Persistency of genuine correlations under particle loss

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In a recent work Brunner and Vertesi [Phys. Rev. A **86**, 042113 (2012)] the question of persistency of entanglement and nonlocality of multiparty systems under particle loss has been addressed. This question is of immense importance considering the practical realization of the information theoretic tasks which make use of the power of quantum correlations. But in the multipartite scenario more interesting cases arise since subsystems can also possess genuineness in correlation which is prevalently inequivalent to the bipartite scenario. In this paper we investigate the persistency of such genuine correlations under particle loss. Keeping in mind the practical importance, considerable attention has been devoted to find the multiparty states which exhibit maximal persistency of genuine correlations.

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I. INTRODUCTION

Correlations play a fundamental role in quantum information science. Entanglement [1], quantum steering [2], and nonlocality [3] are considered as prime features of quantum correlations. Quantum entanglement is a physical phenomenon that occurs where many particles are generated or interacted in such a fashion that the quantum state of each particle cannot describe the full system separately but it can be described holistically only. This kind of quantum state can be used to demonstrate nonlocality where the statistics generated from each subsystem cannot be reproduced by any local realistic theory analogous to classical physics [4]. Bell nonlocal correlation along with entanglement are found to be key resources for many information processing tasks such as teleportation [5], dense coding [6], randomness certification [7], key distribution [8], dimension witness [9], Bayesian game theoretic applications [10], and so forth. Quantum steering [2] is a scenario where one party can remotely prepare the state of another party who are spatially separated, by applying a suitable choice of measurements. This curious feature has also found applications in one-sided device independent cryptography [11]. In a recent work [12] it has been shown that entanglement, steering, and nonlocality are inequivalent notions under general quantum operations.

Entanglement, steering, and nonlocality are well understood in the bipartite scenario. But for more than two parties complexity increases, resulting in the multipartite case being richer in essence [1]. In the recent past a considerable number of attempts have been made to understand the genuine multipartite correlations, which are remarkably different from their bipartite counterpart. An n-partite entangled state will be called genuinely entangled (GE) [13] if and only if the state is not separable with respect to any m partition ($m \leqslant n$) of the subsystems. Being a useful resource for computation [14], simulation [15], and metrology [16], the study of genuine multipartite entanglement is a field of recent attraction. It is even useful for the dinning cryptography problem [17]. Similarly multipartite nonlocality is also not very easy to understand compared to the bipartite cases. Here also one can define a multiparty correlation to be genuinely nonlocal (GNL) if and only if it is incompatible with any local-realistic theory with respect to any bipartite cut. It has been found to be an important resource in a number of information processing tasks [3]. Extension of quantum steering phenomena to the multipartite scenario has also been recently explored [18,19]. Even genuineness of steering found some importance in a number of recent works. So any kind of robustness or preservation to losses of these nonclassical features is really an important issue in case of practical implementation of information processing tasks.

The idea of persistency of entanglement and nonlocal features of quantum correlations under the particle loss scenario (i.e., the minimal number of particles to be lost for those nonclassical features to vanish completely) has gained interest in recent times [20–23]. Here the loss of particles renders the situation where the information about particles becomes inaccessible. For example, one can consider the multiparty quantum cryptography protocols where a number of parties are not willing to cooperate. Hence the idea of persistency of quantum correlations is crucial to the implementation of

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such information theoretic tasks. In [20] persistency of correlation was first defined as the possibility of obtaining residual correlation when a selective measurement is performed on a subsystem. But Brunner and Vertesi [21] have investigated the persistency of entanglement and nonlocality in a stronger scenario where a subsystem is lost completely in comparison to the earlier definition of persistency as in [20] for a number of special multipartite classes such as cluster states. The authors have also discussed the possibility of maximal persistency of entanglement and nonlocality for **W** class states. Recently Diviánszky *et al.* [22] have provided a simple upper bound on the persistency of nonlocality for **W** states and any permutation-symmetric state with two settings per party. A similar notion of persistency can also be defined for quantum steering, a weaker notion of nonlocality. This is important since the persistency of steering is required while considering a star-type network for one-sided device independent quantum key distribution under particle loss [11].

Keeping in mind the usefulness of genuine correlation in a multiparty scenario a pertinent question is of the persistence of GE along with GNL and similarly for genuine steering (GS) in different classes of multiparty states. In this paper we define the notion of persistency of genuine correlation and hence study the capacity of different classes of multiparty states to persistent genuineness. We further investigate the possibility of achieving maximum persistency of correlations within these classes of multipartite states, which is of practical interest.

In the following sections we first (Sec. II) provide a motivation for considering the concept of persistency for genuine correlations. In Sec. III we briefly provide the relevant definitions and notations. Section IV consists of our results regarding the persistency of genuine correlation for a varied class of multipartite states and the possibility of achieving maximal persistency for certain classes of multiparty states. We conclude with Sec. V where we discuss the implications of our paper in understanding multipartite correlations and further scopes for generalization of the results presented in this paper.

II. MOTIVATION

Given that one can quantify genuine correlation of any multiparty state, then a common intuition while studying persistency of correlation under particle loss is that whichever multiparty state has the higher amount of genuine entanglement will be more persistent. For example, when one takes the entanglement measure to be the entropy of entanglement, or the Tsallis and Rényi entropy of entanglement, the four qubit $|M\rangle$ [24] and $|L\rangle$ [24] states maximize the average entropy of entanglement. As we show later, both the states have persistency of genuine entanglement higher than that of the four qubit Greenberger-Horne-Zeilinger (GHZ) state, which has a lesser amount of entanglement according to that measure. But what we find out in this paper is quite contrary to this intuition. We show that there exist states with less entanglement (in the sense of a valid measure of entanglement) that can have maximal persistency of entanglement whereas a class of higher entangled states has minimal persistency.

In case of multiparty nonlocality another important notion is that of monogamy. This states that all the reduced systems of a parent multiparty nonlocally correlated state obtained by tracing out every other party cannot show nonlocality [25–27]. While considering particle loss, the residual systems that can be achieved by tracing out every other party in the system are either nonlocal or not. If all of them show some nonlocality (by violation of some nonlocal inequality) then the notion of monogamy fails but the persistency of nonlocality is maintained. Thus one can consider the concept of monogamy for multiparty states as complementary to the persistency of nonlocality.

Another question that naturally arises in the GNL scenario is whether the possibility of performing local filtering operations can strictly enlarge the genuine nonlocality-persistent states on par with similar results obtained by the authors of [21] in case of nonlocality and hidden nonlocality. We answer this question in the affirmative and also present examples where the persistency of genuine nonlocality is 1, i.e., minimum possible, but when allowed to perform local filtering operations the persistence of "hidden" genuine nonlocality can be maximum.

From a practical perspective the question of achieving maximum persistency is very crucial. To understand this let us consider a star network. A simple futuristic banking system is an example of a star network, where the central body bank tries to maintain quantum correlation with multiple customers. Now it is quite unexpected that since one of the customers leaves the system by closing her account the existing quantum correlation between the bank and other customers gets destroyed. So, the multipartite state shared in the star network should be something which has a higher persistency under particle loss. In this sense achieving maximum persistency is ideal.

With these motivations in mind we move on to present our results. But before that let us discuss a few definitions and relevant tools.

