BOTT-SAMELSON VARIETIES AND POISSON ORE EXTENSIONS

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Abstract. We show that associated to any n-dimensional Bott-Samelson variety of a complex semi-simple Lie group G, one has 2^n naturally defined Poisson brackets on the polynomial algebra $A = \mathbb{C}[z_1, \ldots, z_n]$, each an *iterated Poisson Ore extension* and one of them a *symmetric Poisson CGL extension* in the sense of Goodearl-Yakimov. We express all the Poisson brackets in terms of root strings and the structure constants of the Lie algebra of G. It follows that the coordinate rings of all *generalized Bruhat cells* can be presented as symmetric Poisson CGL extensions. The paper establishes the foundation on generalized Bruhat cells and sets up the stage for their applications to integrable systems, toric degenerations of Poisson varieties, cluster algebras and total positivity, some of which are discussed in the Introduction.

1. Introduction

1.1. **Introduction.** Poisson Ore extensions were introduced in [37] as Poisson analogs of Ore extensions in the theory of non-commutative rings. Let $\mathbf{k}$ be any field. A polynomial Poisson $\mathbf{k}$-algebra

$$A = (\mathbf{k}[z_1, z_2, \ldots, z_n],\ \{\ ,\ \})$$

is called an *iterated Poisson Ore extension (of* $\mathbf{k}$*)* if for each $1 \leq i \leq n-1$,

$$\{z_i,\ \mathbf{k}[z_{i+1}, \ldots, z_n]\} \subset z_i\mathbf{k}[z_{i+1}, \ldots, z_n] + \mathbf{k}[z_{i+1}, \ldots, z_n]. \tag{1}$$

When a split $\mathbf{k}$-torus $\mathbb{T}$ acts rationally on such an iterated Poisson Ore extension, one may impose certain compatibility conditions between the $\mathbb{T}$-action and the iterations, leading to the notion of *iterated* $\mathbb{T}$*-Poisson Ore extensions* (see §5.2). Iterated $\mathbb{T}$-Poisson Ore extensions were studied in [18, 27] for their $\mathbb{T}$-invariant Poisson prime ideals and are shown to arise as semi-classical limits of certain quantum coordinate rings.

By imposing additional symmetry conditions on iterated $\mathbb{T}$-Poisson Ore extensions, K. Goodearl and M. Yakimov introduce and study in [20, 23] *symmetric Poisson CGL extensions*, which are named after G. Cauchon, K. Goodearl, and E. Letzter and are Poisson analogs of (non-commutative) CGL extensions studied in [28]. In particular, it is shown in [23] that a presentation of a polynomial Poisson algebra A as a symmetric Poisson CGL extension gives rise to a cluster structure on A compatible with the Poisson structure in the sense of Gekhtman-Shapiro-Vainshtein [16]. CGL extensions and Poisson CGL extensions are the starting points of the remarkable body of work of K. Goodearl and M. Yakimov on quantum and classical cluster algebras related to Lie theory [20, 21, 22, 23] and especially on the quantum and classical Berenstein-Zelevinsky conjectures on the equality of the cluster algebras and the upper cluster algebras defined by the Berenstein-Fomin-Zelevinsky initial seeds associated to double Bruhat cells in complex semi-simple Lie groups [2].

In this paper, we give a systematic construction of a class of iterated T-Poisson Ore extensions associated to any connected complex semi-simple Lie group G and a maximal torus $T \subset G$. Briefly, associated to any sequence $\mathbf{u} = (s_1, s_2, \ldots, s_n)$ of simple reflections in the Weyl group W of G, one has the n-dimensional Bott-Samelson variety $Z_{\mathbf{u}}$ and a so-called *standard Poisson structure* π_n on $Z_{\mathbf{u}}$ (see §2.2). On the other hand, $Z_{\mathbf{u}}$ has a natural atlas consisting of 2^n coordinate charts, one chart $\mathcal{O}^\gamma$ for each *subexpression* γ of $\mathbf{u}$, and all parametrized by $\mathbb{C}^n$ (see §3.1). For each coordinate chart $\mathcal{O}^\gamma$, we prove that the restriction of π_n to $\mathcal{O}^\gamma$ gives rise to an iterated T-Poisson Ore extension $(\mathbb{C}[z_1, \ldots, z_n], \{\, , \}_\gamma)$, and we give the explicit formulas for $\{\, , \}_\gamma$ in terms of the root strings and the structure constants of the Lie algebra $\mathfrak{g}$ of G. For $\gamma = \mathbf{u}$, we show that $(\mathbb{C}[z_1, \ldots, z_n], \{\, , \}_{\mathbf{u}})$ is a symmetric Poisson CGL extension. Consequently, we prove that the coordinate rings of all generalized Bruhat cells, introduced in [31, §1.3], have natural presentations as symmetric Poisson CGL extensions.

The origin of the standard Poisson structures Bott-Samelson varieties is the so-called *standard multiplicative Poisson structure* $\pi_{\rm st}$ on G, and the Poisson Lie group $(G, \pi_{\rm st})$ is the semi-classical limit of the much studied quantum group [7, 10, 12] associated to G. Results of this paper have applications to quantum groups, integrable systems, cluster algebras, total positivity, and toric degenerations of some Poisson varieties associated to G. In the next §1.2, we give more details on the results of the paper. In §1.3 - §1.5 of the Introduction, we discuss applications.

1.2. **Bott-Samelson varieties and iterated Poisson Ore extensions.** Let G again be a connected complex semi-simple Lie group with a fixed Borel subgroup B and a maximal torus $T \subset B$, and let $\mathfrak{g}$, $\mathfrak{b}$, and $\mathfrak{h}$ be the respective Lie algebras of G, B, and T. Let $\Delta_+ \subset \mathfrak{h}^*$ be the set of positive roots determined by $\mathfrak{b}$ and $\Gamma \subset \Delta_+$ the set of simple roots. Let $W = N_G(T)/T$ be the Weyl group, where $N_G(T)$ is the normalizer of T in G. For $\alpha \in \Gamma$, let $s_\alpha \in W$ be the corresponding simple reflection.

Let $\mathbf{u} = (s_1, s_2, \cdots, s_n)$ be any sequence of simple reflections, i.e, any word, in W, and for $1 \leq i \leq n$, let $P_{s_i} = B \cup Bs_iB$, the parabolic subgroup of G associated to B and s_i. Consider the product manifold $P_{s_1} \times \ldots \times P_{s_n}$ with the right action of B^n by

$$(p_1, p_2, \ldots, p_n) \cdot (b_1, b_2, \ldots, b_n) = (p_1 b_1,\, b_1^{-1} p_2 b_2,\, \ldots,\, b_{n-1}^{-1} p_n b_n), \qquad p_i \in P_{s_i},\, b_i \in B.$$

The quotient space, denoted by $Z_{\mathbf{u}} = P_{s_1} \times_B \ldots \times_B P_{s_n}/B$, is the Bott-Samelson variety associated to $\mathbf{u}$. For $(p_1, \ldots, p_n) \in P_{s_1} \times \ldots \times P_{s_n}$, let $[p_1, \ldots, p_n] \in Z_{\mathbf{u}}$ denote the image of $(p_1, \ldots, p_n)$ in $Z_{\mathbf{u}}$. Multiplication in the group G gives a well-defined map

$$\mu:\ Z_{\mathbf{u}} \longrightarrow G/B: \quad \mu([p_1,\, p_2,\, \ldots,\, p_n]) = p_1 p_2 \cdots p_n B/B. \tag{2}$$

When $\mathbf{u}$ is a reduced word, μ is a resolution of singularities of the Schubert variety $\overline{BuB/B}$ in G/B, where $u = s_1 s_2 \cdots s_n \in W$. Bott-Samelson varieties have been studied extensively in the literature and play an important role in geometric representation theory. See, for example, [3, 4] and the references therein.

It is well-known (see, for example, [7, §1.5] or [12, §4.4]) that the choice of the pair (B, T), together with that of a symmetric non-degenerate invariant bilinear form $\langle\, , \rangle$ on $\mathfrak{g}$, give rise to a *standard multiplicative holomorphic Poisson structure* $\pi_{\rm st}$, making $(G, \pi_{\rm st})$ into the standard complex semi-simple Poisson Lie group (see §2.1). Every parabolic subgroup of G containing B is a Poisson submanifold of $(G, \pi_{\rm st})$. Consequently, for any sequence $\mathbf{u} = (s_1, \ldots, s_n)$ of simple reflections in W, the restriction to $P_{s_1} \times \ldots \times P_{s_n} \subset G^n$ of the n-fold product Poisson structure $\pi_{\rm st}^n = \pi_{\rm st} \times \ldots \times \pi_{\rm st}$ on G^n projects to a well-defined Poisson structure, denoted by π_n, on the Bott-Samelson variety $Z_{\mathbf{u}}$ (see §2.2 for details). We refer to π_n as *a standard Poisson structure on* $Z_{\mathbf{u}}$.

Fixing root vectors $\{e_\alpha : \alpha \in \Gamma\}$ and extending them to a Chevalley basis of $\mathfrak{g}$, one obtains an atlas

$$\mathcal{A} = \{(\Phi^\gamma : \mathbb{C}^n \longrightarrow \mathcal{O}^\gamma \subset Z_{\mathbf{u}}) : \ \gamma \in \Upsilon_{\mathbf{u}}\}, \tag{3}$$

on $Z_{\mathbf{u}}$, where $\Upsilon_{\mathbf{u}}$ is the set of all the 2^n *subexpressions* of $\mathbf{u}$ (see §3.1). While referring to §3.1 for the precise definition of the parametrization $\Phi^\gamma : \mathbb{C}^n \to \mathcal{O}^\gamma$ for an arbitrary $\gamma \in \Upsilon_{\mathbf{u}}$, we point out here that for $\gamma = \mathbf{u}$,

$$\mathcal{O}^{\mathbf{u}} = \varpi(Bs_1B \times Bs_2B \times \cdots \times Bs_nB) \subset Z_{\mathbf{u}}, \tag{4}$$

where $\varpi : P_{s_1} \times \cdots \times P_{s_n} \to Z_{\mathbf{u}}$ is the projection. The coordinate chart $\Phi^{\mathbf{u}} : \mathbb{C}^n \to \mathcal{O}^{\mathbf{u}}$ will play a very special role in this paper and for applications of the results in this paper.

In §3.2, we give our first formula (Lemma 3.2) of the Poisson structure π_n in each coordinate chart $\Phi^\gamma : \mathbb{C}^n \to \mathcal{O}^\gamma$ in terms of certain vector fields on Bott-Samelson subvarieties of $Z_{\mathbf{u}}$. It is also shown in §3.3 that π_n is log-canonical in some of the coordinate charts. The first main result of the paper is Theorem 4.14, in which we further express the vector fields in Lemma 3.2 in terms of root strings and the structure constants of $\mathfrak{g}$. In particular, π_n is algebraic in every coordinate chart $\Phi^\gamma : \mathbb{C}^n \to \mathcal{O}^\gamma$.

For $\gamma \in \Upsilon_{\mathbf{u}}$, let $\{\, ,\, \}_\gamma$ be the Poisson bracket on the polynomial algebra $\mathbb{C}[z_1, \ldots, z_n]$ defined by π_n through the parametrization $\Phi^\gamma : \mathbb{C}^n \to \mathcal{O}^\gamma$. As consequences of Theorem 4.14, we prove in §5 the following prominent features of the Poisson polynomial algebras $(\mathbb{C}[z_1, \ldots, z_n], \{\, ,\, \}_\gamma)$ for $\gamma \in \Upsilon_{\mathbf{u}}$:

1) The Poisson polynomial algebra $(\mathbb{C}[z_1, \ldots, z_n], \{\, ,\, \}_\gamma)$ is an *iterated T-Poisson Ore extension* and is of the form

$$\{z_i,\ z_k\}_\gamma = c_{i,k} z_i z_k + b_i(z_k), \qquad 1 \leq i < k \leq n, \tag{5}$$

where $c_{i,k} \in \mathbb{C}$ and b_i is a derivation of $\mathbb{C}[z_{i+1}, \ldots, z_k]$. When $\gamma = \mathbf{u}$, the iterated T-Poisson Ore extension $(\mathbb{C}[z_1, \ldots, z_n], \{\, ,\, \}_{\mathbf{u}})$ is a *symmetric Poisson CGL extension* in the terminology of [23], and in particular

$$b_i(z_k) \in \mathbb{C}[z_{i+1}, \ldots, z_{k-1}], \qquad \forall\ 1 \leq i < k \leq n. \tag{6}$$

Precise statements are given in Theorem 5.12. Details on the constants $c_{i,k}$ and the polynomials $b_i(z_k)$ for the case of $\gamma = \mathbf{u}$ are summarized in Theorem 5.15.

2) Choose the bilinear form $\langle\, ,\, \rangle$ on $\mathfrak{g}$ such that $\frac{\langle \alpha, \alpha \rangle}{2} \in \mathbb{Z}$ for each root α. Then for any $\gamma \in \Upsilon_{\mathbf{u}}$ and $1 \leq i < k \leq n$, the polynomial $\{z_i, z_k\}_\gamma \in \mathbb{C}[z_1, \ldots, z_n]$ has integer coefficients, so $\{\, ,\, \}_\gamma$ defines a Poisson bracket on $\mathbb{Z}[z_1, \ldots, z_n]$. Consequently, each $\gamma \in \Upsilon_{\mathbf{u}}$ gives rise to an iterated Poisson Ore extension $(\mathbf{k}[z_1, \ldots, z_n], \{\, ,\, \}_\gamma)$ of any field $\mathbf{k}$ of arbitrary characteristic. In particular, when the shortest roots α satisfy $\langle \alpha, \alpha \rangle = 2$, associated to $\gamma = \mathbf{u}$ one then has a symmetric Poisson CGL extension of any field $\mathbf{k}$ with $\mathrm{char}(\mathbf{k}) \neq 2, 3$. See Theorem 5.21 and Remark 5.22.

We also remark that our explicit formula for $\{\, ,\, \}_\gamma$ in Theorem 5.12 made it possible for the first author to write a computer program using GAP [15] which computes the Poisson bracket $\{\, ,\, \}_\gamma$ on $\mathbb{Z}[z_1, \ldots, z_n]$ for *any triple* $(G, \mathbf{u}, \gamma)$, where G is any connected complex simple Lie group (the results only depend on the isogeny class of G), $\mathbf{u}$ is any length n sequence of simple reflections in the Weyl group of G, and γ is any subexpression of $\mathbf{u}$. Some examples, such as when $\mathfrak{g}$ is of type G_2, are given in §5.5.

When $\mathbf{u} = (s_1, s_2, \ldots, s_n)$ is a reduced word of $u = s_1 s_2 \cdots s_n \in W$, the map μ in (2) restricts to an isomorphism between $\mathcal{O}^{\mathbf{u}}$ and the Bruhat (or Schubert) cell BuB/B in G/B. On the other hand, as B is a Poisson Lie subgroup of $(G, \pi_{\rm st})$, the Poisson structure $\pi_{\rm st}$ on G projects to a well-defined Poisson structure on G/B, denoted as $\pi_{G/B}$, with respect to which BuB/B is a Poisson submanifold [19]. It then follows from the definition of π_n and the multiplicativity of $\pi_{\rm st}$ that

$$\mu|_{\mathcal{O}^{\mathbf{u}}}: \ (\mathcal{O}^{\mathbf{u}}, \ \pi_n) \longrightarrow (BuB/B, \ \pi_{G/B}) \tag{7}$$

is a Poisson isomorphism. Referring to the coordinates $(z_1, \ldots, z_n)$ on $\mathcal{O}^{\mathbf{u}}$ as the *Bott-Samelson coordinates* on BuB/B (defined by the reduced expression $u = s_1 s_2 \cdots s_n$ via $\mu|_{\mathcal{O}^{\mathbf{u}}}$), Theorem 5.12 then says that the coordinate ring of $(BuB/B, \pi_{G/B})$ becomes a symmetric Poisson CGL extension in the Bott-Samelson coordinates on BuB/B.

