their position in liquid risky assets to decrease their total risks. With more frequent trading opportunities, illiquid assets can be treated as substitutes for liquid risky assets. Therefore, for a larger λ_1, decreases in $\theta(x,1)$ are larger. This effect is especially large for $\lambda_1 = 5$. We can see $\theta(x,1) \to 0$ as $x \to 0.8$.

5.3 Numerical results with correlated assets

In this section, we recalculate our model for $\rho \in \{-0.8, 0, 0.8\}$, *i.e* we consider the situations where illiquid assets and liquid risky assets are negatively correlated, independent, or positively correlated. We only show the results for the illiquid state, because in the liquid state all of the value functions and control policies are constant; therefore, it is the non-constant case that is the most interesting.

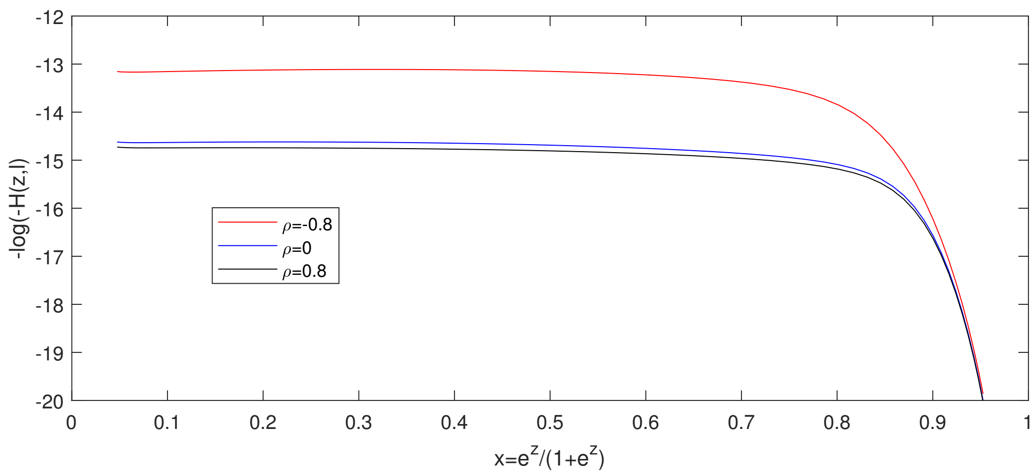

Figure 5.6: Value Function $w.r.t$ Different ρ

Figure 5.6, Figure 5.7, and Figure 5.8 show the value function and control policies under the three different cases. We can see that in the negatively correlated case, investors have slightly better expected discounted utilities, higher consumption rates, and a higher proportion of liquid wealth invested in risky assets. Note that in our basic setting, $\bar{k} = 0.15, \bar{\mu} = 0.1, \bar{\psi} = 0.2, \bar{\sigma} = 0.18$, and $\bar{r} = 0.03$, which means that the Sharpe ratio of the illiquid assets is more than twice that of the liquid risky assets in the regime $l = 1$. Hence, this can create an incentive to leverage illiquid assets by hedging with liquid assets, if these two assets are strongly positively correlated. This trend is verified by Figure 5.8. With ρ increasing from negative to positive, θ_x decreases and investors put more wealth in illiquid assets to take advantage of the difference in the Sharpe ratio. Note that for the extreme cases on the graph for $x = 0$ or 1, three curves coincide, which is consistent with our analytic analysis shown in Theorem 3.4.

Recall from the results of the Merton-two assets model, if the two assets are strongly positively