III. DEFINITIONS AND TOOLS

Let ρ be a quantum state of N systems. Taking partial trace over $k < N$ systems $j_1, \dots, j_k \in \{1, \dots, N\}$, let the reduced state be denoted as

$$\rho_{(j_1,\dots,j_k)} = \mathrm{Tr}_{j_1,\dots,j_k}(\rho). \tag{1}$$

Definition 1 [21]. The strong persistency of entanglement of any quantum state ρ, denoted by $P_E(\rho)$, is defined as the minimal number of parties k (say) such that the corresponding reduced state $\rho_{(j_1,\dots,j_k)}$ becomes fully separable, for at least one set of subsystems $\{j_1, \dots, j_k\}$.

In [21] the authors defined this stronger notion of persistency, and tried to relate it with the slightly different concept of persistency of entanglement introduced in [20]. Throughout this paper we have adopted this "stronger" notion of persistency, which deals with the complete loss of information of particles.

While checking for persistency of entanglement, when one considers mixed multipartite states there does not exist any necessary and sufficient criterion to detect entanglement. But in literature there are certain sufficient conditions [13,28] which can be used to witness entanglement conclusively. For our purpose we make use of the criterion in [28] to detect the presence of entanglement for more than two parties.

Definition 2 [21]. The persistency of nonlocality of any quantum state ρ, denoted by $P_{\rm NL}(\rho)$, is defined by demanding that at least one of the reduced states, say $\rho_{(j_1,\ldots,j_k)}$, becomes local, i.e., that the probability distributions obtained from local measurements on $\rho_{(j_1,\ldots,j_k)}$ do not violate any Bell inequality. Formally this means that the probability distribution

$$p(a_1\ldots a_N|x_1\ldots x_N)=\mathrm{Tr}\big(\rho M_{a_1}^{x_1}\otimes\ldots\otimes M_{a_N}^{x_N}\big) \quad (2)$$

admits a hidden variable model for general local measurement operators $M_{a_i}^{x_i}$, with $M_{a_i}^{x_i}=\mathbb{1}$ if $i=j_1,\ldots,j_k$ (the systems that have been traced out) and $\sum_{a_i}M_{a_i}^{x_i}=\mathbb{1}$ otherwise. Here x_i and a_i denote the measurement setting and its outcome, respectively, of party i.

To detect nonlocality of reduced tripartite states, we have considered the whole set of 46 facet inequalities of the Bell-local polytope which serves as necessary and sufficient conditions in the two input–two output Bell scenario (see supplementary material of [29]). On the other hand, for detecting bipartite nonlocality, Bell–Clauser-Horne-Shimony-Holt [30] and I_{3322} inequalities [29,31] are considered as these are the only possible inequivalent facets for the three input–two output Bell scenario.

In this context one can also consider the concept of hidden nonlocality [32].

Definition 3 [21]. One can also demand that at least one of the reduced states, say $\rho_{(j_1,\ldots,j_k)}$, is local even after the remaining parties have performed a local filtering. In this case, persistency of nonlocality is denoted by $P_{\rm HNL}(\rho)$.

For any state ρ, the above three notions of persistency maintain the following relation:

$$N-1\geqslant P_E(\rho)\geqslant P_{\rm HNL}(\rho)\geqslant P_{\rm NL}(\rho)\geqslant 1. \quad (3)$$

The second inequality comes from the fact that (i) entanglement is necessary for having quantum nonlocality and (ii) there exist entangled states which are local [33]. The third inequality follows from the fact that there exist local quantum states featuring hidden nonlocality [32].

In a similar spirit we define the concept of persistency of steering as follows.

Definition 4. The persistency of steering of ρ, $P_S(\rho)$, is defined as the minimal k such that the reduced state $\rho_{(j_1,\ldots,j_k)}$ becomes fully unsteerable, for at least one set of subsystems $j_1,\ldots,j_k$.

Then the revised hierarchy of the persistency of a state ρ will be given by

$$N-1\geqslant P_E(\rho)\geqslant P_S(\rho)\geqslant P_{\rm NL}(\rho)\geqslant 1. \quad (4)$$

The second inequality follows from the fact that entanglement is necessary for having quantum steering and there exist entangled states which are unsteerable [12]. The third inequality follows since quantum steering is necessary for having nonlocality and there exist steerable states which are local [2,12].

The notions described above have a large impact from an operational angle, specifically for characterizing robustness of multipartite quantum correlations when the corresponding quantum state is subjected to particle loss. In this context it will be interesting to focus on $P_{\rm NL}(\rho)$ and $P_S(\rho)$ as they represent a lower bound on $P_E(\rho)$ that can be obtained in a device independent and semi-device independent way, respectively.

In case of more than two parties one has different notions of correlation. The concept of genuine correlation provides an interesting paradigm to understand correlation in the multipartite scenario by excluding the possibility of bipartite correlations. At this point let us provide the definitions for genuine entanglement, nonlocality, and steering. For simplicity we define these notions for three parties but the reader can easily extend these definitions for a higher number of parties.

Definition 5. A quantum state is biseparable if it can be written as $\rho_{ABC}=\sum_\lambda p^\lambda_{A(BC)}\rho^\lambda_A\otimes\rho^\lambda_{BC}+\sum_\mu p^\mu_{B(AC)}\rho^\mu_B\otimes\rho^\mu_{AC}+\sum_\nu p^\nu_{C(AB)}\rho^\nu_C\otimes\rho^\nu_{AB}$ where $p^\lambda_{A(BC)}$, $p^\mu_{B(AC)}$, and $p^\nu_{C(AB)}$ are probability distributions. Finally a state is genuine multipartite entangled, if it is not biseparable.

There does not exist any necessary and sufficient criterion to detect genuine entanglement of mixed multipartite states. But several sufficient conditions for witnessing genuine entanglement have been proposed [13,28]. Here we make use of the sufficient criteria given in [28] for our purpose.

Now let us consider the correlation scenario among three parties with inputs and outputs as $\{X,Y,Z\}$ and $\{a,b,c\}$, respectively. Then one can have the following definition for genuine nonlocal correlations [34].

Definition 6. Suppose that $P(abc|XYZ)$ can be written in the form

$$\begin{aligned}P(abc|XYZ)=&\sum_\lambda q_\lambda P_\lambda(ab|XY)P_\lambda(c|Z)\\&+\sum_\mu q_\mu P_\mu(ac|XZ)P_\mu(b|Y)\\&+\sum_\nu q_\nu P_\nu(bc|YZ)P_\nu(a|X),\end{aligned} \quad (5)$$

where the bipartite terms are nonsignaling and $\{q_\lambda\}$, $\{q_\mu\}$, and $\{q_\nu\}$ are valid probability distributions. Then the correlations are NS_2 local. Otherwise, we say that they are genuinely three-way nonlocal.

To check the genuine three-way nonlocality of reduced tripartite states, we have used the necessary and sufficient criteria provided by the whole set of 185 facet inequalities of the NS_2 local polytope in the presence of binary input and output (see Supplementary Material of [34]).

In a recent work [19] the concept of genuine steering among three parties has been defined as follows.