In applications, however, it is crucial that we have a symmetric Poisson CGL extension associated to an arbitrary (i.e., not necessarily reduced) word$\mathbf{u} = (s_1, s_2, \ldots, s_n)$ in W, as we will see in §1.4 when we apply Theorem 5.12 to generalized Bruhat cells.

In the remaining §1.3 - §1.5 of the Introduction, we discuss applications of results in this paper to quantum groups and, via generalized Bruhat cells, to integrable systems, cluster algebras, total positivity, and toric degenerations of Poisson varieties.

1.3. **The Poisson analog of the Levendorskii-Soibelman strengthening law.** Consider again the case when $\mathbf{u} = (s_1, s_2, \ldots, s_n)$ is reduced, and let $u = s_1 s_2 \cdots s_n \in W$. It is known (see [40, Lemma 4.3]) that $(BuB/B, -\pi_{G/B})$ is the semi-classical analog of the quantum Schubert cell $\mathcal{U}^-[u]$ introduced by De Concini-Kac-Procesi [8] and Lusztig [33], and that the coordinates $(z_1, \ldots, z_n)$ on $\mathcal{O}^{\mathbf{u}}$, now regarded as regular functions on BuB/B via the isomorphism $\mu|_{\mathcal{O}^{\mathbf{u}}}: \mathcal{O}^{\mathbf{u}} \cong BuB/B$, are the semi-classical analogs of the Lusztig root vectors $F_{\beta_1}, \ldots, F_{\beta_n} \in \mathcal{U}^-[u]$, where $\beta_l = s_1 s_2 \cdots s_{l-1}(\alpha_l)$ for $1 \leq l \leq n$ (see [22, §9.2]). Recall [22, §9.2] [5, I.6.10] that the Lusztig root vectors $F_{\beta_1}, \ldots, F_{\beta_n}$ satisfy the *Levendorskii-Soibelman straightening law*

$$F_{\beta_i} F_{\beta_k} - q^{-\langle \beta_i, \beta_k \rangle} F_{\beta_k} F_{\beta_i} = \sum_{(j_{i+1}, \ldots, j_{k-1}) \in \mathcal{J}_{i,k}} \xi_{j_{i+1}, \ldots, j_{k-1}} F_{\beta_{i+1}}^{j_{i+1}} \cdots F_{\beta_{k-1}}^{j_{k-1}}, \quad 1 \leq i < k \leq n,$$

where $\mathcal{J}_{i,k}$ is a finite subset of $\mathbb{N}^{k-i-1}$ and $\xi_{j_{i+1}, \ldots, j_{k-1}} \in \mathbb{Q}[q, q^{-1}]$ is non-zero for $(j_{i+1}, \ldots, j_{k-1}) \in \mathcal{J}_{i,k}$. The fact that the Poisson bracket $\{\, , \,\}_{\mathbf{u}}$ is of the form

$$\{z_i, \, z_k\}_{\mathbf{u}} = -\langle \beta_i, \beta_k \rangle z_i z_k + b_i(z_k), \quad 1 \leq i < k \leq n,$$

with $b_i(z_k) \in \mathbb{C}[z_{i+1}, \ldots, z_{k-1}]$ is thus the Poisson analog of the Levendorskii-Soibelman straightening law. However, while we express all the polynomials $b_i(z_k)$ explicitly in terms of roots strings and structure constants of $\mathfrak{g}$ in Theorem 5.15, there are no such general explicit descriptions, as far as we know, neither for the index set $\mathcal{J}_{i,k}$ nor for the non-zero $\xi_{j_{i+1}, \ldots, j_{k-1}} \in \mathbb{Q}[q, q^{-1}]$ when $(j_{i+1}, \ldots, j_{k-1}) \in \mathcal{J}_{i,k}$ themselves (see, however, [9, Appendix, (A4)-(A8)] for some concrete formulas when u is the longest element in W for rank 2 groups, and [32, 34] for some other special cases). It would thus be very interesting to seek for a *quantization* of the formulas in Theorem 5.15 to obtain explicit expressions of the Levendorskii-Soibelman straightening law, and see in particular how the q-analogs of the binomial coefficients in Theorem 5.15 may appear in such formulas. Partial results in this direction have been obtained in [35].

1.4. **Symmetric Poisson CGL extensions associated to generalized Bruhat cells.** With the notation as in §1.1, for any integer $n \geq 1$, let B^n act on G^n by

$$(g_1, g_2 \dots g_n) \cdot (b_1, b_2, \dots, b_n) = (g_1 b_1,\, b_1^{-1} g_2 b_2,\, \dots,\, b_{n-1}^{-1} g_n b_n), \qquad g_i \in G,\, b_i \in B,$$

and denote the quotient manifold by

$$F_n = G \times_B \cdots \times_B G/B. \tag{8}$$

It is shown in [30, §7.1] (see also §2.2) that the n-fold product Poisson structure $\pi_{\rm st}^n$ on G^n projects to a well-defined Poisson structure on F_n, which will also be denoted by π_n. Note that for any sequence $\mathbf{u} = (s_1, \dots, s_n)$ of simple reflections in W, the Bott-Samelson variety $Z_{\mathbf{u}}$ is isomorphic to a closed submanifold of F_n under the embedding $P_{s_1} \times \cdots \times P_{s_n} \subset G^n$. As $P_{s_1} \times \cdots \times P_{s_n}$ is a Poisson submanifold of G^n with respect to $\pi_{\rm st}^n$, it follows from the definitions that $Z_{\mathbf{u}}$, with the Poisson structure π_n defined in §1.1, is a Poisson submanifold of (F_n, π_n). Note also that the T-action on the first factor of G^n by left translation descends to a T-action on F_n preserving π_n.

For an arbitrary sequence $\mathbf{u} = (u_1, \dots, u_n)$ in the Weyl group W, where the u_i's are not necessarily simple reflections, the image of $Bu_1B \times \cdots \times Bu_nB$ in F_n, denoted as

$$B\mathbf{u}B/B = (Bu_1B) \times_B \cdots \times_B (Bu_nB)/B \subset F_n,$$

is called a *generalized Bruhat cell* in [31, §1.3]. It follows from the Bruhat decomposition $G = \bigsqcup_{u \in W} BuB$ of G that one has the decomposition

$$F_n = \bigsqcup_{\mathbf{u} \in W^n} B\mathbf{u}B/B \tag{9}$$

of F_n into the disjoint union of generalized Bruhat cells. As each BuB, where $u \in W$, is a Poisson submanifold of G with respect to $\pi_{\rm st}$, each generalized Bruhat cell $B\mathbf{u}B/B$ is a T-invariant Poisson submanifold of (F_n, π_n).

A generalized Bruhat cell of the form $B(s_1, \dots, s_n)B/B \subset F_n$, where each s_i is a simple reflection, is said to be of *Bott-Samelson type* [31, §1.3]. In the notation of the current paper, a generalized Bruhat cell $B(s_1, \dots, s_n)B/B$ in F_n of Bott-Samelson type is nothing but the affine chart $\mathcal{O}^{\mathbf{u}}$ in the Bott-Samelson variety $Z_{\mathbf{u}} \subset F_n$, where $\mathbf{u} = (s_1, \dots, s_n)$. See (4). Given an arbitrary $\mathbf{u} = (u_1, \dots, u_n) \in W^n$, choose any reduced decomposition $u_i = s_{i,1} s_{i,2} \cdots s_{i,l(u_i)}$ for each u_i, where $l : W \to \mathbb{N}$ is the length function of W, and consider the sequence

$$\tilde{\mathbf{u}} = (s_{1,1},\, \dots,\, s_{1,l(u_1)},\, s_{2,1},\, \dots,\, s_{2,l(u_2)},\, \dots, s_{n,1},\, \dots,\, s_{n,l(u_n)})$$

of simple reflections of length $l(\mathbf{u}) = l(u_1) + \cdots + l(u_n)$. Then the multiplication map on G induces a T-equivariant Poisson isomorphism

$$(Z_{\tilde{\mathbf{u}}},\, \pi_{l(\mathbf{u})}) \supset (\mathcal{O}^{\tilde{\mathbf{u}}},\, \pi_{l(\mathbf{u})}) = (B\tilde{\mathbf{u}}B/B,\, \pi_{l(\mathbf{u})}) \xrightarrow{\sim} (B\mathbf{u}B/B,\, \pi_n) \subset (F_n,\, \pi_n) \tag{10}$$

(see [31, §1.3]). Thus, as Poisson manifolds, every generalized Bruhat cell $B\mathbf{u}B/B$ is Poisson isomorphic to one of Bott-Samelson type. We will refer to the coordinates $(z_1, \dots, z_{l(\mathbf{u})})$ on $\mathcal{O}^{\tilde{\mathbf{u}}} = B\tilde{\mathbf{u}}B/B$ defined in §3.1 of the present paper as *Bott-Samelson coordinates* on $B\mathbf{u}B/B$ (via the isomorphism in (10)). Theorem 5.12, applied to $\mathcal{O}^{\tilde{\mathbf{u}}}$, then immediately leads to the following conclusion on generalized Bruhat cells.

Theorem 1.1. *For any generalized Bruhat cell $B\mathbf{u}B/B$, where $\mathbf{u} = (u_1, \ldots, u_n) \in W^n$, the standard Poisson structure π_n on $B\mathbf{u}B/B$ makes its coordinate ring into a symmetric Poisson CGL extension in any Bott-Samelson coordinates $(z_1, \ldots, z_{l(\mathbf{u})})$ on $B\mathbf{u}B/B$; the corresponding Poisson bracket on $\mathbb{C}[z_1, \ldots, z_{l(\mathbf{u})}]$ is explicitly given in Theorem 5.15 (applied to $\tilde{\mathbf{u}}$).*

We also point out that for an arbitrary generalized Bruhat cell $B\mathbf{u}B/B$, where $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in W^n$, the T-orbits of symplectic leaves of π_n in $B\mathbf{u}B/B$, also called *T-leaves*, are described in [31, Theorem 1.1]. Namely, the T-leaves of π_n in $B\mathbf{u}B/B$ are precisely all the submanifolds of $B\mathbf{u}B/B$ of the form

$$R_w^{\mathbf{u}} = \{[g_1, g_2, \ldots, g_n] \in B\mathbf{u}B/B : \ g_1 g_2 \cdots g_n \in B_- wB\},$$

where $w \in W$, and $w \leq u_1 * u_2 * \cdots * u_n$, with $*$ being the monoidal product on W. Here B_- is the Borel subgroup of G such that $B \cap B_- = T$. Moreover, the dimension of every symplectic leaf of π_n in $R_w^{\mathbf{u}}$ is shown in [31, Theorem 1.1] to be equal to

$$l(\mathbf{u}) - l(w) - \dim \ker(1 + u_1 u_2 \cdots u_n w^{-1}),$$

where $1 + u_1 u_2 \cdots u_n w^{-1}$ denotes the linear operator on the Lie algebra $\mathfrak{h}$ of T given by $x \mapsto x + u_1 u_2 \cdots u_n w^{-1}(x)$, $x \in \mathfrak{h}$. The *leaf-stabilizer subalgebra of* $\mathfrak{h}$ in $R_w^{\mathbf{u}}$ is also explicitly described in [31, Theorem 1.1]. We refer to [31, Theorem 1.1] for more detail.

We regard Theorem 1.1 and [31, Theorem 1.1] as two basic results on the standard Poisson structures on generalized Bruhat cells. As we will explain in the next §1.5, iterated Poisson Ore extensions, or the more restrictive symmetric Poisson CGL extensions, can be associated with many important varieties in Lie theory through generalized Bruhat cells. These results set the foundation for applications of generalized Bruhat cells to integrable systems, cluster algebras, total positivity, and toric degenerations of Poisson varieties.

1.5. **Other symmetric Poisson CGL extensions through generalized Bruhat cells.** Recall [13] that double Bruhat cell in G are defined as $G^{u,v} = (BuB) \cap (B_- vB_-)$, where $u, v \in W$. Fomin and Zelevinsky introduced in [13] certain regular functions on $G^{u,v}$, called *twisted generalized minors*, which play crucial roles in the theory of total positivity and cluster algebra structures [2] on $G^{u,v}$. For $u = v$, Kogan and Zelevinsky introduced in [26] an integrable system on the Poisson manifold $(G^{u,u}, \pi_{\rm st})$ formed by some twisted generalized minors on $G^{u,u}$. In [29], a certain open Poisson embedding

$$F^{u,v}: \quad (G^{u,v}, \ \pi_{\rm st}) \longrightarrow (T \times (B(v^{-1}, u)B/B), \ 0 \bowtie \pi_2), \tag{11}$$

called the *Fomin-Zelevinsky embedding*, is introduced, where $0 \bowtie \pi_2$ is the sum of the product Poisson structure $0 \times \pi_2$ and a *mixed term* defined using the T-action on the generalized Bruhat cell $B(v^{-1}, u)B/B$ by left translation. Defining *Bott-Samelson coordinates* on $G^{u,v}$ to be the combination of any (algebraic) coordinates on T and Bott-Samelson coordinates on $B(v^{-1}, u)B/B$ (defined using reduced words for u and v as in §3.1 of the present paper), it is shown in [29] that all the Fomin-Zelevinsky twisted generalized minors on $G^{u,v}$ become certain distinguished polynomials in the Bott-Samelson coordinates. The fact that the Poisson structure π_2 on $B(v^{-1}, u)B/B$ is a symmetric Poisson CGL extension in the Bott-Samelson coordinates is then used in [29] to prove that the Hamiltonian vector fields of all the Fomin-Zelevinsky twisted generalized minors on every $G^{u,v}$ are *complete* in the sense that all of their integral curves are defined on the whole of $\mathbb{C}$. Consequently all the Hamiltonian flows of the Kogan-Zelevinsky integrable system on each $G^{u,u}$ are defined on the whole of $\mathbb{C}$.

In general, we say that an n-dimensional complex algebraic Poisson manifold (P, π) is an iterated Poisson Ore extension (of a point) (resp. a symmetric Poisson CGL extension (of a point)) if there exists an isomorphism $P \cong \mathbb{C}^n$ through which the coordinate ring of (P, π) becomes an iterated Poisson Ore extension (resp. a symmetric Poisson CGL extension) of $\mathbb{C}$. A Poisson manifold (Z, π_Z) is said to be *paved (resp. covered) by iterated Poisson Ore extensions* if it is the disjoint union of (resp. has an open cover by) iterated Poisson Ore extensions. Theorem 5.12, then, says that every Bott-Samelson variety $Z_{\mathbf{u}}$ with the standard Poisson structure π_n is *covered* by iterated Poisson Ore extensions, while the Poisson manifold (F_n, π_n) is *paved* by symmetric Poisson CGL extensions, namely by the generalized Bruhat cells $B\mathbf{u}B/B \subset F_n$, as in (9). In [41], results in this paper on generalized Bruhat cells are used to show that every orbit in the double flag variety $G/B \times G/B_-$ for the diagonal G-action is covered by symmetric Poisson CGL extensions. Note that examples are the closed orbit, which is isomorphic to G/B, and the open orbit, which is isomorphic to G/T.