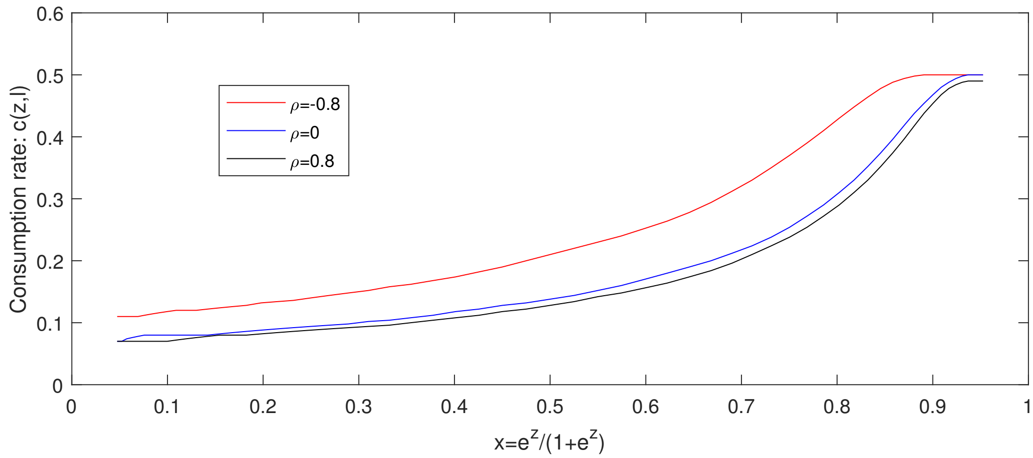

Figure 5.7: Consumption Policy $w.r.t$ Different ρ

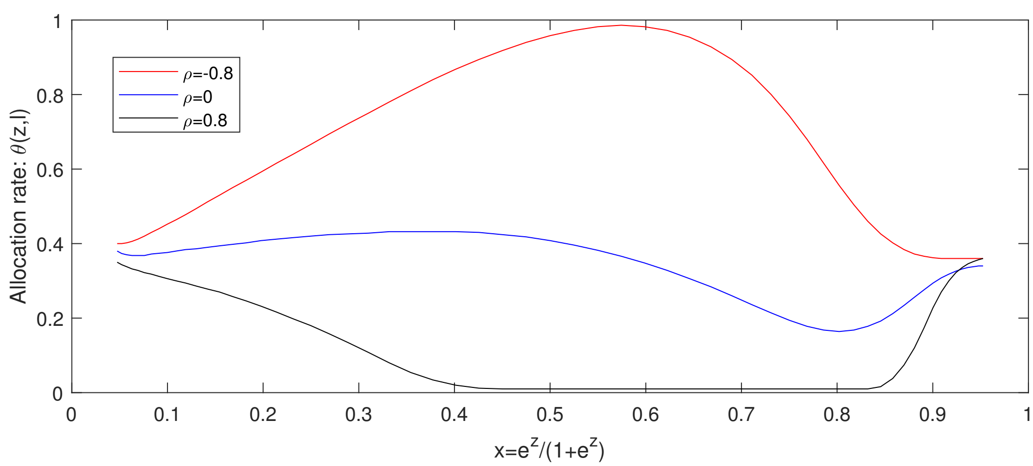

Figure 5.8: Asset Allocation $w.r.t$ Different ρ

correlated, the different in the Sharpe ratio can result in arbitrage trading opportunities. However, the situation is different in our model, because liquidity risks exist for illiquid assets and they are not close substitutes for liquid assets, even though they have a high positive correlation.

Another issue is the consumption smoothing effect. In Figure 5.7, the three consumption policies are not as different as the allocation policies, which means investors do not lower their consumption to make use of the larger Sharpe ratio of illiquid assets. This is further evidence of the effect of consumption policies.

6 Concluding remarks

This paper extends from the Ang et al. (2014) by introducing a more complex regime switching model, with different independent Poisson processes to model the price jumps of illiquid assets. In the illiquid state, investors have another liquidity risk such that they can only consume illiquid wealth during some random trading opportunities. Another factor we introduce is heritage, modeled by the expected discounted utility at the time of investors's death. The opportunities provided by market switches can be treated as another type of trading opportunity for illiquid assets, which leads to higher utility than in Ang et al. (2014). Compared to the classical Merton problem in Merton (1969), our analysis suggests that liquidity risk implies a large reduction in consumption, because investors need to hold sufficient liquid assets to meet future consumption. As illiquid assets increase, investors behave in a more risk-averse way, fearing reduced consumption from liquid wealth in the future. In addition, we discuss the effect of the intensity of regime switching, which further explains the effect of consumption smoothing.

In this paper, we assume there are no transaction costs. To be more consistent with reality, we could extend our study by adding transaction costs when investors rebalance their portfolio. There are two methods to model transaction costs. One is to introduce a fixed amount cost, which is easier to handle but is not realistic. The other method is to assume transaction costs are a function of the trading values. A mixture of the two formulations would be more realistic but harder to deal with.

Another direction for further research is consider a general regime switching model. Keykhaei (2018) discusses the effect of a bankruptcy state under mean-variance formulations. During that state, investors can only receive a random fraction of the wealth they invest in risky assets. Bankruptcy is another important result of the liquidity risk to explore in our future research. In addition to mean-variance analysis, Levy and Kaplanski (2015) employ second degree stochastic dominance rule to discuss the market oscillation in a regime-switching model. Another issue is that the transition intensities are constants in this paper. We can extend the work by assuming them to be correlated with the regime state or the asset price. For example, when the illiquid asset price is sufficiently high, the possibility of a down jump in price is bigger than in normal times. In addition to the transition intensities, we can let the jump size be dependent on the asset price, which will create more difficulties for obtaining explicit solutions. Nevertheless, a numerical algorithm can always find a viable solution.

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Appendix

The appendix is contained in a separate file of supplementary materials.

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