Definition 7. Suppose that $P(abc|XYZ)$ cannot be explained by the following nonlocal local hidden states (LHS)-local hidden variables (LHV) (NLHS) model:

$$\begin{aligned}P(abc|XYZ)=&\sum_\lambda p_\lambda P\big(ab|XY,\rho^\lambda_{AB}\big)P_\lambda(c|Z)\\&+\sum_\lambda q_\lambda P\big(a|X,\rho^\lambda_A\big)P\big(b|Y,\rho^\lambda_B\big)P_\lambda(c|Z),\end{aligned} \quad (6)$$

where $P(ab|XY,\rho_\lambda)$ denotes the nonlocal probability distribution arising from two qubit state ρ^λ_{AB}, $P(a|X,\rho^\lambda_A)$ and $P(b|Y,\rho^\lambda_B)$ are the distributions arising from qubit states ρ^λ_A and ρ^λ_B, and $\{p_\lambda\}$ and $\{q_\lambda\}$ are probability distributions. Then the quantum correlation exhibits genuine steering from Charlie to Alice and Bob. This definition also holds for any party permutation.

Based on the definitions of genuine correlation presented above, one can also define the following quantities regarding the persistency of genuineness in correlations.

Definition 8. Persistency of genuine entanglement ($P_{\rm GE}$), nonlocality ($P_{\rm GNL}$), and steering ($P_{\rm GS}$) for a quantum state is defined as the minimum number of particles lost so that at least one of the reduced states is no longer genuinely entangled, nonlocal, and steerable respectively.

Definition 9. Persistency of genuine nonlocality under local filtering operation ($P_{\rm HGNL}$) for a quantum state is defined as the minimum number of particles lost so that at least one of the reduced states is no longer genuinely nonlocal under local filtering operations.

One can clearly see that $P_{\rm GE} \geqslant P_{\rm GS} \geqslant P_{\rm GNL}$. The first inequality follows from the fact that genuine entanglement is necessary for genuine steering. The second inequality comes from the requirement that genuine steering is necessary for genuine nonlocality. Thus, operationally, persistency of genuine nonlocality and genuine steering of a state provide a lower bound to the persistency of genuine entanglement of the state in a device independent and semi-device independent way, respectively.

IV. RESULTS

In this section we present our results regarding persistency of genuine correlations for a number of classes of multiparty states. These states are important for different information theoretic tasks. Hence the robustness of genuine correlation under particle loss for these states is of practical importance.

A. Study of a generic class of four qubit states regarding persistency of multipartite correlation

A generic class of four qubit states is one of the nine groups into which four qubit states can be classified [35]. This class of states is dense in the space of four qubits $\mathcal{H}_4 \equiv \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ under the action of stochastic local operations with classical communication. The generic class is given by

$$\mathcal{A} \equiv \{z_0u_0 + z_1u_1 + z_2u_2 + z_3u_3|\ z_0, z_1, z_2, z_3 \in \mathbb{C}\}$$

where

$$u_0 \equiv |\phi^+\rangle|\phi^+\rangle, \quad u_1 \equiv |\phi^-\rangle|\phi^-\rangle,$$
$$u_2 \equiv |\psi^+\rangle|\psi^+\rangle, \quad u_3 \equiv |\psi^-\rangle|\psi^-\rangle.$$

A pure state of four qubits $|\psi\rangle \in \mathcal{A}$ can be written in computational basis as the following:

$$|\psi\rangle = \frac{z_0+z_3}{2}(|0000\rangle + |1111\rangle) + \frac{z_0-z_3}{2}(|0011\rangle + |1100\rangle) + \frac{z_1+z_2}{2}(|0101\rangle + |1010\rangle) + \frac{z_1-z_2}{2} \times (|0110\rangle + |1001\rangle).$$

The four qubit entanglement monotone that is invariant under any permutation of the four qubits, the Wong-Christensen four-tangle [36], is defined as the following. Letting $|\psi\rangle \in \mathcal{H}_4 \equiv \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, the four-tangle is defined by [36]

$$\tau_{ABCD} \equiv |\langle\psi|\sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y|\psi^*\rangle|^2. \tag{7}$$

As a measure for pure bipartite entanglement we first take the tangle or the square of the I concurrence [37]. Denoting the four qubits by A, B, C, and D, one can define [24]

$$\tau_1 \equiv \tfrac{1}{4}(\tau_{A(BCD)} + \tau_{B(ACD)} + \tau_{C(ABD)} + \tau_{D(ABC)}), \tag{8}$$

$$\tau_2 \equiv \tfrac{1}{3}(\tau_{(AB)(CD)} + \tau_{(AC)(BD)} + \tau_{(AD)(BC)}), \tag{9}$$

where $\tau_{A(BCD)}$ is the tangle between qubit A and qubits B, C, and D. Similarly, $\tau_{(AB)(CD)}$ is the tangle between qubits A and B and qubits C and D. The reader can note that the maximum value possible for τ_1 is 1 and the maximum value possible for τ_2 is 3/2.

1. $\tau_{\min}$ and $\mathcal{M}$ classes

A normalized state $|\psi\rangle \in \mathcal{H}_4$ is maximally entangled (i.e., $\tau_2(|\psi\rangle) = 4/3$) if and only if up to local unitary $|\psi\rangle \in \mathcal{M}$, where $\mathcal{M}$ is the set of states in $\mathcal{A}$ with zero four-tangle [38].

As defined earlier a pure state $\psi = \sum_{j=0}^3 z_ju_j$ in $\mathcal{A}$ depends on four complex parameters z_j $(j = 0, 1, 2, 3)$. The condition that the four-tangle $\tau_{ABCD}(\psi) = |\sum_{j=0}^3 z_j^2|^2 = 0$ implies that the states in the maximally entangled class $\mathcal{M}$ are characterized by four *real* parameters. The reduction in number of parameters is due to the normalization condition and ignoring the global phase. When written in its polar form $z_j = \sqrt{p_j}e^{i\theta_j}$ (with non-negative p_j and $\theta_j \in [0, 2\pi]$) one can denote the class $\mathcal{M}$ as follows:

$$\mathcal{M} = \left\{ \sum_{j=0}^{3} \sqrt{p_j}e^{i\theta_j}u_j \,\middle|\, \sum_{j=0}^{3} p_j = 1,\ \sum_{j=0}^{3} p_je^{2i\theta_j} = 0 \right\}. \tag{10}$$

For example, cluster states [20] and $|M\rangle$ [24] and $|L\rangle$ [24] states belong to this class.

Another important set of the states in $\mathcal{A}$, denoted by $\mathcal{T}_{\min}$, with the minimum possible value $\tau_2 = 1$, can be characterized as follows:

$$\mathcal{T}_{\min} \equiv \{\psi \in \mathcal{A}|\tau_2(\psi) = 1\} = \left\{ \sum_{j=0}^{3} x_ju_j \,\middle|\, \sum_{j=0}^{3} x_j^2 = 1,\ x_j \in \mathbb{R} \right\}. \tag{11}$$

For example, the four qubit GHZ state and well-known Dicke state $|D_4^2\rangle$ [39] belong to $\mathcal{T}_{\min}$. In this sense, the GHZ state and $|D_4^2\rangle$ are states in $\mathcal{A}$ with the least amount of entanglement.