For an n-dimensional smooth Poisson variety (Z, π) covered by iterated Poisson Ore extensions, one can regard (Z, π) as being glued together by *Poisson Ore charts.* On the other hand, the changes of coordinates between these coordinate charts are, in general, highly non-trivial birational maps from $\mathbb{C}^n$ to itself. See Example 4.12 for an example for Bott-Samelson varieties. It seems a miracle that such complicated birational maps from $\mathbb{C}^n$ to $\mathbb{C}^n$ in fact transform one iterated Poisson Ore extension to another (see [11, Appendix A] for some direct computations related to Example 4.12). It would be very interesting to see whether the changes of coordinates between two arbitrary Poisson Ore charts are compositions of some simpler *one-step mutations of iterated Poisson Ore extensions.* Results in this paper on Bott-Samelson varieties provide testing ground for answering such questions.

For Poisson varieties (Z, π) that can be covered by symmetric Poisson CGL extensions, such as the diagonal G-orbits in the double flag variety, one may consider the cluster structure, and the resulting total positivity [14], on each Poisson Ore chart defined by the corresponding symmetric Poisson CGL extension using the Goodearl-Yakimov theory [23], and one can ask how they glue together to give some *global cluster structure and global total positivity* on Z. These questions will be investigated elsewhere. We point out for now that Theorem 1.1 and Theorem 5.12 from this paper on the symmetric Poisson CGL extensions associated to generalized Bruhat cells will be crucial for such a project.

Finally, we remark that the symmetric Poisson CGL extension $(\mathbb{C}[z_1, z_2, \ldots, z_n], \{\,,\,\}_{\mathbf{u}})$ associated to the affine chart $\mathcal{O}^{\mathbf{u}}$ of the Bott-Samelson variety $Z_{\mathbf{u}}$ is also intimately related to toric degenerations of $Z_{\mathbf{u}}$ through tropical geometry. More precisely, consider $(Z_{\mathbf{u}}, \pi_n)$ with its cover $\mathcal{A} = \{(\Phi^\gamma : \mathbb{C}^n \longrightarrow \mathcal{O}^\gamma) : \gamma \in \Upsilon_{\mathbf{u}}\}$ by iterated Poisson Ore extensions. It is shown in [36] one has an isomorphism of cones

$$\mathcal{C}_{\rm toric}(Z_{\mathbf{u}}, \mathcal{A}) \cong \mathcal{C}_{\rm log-can}(\mathbb{C}[z_1, z_2, \ldots, z_n], \{\,,\,\}_{\mathbf{u}}), \tag{12}$$

by an element in $GL(n, \mathbb{Z})$, where $\mathcal{C}_{\rm toric}(Z_{\mathbf{u}}, \mathcal{A})$ is a certain *toric degeneration cone*, whose integer points are the "directions" in which the Bott-Samelson variety $Z_{\mathbf{u}}$ can be degenerated to a toric variety via re-scalings of the coordinates in the coordinate charts in $\mathcal{A}$, and $\mathcal{C}_{\log-\mathrm{can}}(\mathbb{C}[z_1, z_2, \ldots, z_n], \{\, ,\, \}_{\mathbf{u}})$ is the *log-canonical degeneration cone*, introduced in [1] by A. Alexseev and I. Davydenkova, of the polynomial Poisson algebra $(\mathbb{C}[z_1, z_2, \ldots, z_n], \{\, ,\, \}_{\mathbf{u}})$, whose integer points are the "directions" in which the Poisson bracket $\{\, ,\, \}_{\mathbf{u}}$ can be degenerated to its log-canonical term. The proof of (12) in [36] uses in a very essential way the explicit formulas for the Poisson bracket $\{\, ,\, \}_{\mathbf{u}}$ given in Theorem 5.15.

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1.7. **Notation.** Continuing with the notation from §1.2, let $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ be the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. For $\alpha \in \Delta$, let h_α be the unique element in $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(h_\alpha) = 2$, and let $\alpha^\vee : \mathbb{C}^\times \to T$ be the co-character of T defined by h_α. Let $\Delta_+ \subset \Delta$ be the set of positive roots determined by $\mathfrak{b}$, and let $\mathfrak{b}_- = \mathfrak{h} + \sum_{\alpha\in\Delta_+} \mathfrak{g}_{-\alpha}$. The Borel subgroup of G with Lie algebra $\mathfrak{b}_-$ is denoted by B_-.

Let $\alpha \in \Delta_+$. If $e_\alpha \in \mathfrak{g}_\alpha$ and $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ are such that $[e_\alpha, e_{-\alpha}] = h_\alpha$, we call $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ an $\mathfrak{sl}(2,\mathbb{C})$-triple for α. Clearly, any non-zero $e_\alpha \in \mathfrak{g}_\alpha$ uniquely determines an $\mathfrak{sl}(2,\mathbb{C})$-triple $\{h_\alpha, e_\alpha, e_{-\alpha}\}$, and every other $\mathfrak{sl}(2,\mathbb{C})$-triple for α is of the form $\{h_\alpha, \lambda e_\alpha, \lambda^{-1} e_{-\alpha}\}$ for a unique $\lambda \in \mathbb{C}$. Given an $\mathfrak{sl}(2,\mathbb{C})$-triple $\{h_\alpha, e_\alpha, e_{-\alpha}\}$, let θ_α denote both the Lie algebra homomorphism $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ defined by

$$\theta_\alpha : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longmapsto h_\alpha, \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longmapsto e_\alpha, \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \longmapsto e_{-\alpha},$$

and the corresponding Lie group homomorphism $\mathrm{SL}(2,\mathbb{C}) \to G$, so that

$$\alpha^\vee(t) = \theta_\alpha\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right), \qquad t \in \mathbb{C}^\times,$$

and one also has the one-parameter subgroups $u_{\pm\alpha} : \mathbb{C} \to G$ given by

$$u_\alpha(z) = \theta_\alpha\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) = \exp(z e_\alpha), \quad u_{-\alpha}(z) = \theta_\alpha\left(\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\right) = \exp(z e_{-\alpha}), \qquad z \in \mathbb{C}.$$

Let $W = N_G(T)/T$ be again the Weyl group of (G, T). For $\alpha \in \Delta_+$, let $s_\alpha \in W$ be the reflection in W determined by α, and if $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ is an $\mathfrak{sl}(2,\mathbb{C})$-triple for α, let $\dot{s}_\alpha$ be the representative of s_α in $N_G(T)$ given by

$$\dot{s}_\alpha = u_\alpha(-1) u_{-\alpha}(1)\, u_\alpha(-1) \in N_G(T). \tag{13}$$

For a complex algebraic torus $\mathbb{T}$ with Lie algebra $\mathfrak{t}$ and for $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^\times)$, the differential at the identity element of $\mathbb{T}$, which is an element in $\mathfrak{t}^*$, is also denoted by λ. The values of λ on $t \in \mathbb{T}$ and on $x \in \mathfrak{t}$ are respectively denoted as $t^\lambda \in \mathbb{C}^\times$ and $\lambda(x) \in \mathbb{C}$. For a vector space V and $u, v \in V$, we also use the convention that

$$u \wedge v = u \otimes v - v \otimes u \in \wedge^2 V \subset V \otimes V.$$

2. Definition of the Poisson structure π_n on $Z_{\mathbf{u}}$

2.1. **The standard semi-simple Poisson Lie group** $(G, \pi_{\rm st})$**.** Recall from [7, 12] that a Poisson bivector field π_L on a Lie group L is said to be multiplicative if

$$(L \times L,\ \pi_L \times \pi_L) \longrightarrow (L,\ \pi_L):\ \ (l_1,\ l_2) \longmapsto l_1 l_2, \quad l_1, l_2 \in L,$$

is a Poisson map, where $\pi_L \times \pi_L$ is the product Poisson structure on $L \times L$. A Poisson Lie group is a pair (L, π_L), where L is a Lie group and π_L is a multiplicative Poisson bivector field on L. A Poisson Lie subgroup of a Poisson Lie group (L, π_L) is a Lie subgroup L_1 of L which is also a Poisson submanifold with respect to π_L, and in this case $(L_1, \pi_L|_{L_1})$, or simply denoted as (L_1, π_L), is a Poisson Lie group.

Let G be a connected complex semi-simple Lie group and let the notation be as in §1.7. Fix, furthermore, a symmetric non-degenerate invariant bilinear form $\langle\, ,\, \rangle$ on $\mathfrak{g}$, and denote also by $\langle\, ,\, \rangle$ the induced bilinear form on $\mathfrak{h}^*$. Define $\Lambda \in \wedge^2\mathfrak{g}$ by

$$\Lambda = \sum_{\alpha \in \Delta_+} \frac{\langle \alpha, \alpha\rangle}{2} e_{-\alpha} \wedge e_\alpha \in \wedge^2\mathfrak{g},$$

where for each $\alpha \in \Delta_+$, $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ is an $\mathfrak{sl}(2, \mathbb{C})$-triple for α. Note that for any $\alpha \in \Delta_+$, the element $e_{-\alpha} \wedge e_\alpha \in \wedge^2\mathfrak{g}$ stays the same if the $\mathfrak{sl}(2, \mathbb{C})$-triple $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ is changed to $\{h_\alpha, \lambda e_\alpha, \frac{1}{\lambda} e_{-\alpha}\}$ for $\lambda \in \mathbb{C}^\times$. Consequently, the element $\Lambda \in \wedge^2\mathfrak{g}$ depends on $\langle\, ,\, \rangle$ but *not* on the choices of the $\mathfrak{sl}(2, \mathbb{C})$-triples for the positive roots. Let $\pi_{\rm st}$ be the bivector field on G given by

$$\pi_{\rm st}(g) = l_g(\Lambda) - r_g(\Lambda), \quad g \in G,$$

where for $g \in G$, l_g and r_g respectively denote the left and right translations on G by g. Then $(G, \pi_{\rm st})$ is a Poisson Lie group, called a *standard complex semi-simple Poisson Lie group* [12, §4.4]. Moreover, the Poisson structure $\pi_{\rm st}$ is invariant under the action of T by left translation, and the T-orbits of symplectic leaves, also called T-leaves, of $\pi_{\rm st}$ are precisely the so-called double Bruhat cells $(BuB) \cap (B_-vB_-)$, where $u, v \in W$ (see [24, 26]). In particular, every BuB, where $u \in W$, is a Poisson submanifold of $(G, \pi_{\rm st})$, and every parabolic subgroup P of G containing B, being a union of (B, B)-double cosets in G, is a Poisson Lie subgroup of $(G, \pi_{\rm st})$. Similar statements hold if B is replaced by B_-.

We state another important property of $(G, \pi_{\rm st})$: let α be a simple root and consider the group homomorphism $\theta_\alpha : SL(2, \mathbb{C}) \to G$ in §1.7 corresponding to any $sl(2, \mathbb{C})$-triple $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ for α. Equip $SL(2, \mathbb{C})$ with the multiplicative Poisson structure

$$\pi_{SL(2,\mathbb{C})}(g) = l_g(\Lambda_0) - r_g(\Lambda_0), \quad g \in SL(2, \mathbb{C}), \tag{14}$$

where $\Lambda_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \wedge^2\mathfrak{sl}(2, \mathbb{C})$. Then [26]

$$\theta_\alpha :\ \left(SL(2, \mathbb{C}),\ \frac{\langle \alpha, \alpha\rangle}{2}\pi_{SL(2,\mathbb{C})}\right) \longrightarrow (G,\ \pi_{\rm st}) \tag{15}$$

is a Poisson map. It follows that $\theta_\alpha(SL(2,\mathbb{C}))$ is a Poisson Lie subgroup of $(G, \pi_{\rm st})$. Moreover, let $g = u_{-\alpha}(z)$ and $g' = u_\alpha(z)\dot{s}_\alpha$, where $z \in \mathbb{C}$. Then

$$\pi_{\rm st}(g) = \frac{\langle \alpha, \alpha \rangle}{2} l_g(zh_\alpha \wedge e_{-\alpha}), \tag{16}$$

$$\pi_{\rm st}(g') = \frac{\langle \alpha, \alpha \rangle}{2} l_{g'}(zh_\alpha \wedge e_{-\alpha} - 2e_\alpha \wedge e_{-\alpha}) = \frac{\langle \alpha, \alpha \rangle}{2} r_{g'}(ze_\alpha \wedge h_\alpha + 2e_\alpha \wedge e_{-\alpha}). \tag{17}$$

2.2. **The definition of the Poisson structure π_n on $Z_{\mathbf{u}}$.** Recall that given a Poisson Lie group (L, π_L) and a Poisson manifold (Y, π_Y), a left Lie group action $\sigma: L \times Y \to Y$ of L on Y is said to be a Poisson action if σ is a Poisson map from the product Poisson manifold $(L \times Y,\ \pi_L \times \pi_Y)$ to (Y, π_Y). Right Poisson actions of Poisson Lie groups are similarly defined.

Let (Q, π_Q) be a Poisson Lie group, let (X, π_X) be a Poisson manifold with a right Poisson action by (Q, π_Q), and let (Y, π_Y) a Poisson manifold with a left Poisson action by (Q, π_Q). Define the right action of Q on $X \times Y$ by

$$(x,\, y) \cdot q = (xq,\, q^{-1}y), \quad x \in X,\, y \in Y,\, q \in Q,$$

and assume that the quotient space of $X \times Y$ by Q, denoted by $X \times_Q Y$, is a smooth manifold. Then (see [30, §7.1] and [38]) the direct product Poisson structure $\pi_X \times \pi_Y$ on $X \times Y$ projects to a well-defined Poisson structure on $X \times_Q Y$.

Example 2.1. Let (Q, π_Q) be a closed Poisson Lie subgroup of a Poisson Lie group (L, π_L), and let (Y, π_Y) be a Poisson manifold with a left Poisson action by (Q, π_Q). Consider the quotient manifold $Z = L \times_Q Y$, where Q acts on L by right translation. Then Z has the Poisson structure π_Z that is the projection to Z of the direct product Poisson structure $\pi_L \times \pi_Y$ on $L \times Y$. Denoting the image in Z of $(l, y) \in L \times Y$ by $[l, y]$, it follows from the multiplicativity of π_L that the left action of L on Z given by

$$l \cdot [l_1,\, y] = [ll_1,\, y], \quad l,\, l_1 \in L,\, y \in Y, \tag{18}$$

is a Poisson action of the Poisson Lie group (L, π_L) on $(Z,\, \pi_Z)$. Moreover, since $\pi_L(e) = 0$, where e is the identity element of L, the inclusion $Y \hookrightarrow L \times Y, y \mapsto (e, y)$, $y \in Y$, is a Poisson embedding of (Y, π_Y) into $(L \times Y,\, \pi_L \times \pi_Y)$. Consequently,

$$Y \hookrightarrow Z, \;\; y \longmapsto [e,\, y], \quad y \in Y,$$

is a Poisson embedding of (Y, π_Y) into the Poisson manifold (Z, π_Z). ◇

Consider now the standard semi-simple Poisson Lie group $(G, \pi_{\rm st})$ in §2.1. Let $\mathbf{u} = (s_1, \ldots, s_n)$ be any sequence of simple reflections in the Weyl group W. Then for each $1 \leq i \leq n$, the parabolic subgroup $P_{s_i} = B \cup Bs_iB$ is a Poisson Lie subgroup of $(G, \pi_{\rm st})$. By taking $(L, \pi_L) = (P_{s_i}, \pi_{\rm st})$ and $Q = B$ in Example 2.1 and repeat the construction therein, one sees that the direct product Poisson structure $\pi_{\rm st}^n$, regarded as a Poisson structure on the product manifold $P_{s_1} \times \cdots \times P_{s_n}$, projects to a well-defined Poisson structure, denoted by π_n, on the Bott-Samelson variety $Z_{\mathbf{u}}$. It also follows from Example 2.1 that the left action of P_{s_1} on $Z_{\mathbf{u}}$ given by

$$p \cdot [p_1, p_2, \ldots, p_n] = [pp_1,\, p_2, \ldots,\, p_n], \quad p \in P_{s_1},\, p_j \in P_{s_j},\; 1 \leq j \leq n, \tag{19}$$

is a Poisson action of the Poisson Lie group $(P_{s_1}, \pi_{\rm st})$ on $(Z_{\mathbf{u}}, \pi_n)$. In particular, since $\pi_{\rm st}(t) = 0$ for $t \in T$, the action of T on $Z_{\mathbf{u}}$ via (19) preserves π_n.