2. Persistency of entanglement and genuine entanglement

a. P_E and P_{GE} of the $\tau_{\min}$ class. From the criteria presented in [28] one can provide the following conditions for $P_{\rm GE}$ and P_E for the four qubit states in the $\tau_{\min}$ class.

Condition 10. $P_{\rm GE} > 1$ if

$$2|x_2^2 - x_3^2| > 2[x_2^2 + x_3^2] + (x_0 + x_1)^2 + 2[x_0^2 - x_1^2] - 8\min\{|x_0|, |x_1|\}\max\{|x_2|, |x_3|\}.$$

Condition 11. $P_E > 1$ if

(1) for $\mathrm{sgn}(x_0x_1) = 1$

$$|x_2^2 - x_3^2| > [x_0^2 - x_1^2] + 4\min\{|x_0|, |x_1|\}\max\{|x_2|, |x_3|\},$$

(2) for $\mathrm{sgn}(x_0x_1) = -1$

$$\left|x_2^2 - x_3^2\right| > \left[x_0^2 - x_1^2\right] - 4\min\{|x_0|, |x_1|\}\max\{|x_2|, |x_3|\}.$$

At this point one might wonder whether the $\tau_{\min}$ class contains states with minimal persistency of entanglement. Let us consider the following example of the state $|\mathrm{GHZ}_4\rangle = \frac{1}{\sqrt{2}}[|0000\rangle + |1111\rangle]$. It is straightforward to show that both P_E and P_{GE} of $|\mathrm{GHZ}_4\rangle$ are 1. This implies that even for the same value of the entanglement measure τ_2 throughout the $\tau_{\min}$ class there exist states which have different capabilities of persisting entanglement and genuine entanglement.

b. P_E and P_{GE} of the $\mathcal{M}$ class. Intuitively it can be expected that $\mathcal{M}$ class states being maximally entangled might have greater persistency of entanglement compared to the states in the $\tau_{\min}$ class. Let us take the example of cluster states. Cluster states [20] form a class of multiparty entangled quantum states with surprising and useful properties. The main interest in these states draws from their role as a universal resource in the one-way quantum computer [14]: Given a collection of sufficiently many particles that are prepared in a cluster state, one can realize any quantum computation by simply measuring the particles, one by one, in a specific order and basis. By the measurements, one exploits correlations in quantum mechanics which are rich enough to allow for universal logical processing. A four party cluster state is given by the following:

$$\eta_4 = \tfrac{1}{2}[|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle].$$

The tripartite reduced states can be written in a biseparable form [21]. Thus $P_{\mathrm{GE}} = 1$ for cluster states. This is in contrast to the states in the $\tau_{\min}$ class which are less entangled according to the measure τ_2 but can have $P_{\mathrm{GE}} > 1$ according to Conditions 10 and 11.

3. Persistency of nonlocality and genuine nonlocality

Now let us come to the question of persistence of nonlocality for states in the $\tau_{\min}$ class.

Theorem 12. $P_{\mathrm{NL}}(\rho) = 1$ for all four qubit states $\rho \in \tau_{\min}$.

Proof. Let us consider a four qubit state $\rho \in \tau_{\min}$. Upon loss of the ith particle the reduced states are

$$\rho_i^3 = \left|\psi_i^3\right\rangle\!\left\langle\psi_i^3\right| + \left|\phi_i^3\right\rangle\!\left\langle\phi_i^3\right|$$

for $i = 1, 2, 3, 4$ where $|\psi_i^3\rangle = |\tilde{W}_i^3\rangle + \frac{x_0+x_1}{2}|111\rangle$, $|\phi_i^3\rangle = \sigma_x^{\otimes 3}[|\tilde{W}_i^3\rangle + \frac{x_0+x_1}{2}|111\rangle]$, and

$$\left|\tilde{W}_1^3\right\rangle = \frac{x_2 - x_3}{2}|001\rangle + \frac{x_2 + x_3}{2}|010\rangle + \frac{x_0 - x_1}{2}|100\rangle,$$

$$\left|\tilde{W}_2^3\right\rangle = \frac{x_2 + x_3}{2}|001\rangle + \frac{x_2 - x_3}{2}|010\rangle + \frac{x_0 - x_1}{2}|100\rangle,$$

$$\left|\tilde{W}_3^3\right\rangle = \frac{x_0 - x_1}{2}|001\rangle + \frac{x_2 - x_3}{2}|010\rangle + \frac{x_2 + x_3}{2}|100\rangle,$$

$$\left|\tilde{W}_4^3\right\rangle = \frac{x_0 - x_1}{2}|001\rangle + \frac{x_2 + x_3}{2}|010\rangle + \frac{x_2 - x_3}{2}|100\rangle. \quad (12)$$

To check the nonlocality of these reduced tripartite states let us consider all 46 facets of the three qubit local polytope [29]. One can check that all the reduced states ρ_i^3 violate only the fourth facet (the same numbering as in [29] has been used for convenience) for different values of the real parameters $\{x_i\}_{i=0}^3$. At the same time it can also be shown (see Appendix A) that all reduced states cannot violate the fourth facet for a common set of parameter values. This implies that the nonlocality of any $\rho \in \tau_{\min}$ cannot persist upon loss of even one of the particles. Hence the theorem. ■

At this point we make the following observation.

Observation 13. For all four qubit states $\rho \in \tau_{\min}$ monogamy of nonlocality holds for the reduced three qubit states in the sense that depending on the state parameters at most two three qubit reduced states can demonstrate Bell nonlocality (in the two input–two output scenario) simultaneously.

In this connection, note that similar monogamy of Bell-nonlocality conditions was derived in [25] where the authors considered the reduced two qubit states obtained by tracing out particles from n qubit systems.

From Theorem 12 one can immediately arrive at the following corollary regarding the persistency of genuine nonlocality.

Corollary 14. $P_{\mathrm{GNL}}(\rho) = 1$ for all four qubit states $\rho \in \tau_{\min}$.

At this stage a pertinent question would be whether a weaker form of nonlocality can persist upon loss of particles for states in the $\tau_{\min}$ class. We deal with this question in the next subsection considering quantum steering as a weaker form of nonlocality.

4. Persistency of steering and genuine steering

Observation 15. There exist states $\rho \in \tau_{\min}$ such that $P_S(\rho)$ is maximal, i.e., 3.