2.3. $\mathbb{P}^1$**-extensions.** To prepare for the calculation of the Poisson structure π_n in coordinates, we first look at a special case of Example 2.1: let (Y, π_Y) be a Poisson manifold with a left Poisson action σ by the Poisson Lie subgroup $(B, \pi_{\rm st})$ of $(G, \pi_{\rm st})$, and let α be a simple root. One then has the quotient manifold $Z = P_{s_\alpha} \times_B Y$, which fibers over $P_{s_\alpha}/B \cong \mathbb{P}^1$ with fibers diffeomorphic to Y. Let π_Z denote the projection to Z of the product Poisson structure $\pi_{\rm st} \times \pi_Y$ on $P_{s_\alpha} \times Y$. Choose any non-zero $e_\alpha \in \mathfrak{g}_\alpha$, giving rise to the $\mathfrak{sl}(2,\mathbb{C})$-triple $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ for α, and let the notation be as in §1.7. Consider the two open subsets

$$Z_- = \{[u_{-\alpha}(z),\, y] : \, z \in \mathbb{C},\, y \in Y\} \quad \text{ and } \quad Z_+ = \{[u_\alpha(z)\dot{s}_\alpha,\ y] : z \in \mathbb{C}, y \in Y\}$$

of Z with parametrizations

$$\psi_- : \ \mathbb{C} \times Y \longrightarrow Z_-, \ \ \psi_-(z, y) = [u_{-\alpha}(z),\ y],$$
$$\psi_+ : \ \mathbb{C} \times Y \longrightarrow Z_+, \ \ \psi_+(z, y) = [u_\alpha(z)\dot{s}_\alpha,\ y].$$

We will compute $\psi_-^{-1}(\pi_Z)$ and $\psi_+^{-1}(\pi_Z)$ as bi-vector fields on $\mathbb{C} \times Y$. For $x \in \mathfrak{b}$, let η_x be the vector field on Y given by $\eta_x(y) = \frac{d}{dt}|_{t=0} \exp(tx)y$ for $y \in Y$. In the statement of the following Lemma 2.2, we use the obvious way of viewing vector fields on $\mathbb{C}$ and on Y as that on $\mathbb{C} \times Y$.

Lemma 2.2. *With the notation as above, one has*

$$\psi_-^{-1}(\pi_Z)(z, y) = -\frac{\langle \alpha, \alpha \rangle}{2} z \frac{d}{dz} \wedge \eta_{h_\alpha}(y) + \pi_Y(y), \tag{20}$$

$$\psi_+^{-1}(\pi_Z)(z, y) = \frac{\langle \alpha, \alpha \rangle}{2} \frac{d}{dz} \wedge (z\eta_{h_\alpha}(y) - 2\eta_{e_\alpha}(y)) + \pi_Y(y). \tag{21}$$

Proof. For $g \in P_{s_\alpha}$ and $y \in Y$, let

$$\lambda_g : \ \ Z \longrightarrow Z : \ \ [p,\ y'] \longmapsto [gp,\ y'], \quad \ p \in P_{s_\alpha},\, y' \in Y,$$
$$\rho_y : \ \ P_{s_\alpha} \longrightarrow Z : \ \ p \longmapsto [p,\ y], \quad \ p \in P_{s_\alpha}.$$

Fix $z \in \mathbb{C}$ and $y \in Y$, and let $g = u_{-\alpha}(z) \in P_{s_\alpha}$ and $q = [u_{-\alpha}(z),\, y] = \lambda_g([e,\, y]) \in Z$. By Example 2.1, $\pi_Z(q) = \lambda_g(\pi_Z([e, y])) + \rho_y(\pi_{\rm st}(g))$. Using (16), one has

$$\pi_Z(q) = \lambda_g(\pi_Z([e, y])) + \frac{\langle \alpha, \alpha \rangle}{2}(\rho_y l_g)(zh_\alpha \wedge e_{-\alpha}) = \lambda_g(\pi_Z([e, y])) + \frac{\langle \alpha, \alpha \rangle}{2}(\lambda_g \rho_y)(zh_\alpha \wedge e_{-\alpha})$$

and thus

$$(\psi_-^{-1}(\pi_Z))(z, y) = \psi_-^{-1}(\pi_Z(q)) = (\psi_-^{-1} \circ \lambda_g)(\pi_Z([e, y])) + \frac{\langle \alpha, \alpha \rangle}{2}(\psi_-^{-1} \lambda_g \rho_y)(zh_\alpha \wedge e_{-\alpha}).$$

Since the inclusion $(Y, \pi_Y) \hookrightarrow (Z, \pi_Z) : y' \mapsto [e, y']$ is Poisson, $(\psi_-^{-1} \circ \lambda_g)(\pi_Z([e, y])) = \pi_Y(y)$. Direct calculations give

$$(\psi_-^{-1} \lambda_g \rho_y)(h_\alpha) = \eta_{h_\alpha}(y) \quad \text{ and } \quad (\psi_-^{-1} \lambda_g \rho_y)(e_{-\alpha}) = \frac{d}{dz}.$$

One thus has (20). Similarly, for $z \in \mathbb{C}$ and $y \in Y$, letting $g' = u_\alpha(z)\dot{s}_\alpha$ and using (17), one has

$$\psi_+^{-1}(\pi_Z)(z, y) = \pi_Y(y) + \frac{\langle \alpha, \alpha \rangle}{2}(\psi_+^{-1} \lambda_{g'} \rho_y)((zh_\alpha - 2e_\alpha) \wedge e_{-\alpha}).$$

Since $(\psi_+^{-1} \lambda_{g'} \rho_y)(h_\alpha) = \eta_{h_\alpha}$, $(\psi_+^{-1} \lambda_{g'} \rho_y)(e_\alpha) = \eta_{e_\alpha}$, and $(\psi_+^{-1} \lambda_{g'} \rho_y)(e_{-\alpha}) = -\frac{d}{dz}$, one has (21).

Q.E.D.

3. The Poisson structure π_n in affine coordinate charts, I

Throughout §3, we fix a sequence $\mathbf{u} = (s_1, \ldots, s_n)$ of simple reflections in W, and let $Z_{\mathbf{u}}$ be the Bott-Samelson variety associated to $\mathbf{u}$. Recall that Γ denotes the set of all simple roots. For $1 \leq j \leq n$, let $\alpha_j \in \Gamma$ be such that $s_j = s_{\alpha_j}$. To define local coordinates on $Z_{\mathbf{u}}$, we also fix a root vector e_α for each $\alpha \in \Gamma$ and let $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ be the unique element such that $[e_\alpha, e_{-\alpha}] = h_\alpha$. One then (see §1.7) has the one-parameter subgroups $u_{\pm\alpha} : \mathbb{C} \to G$ for each $\alpha \in \Gamma$ and the representative $\dot{s}_\alpha \in N_G(T)$ for the simple reflection $s_\alpha \in W$.

3.1. **Affine coordinate charts on $Z_{\mathbf{u}}$.** Let

$$\Upsilon_{\mathbf{u}} = \{e, s_1\} \times \{e, s_2\} \times \cdots \{e, s_n\},$$

where e denotes the identity element of W. Elements in $\Upsilon_{\mathbf{u}}$ will be called *subexpressions* of $\mathbf{u}$. When $\gamma = \mathbf{u}$, we say that γ is the *full subexpression* of $\mathbf{u}$. For $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$, let $\gamma^0 = e$ and $\gamma^i = \gamma_1\gamma_2\cdots\gamma_i \in W$ for $1 \leq i \leq n$.

The maximal torus T of G acts on $Z_{\mathbf{u}}$ via (19) with

$$Z_{\mathbf{u}}^T = \{[\dot{\gamma}_1, \dot{\gamma}_2, \ldots, \dot{\gamma}_n] : \, (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}\}$$

as the fixed point set, where $\dot{e} = e$. For each $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$, let $\mathcal{O}^\gamma \subset Z_{\mathbf{u}}$ be the image of the embedding $\Phi^\gamma : \mathbb{C}^n \to Z_{\mathbf{u}}$ given by

$$\Phi^\gamma(z_1, \ldots, z_n) = [u_{-\gamma_1(\alpha_1)}(z_1)\dot{\gamma}_1, \, u_{-\gamma_2(\alpha_2)}(z_2)\dot{\gamma}_2, \, \ldots, \, u_{-\gamma_n(\alpha_n)}(z_n)\dot{\gamma}_n]. \tag{22}$$

The parametrization Φ^γ of $\mathcal{O}^\gamma$ by $\mathbb{C}^n$ depends on the choice of the root vectors $\{e_\alpha : \alpha \in \Gamma\}$ for the simple roots, but different choices of such root vectors only result in re-scalings of the coordinate functions (see §5.1 for more discussions). In particular, the open subset $\mathcal{O}^\gamma$ of $Z_{\mathbf{u}}$ is canonically defined. It is also easy to see that each $\mathcal{O}^\gamma$ is T-invariant with

$$t \cdot \Phi^\gamma(z_1, z_2, \ldots, z_n) = \Phi^\gamma\left(t^{-\gamma^1(\alpha_1)}z_1, \, t^{-\gamma^2(\alpha_2)}z_2, \, \ldots, \, t^{-\gamma^n(\alpha_n)}z_n\right), \tag{23}$$

where $t \in T$ and $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$. Note also that $\bigcup_{\gamma \in \Upsilon_{\mathbf{w}}} \mathcal{O}^\gamma = Z_{\mathbf{u}}$, i.e., $Z_{\mathbf{u}}$ is covered by the 2^n T-invariant affine coordinate charts $\{(\Phi^\gamma : \mathbb{C}^n \to \mathcal{O}^\gamma) : \gamma \in \Upsilon_{\mathbf{u}}\}$, which we will also abbreviate as the affine charts $\{\mathcal{O}^\gamma : \gamma \in \Upsilon_{\mathbf{u}}\}$. .

3.2. **The Poisson structure π_n in coordinates, I.** For each $\gamma \in \Upsilon_{\mathbf{u}}$, we now give our first formula for the Poisson structure π_n on $Z_{\mathbf{u}}$ in the coordinates $(z_1, z_2, \ldots, z_n)$ on $\mathcal{O}^\gamma$ given in (22). A more detailed formula, expressing each Poisson bracket $\{z_i, z_k\}$, where $1 \leq i < k \leq n$, as a polynomial with coefficients explicitly in terms of the structure constants of the Lie algebra $\mathfrak{g}$, will be given in §4.

Notation 3.1. For $1 \leq i \leq n-1$, let σ_i be the holomorphic vector field on the Bott-Samelson variety $Z_{(s_{i+1},\ldots,s_n)}$ given by

$$\sigma_i(p) = \frac{d}{dt}|_{t=0}((\exp(te_{\alpha_i})) \cdot p), \qquad p \in Z_{(s_{i+1},\ldots,s_n)}, \tag{24}$$

where $\cdot$ denotes the left action of $B \subset P_{s_i}$ on $Z_{(s_{i+1},\ldots,s_n)}$ by left translation (see (19)). For $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$ and the coordinates $(z_1, \ldots z_n)$ on the affine chart $\mathcal{O}^\gamma$ given in (22), we will also regard $(z_{i+1}, \ldots, z_n)$ as coordinates on $\mathcal{O}^{(\gamma_{i+1},\ldots,\gamma_n)} \subset Z_{(s_{i+1},\ldots,s_n)}$ via the parametrization

$$\mathbb{C}^{n-i} \ni (z_{i+1}, \ldots, z_n) \longmapsto [u_{-\gamma_{i+1}(\alpha_{i+1})}(z_{i+1})\dot{\gamma}_{i+1}, \ldots, u_{-\gamma_n(\alpha_n)}(z_n)\dot{\gamma}_n] \in \mathcal{O}^{(\gamma_{i+1},\ldots,\gamma_n)},$$

and we identify the algebra of regular functions on $\mathcal{O}^{(\gamma_{i+1},\ldots,\gamma_n)}$ as a subalgebra of that on $\mathcal{O}^\gamma$ via the pullback of the projection $\mathcal{O}^\gamma \to \mathcal{O}^{(\gamma_{i+1},\ldots,\gamma_n)}$ given by

$$[u_{-\gamma_1(\alpha_1)}(z_1)\dot{\gamma}_1, \ldots, u_{-\gamma_n(\alpha_n)}(z_n)\dot{\gamma}_n] \longmapsto [u_{-\gamma_{i+1}(\alpha_{i+1})}(z_{i+1})\dot{\gamma}_{i+1}, \ldots, u_{-\gamma_n(\alpha_n)}(z_n)\dot{\gamma}_n].$$

A polynomial in $(z_{i+1}, \ldots, z_n)$ can then be unambiguously regarded as a regular function on both $\mathcal{O}^\gamma$ and $\mathcal{O}^{(\gamma_{i+1},\ldots,\gamma_n)}$. ◇

Lemma 3.2. *Let $\gamma \in \Upsilon_{\mathbf{u}}$. In the coordinates $(z_1, \ldots z_n)$ on the affine chart $\mathcal{O}^\gamma$ given in (22), the Poisson structure π_n on $Z_{\mathbf{u}}$ is given by,*

$$\{z_i, z_k\} = \begin{cases} \langle \gamma^i(\alpha_i), \gamma^k(\alpha_k)\rangle z_i z_k, & \text{if } \gamma_i = e \\ -\langle \gamma^i(\alpha_i), \gamma^k(\alpha_k)\rangle z_i z_k - \langle \alpha_i, \alpha_i\rangle \sigma_i(z_k) & \text{if } \gamma_i = s_i \end{cases}, \quad 1 \le i < k \le n, \tag{25}$$

where $\sigma_i(z_k)$ denotes the action of the vector field σ_i on z_k as a function on $\mathcal{O}^{(\gamma_{i+1},\ldots,\gamma_n)} \subset Z_{(s_{i+1},\ldots,s_n)}$ (see Notation 3.1).