This can be seen in a straightforward way. If one can show that there exist two qubit reduced states which can demonstrate steering, this in turn implies that there exist states in the $\tau_{\min}$ class with maximal persistency of steering. Upon loss of two particles the bipartite reduced states take the following forms:

$$\rho_i^2 = \left|\eta_i^2\right\rangle\!\left\langle\eta_i^2\right| + \left|\xi_i^2\right\rangle\!\left\langle\xi_i^2\right| + \sigma_x^{\otimes 2}\left|\eta_i^2\right\rangle\!\left\langle\eta_i^2\right|\sigma_x^{\otimes 2} + \sigma_x^{\otimes 2}\left|\xi_i^2\right\rangle\!\left\langle\xi_i^2\right|\sigma_x^{\otimes 2}$$

for $i = 1, 2, 3$, where

$$\left|\eta_1^2\right\rangle = \frac{x_0 - x_1}{2}|00\rangle + \frac{x_0 + x_1}{2}|11\rangle,$$

$$\left|\eta_2^2\right\rangle = \frac{x_2 + x_3}{2}|00\rangle + \frac{x_0 + x_1}{2}|11\rangle,$$

$$\left|\eta_3^2\right\rangle = \frac{x_2 - x_3}{2}|00\rangle + \frac{x_0 + x_1}{2}|11\rangle,$$

and

$$\left|\xi_1^2\right\rangle = \frac{x_2 - x_3}{2}|01\rangle + \frac{x_2 + x_3}{2}|10\rangle,$$

$$\left|\xi_2^2\right\rangle = \frac{x_2 + x_3}{2}|01\rangle + \frac{x_0 - x_1}{2}|10\rangle,$$

$$\left|\xi_3^2\right\rangle = \frac{x_2 + x_3}{2}|01\rangle + \frac{x_0 - x_1}{2}|10\rangle.$$

Now there exist states $\rho^4 \equiv \{x_0, x_1, x_2, x_3\}$ such that ρ_i^2 is steerable for $i = 1, 2, 3$ (see Appendix B). The existence of such states can be depicted in the parameter space as shown in Fig. 1. For example Dicke state $|D_4^2\rangle$ [39] belongs to this class and exhibits maximal persistency of steering (see Table I in Appendix C).

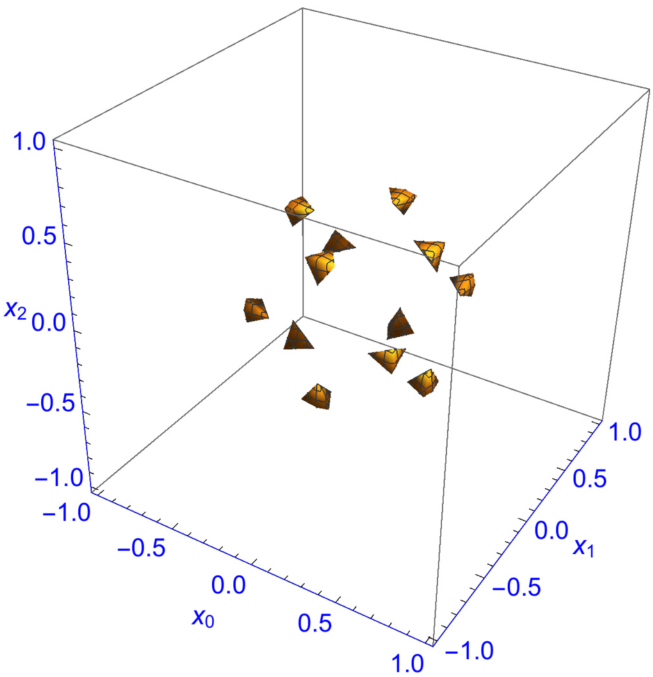

FIG. 1. The shaded regions represent the states belonging to the $\tau_{\min}$ class which exhibit maximum persistency of steering i.e., $P_S = 3$ in the parameter space $\{x_0, x_1, x_2\}$.

B. Achieving maximal persistency

Let us now consider the cases of achieving maximal persistency of correlations. For example, when one considers nonlocality demonstrated by multiparty systems, maximal persistency of nonlocality is interesting since it signifies the robustness of nonlocality under particle loss and the failure of monogamy of nonlocality between distant parties.

1. Maximal persistency of genuine nonlocality

In [21] the authors could not present any state with local dimension 2 which has maximum persistency of nonlocality. This can partly be understood as the strength of monogamy principle for nonlocality [25,26]. This implies that the demonstration of maximal persistency of genuine nonlocality will be harder. But there exist multipartite states with local dimension 2 which can demonstrate maximal persistency of genuine nonlocality when local filtering is allowed. This is to say P_{HGNL} for such states are maximal. Let us consider the following example.

Example 16. An n-partite state $|W^N\rangle$ is given by [40]

$$|W^N\rangle = \frac{1}{\sqrt{N}}[|0\ldots01\rangle + |0\ldots10\rangle + \ldots + |10\ldots0\rangle].$$

Now consider any reduced state of three parties obtained by losing $(N-3)$ parties. These states are of the form

$$\rho(p) = p|W^3\rangle\langle W^3| + (1-p)|000\rangle\langle 000| \qquad (13)$$

where $p = \frac{3}{N}$. These reduced states do not demonstrate genuine nonlocality for two settings per site since they do not violate any of the 185 inequalities given in [34]. Thus persistency of genuine nonlocality for W^N cannot be maximum. Now take the local filtering of the form

$$\begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix}$$

where $0 \leqslant \epsilon \leqslant 1$. After local filtering the state becomes

$$\rho(p,\epsilon) = \frac{p\epsilon^4}{p\epsilon^4 + (1-p)\epsilon^6}|W^3\rangle\langle W^3| + \frac{(1-p)\epsilon^6}{p\epsilon^4 + (1-p)\epsilon^6}|000\rangle\langle 000|. \qquad (14)$$

Now consider the Bell quantity

$$\begin{aligned} B_{16} = {} & \langle A_0B_0\rangle + \langle A_1B_0\rangle + \langle A_0B_1\rangle - \langle A_1B_1\rangle - 2\langle C_0\rangle \\ & + \langle A_0B_0C_0\rangle + \langle A_1B_0C_0\rangle + \langle A_1B_1C_0\rangle \\ & + 2\langle A_1C_1\rangle + 2\langle B_1C_1\rangle \end{aligned} \qquad (15)$$

where $B_{16} \leqslant 4$ is the 16th facet inequality as given in [34]. The maximum value of B_{16} obtainable from a state of the form (14) is

$$B_{16}(p,\epsilon) = \frac{p(4.72678) + 2\epsilon^2(p-1)}{\epsilon^2(1-p) + p}. \qquad (16)$$

By choosing $\epsilon \to 0$, this value can reach up to 4.72678 for any value of p. This implies $P_{\mathrm{HGNL}}(W^N) \geqslant (N-2)$. In [21] it has already been shown that the bipartite reduced states of $|W^N\rangle$ exhibit hidden nonlocality. Thus one has $P_{\mathrm{HGNL}}(W^N) = (N-1)$, i.e., maximal persistency of genuine correlation under local filtering.

But it would be more interesting to find the persistency of genuine nonlocality when local filtering is not considered, since the three qubit reduced state $\rho(p)$ does not violate any of the 185 facets for detecting genuine nonlocality under two measurement settings for all parties. On the basis of this evidence we make the following conjecture.

Conjecture 17. $P_{\mathrm{GNL}}(W^N) < N-2$ for all N-partite **W** states with local dimension 2.

In the next subsection we ask the question whether a weaker form of genuine nonlocality, namely, genuine quantum steering, can achieve maximal persistency.