Proof. Identify $\mathcal{O}^\gamma \cong \mathbb{C} \times \mathcal{O}^{\gamma'}$, where $\gamma' = (\gamma_2, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}'}$ and $\mathbf{u}' = (s_2, \ldots, s_n)$. Equip $\mathcal{O}^{\gamma'}$ with the Poisson structure π_{n-1} on $Z_{(s_2,\ldots,s_n)}$. One has, by Lemma 2.2,

$$\pi_n = \begin{cases} -\frac{\langle\alpha_1,\alpha_1\rangle}{2} z_1 \frac{d}{dz_1} \wedge \eta_1 + \pi_{n-1}, & \text{if } \gamma_1 = e, \\ \frac{\langle\alpha_1,\alpha_1\rangle}{2} \frac{d}{dz_1} \wedge (z_1\eta_1 - 2\sigma_1) + \pi_{n-1}, & \text{if } \gamma_1 = s_1, \end{cases} \tag{26}$$

where η_1 is the holomorphic vector field on $Z_{(s_2,\ldots,s_n)}$ given by

$$\eta_1(q) = \frac{d}{dt}|_{t=1}(\alpha_1^\vee(t) \cdot q), \quad q \in Z_{(s_2,\ldots,s_n)}.$$

By (23), the vector field η_1 is given in the coordinates $(z_2, \ldots, z_n)$ on $\mathcal{O}^{\gamma'}$ by

$$\eta_1 = \sum_{k=2}^n (-\gamma_2 \cdots \gamma_k(\alpha_k))(h_{\alpha_1}) z_k \frac{\partial}{\partial z_k} = -\sum_{k=2}^n \frac{2\langle \gamma^1(\alpha_1), \gamma^k(\alpha_k)\rangle}{\langle \alpha_1, \alpha_1\rangle} z_k \frac{\partial}{\partial z_k}.$$

Lemma 3.2) now follows by repeatedly using (26).

Q.E.D.

Example 3.3. Consider $G = SL(3, \mathbb{C})$ with the standard choices of B and B_- consisting respectively of upper triangular and lower triangular matrices in $SL(3, \mathbb{C})$, and let the bilinear form $\langle\, , \,\rangle$ on $\mathfrak{sl}(3, \mathbb{C})$ be given by $\langle X, Y\rangle = \mathrm{tr}(XY)$ for $X, Y \in \mathfrak{sl}(3, \mathbb{C})$. Denote the two simple roots by α_1 and α_2 and choose root vectors $e_{\alpha_1} = E_{12}$ and $e_{\alpha_2} = E_{23}$, where E_{ij} has 1 at the (i, j)-entry and 0 everywhere else. Let $\mathbf{u} = (s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_1})$. Using Lemma 3.2, one can compute directly the Poisson structure π_3 on $Z_{\mathbf{u}}$ in any of the eight affine coordinate charts. For example, for $\gamma = \mathbf{u}$, one has

$$\{z_1, z_2\} = -z_1 z_2, \quad \{z_1, z_3\} = z_1 z_3 - 2, \quad \{z_2, z_3\} = -z_2 z_3, \tag{27}$$

and for $\gamma = (s_{\alpha_1}, e, e) \in \Upsilon_{\mathbf{u}}$, one has

$$\{z_1, z_2\} = z_1 z_2, \quad \{z_1, z_3\} = -2z_1 z_3 + 2z_3^2, \quad \{z_2, z_3\} = -z_2 z_3. \tag{28}$$

◇

3.3. **Some log-canonical charts for π_n.** Let $\gamma \in \Upsilon_{\mathbf{u}}$. We say that the affine coordinate chart $\mathcal{O}^\gamma$ of $Z_{\mathbf{u}}$ is *log-canonical* for the Poisson structure π_n, or that the Poisson structure π_n is *log-canonical* in the affine coordinate chart $\mathcal{O}^\gamma$, if the Poisson brackets between the coordinate functions $(z_1, z_2, \ldots, z_n)$ on $\mathcal{O}^\gamma$ have the form $\{z_i, z_k\} = \lambda_{ik} z_i z_k$ for some $\lambda_{ik} \in \mathbb{C}$ for each pair $1 \leq i < k \leq n$. By Lemma 3.2, π_n is log-canonical in $\mathcal{O}^\gamma$ if and only if

$$\{z_i, z_k\} = \epsilon_i \langle \gamma^i(\alpha_i), \gamma^k(\alpha_k)\rangle z_i z_k, \qquad 1 \leq i < k \leq n,$$

where $\epsilon_i = 1$ if $\gamma_i = e$ and $\epsilon_i = -1$ if $\gamma_i = s_i$. The following Lemma 3.4, which follows trivially from Lemma 3.2, says that π_n is log-canonical in the affine chart $\mathcal{O}^{(e,e,\ldots,e)}$.

Lemma 3.4. *In the coordinates $(z_1, z_2, \ldots, z_n)$ on $\mathcal{O}^{(e,e,\ldots,e)}$, one has*

$$\{z_i,\, z_k\} = \langle \alpha_i,\, \alpha_k\rangle z_i z_k, \qquad \forall\, 1 \leq i < k \leq n.$$

To exhibit other log-canonical affine coordinate charts for π_n, we make the following observation on the functions $\sigma_i(z_k)$, $1 \leq i < k \leq n$, in Lemma 3.2.

Lemma 3.5. *Let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$, and let $1 \leq i \leq n$. If $\gamma_i = s_i$ and if $k > i$ is such that $s_j \neq s_i$ for all $i+1 \leq j \leq k$, then $\sigma_i(z_k) = 0$.*

Proof. For $i+1 \leq j \leq n$, let $z_j \in \mathbb{C}$ and $p_j = u_{-\gamma_j(\alpha_j)}(z_j)\dot{\gamma}_j$. For $t \in \mathbb{C}$, consider

$$[u_{\alpha_i}(t) p_{i+1},\; p_{i+2},\; \ldots,\; p_n] \in Z_{(s_{i+1},\ldots,s_n)}.$$

For each $i+1 \leq j \leq k$, since $p_{i+1}p_{i+2}\cdots p_j$ lies in the Levi subgroup of the parabolic subgroup of G determined by the set of simple roots in $\{\alpha_{i+1}, \ldots, \alpha_j\}$ which does not contain α_i, one has

$$g_j := (p_{i+1}p_{i+2}\cdots p_j)^{-1} u_{\alpha_i}(t) p_{i+1}p_{i+2}\cdots p_j \in N,$$

where N is the unipotent subgroup of G with Lie algebra $\mathfrak{n} = \sum_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$. Thus

$$\begin{aligned}
[u_{\alpha_i}(t)p_{i+1},\; p_{i+2},\; \ldots,\; p_n] &= [p_{i+1},\; g_{i+1}p_{i+2},\; p_{i+3},\; \ldots,\; p_n] \\
&= [p_{i+1},\; p_{i+2},\; \ldots,\; g_{k-1}p_k,\; p_{k+1},\; \ldots,\; p_n] \\
&= [p_{i+1},\; p_{i+2},\; \ldots,\; p_k,\; g_k p_{k+1},\; \ldots,\; p_n].
\end{aligned}$$

It follows from the definition of the vector field σ_i that $\sigma_i(z_k) = 0$.

Q.E.D.

The next Lemma 3.6, which follows directly from Lemma 3.2 and Lemma 3.5, exhibits a log-canonical affine chart for π_n associated to each $s \in \{s_1, \ldots, s_n\}$.

Lemma 3.6. *Let $s \in \{s_1, s_2, \ldots, s_n\}$ and let $i_0 = \max\{i : 1 \leq i \leq n,\, s_i = s\}$. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ be such that $\gamma_{i_0} = s$ and $\gamma_i = e$ for all $i \neq i_0$. Then in the coordinates $(z_1, z_2, \ldots, z_n)$ on $\mathcal{O}^\gamma$ given in (22) and for all $1 \leq i < k \leq n$, one has*

$$\{z_i, z_k\} = \begin{cases} \langle \alpha_i, \alpha_k\rangle z_i z_k, & 1 \leq i < k < i_0 \text{ or } i_0 < i < k \leq n, \\ \langle \alpha_i, s(\alpha_k)\rangle z_i z_k, & 1 \leq i \leq i_0 \leq k \leq n. \end{cases} \tag{29}$$

The following Corollary 3.7 also follows directly from Lemma 3.2 and Lemma 3.5.

Corollary 3.7. *If $\mathbf{u} = (s_1, s_2, \ldots, s_n)$ is such that $s_i \neq s_j$ for all $i \neq j$, then the Poisson structure π_n on $Z_{\mathbf{u}}$ is log-canonical in every one of the 2^n affine coordinate charts $\{\mathcal{O}^\gamma : \gamma \in \Upsilon_{\mathbf{u}}\}$.*

4. The Poisson structure π_n in affine coordinates charts, II

Throughout §4, fix a sequence $\mathbf{u} = (s_1, \ldots, s_n)$ of simple reflections, and let $Z_{\mathbf{u}}$ be the corresponding Bott-Samelson variety. To better understand the Poisson structure π_n in the coordinates $(z_1, z_2, \ldots, z_n)$ on the affine chart $\mathcal{O}^\gamma$ defined in §3.1, where $\gamma \in \Upsilon_{\mathbf{u}}$, one needs to compute more explicitly the vector field σ_i in Lemma 3.2 on the Bott-Samelson variety $Z_{(s_{i+1},\ldots,s_n)}$ for $1 \leq i \leq n-1$. For $x \in \mathfrak{b}$, define the vector field σ_x on $Z_{\mathbf{u}}$ by

$$\sigma_x(p) = \frac{d}{dt}|_{t=0}((\exp tx) \cdot p), \qquad p \in Z_{\mathbf{u}}, \tag{30}$$

where $\cdot$ denotes the left action of $B \subset P_{s_1}$ on $Z_{\mathbf{u}}$ given in (19). Using some facts on root strings of the root system of $\mathfrak{g}$ reviewed in §4.1, and for any $\beta \in \Delta_+$ and $e_\beta \in \mathfrak{g}_\beta$, we give in §4.2 an explicit formula for σ_{e_β} in the coordinates $(z_1, z_2, \ldots, z_n)$ on each affine chart $\mathcal{O}^\gamma$ of $Z_{\mathbf{u}}$. The formula for σ_{e_β}, given in Theorem 4.10, is expressed explicitly in terms of the root strings and the structure constants of $\mathfrak{g}$. As a consequence (see Theorem 4.14), the Poisson structure π_n can also be expressed in each affine coordinate chart $\mathcal{O}^\gamma$ in terms of root strings and the structure constants of $\mathfrak{g}$. We believe that our formula for the vector fields σ_{e_β} is of interest irrespective of the Poisson structure π_n.

4.1. **Some lemmas on root strings.** In §4.1, let

$$\{h_\alpha\}_{\alpha\in\Gamma} \cup \{e_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}\}_{\alpha\in\Delta_+} \tag{31}$$

be any basis of $\mathfrak{g}$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$ for each $\alpha \in \Delta_+$. One then has the Lie group homomorphism $\theta_\alpha : SL(2,\mathbb{C}) \to G$ for each $\alpha \in \Delta_+$. Let the notation be as in §1.7. For $\alpha, \beta \in \Delta$ such that $\alpha + \beta \in \Delta$, let $N_{\alpha,\beta} \neq 0$ be such that $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$.

Lemma 4.1. *For $\alpha \in \Delta_+$, one has*

$$u_\alpha(t)u_\alpha(z)\dot{s}_\alpha = u_\alpha(t+z)\dot{s}_\alpha, \qquad t, z \in \mathbb{C}, \tag{32}$$

$$u_\alpha(t)u_{-\alpha}(z) = u_{-\alpha}\left(\frac{z}{1+tz}\right)u_\alpha(t(1+tz))\alpha^\vee(1+tz), \qquad t, z \in \mathbb{C}, 1+tz \neq 0, \tag{33}$$

$$u_{-\alpha}(t) = u_\alpha\left(\frac{1}{t}\right)\dot{s}_\alpha u_\alpha(t)\alpha^\vee(t), \qquad t \in \mathbb{C}^\times. \tag{34}$$

For $\alpha, \beta \in \Gamma$ and $\alpha \neq \beta$, one has .

$$u_\beta(t)\beta^\vee(t)u_{-\alpha}(z) = u_{-\alpha}\left(t^{\frac{-2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}} z\right)u_\beta(t)\beta^\vee(t), \qquad t \in \mathbb{C}^\times,\ z \in \mathbb{C}. \tag{35}$$

Proof. Identity (32) is clear. Identities (33) and (34) follow from computations in $\mathrm{SL}(2,\mathbb{C})$, and (35) follows from the fact that the two root subgroups corresponding to $-\alpha$ and β commute.

Q.E.D.

Let α and β be two linearly independent roots, $\alpha \in \Delta_+$, and let $\{\beta + j\alpha : -p \leq j \leq q\}$, where p and q are non-negative integers, be the α-string through β. Then the subspace

$$L = \sum_{j=-p}^{q} \mathfrak{g}_{\beta+j\alpha}$$

of $\mathfrak{g}$ becomes an $SL(2,\mathbb{C})$-module via the group homomorphism $\theta_\alpha : SL(2,\mathbb{C}) \to G$ and the adjoint representation of G on $\mathfrak{g}$. On the other hand, let L^{p+q} be the vector space of homogeneous polynomials in (x, y) of degree $p + q$ with the action of $\mathrm{SL}(2,\mathbb{C})$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f\right)(x,y) = f\left((x,y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = f\,(ax+cy,\, bx+dy)\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}).$$

Let $\{u_0, \ldots, u_{p+q}\}$ be the basis of L^{p+q} given by

$$u_i = \varepsilon_0\varepsilon_1 \cdots \varepsilon_{i-1}\binom{p+q}{i} x^i y^{p+q-i}, \qquad 0 \le i \le p+q, \tag{36}$$

where for $0 \le j \le p + q - 1$, $\varepsilon_j \in \mathbb{C}$ is defined by

$$\varepsilon_j = \frac{j+1}{N_{\alpha,\beta-(p-j)\alpha}}, \tag{37}$$

and it is understood that $\varepsilon_0\varepsilon_1 \cdots \varepsilon_{i-1} = 1$ when $i = 0$ in (36).

Lemma 4.2. *With the notation as above, the linear map*

$$\chi :\ L \longrightarrow L^{p+q} :\ \ \chi(e_{\beta+j\alpha}) = u_{p+j}, \quad -p \le j \le q, \tag{38}$$

is an $\mathrm{SL}(2,\mathbb{C})$*-equivariant isomorphism.*

Proof. The two irreducible representations of $SL(2,\mathbb{C})$ on L and on L^{p+q}, being of the same dimension, must be isomorphic, and by Schur's lemma, there is a unique $SL(2,\mathbb{C})$-equivariant isomorphism $\chi : L \to L^{p+q}$ such that $\chi(e_{\beta-p\alpha}) = u_0$. Straightforward calculations show that χ must be given as in (38). See also [6, Lemma 6.2.2].

Q.E.D.

The following Lemma 4.3 is the key to the proof of Theorem 4.10 in §4.2.