2. Maximal persistency of genuine steering

Here we present a four qubit state which exhibits maximum persistency of genuine steering. We present our argument below. Let us consider the state $|W^4\rangle$. Remember that this state does not achieve maximal persistency of nonlocality or genuine nonlocality. Upon loss of one particle the three qubit reduced state takes the form (15), where $p = \frac{3}{4}$. This state violates the following three-setting genuine steering inequality (see Appendix D):

$$|\langle D_0C_0\rangle + \langle D_1C_1\rangle + \langle D_2C_2\rangle| \leqslant 3 \qquad (17)$$

where

$$D_0 = A_0B_0 + A_1B_1 + A_2B_2, \qquad (18)$$

$$D_1 = A_0B_2 - A_1B_0 + A_2B_1, \qquad (19)$$

$$D_2 = A_0B_1 - A_1B_2 + A_2B_0, \qquad (20)$$

and $\{A_i\}$, $\{B_j\}$, and $\{C_k\}$ for $(i, j, k = 0, 1, 2)$ are measurement settings of Alice, Bob, and Charlie, respectively. Thus one has $P_{GS}(W^4) \geqslant 2$. Now for loss of two parties the two qubit reduced state of $|W^4\rangle$ is of the form

$$\rho^2(p) = p|W^2\rangle\langle W^2| + (1-p)|00\rangle\langle 00| \tag{21}$$

where $p = \frac{1}{2}$. We know that this state has a local model under projective measurements [41]. But nonetheless this state exhibits steering because it violates a sufficient criterion [42] for steering. Hence $|W^4\rangle$ has maximum persistency of genuine steering, i.e., $P_{GS}(W^4) = 3$. This trivially implies the persistency of steering of the $|W^4\rangle$ state is also maximum. Note that this is a typical example where $P_S(\rho) > P_{NL}(\rho)$.

3. Maximal persistency of genuine entanglement

Now let us come to the question of maximum persistency of genuine entanglement. Let us consider the four qubit $\tau_{\min}$ class of states. States belonging to this class will have maximum persistency of genuine entanglement i.e., $P_{GE} = 3$ under the following conditions.

Condition 18. $P_{GE} > 1$ and $S_i > 0$ for $i = 1, 2, 3$ where

$$\begin{aligned}
S_1 &= 2\max\Big\{\big|-\tfrac{1}{2}(x_0+x_1)(-x_2+x_3)\big| - \tfrac{1}{4}((x_0-x_1)^2+(x_2+x_3)^2),\\
&\quad \big|\tfrac{1}{2}(x_0-x_1)(x_2+x_3)\big| - \tfrac{1}{4}((x_0+x_1)^2+(x_2-x_3)^2)\Big\},\\
S_2 &= 2\max\Big\{\big|\tfrac{1}{2}(x_0+x_1)(x_2+x_3)\big| - \tfrac{1}{4}((x_0-x_1)^2+(x_2-x_3)^2)\\
&\quad \big|-\tfrac{1}{2}(x_0-x_1)(-x_2+x_3)\big| - \tfrac{1}{4}((x_0+x_1)^2+(x_2+x_3)^2)\Big\},\\
S_3 &= 2\max\Big\{\big|\tfrac{1}{2}(x_2+x_3)(x_2-x_3)\big| - \tfrac{1}{4}((x_0-x_1)^2+(x_0+x_1)^2)\\
&\quad \big|\tfrac{1}{2}(x_0+x_1)(x_0-x_1)\big| - \tfrac{1}{4}((x_2+x_3)^2+(x_2-x_3)^2)\Big\}.
\end{aligned} \tag{22}$$

We depict the states with maximum persistency of genuine entanglement in the $\tau_{\min}$ class in the parameter space in Fig. 2. For example, the four qubit Dicke state $|D_4^2\rangle$ [39] belongs to the $\tau_{\min}$ class and it satisfies all the above conditions and hence $P_{GE}(D_4^2) = 3$.

The conditions for states in the $\tau_{\min}$ class to have maximum persistency of entanglement are the following:

Condition 19. $S_i > 0$ for $i = 1, 2, 3$.

The states with $P_E = 3$ belonging to the $\tau_{\min}$ class have been shown in Fig. 3.

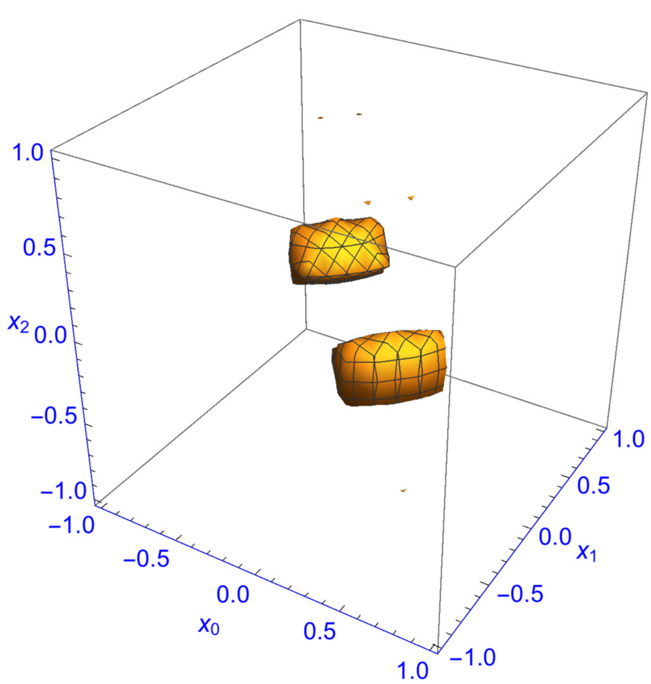

FIG. 2. The shaded regions represent the states belonging to the $\tau_{\min}$ class which exhibit maximum persistency of genuine entanglement, i.e., $P_{GE} = 3$ in the parameter space $\{x_0, x_1, x_2\}$.

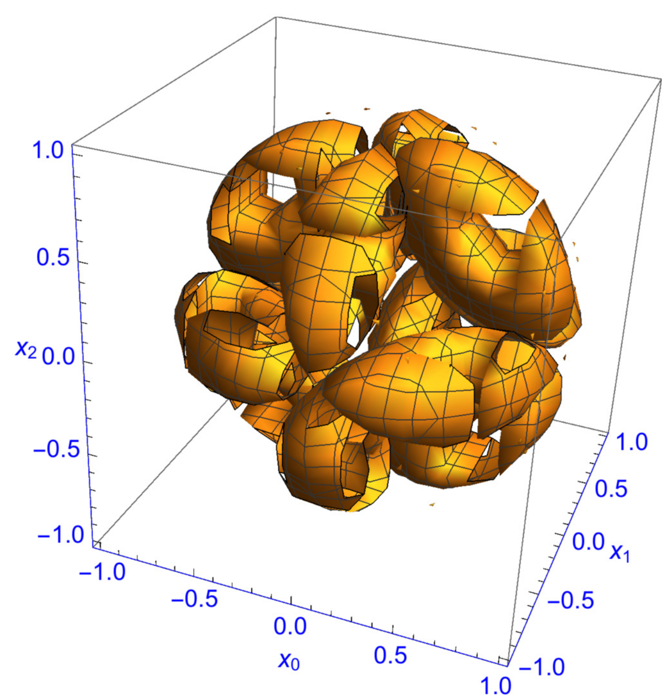

FIG. 3. The shaded regions represent the states belonging to the $\tau_{\min}$ class which exhibit maximum persistency of entanglement, i.e., $P_E = 3$ in the parameter space $\{x_0, x_1, x_2\}$. Comparison of the shaded regions in this figure with those of Fig. 2 shows that these regions contain the regions given in Fig. 2.