Lemma 4.3. *Let* $\alpha \in \Delta_+$ *and* $\beta \in \Delta$ *be linearly independent, and let* $\{\beta + j\alpha : -p \le j \le q\}$ *be the* α*-string through* β*. Then for any* $t \in \mathbb{C}$*, one has*

$$\mathrm{Ad}_{(u_\alpha(t)\dot{s}_\alpha)^{-1}}(e_\beta) = \sum_{j=0}^{q} (-1)^p \frac{\varepsilon_0\varepsilon_1 \cdots \varepsilon_{p-1}}{\varepsilon_0\varepsilon_1 \cdots \varepsilon_{q-j-1}} \binom{p+j}{j} t^j e_{s_\alpha(\beta)-j\alpha}, \tag{39}$$

$$\mathrm{Ad}_{(u_{-\alpha}(t))^{-1}}(e_\beta) = \sum_{j=0}^{p} (-1)^j \varepsilon_{p-j}\varepsilon_{p-j+1} \cdots \varepsilon_{p-1} \binom{q+j}{j} t^j e_{\beta-j\alpha}. \tag{40}$$

Proof. By Lemma 4.2, one has

$$\begin{aligned}
\chi\left(\mathrm{Ad}_{(u_\alpha(t)\dot{s}_\alpha)^{-1}}(e_\beta)\right) &= \begin{pmatrix} 0 & 1 \\ -1 & t \end{pmatrix} \cdot u_p \\
&= \varepsilon_0\varepsilon_1 \cdots \varepsilon_{p-1}\binom{p+q}{p}(-y)^p(x+ty)^q \\
&= \varepsilon_0\varepsilon_1 \cdots \varepsilon_{p-1}\binom{p+q}{p}(-y)^p\left(\sum_{j=0}^{q}\binom{q}{j} t^j y^j x^{q-j}\right) \\
&= \sum_{j=0}^{q} (-1)^p \frac{\varepsilon_0\varepsilon_1 \cdots \varepsilon_{p-1}}{\varepsilon_0\varepsilon_1 \cdots \varepsilon_{q-j-1}} \binom{p+j}{j} t^j u_{q-j}.
\end{aligned}$$

It follows that

$$\mathrm{Ad}_{(u_\alpha(t)\dot{s}_\alpha)^{-1}}(e_\beta) = \sum_{j=0}^{q} (-1)^p \frac{\varepsilon_0\varepsilon_1\cdots\varepsilon_{p-1}}{\varepsilon_0\varepsilon_1\cdots\varepsilon_{q-j-1}} \binom{p+j}{j} t^j e_{\beta+(q-p-j)\alpha}.$$

As (see for example, [25, Proposition 25.1]) $\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} = p-q$, one has, for any $j \in \mathbb{Z}$,

$$s_\alpha(\beta) - j\alpha = \beta - \frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha - j\alpha = \beta + (q-p-j)\alpha,$$

from which (39) follows. One proves (40) similarly (see also [6, Lemma 6.2.1]).

Q.E.D.

To unify the two formulas in (39) and (40), for $\alpha \in \Delta_+$, $\kappa \in \{s_\alpha, e\}$, and $t \in \mathbb{C}$, let

$$p_{\kappa,\alpha}(t) = u_{-\kappa(\alpha)}(t)\dot{\kappa} \in G, \tag{41}$$

and for $\beta \in \Delta$, $\beta \neq \pm\alpha$, as in Lemma 4.3, let

$$c_{\alpha,\beta}^{\kappa,j} = (-1)^p \frac{\varepsilon_0\varepsilon_1\cdots\varepsilon_{p-1}}{\varepsilon_0\varepsilon_1\cdots\varepsilon_{q-j-1}} \binom{p+j}{j}, \qquad j = 0,\ldots,q \text{ and } \kappa = s_\alpha, \tag{42}$$

$$c_{\alpha,\beta}^{\kappa,j} = (-1)^j \varepsilon_{p-j}\varepsilon_{p-j+1}\cdots\varepsilon_{p-1} \binom{q+j}{j}, \qquad j = 0,\ldots,p \text{ and } \kappa = e. \tag{43}$$

Lemma 4.3 can now be reformulated as follows.

Lemma 4.4. *Let $\alpha \in \Delta_+$ and $\beta \in \Delta$ be linearly independent. Then for $\kappa \in \{s_\alpha, e\}$ and $t \in \mathbb{C}$,*

$$\mathrm{Ad}_{(p_{\kappa,\alpha}(t))^{-1}}(e_\beta) = \sum_{\substack{j\geq 0,\\ \kappa(\beta)-j\alpha\in\Delta}} c_{\alpha,\beta}^{\kappa,j}\, t^j\, e_{\kappa(\beta)-j\alpha}. \tag{44}$$

Proof. Let $j \in \mathbb{Z}$ and $j \geq 0$. When $\kappa = e$, $\kappa(\beta) - j\alpha \in \Delta$ if and only if $\beta - j\alpha \in \Delta$, which is the same as $0 \leq j \leq p$. When $\kappa = s_\alpha$, $\kappa(\beta) - j\alpha \in \Delta$ if and only if $s_\alpha(\beta + j\alpha) \in \Delta$, which is the same as $\beta + j\alpha \in \Delta$, which, in turn, is the same as $0 \leq j \leq q$.

Q.E.D.

Remark 4.5. Recall that a basis $\{h_\alpha\}_{\alpha\in\Gamma} \cup \{e_\alpha \in \mathfrak{g}_\alpha\}_{\alpha\in\Delta}$ of $\mathfrak{g}$ is said to be a Chevalley basis if $[e_\alpha, e_{-\alpha}] = h_\alpha$ for all $\alpha \in \Delta$, and if for all $\alpha, \beta \in \Delta$ such that $\alpha + \beta \in \Delta$, one has $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$. If $\{h_\alpha\}_{\alpha\in\Gamma} \cup \{e_\alpha \in \mathfrak{g}_\alpha\}_{\alpha\in\Delta}$ is a Chevalley basis of $\mathfrak{g}$, by [6, Theorem 4.1.2] and [25, Theorem 25.2], $N_{\alpha,\beta} = \pm(p+1)$ for any roots α and β such that $\alpha+\beta \in \Delta$, where p is the largest non-negative integer such that $\beta - p\alpha \in \Delta$. Thus, for α and β as in Lemma 4.3 and for every $0 \leq j \leq p+q-1$, one has $\varepsilon_j = \pm 1$, and consequently all the coefficients $c_{\alpha,\beta}^{\kappa,j}$ appearing in (44) are integers. ⋄

4.2. **The vector field σ_{e_β} in coordinates.** Fix again $\mathbf{u} = (s_1, \ldots, s_n) = (s_{\alpha_1}, \ldots, s_{\alpha_n})$ be a sequence of simple reflections, and let $Z_\mathbf{u}$ be the corresponding Bott-Samelson variety. Let $\{e_\alpha \in \mathfrak{g}_\alpha : \alpha \in \Gamma\}$ be a set of root vectors for the simple roots, and extend it to a basis $\{h_\alpha\}_{\alpha\in\Gamma} \cup \{e_\alpha \in \mathfrak{g}_\alpha\}_{\alpha\in\Delta}$ of $\mathfrak{g}$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$ for all $\alpha \in \Delta$. Recall from (30) that for $x \in \mathfrak{b}$, σ_x is the vector field on $Z_{\mathbf{u}}$ generating the action of B on $Z_{\mathbf{u}}$ in the direction of x. For $\beta \in \Delta_+$, we then have the vector field σ_{e_β} on $Z_{\mathbf{u}}$ given by

$$\sigma_{e_\beta}(p) = \frac{d}{dt}|_{t=0}((\exp te_\beta) \cdot p), \qquad p \in Z_{\mathbf{u}}. \tag{45}$$

On the other hand, recall that the choice $\{e_\alpha : \alpha \in \Gamma\}$ gives rise to coordinates $(z_1, \ldots, z_n)$ on the affine chart $\mathcal{O}^\gamma$ for $\gamma \in \Upsilon_{\mathbf{u}}$ via (22). For each $\gamma \in \Upsilon_{\mathbf{u}}$, we now use the results in §4.1 to compute the vector fields σ_{e_β}, $\beta \in \Delta_+$, in the coordinates $(z_1, \ldots, z_n)$ on $\mathcal{O}^\gamma$ in terms of root strings and structure constants of $\mathfrak{g}$ in the basis $\{h_\alpha\}_{\alpha\in\Gamma} \cup \{e_\alpha \in \mathfrak{g}_\alpha\}_{\alpha\in\Delta}$ of $\mathfrak{g}$.

For $x \in \mathfrak{b}$ and $1 \leq k \leq n$, consider also the vector field $\sigma_x^{(k)}$ on the Bott-Samelson variety $Z_{(s_k,\ldots,s_n)}$ defined by

$$\sigma_x^{(k)}(p) = \frac{d}{dt}|_{t=0}((\exp tx) \cdot p), \qquad p \in Z_{(s_k,\ldots,s_n)}, \tag{46}$$

where $\cdot$ is the left action of B on $Z_{(s_k,\ldots,s_n)}$ in (19). Note that $\sigma_x = \sigma_x^{(1)}$ for $x \in \mathfrak{b}$ and that for $1 \leq i \leq n-1$, $\sigma_i = \sigma_{e_{\alpha_i}}^{(i+1)}$ for the vector field σ_i on $Z_{(s_{i+1},\ldots,s_n)}$ defined in Notation 3.1.

Fix $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$ and let again $(z_1, \ldots, z_n)$ be the coordinates on $\mathcal{O}^\gamma \subset Z_{\mathbf{u}}$. Recall from Notation 3.1 that for $1 \leq k \leq n$, we regard polynomials in $(z_k, \ldots, z_n)$ as regular functions on both $\mathcal{O}^{(\gamma_k,\ldots,\gamma_n)} \subset Z_{(s_k,\ldots,s_n)}$ and on $\mathcal{O}^\gamma$. In particular, for $x \in \mathfrak{b}$ and $k \leq j \leq n$, $\sigma_x^{(k)}(z_j)$, the action of $\sigma_x^{(k)}$ on z_j as a function on $\mathcal{O}^{(\gamma_k,\ldots,\gamma_n)}$, is also regarded as a regular function on $\mathcal{O}^\gamma$. We now a recursive formula for σ_{e_β} as a vector field on $\mathcal{O}^\gamma$.

Lemma 4.6. *Let $\beta \in \Delta_+$ and $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$.*

1) $\beta = \alpha_1$ and $\gamma_1 = s_1$. In this case, $\sigma_{e_\beta}(z_1) = 1$ and $\sigma_{e_\beta}(z_k) = 0$ for all $k \geq 2$;

2) $\beta = \alpha_1$ and $\gamma_1 = e$. In this case, $\sigma_{e_\beta}(z_1) = -z_1^2$ and for $k \geq 2$,

$$\sigma_{e_\beta}(z_k) = \sigma_{e_\beta}^{(2)}(z_k) + z_1\sigma_{h_{\alpha_1}}^{(2)}(z_k);$$

3) $\beta \neq \alpha_1$. In this case, $\sigma_{e_\beta}(z_1) = 0$ and for $k \geq 2$,

$$\sigma_{e_\beta}(z_k) = \sum_{\substack{j\geq 0,\\ \gamma_1(\beta)-j\alpha_1\in\Delta_+}} c_{\alpha_1,\beta}^{\gamma_1,j}\, z_1^j\, \sigma_{e_{\gamma_1(\beta)-j\alpha_1}}^{(2)}(z_k).$$

Proof. Cases 1) and 2) follow from (32) and (33) respectively. Case 3) follows from Lemma 4.4 and the fact that, as $\beta \in \Delta_+$ and $\beta \neq \alpha_1$, all the roots in the α_1-string through $\gamma_1(\beta)$ are positive.

Q.E.D.

To combine the cases in Lemma 4.6, we note that when $\beta = \alpha_1$,

$$\{j_1 \geq 0 : \gamma_1(\beta) - j_1\alpha_1 \in \Delta_+\} = \begin{cases} \emptyset, & \text{if } \gamma_1 = s_1, \\ \{0\}, & \text{if } \gamma_1 = e. \end{cases}$$

For $\alpha \in \Gamma$, also set

$$c_{\alpha,\alpha}^{e,0} = 1. \tag{47}$$

We can now reformulate Lemma 4.6 as follows.

Lemma 4.7. *Let $\beta \in \Delta_+$ and $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$. Then*

$$\sigma_{e_\beta}(z_1) = \begin{cases} 1, & \text{if } \beta = \alpha_1 \text{ and } \gamma_1 = s_1, \\ -z_1^2, & \text{if } \beta = \alpha_1 \text{ and } \gamma_1 = e, \\ 0, & \text{if } \beta \neq \alpha_1, \end{cases} \tag{48}$$

and for $2 \leq k \leq n$,

(49)

$$\sigma_{e_\beta}(z_k) = \sum_{\substack{j_1 \geq 0, \\ \gamma_1(\beta) - j_1\alpha_1 \in \Delta_+}} c_{\alpha_1,\beta}^{\gamma_1,j_1} z_1^{j_1} \sigma^{(2)}_{e_{\beta_{\gamma_1(\beta)-j_1\alpha_1}}}(z_k) + \begin{cases} z_1 \sigma^{(2)}_{h_{\alpha_1}}(z_k), & \text{if } \beta = \alpha_1 \text{ and } \gamma_1 = e, \\ 0, & \text{otherwise.} \end{cases}$$

To obtain a closed formula for the vector field σ_{e_β} on $Z_{\mathbf{u}}$, we introduce more notation. Let $\mathbb{N}$ denote the set of non-negative integers.

Notation 4.8. For $\beta \in \Delta_+$ and $(j_1, \ldots, j_n) \in \mathbb{N}^n$, let $\beta_{(j_1)} = \gamma_1(\beta) - j_1\alpha_1 \in \mathfrak{h}^*$, and for $2 \leq k \leq n$, let

$$\begin{aligned} \beta_{(j_1,\ldots,j_k)} &= \gamma_k(\beta_{(j_1,\ldots,j_{k-1})}) - j_k\alpha_k \\ &= \gamma_k\gamma_{k-1}\cdots\gamma_2\gamma_1(\beta) - j_1\gamma_k\gamma_{k-1}\cdots\gamma_2(\alpha_1) - \ldots - j_{k-1}\gamma_k(\alpha_{k-1}) - j_k\alpha_k \in \mathfrak{h}^*, \end{aligned}$$

$$J_k = \left\{(j_1, \ldots, j_{k-1}) \in \mathbb{N}^{k-1} : \beta_{(j_1,\ldots,j_l)} \in \Delta_+, \forall\, 1 \leq l \leq k-1, \text{ and } \beta_{(j_1,\ldots,j_{k-1})} = \alpha_k\right\}.$$

For $2 \leq k \leq n$ and for $(j_1, \ldots, j_{k-1}) \in J_k$, let

$$c^\gamma_{j_1,\ldots,j_{k-1}} = c^{\gamma_1,j_1}_{\alpha_1,\beta} \cdots c^{\gamma_{k-1},j_{k-1}}_{\alpha_{k-1},\beta_{(j_1,\ldots,j_{k-2})}} \neq 0. \tag{50}$$

Here it is understood that $\beta_{(j_1,\ldots,j_{k-2})} = \beta$ if $k = 2$. Also note that for $k \geq 2$ and $1 \leq i \leq k-1$, $c^{\gamma_i,j_i}_{\alpha_i,\beta_{(j_1,\ldots,j_{i-1})}}$ is defined in (42) and (43) when $\beta_{(j_1,\ldots,j_{i-1})} \neq \alpha_i$. When $\beta_{(j_1,\ldots,j_{i-1})} = \alpha_i$, one has $\gamma_i(\beta_{(j_1,\ldots,j_{i-1})}) - j_i\alpha_i \in \Delta_+$ only when $\gamma_i = e$ and $j_i = 0$, and in this case $c^{\gamma_i,j_i}_{\alpha_i,\beta_{(j_1,\ldots,j_{i-1})}} = 1$ as defined in (47).

For each $1 \leq k \leq n$, introduce two functions $\phi^\gamma_\beta(z_1, \ldots, z_{k-1})$ and $\psi^\gamma_\beta(z_1, \ldots, z_{k-1})$ as follows: for $k = 1$, let

$$\phi^\gamma_\beta(z_1, \ldots, z_{k-1}) = \begin{cases} 1 & \text{if } \beta = \alpha_1, \\ 0 & \text{if } \beta \neq \alpha_1, \end{cases} \quad \text{and} \quad \psi^\gamma_\beta(z_1, \ldots, z_{k-1}) = 0, \tag{51}$$

and for $2 \leq k \leq n$, let

$$\phi^\gamma_\beta(z_1, \ldots, z_{k-1}) = \sum_{(j_1,\ldots,j_{k-1}) \in J_k} c^\gamma_{j_1,\ldots,j_{k-1}} z_1^{j_1} z_2^{j_2} \cdots z_{k-1}^{j_{k-1}}, \tag{52}$$

$$\psi^\gamma_\beta(z_1, \ldots, z_{k-1}) = -\sum_{1 \leq i \leq k-1,\, \gamma_i = e} \frac{2\langle \gamma^i(\alpha_i), \gamma^k(\alpha_k)\rangle}{\langle \gamma^i(\alpha_i), \gamma^i(\alpha_i)\rangle} z_i \phi^\gamma_\beta(z_1, \ldots z_{i-1}), \tag{53}$$

where recall that $\gamma^i = \gamma_1\gamma_2\cdots\gamma_i$ for $1 \leq i \leq n$, and the function $\phi^\gamma_\beta(z_1, \ldots, z_{k-1})$ (resp. $\psi^\gamma_\beta(z_1, \ldots, z_{k-1})$) is defined to be 0 if the index set for the summation on the right hand side of (52) (resp. (53)) is empty. ◇

Remark 4.9. Since a root string can have length at most 4, it follows from (52) and (53) that the powers of any coordinate z_i in the polynomials $\phi^\gamma_\beta(z_1, \ldots, z_{k-1})$ and $\psi^\gamma_\beta(z_1, \ldots, z_{k-1})$ can be at most 3 (and 1 when $\mathfrak{g}$ is simply-laced). ◇

The following Theorem 4.10 is our main result for the vector field σ_{e_β}.

Theorem 4.10. *Let $\beta \in \Delta_+$ and let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$. The vector field σ_{e_β} acts on the coordinate functions $(z_1, \ldots, z_n)$ on the affine chart $\mathcal{O}^\gamma$ as follows: for $1 \leq k \leq n$,*

$$\sigma_{e_\beta}(z_k) = \begin{cases} \phi_\beta^\gamma(z_1, \ldots, z_{k-1}) + \psi_\beta^\gamma(z_1, \ldots, z_{k-1})z_k, & \text{if } \gamma_k = s_k, \\ -\phi_\beta^\gamma(z_1, \ldots, z_{k-1})z_k^2 + \psi_\beta^\gamma(z_1, \ldots, z_{k-1})z_k, & \text{if } \gamma_k = e. \end{cases} \tag{54}$$

Proof. When $k = 1$, Theorem 4.10 holds by (51) and by Lemma 4.7. Let $k \geq 2$. Let

$$J_k' = \left\{(j_1, \ldots, j_{k-1}) \in \mathbb{N}^{k-1} : \beta_{(j_1,\ldots,j_l)} \in \Delta_+, \forall\, 1 \leq l \leq k-1\right\},$$

and define $c^\gamma_{j_1,\ldots,j_{k-1}} \in \mathbb{C}^\times$ for $(j_1, \ldots, j_{k-1}) \in J_k'$ as in (50). Then by Lemma 4.7,

$$\sigma_{e_\beta}(z_k) = \sum_{j_1 \in J_2'} c^{\gamma_1, j_1}_{\alpha_1,\beta} z_1^{j_1} \sigma^{(2)}_{e_{\beta_{(j_1)}}}(z_k) + \begin{cases} z_1\sigma^{(2)}_{h_{\alpha_1}}(z_k), & \text{if } \beta = \alpha_1 \text{ and } \gamma_1 = e, \\ 0, & \text{otherwise.} \end{cases} \tag{55}$$

Applying (55) to $\sigma^{(2)}_{e_{\beta_{(j_1)}}}(z_k)$ and repeating the process, one sees using the definition of $\phi_\beta^\gamma(z_1, \ldots, z_{i-1})$ for $1 \leq i \leq k-1$ that

$$\begin{aligned} \sigma_{e_\beta}(z_k) = & \sum_{(j_1,\ldots,j_{k-1}) \in J_k'} c^\gamma_{j_1,\ldots,j_{k-1}} z_1^{j_1} \cdots z_{k-1}^{j_{k-1}} \sigma^{(k)}_{e_{\beta_{(j_1,\ldots,j_{k-1})}}}(z_k) \\ & + \sum_{1 \leq i \leq k-1,\, \gamma_i = e} \phi_\beta^\gamma(z_1, \ldots, z_{i-1}) z_i \sigma^{(i+1)}_{h_{\alpha_i}}(z_k). \end{aligned}$$

Let $z_k' = 1$ if $\gamma_k = s_k$ and $z_k' = -z_k^2$ if $\gamma_k = e$. By Lemma 4.6, for $(j_1, \ldots, j_{k-1}) \in J_k'$, one has $\sigma^{(k)}_{e_{\beta_{(j_1,\ldots,j_{k-1})}}}(z_k) = 0$ unless $\beta_{(j_1,\ldots,j_{k-1})} = \alpha_k$, in which case $\sigma^{(k)}_{e_{\beta_{(j_1,\ldots,j_{k-1})}}}(z_k) = z_k'$. Thus

$$\begin{aligned} \sigma_{e_\beta}(z_k) &= \sum_{(j_1,\ldots,j_{k-1}) \in J_k} c^\gamma_{j_1,\ldots,j_{k-1}} z_1^{j_1} \cdots z_{k-1}^{j_{k-1}} z_k' + \sum_{1 \leq i \leq k-1,\, \gamma_i = e} \phi_\beta^\gamma(z_1, \ldots, z_{i-1}) z_i \sigma^{(i+1)}_{h_{\alpha_i}}(z_k) \\ &= \phi_\beta^\gamma(z_1, \ldots, z_{k-1}) z_k' + \sum_{1 \leq i \leq k-1,\, \gamma_i = e} \phi_\beta^\gamma(z_1, \ldots, z_{i-1}) z_i \sigma^{(i+1)}_{h_{\alpha_i}}(z_k) \end{aligned}$$

On the other hand, for each $1 \leq i \leq k-1$ with $\gamma_i = e$,

$$\sigma^{(i+1)}_{h_{\alpha_i}}(z_k) = -\frac{2\langle \alpha_i, \gamma_{i+1} \cdots \gamma_k(\alpha_k)\rangle}{\langle \alpha_i, \alpha_i \rangle} z_k = -\frac{2\langle \gamma^i(\alpha_i), \gamma^k(\alpha_k)\rangle}{\langle \gamma^i(\alpha_i), \gamma^i(\alpha_i)\rangle} z_k.$$

It follows that

$$\sigma_{e_\beta}(z_k) = \phi_\beta^\gamma(z_1, \ldots, z_{k-1}) z_k' + \psi_\beta^\gamma(z_1, \ldots, z_{k-1}) z_k.$$

Q.E.D.

Remark 4.11. In the context of Theorem 4.10, for a given $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$ and $1 \leq k \leq n$, let $\gamma' = (\gamma_1, \ldots, \gamma_{k-1}, \gamma_k s_k, \gamma_{k+1}', \ldots, \gamma_n') \in \Upsilon_{\mathbf{u}}$, where $\gamma_j' \in \{e, s_j\}$ are arbitrary for $k+1 \leq j \leq n$, and let $(z_1', \ldots, z_n')$ be the coordinates on $\mathcal{O}^{\gamma'}$. Then $z_j = z_j'$ for $1 \leq j \leq k-1$, and $z_k' = 1/z_k$. By (52) and (53),

$$\phi_\beta^\gamma(z_1, \ldots, z_{k-1}) = \phi_\beta^{\gamma'}(z_1, \ldots, z_{k-1}) \quad \text{and} \quad \psi_\beta^\gamma(z_1, \ldots, z_{k-1}) = -\psi_\beta^{\gamma'}(z_1, \ldots, z_{k-1}).$$

One can thus derive one case of the formula (54) from the other case using the change of coordinates $z_k' = 1/z_k$. ◇

Example 4.12. Let β be a simple root and let $\gamma = (e, e, \ldots, e) \in \Upsilon_{\mathbf{u}}$. Then in the affine chart $\mathcal{O}^{(e,e,\ldots,e)}$ with coordinates $(z_1, \ldots, z_n)$ given in (22), the vector field σ_{e_β} is given by

$$\sigma_{e_\beta}(z_k) = -\frac{2\langle \beta, \alpha_k\rangle}{\langle \beta, \beta\rangle}\left(\sum_{1\leq i\leq k-1,\, \alpha_i=\beta} z_i\right) z_k + \begin{cases} 0, & \text{if } \alpha_k \neq \beta, \\ -z_k^2, & \text{if } \alpha_k = \beta, \end{cases} \qquad 1 \leq k \leq n. \tag{56}$$

Indeed, let $1 \leq k \leq n$. By Theorem 4.10, one has,

$$\sigma_{e_\beta}(z_k) = -\phi_\beta^\gamma(z_1, \ldots, z_{k-1})z_k^2 + \psi_\beta^\gamma(z_1, \ldots, z_{k-1})z_k.$$

As β is a simple root, one sees from the definition of ϕ_β^γ that $\phi_\beta^\gamma(z_1, \ldots, z_{k-1}) = 1$ if $\alpha_k = \beta$ and $\phi_\beta^\gamma(z_1, \ldots, z_{k-1}) = 0$ if $\alpha_k \neq \beta$. It follows from the definition of ψ_β^γ that

$$\psi_\beta^\gamma(z_1, \ldots, z_{k-1}) = -\frac{2\langle \beta, \alpha_k\rangle}{\langle \beta, \beta\rangle}\left(\sum_{1\leq i\leq k-1,\, \alpha_i=\beta} z_i\right).$$

This proves (56). Applying Lemma 3.2 and (56), one sees that in the affine chart $\mathcal{O}^{(s_1,e,\ldots,e)}$, the Poisson structure Π is given by

$$\{z_i, z_k\} = \langle \alpha_i, \alpha_k\rangle z_i z_k, \quad \text{if } 2 \leq i < k \leq n,$$

$$\{z_1, z_k\} = \begin{cases} -\langle \alpha_1, \alpha_k\rangle \left(z_1 - 2\sum_{2\leq i\leq k-1,\, \alpha_i=\alpha_1} z_i\right) z_k, & \text{if } 2 \leq k \leq n \text{ and } \alpha_k \neq \alpha_1, \\ -\langle \alpha_1, \alpha_1\rangle \left(z_1 - z_k - 2\sum_{2\leq i\leq k-1,\, \alpha_i=\alpha_1} z_i\right) z_k, & \text{if } 2 \leq k \leq n \text{ and } \alpha_k = \alpha_1. \end{cases}$$

On the other hand, by Lemma 3.4, in the coordinates $(\xi_1, \ldots, \xi_n)$ on $\mathcal{O}^{(e,e,\ldots,e)}$ given by

$$(\xi_1, \xi_2, \ldots, \xi_n) \longmapsto [u_{-\alpha_1}(\xi_1),\, u_{-\alpha_2}(\xi_2),\, \ldots,\, u_{-\alpha_n}(\xi_n)],$$

the Poisson structure π_n is given by $\{\xi_i, \xi_k\} = \langle \alpha_i, \alpha_k\rangle \xi_i \xi_k$ for all $1 \leq i < k \leq n$. It is easy to see that on the intersection $\mathcal{O}^{(e,e,\ldots,e)} \cap \mathcal{O}^{(s_1,e,\ldots,e)}$, the changes between the coordinates $(\xi_1, \xi_2, \ldots, \xi_n)$ on $\mathcal{O}^{(e,e,\ldots,e)}$ and the coordinates $(z_1, z_2, \ldots, z_n)$ on $\mathcal{O}^{(s_1,e,\ldots,e)}$ are given by $z_1 = 1/\xi_1$, and for $2 \leq k \leq n$,

$$z_k = \begin{cases} \xi_k \left(\sum_{\substack{\alpha_i=\alpha_1 \\ 1\leq i\leq k-1}} \xi_i\right)^{\frac{-2\langle \alpha_1, \alpha_k\rangle}{\langle \alpha_1, \alpha_1\rangle}} & \text{if } \alpha_k \neq \alpha_1, \\ \xi_k \left(\sum_{\substack{\alpha_i=\alpha_1 \\ 1\leq i\leq k-1}} \xi_i\right)^{-1} \left(\xi_k + \sum_{\substack{\alpha_i=\alpha_1 \\ 1\leq i\leq k-1}} \xi_i\right)^{-1} & \text{if } \alpha_k = \alpha_1. \end{cases}$$

It is remarkable (see [11] for some details of the calculations) that these changes of coordinates indeed change the quadratic Poisson structure expressed in the coordinates $(z_1, \ldots, z_n)$ to the log-canonical one in the coordinates $(\xi_1, \ldots, \xi_n)$. ◇

4.3. **The Poisson structure π_n in coordinates, II.** Let again $\{e_\alpha \in \mathfrak{g}_\alpha : \alpha \in \Gamma\}$ be a set of root vectors for the simple roots, which gives rise to the coordinates $(z_1, \ldots, z_n)$ on each affine chart $\mathcal{O}^\gamma$ via (22). Recall from Lemma 3.2 that the Poisson structure π_n can be expressed in the coordinates $(z_1, \ldots, z_n)$ on $\mathcal{O}^\gamma$ in terms of the vector fields σ_i, $1 \leq i \leq n-1$ on the Bott-Samelson variety $Z_{(s_{i+1},\ldots,s_n)}$, given in (24). We now apply Theorem 4.10 to the vector fields σ_i.