V. DISCUSSIONS

In a couple of recent studies [20,21] the concepts of persistency of entanglement and nonlocality were introduced. This new concept is fundamental to the understanding of quantum correlations and at the same time important from a practical perspective since it deals with the scenario where information about some of the parties can be completely lost. In particular, to date various experimental investigations of behavior of quantum entanglement of multiqubit states [23] under particle loss have been investigated.

Besides defining the same notion for quantum steering, our paper extends the concept of persistency to genuine correlations which are inherently multipartite in nature. We also discuss the possibility of achieving maximum persistency of genuine correlations with several important classes of multipartite states. As we have emphasized in the subsequent sections, maximum persistency of correlation becomes indispensable in certain multiparty quantum cryptography protocols.

It is also instructive to relate these notions of persistency with slightly different concepts of persistency of a selected property: the maximum number of parties that can be traced out from a quantum state ρ so that the selected property is still present in at least one of the reduced states. This issue will be addressed in our future work.

Now we point out some of the questions which this paper leaves open. A thorough understanding of the persistency of correlation for the four qubit states can enable one to classify the whole class of four qubit states in terms of persistency. Moreover, by using our method, one can also extend the study of persistency of correlation for multiparty systems of higher dimension (>2). Another interesting question is to find multiparty qubit states which have maximal persistency of genuine nonlocality.

ACKNOWLEDGMENTS

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APPENDIX A: VIOLATION OF THE FOURTH FACET AS IN [29]

In the main text (Theorem 12), it is shown that all the reduced states given in Eq. (12) cannot violate the fourth facet for a common set of state parameter values. Here, we provide all details for the discussion in Theorem 12.

The maximum violation value for each reduced state (12) for the fourth facet in [29] is as follows:

$$\begin{aligned} B_{\text{Max}}(\rho_1^3) = \max\Big[&2\sqrt{\big((-1+2x_1^2\big)^2+\big(-1+2x_0^2\big)^2}\\ &2\sqrt{\big(-1+2x_0^2+2x_1^2\big)^2+\big(1-2x_0^2\big)^2}\\ &2\sqrt{\big(-1+2x_0^2+2x_1^2\big)^2+\big(-1+2x_1^2\big)^2}\Big], \end{aligned} \tag{A1}$$

$$\begin{aligned} B_{\text{Max}}(\rho_2^3) = \max\Big[&2\sqrt{\big((1-2x_0^2-2x_2^2\big)^2+\big(1-2x_1^2-2x_2^2\big)^2}\\ &2\sqrt{\big(1-2x_1^2-2x_0^2\big)^2+\big(1-2x_2^2-2x_0^2\big)^2}\\ &2\sqrt{\big(1-2x_0^2-2x_1^2\big)^2+\big(1-2x_2^2-2x_1^2\big)^2}\Big], \end{aligned} \tag{A2}$$

$$\begin{aligned} B_{\text{Max}}(\rho_3^3) = \max[&4\sqrt{(x_0x_2+x_1x_3)^2+(x_1x_2+x_0x_3)^2}\\ &4\sqrt{(x_0x_1+x_2x_3)^2+(x_1x_2+x_0x_3)^2}\\ &4\sqrt{(x_0x_2+x_1x_3)^2+(x_1x_0+x_2x_3)^2}], \end{aligned} \tag{A3}$$

$$\begin{aligned} B_{\text{Max}}(\rho_4^3) = \max[&4\sqrt{(x_0x_1-x_2x_3)^2+(x_0x_2-x_1x_3)^2}\\ &4\sqrt{(x_0x_2-x_1x_3)^2+(x_1x_2-x_0x_3)^2}\\ &4\sqrt{(x_1x_2-x_0x_3)^2+(x_1x_0-x_2x_3)^2}. \end{aligned} \tag{A4}$$

All these reduced states exhibit nonlocality only when each of $B_{\text{Max}}(\rho_i^3) > 2 (i = 1, 2, 3, 4)$. It is impossible that all reduced states violate fourth facet inequality. Hence the result $P_{\text{NL}} = 1$.

APPENDIX B: EXISTENCE OF MAXIMALLY PERSISTENT STEERABLE STATES

In this section, we present the existence of states with $P_S = 3$.

The following conditions on the state parameters are obtained from the conditions given as Eq. (22) in [42].

(1) Steering condition for ρ_1^2:

$$\begin{aligned} S(\rho_1^2) := \max\Big[&\big|-1+2x_0^2+2x_2^2\big|+\big|-1+2x_1^2+2x_2^2\big|\\ &-\frac{2}{\pi}\Big(\sqrt{1-\big(-1+2x_0^2+2x_1^2\big)^2}\\ &+\sqrt{1-\big(-1+2x_0^2+2x_1^2\big)^2}\Big),\\ &\big|-1+2x_0^2+2x_2^2\big|+\big|-1+2x_0^2+2x_1^2\big|\\ &-\frac{2}{\pi}\Big(2\sqrt{(1)^2-\big(-1+2x_1^2+2x_2^2\big)^2}\Big)\\ &\big|-1+2x_1^2+2x_2^2\big|+\big|-1+2x_0^2+2x_1^2\big|\\ &-\frac{2}{\pi}\Big(2\sqrt{1-\big(-1+2x_0^2+2x_2^2\big)^2}\Big)\Big]. \end{aligned} \tag{B1}$$

(2) Steering condition for ρ_2^2:

$$\begin{aligned} S(\rho_2^2) := \max\Bigg[&|2(x_0x_2+x_1x_3)|+|-2(x_1x_2+x_0x_3)\\ &-\frac{2}{\pi}(\sqrt{1-(2(x_0x_1+x_2x_3))^2}\\ &+\sqrt{1-(2(x_0x_1+x_2x_3))^2})\\ &|2(x_0x_2-x_1x_3)|+|2(x_0x_1+x_2x_3)|\\ &-\frac{2}{\pi}(2\sqrt{(1)^2-(-2(x_1x_2+x_0x_3))^2}) \end{aligned}$$

TABLE I. Persistency of entanglement (P_E), steering (P_S), nonlocality ($P_{\rm NL}$), genuine entanglement ($P_{\rm GE}$), genuine steering ($P_{\rm GS}$), and genuine nonlocality ($P_{\rm GNL}$) for various important classes of four qubit states. Red colored values are optimal in the sense that these values follow from necessary and sufficient conditions.

States	P_E	P_S	$P_{\rm NL}$	$P_{\rm GE}$	$P_{\rm GS}$	$P_{\rm GNL}$
Cluster state [20]	2	2	2	1	1	1
$\|L\rangle$ [24]	2	2	2	2	2	1
$\|W^4\rangle$ [40]	3	3	2	3	3	1
$\|D_4^2\rangle$ [39]	3	3	1	3	1	1
$\|M\rangle$ [24]	2	2	2	2	1	1
$\|BSSB_4\rangle$ [43]	2	2	2	2	1	1
$\|YC\rangle$ [44]	2	2	2	1	1	1
$\|D_4^3\rangle$ [39]	3	3	2	3	3	1
$\|\chi_4\rangle$ [13]	2	2	2	2	2	1
Singlet state [45]	2	1	1	2	1	1

$$|-2(x_1x_2+x_0x_3)|+|2(x_0x_1+x_2x_3)| - \frac{2}{\pi}\left(2\sqrt{1-(2(x_0x_2+x_1x_3))^2}\right)\Bigg]. \tag{B2}$$

(3) Steering condition for ρ_3^2:

$$\begin{aligned} S(\rho_3^2) := \max\Bigg[& |2x_0x_2-2x_1x_3|+|-2x_1x_2+2x_0x_3| \\ & -\frac{2}{\pi}\left(\sqrt{1-(2x_0x_1-2x_2x_3)^2}\right. \\ & \left.+\sqrt{1-(2x_0x_1-2x_2x_3)^2}\right) \\ & |2x_0x_2-2x_1x_3|+|2x_0x_1-2x_2x_3| \\ & -\frac{2}{\pi}\left(2\sqrt{1-(-2x_1x_2+2x_0x_3)^2}\right) \\ & |-2x_1x_2+2x_0x_3|+|2x_0x_1-2x_2x_3| \\ & -\frac{2}{\pi}\left(2\sqrt{1-(2x_0x_2-2x_1x_3)^2}\right)\Bigg]. \end{aligned} \tag{B3}$$

All of these three reduced states exhibit steering if $S(\rho_1^2)>0$, $S(\rho_2^2)>0$, and $S(\rho_3^2)>0$.

APPENDIX C: TABLE FOR PERSISTENCY OF CORRELATIONS FOR VARIOUS FOUR QUBIT STATES

Here, we obtain the persistency of different correlations for various important class of four qubit states (see Table I).

APPENDIX D: A GENUINE STEERING INEQUALITY FOR THREE SETTINGS PER SITE

In this section, we provide a complete proof of the inequality presented in Eq. (17) of the main text.

Theorem 20. If any given quantum correlation violates the steering inequality,

$$|\langle (A_0B_0+A_1B_1+A_2B_2)C_0\rangle + \langle (A_0B_2-A_1B_0+A_2B_1)C_1\rangle + \langle (A_0B_1-A_1B_2+A_2B_0)C_2\rangle| \leqslant 3, \tag{D1}$$

then the correlation exhibits genuine tripartite steering from Charlie to Alice and Bob, where $\{A_i\}$, $\{B_j\}$, and $\{C_k\}$ for $(i,j,k=0,1,2)$ are two-output measurement settings of Alice, Bob, and Charlie, respectively. Here measurements of Alice and Bob demonstrate Einstein-Podolsky-Rosen (EPR) steering without Bell nonlocality while measurements of Charlie are uncharacterized.

Before proving the theorem, we first prove the following lemma.

Lemma 21. Any LHS-LHS model satisfies the following inequality:

$$|\langle A_0B_0+A_1B_1+A_2B_2\rangle| \leqslant 1. \tag{D2}$$

Proof. Let us denote

$$D_0=\langle A_0B_0+A_1B_1+A_2B_2\rangle. \tag{D3}$$

For any separable state (due to linearity of the quantity D_0, without loss of generality one can consider product states $\rho_{AB}=\rho_A\bigotimes\rho_B$ only for this purpose),

$$|D_0|=|\overrightarrow{v_A}.\overrightarrow{v_B}|, \tag{D4}$$

where $\overrightarrow{v_{A/B}}=(\langle A_0/B_0\rangle,\langle A_1/B_1\rangle,\langle A_2/B_2\rangle)$. By the Cauchy Schwarz inequality, we get

$$|D_0|\leqslant|\overrightarrow{v_A}||\overrightarrow{v_B}|. \tag{D5}$$

Now,

$$|\overrightarrow{v_A}|=\sqrt{\sum_{i=0}^{2}\langle A_i\rangle^2}. \tag{D6}$$

Again,

$$\langle A_i\rangle=\mathrm{Tr}(A_i\rho_A), \tag{D7}$$

where $\rho_A=\mathrm{Tr}_B(\rho_{AB})$. After simple calculation, we get

$$\langle A_i\rangle=\overrightarrow{n_i}.\overrightarrow{r}, \tag{D8}$$

where $\overrightarrow{r}$ denotes the Bloch vector corresponding to the state ρ_A and $\overrightarrow{n_i}$ characterizes the measurement setting $A_i=\overrightarrow{n_i}.\overrightarrow{\sigma}$. Using this relation [Eq. (D8)], Eq. (D6) gets simplified:

$$|\overrightarrow{v_A}|=\sqrt{\sum_{i=0}^{2}(\overrightarrow{n_i}.\overrightarrow{r})^2}. \tag{D9}$$

This on simplification becomes

$$|\overrightarrow{v_A}|=|\overrightarrow{r}|\leqslant 1. \tag{D10}$$

Similarly it can be shown that $|\overrightarrow{v_B}|\leqslant 1$. Hence Eq. (D5) becomes

$$|D_0|\leqslant 1. \tag{D11}$$

■

Proof of Theorem. Before we start with the proof we introduce the following notations:

$$D_0=\langle A_0B_0+A_1B_1+A_2B_2\rangle, \tag{D12}$$

where $A_i=\overrightarrow{v_i^A}.\overrightarrow{\sigma}$, $B_i=\overrightarrow{v_i^B}.\overrightarrow{\sigma}$, and $C_i=\overrightarrow{v_i^C}.\overrightarrow{\sigma}$ with $\overrightarrow{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$ denote the Pauli observables. The other two expressions D_1 [Eq. (D13)] and D_2 [Eq. (D14)],

$$D_1 = \langle A_0B_2 - A_1B_0 + A_2B_1\rangle, \quad \text{(D13)}$$

$$D_2 = \langle A_0B_1 - A_1B_2 + A_2B_0\rangle, \quad \text{(D14)}$$

can be obtained from D_0 [Eq. (D12)] under some specific relabeling of inputs and outputs.

(1) For D_1: $a \rightarrow a \bigoplus_2 x$, where $a \in \{0, 1\}$ and $x \in \{0, 1, 2\}$ and $y \rightarrow y \bigoplus_3 2$ where $y \in \{0, 1, 2\}$. $\bigoplus_j$ denotes addition modulo j for any positive integer j.

(2) For D_2: $a \rightarrow a \bigoplus_2 x$ and $y \rightarrow y \bigoplus_3 1$.

With these notations, Eq. (D1) becomes modified as

$$\left|\sum_{i=0}^{2}\langle D_iC_i\rangle\right| \leqslant 3. \quad \text{(D15)}$$

Now, as Alice, Bob, and Charlie are not in the same laboratory, Alice and Bob do not know which version of the game to play. So they play the average game $\sum_{i=0}^{2} D_i$. Now there are two possible cases.

(1) Alice and Bob share a separable state.

(2) Alice and Bob share a EPR-steerable state.

For case 1, let correlations of Alice and Bob admit a LHS-LHS model, i.e., they share a separable state. Then, by the lemma, we get $D_i \leqslant 1, \forall i \in \{0, 1, 2\}$. Hence Eq. (D15) is satisfied.

For case 2, now consider the case where the correlations do not admit a LHS-LHV model (i.e., if the state is EPR steerable). By quantum predictions, the algebraic maximum of the game is 3. For instance, if Alice and Bob share the entangled state $|\psi^+\rangle$, then for a particular measurement setting $D_0 = 3$ whereas both D_1 and $D_2 = 0$. Hence the theorem. ■

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