To this end, extend the set $\{e_\alpha \in \mathfrak{g}_\alpha : \alpha \in \Gamma\}$ to a basis $\{h_\alpha\}_{\alpha\in\Gamma} \cup \{e_\alpha \in \mathfrak{g}_\alpha\}_{\alpha\in\Delta}$ of $\mathfrak{g}$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$ for all $\alpha \in \Delta$. Fix $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$. For $1 \leq i < k \leq n$, define two polynomials in the variables $(z_{i+1}, \ldots, z_{k-1})$ by

$$\phi^{\gamma}_{i,k}(z_{i+1}, \ldots, z_{k-1}) \stackrel{\text{def}}{=} \phi^{(\gamma_{i+1},\ldots,\gamma_n)}_{\alpha_i}(z_{i+1}, \ldots, z_{k-1}), \tag{57}$$

$$\psi^{\gamma}_{i,k}(z_{i+1}, \ldots, z_{k-1}) \stackrel{\text{def}}{=} \psi^{(\gamma_{i+1},\ldots,\gamma_n)}_{\alpha_i}(z_{i+1}, \ldots, z_{k-1}) \tag{58}$$

by taking $\beta = \alpha_i$ and replacing $\mathbf{u}$ by $(s_{i+1}, \ldots, s_n)$ and γ by $(\gamma_{i+1}, \ldots, \gamma_n)$ in (52) and (53). Here recall that when $k = i+1$, it is understood that $\mathbb{C}[z_{i+1}, \ldots, z_{k-1}] = \mathbb{C}$. Let $1 \leq i \leq n-1$. By Theorem 4.10, the vector field σ_i is given in the coordinates $(z_{i+1}, \ldots, z_n)$ on the affine chart $\mathcal{O}^{(\gamma_{i+1},\ldots,\gamma_n)}$ of $Z_{(s_{i+1},\ldots,s_n)}$ by

$$\sigma_i(z_k) = \begin{cases} \phi^{\gamma}_{i,k}(z_{i+1}, \ldots, z_{k-1}) + \psi^{\gamma}_{i,k}(z_{i+1}, \ldots, z_{k-1})z_k, & \text{if } \gamma_k = s_k, \\ -\phi^{\gamma}_{i,k}(z_{i+1}, \ldots, z_{k-1})z_k^2 + \psi^{\gamma}_{i,k}(z_{i+1}, \ldots, z_{k-1})z_k, & \text{if } \gamma_k = e, \end{cases} \quad i < k \leq n. \tag{59}$$

Lemma 4.13. *The polynomials $\phi^{\gamma}_{i,k}(z_{i+1}, \ldots, z_{k-1})$ and $\psi^{\gamma}_{i,k}(z_{i+1}, \ldots, z_{k-1})$, where $\gamma \in \Upsilon_{\mathbf{u}}$ and $1 \leq i < k \leq n$, are independent of the extension of $\{e_\alpha : \alpha \in \Gamma\}$ to the basis $\{h_\alpha\}_{\alpha\in\Gamma} \cup \{e_\alpha \in \mathfrak{g}_\alpha\}_{\alpha\in\Delta}$ of $\mathfrak{g}$.*

Proof. The coordinates $(z_1, \ldots, z_n)$ on $\mathcal{O}^\gamma$ and the definition of the vector fields σ_i, $1 \leq i \leq n-1$, on $Z_{(s_{i+1},\ldots,s_n)}$ depend only on the choice of $\{e_\alpha : \alpha \in \Gamma\}$ and not on its extension to the basis $\{h_\alpha\}_{\alpha\in\Gamma} \cup \{e_\alpha \in \mathfrak{g}_\alpha\}_{\alpha\in\Delta}$ of $\mathfrak{g}$.

Q.E.D.

The following Theorem 4.14, which expresses more explicitly the formula for the Poisson structure π_n on $Z_{\mathbf{u}}$ in the affine coordinates given in Lemma 3.2, is a combination of Lemma 3.2 and Theorem 4.10.

Theorem 4.14. *Let $\{e_\alpha : \alpha \in \Gamma\}$ be any choice of a set of root vectors for the simple roots and let $\gamma \in \Upsilon_{\mathbf{u}}$. Then in the coordinates $(z_1, \ldots, z_n)$ on the affine chart $\mathcal{O}^\gamma$ of $Z_{\mathbf{u}}$ determined by $\{e_\alpha : \alpha \in \Gamma\}$, the Poisson structure π_n is given by*

$$\{z_i, z_k\} = \begin{cases} \langle \gamma^i(\alpha_i),\, \gamma^k(\alpha_k)\rangle z_i z_k, & \text{if } \gamma_i = e \\ -\langle \gamma^i(\alpha_i),\, \gamma^k(\alpha_k)\rangle z_i z_k - \langle \alpha_i, \alpha_i\rangle \sigma_i(z_k) & \text{if } \gamma_i = s_i \end{cases}, \quad 1 \leq i < k \leq n, \tag{60}$$

where for $1 \leq i < k \leq n$, $\sigma_i(z_k) \in \mathbb{C}[z_{i+1}, \ldots, z_k]$ is given in (59). In particular, when $\gamma = \mathbf{u}$ is the full subexpression, $\sigma_i(z_k) \in \mathbb{C}[z_{i+1}, \ldots, z_{k-1}]$ for all $1 \leq i < k \leq n$.

5. The polynomial Poisson algebras $(\mathbb{C}[z_1, \ldots, z_n], \{\,,\,\}_\gamma)$

Throughout §5, fix a Bott-Samelson variety $Z_{\mathbf{u}}$ with $\mathbf{u} = (s_1, \ldots, s_n) = (s_{\alpha_1}, \ldots, s_{\alpha_n})$ and $\alpha_i \in \Gamma$ for $1 \leq i \leq n$,

The coordinates $(z_1, \ldots, z_n)$ on the affine charts $\mathcal{O}^\gamma$ of $Z_{\mathbf{u}}$, where $\gamma \in \Upsilon_{\mathbf{u}}$, depend on the choice of the set $\{e_\alpha : \alpha \in \Gamma\}$ of root vectors for the simple roots. A different choice of such a set gives rise to re-scalings of the coordinates and thus may result in a different Poisson bracket on the polynomial algebra of the coordinate functions. We show in §5.1 that this is not the case.

5.1. **Re-scaling of coordinates.** Let $\{e_\alpha : \alpha \in \Gamma\}$ and $\{e'_\alpha : \alpha \in \Gamma\}$ be two sets of choices of root vectors for the simple roots. For $\alpha \in \Gamma$, let $u_{\pm\alpha}, u'_{\pm\alpha} : \mathbb{C} \to G$ be the one-parameter subgroups of G respectively determined by the $\mathfrak{sl}(2)$-triples $\{e_\alpha, e_{-\alpha}, h_\alpha\}$ and $\{e'_\alpha, e'_{-\alpha}, h_\alpha\}$ (see §1.7), and let

$$\dot{s}_\alpha = u_\alpha(-1)u_{-\alpha}(1)\, u_\alpha(-1) \in N_G(T) \quad \text{and} \quad \dot{s}'_\alpha = u'_\alpha(-1)u'_{-\alpha}(1)\, u'_\alpha(-1) \in N_G(T).$$

For $z \in \mathbb{C}$, and $\kappa \in \{e, s_\alpha\}$, let

$$p_{\kappa,\alpha}(z) = u_{-\kappa(\alpha)}(z)\dot{\kappa} \in P_{s_\alpha} \quad \text{and} \quad p'_{\kappa,\alpha}(z) = u'_{-\kappa(\alpha)}(z)\dot{\kappa}' \in P_{s_\alpha},$$

where recall that $\dot{e} = \dot{e}' = e \in G$. For each $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$, one then has two sets of coordinates $(z_1, \ldots, z_n)$ and $(z'_1, \ldots, z'_n)$ on $\mathcal{O}^\gamma$, respectively by

$$\mathbb{C}^n \ni (z_1, \ldots, z_n) \longmapsto [p_{\gamma_1,\alpha_1}(z_1),\ \ldots,\ p_{\gamma_n,\alpha_n}(z_n)] \in \mathcal{O}^\gamma, \tag{61}$$

$$\mathbb{C}^n \ni (z'_1, \ldots, z'_n) \longmapsto [p'_{\gamma_1,\alpha_1}(z'_1),\ \ldots,\ p'_{\gamma_n,\alpha_n}(z'_n)] \in \mathcal{O}^\gamma. \tag{62}$$

The main result of §5.1 is the following Proposition 5.1.

Proposition 5.1. *Let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$ and let the two sets of coordinates $(z_1, \ldots, z_n)$ and $(z'_1, \ldots, z'_n)$ on $\mathcal{O}^\gamma$ be given as in* (61) *and* (62)*. For $1 \le i < k \le n$, let $\{z_i, z_k\} = f_{i,k}(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]$. Then*

$$\{z'_i,\ z'_k\} = f_{i,k}(z'_1,\ \ldots,\ z'_n), \qquad 1 \le i < k \le n.$$

Remark 5.2. It is easy to see that the two sets of coordinates are related by re-scalings, i.e., there exist $\delta_1, \ldots, \delta_n \in \mathbb{C}^\times$ such that $z'_i = \delta_i z_i$ for each $1 \le i \le n$. One thus has

$$\{z'_i,\ z'_k\} = \delta_i\delta_k\{z_i,\ z_k\} = \delta_i\delta_k f_{i,k}(z_1,\ \ldots,\ z_n) = \delta_i\delta_k f_{i,k}(\delta_1^{-1}z'_1,\ \ldots,\ \delta_n^{-1}z'_n),$$

for all $1 \le i < k \le n$. Proposition 5.1 states that the polynomials $f_{i,k}$ satisfy

$$\delta_i\delta_k f_{i,k}(\delta_1^{-1}z'_1,\ \ldots,\ \delta_n^{-1}z'_n) = f_{i,k}(z'_1,\ \ldots,\ z'_n), \qquad 1 \le i < k \le n.$$

We will show in Lemma 5.4 that the re-scaling of the coordinates comes from the action of an element $t \in T$, from which Proposition 5.1 will follow. ◇

Lemma 5.3. *Let $\alpha \in \Gamma$ and let $\lambda_\alpha \in \mathbb{C}^\times$ be such that $e'_\alpha = \lambda_\alpha e_\alpha$. Then for $\kappa \in \{e, s_\alpha\}$ and $z \in \mathbb{C}$, one has*

$$p'_{\kappa,\alpha}(z) = \begin{cases} p_{\kappa,\alpha}(\lambda_\alpha z)\alpha^\vee(1/\lambda_\alpha), & \kappa = s_\alpha, \\ p_{\kappa,\alpha}(z/\lambda_\alpha), & \kappa = e. \end{cases} \tag{63}$$

Proof. Let $\theta_\alpha, \theta'_\alpha : SL(2,\mathbb{C}) \to G$ be the Lie group homomorphisms respectively determined by the $\mathfrak{sl}(2)$-triples $\{e_\alpha, e_{-\alpha}, h_\alpha\}$ and $\{e'_\alpha, e'_{-\alpha}, h_\alpha\}$, where note that $e'_{-\alpha} = \lambda_\alpha^{-1} e_{-\alpha}$ (see §1.7). Choose either one of the two square roots of λ_α in $\mathbb{C}^\times$ and denote it by $\sqrt{\lambda_\alpha}$. Then

$$\theta'_\alpha = \mathrm{Ad}_{\alpha^\vee(\sqrt{\lambda_\alpha})} \circ \theta_\alpha,$$

where $\mathrm{Ad}_{\alpha^\vee(\sqrt{\lambda_\alpha})} : G \to G$ denotes conjugation by $\alpha^\vee(\sqrt{\lambda_\alpha}) \in T$. It follows that

$$\dot{s}'_\alpha = \mathrm{Ad}_{\alpha^\vee(\sqrt{\lambda_\alpha})}(\dot{s}_\alpha) = \dot{s}_\alpha\, \alpha^\vee(1/\lambda_\alpha), \tag{64}$$

and thus

$$p'_{\kappa,\alpha}(z) = \mathrm{Ad}_{\alpha^\vee(\sqrt{\lambda_\alpha})}(p_{\kappa,\alpha}(z)) = \begin{cases} p_{\kappa,\alpha}(\lambda_\alpha z)\alpha^\vee(1/\lambda_\alpha), & \kappa = s_\alpha, \\ p_{\kappa,\alpha}(z/\lambda_\alpha), & \kappa = e. \end{cases}$$

Q.E.D.

For $\alpha \in \Gamma$, let $\lambda_\alpha \in \mathbb{C}^\times$ be as in Lemma 5.3. Choose any $t \in T$ such that

$$t^\alpha = \lambda_\alpha, \quad \forall\, \alpha \in \Gamma. \tag{65}$$

Such an element indeed exists, as it can be taken to be any of the preimages in $T \subset G$ of the unique such element in the maximal torus $T/Z(G)$ of $G_{\text{ad}} \stackrel{\text{def}}{=} G/Z(G)$, where $Z(G)$ is the center of G. Recall from (19) that $\cdot$ denotes the left action of B on $Z_{\mathbf{u}}$.

Lemma 5.4. *For any $t \in T$ satisfying (65) and for any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$, one has*

$$t \cdot [p_{\gamma_1,\alpha_1}(z_1),\ \ldots,\ p_{\gamma_n,\alpha_n}(z_n)] = [p'_{\gamma_1,\alpha_1}(z_1),\ \ldots,\ p'_{\gamma_n,\alpha_n}(z_n)], \qquad (z_1, \ldots, z_n) \in \mathbb{C}^n. \tag{66}$$

Proof. We prove Lemma 5.4 by induction on n. When $n = 1$, $t^{-\gamma_1(\alpha_1)} = t^{\alpha_1} = \lambda_{\alpha_1}$ if $\gamma_1 = s_1$ and $t^{-\gamma_1(\alpha_1)} = t^{-\alpha_1} = 1/\lambda_{\alpha_1}$ if $\gamma_1 = e$, so by Lemma 5.3,

$$t \cdot [p_{\gamma_1,\alpha_1}(z_1)] = [p_{\gamma_1,\alpha_1}(t^{-\gamma_1(\alpha_1)} z_1)] = [p'_{\gamma_1,\alpha_1}(z_1)].$$

Let $n \geq 2$ and assume that Lemma 5.4 holds for $n - 1$. Then

$$\begin{aligned} t \cdot [p_{\gamma_1,\alpha_1}(z_1),\ \ldots,\ p_{\gamma_n,\alpha_n}(z_n)] \\ = [p_{\gamma_1,\alpha_1}(t^{-\gamma_1(\alpha_1)} z_1),\ t^{\gamma_1} p_{\gamma_2,\alpha_2}(z_2),\ p_{\gamma_3,\alpha_3}(z_3),\ \ldots,\ p_{\gamma_n,\alpha_n}(z_n)]. \end{aligned}$$

Here for $\kappa \in \{e, s_\alpha\}$, we set $t^\kappa = \dot{\kappa}^{-1} t \dot{\kappa} \in T$. If $\gamma_1 = e$, then $p_{\gamma_1,\alpha_1}(t^{-\gamma_1(\alpha_1)} z_1) = p_{\gamma_1,\alpha_1}(z_1/\lambda_{\alpha_1}) = p'_{\gamma_1,\alpha_1}(z_1)$, so (66) holds by the induction assumption. Assume that $\gamma_1 = s_1$. Then by Lemma 5.3,

$$\begin{aligned} t \cdot [p_{\gamma_1,\alpha_1}(z_1),\ \ldots,\ p_{\gamma_n,\alpha_n}(z_n)] \\ = [p'_{\gamma_1,\alpha_1}(z_1),\ \alpha_1^\vee(\lambda_{\alpha_1}) t^{s_1} p_{\gamma_2,\alpha_2}(z_2),\ p_{\gamma_3,\alpha_3}(z_3),\ \ldots,\ p_{\gamma_n,\alpha_n}(z_n)]. \end{aligned}$$

Consider now the element $\alpha_1^\vee(\lambda_{\alpha_1}) t^{s_1} \in T$. For every $\alpha \in \Gamma$, one has

$$(\alpha_1^\vee(\lambda_{\alpha_1}) t^{s_1})^\alpha = \lambda_{\alpha_1}^{\frac{2\langle \alpha, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle}} t^{s_1(\alpha)} = t^{\frac{2\langle \alpha, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle} \alpha_1 + s_1(\alpha)} = t^\alpha = \lambda_\alpha.$$

By the induction assumption, one then has

$$\begin{aligned} \alpha_1^\vee(\lambda_{\alpha_1}) t^{s_1} \cdot [p_{\gamma_2,\alpha_2}(z_2), p_{\gamma_3,\alpha_3}(z_3),\ \ldots,\ p_{\gamma_n,\alpha_n}(z_n)] \\ = [p'_{\gamma_2,\alpha_2}(z_2),\ \ldots,\ p'_{\gamma_n,\alpha_n}(z_n)] \in Z_{(s_1, \ldots, s_n)}, \end{aligned}$$

and hence (66) holds.

Q.E.D.

Proof of Proposition 5.1: Let $(z_1, \ldots, z_n), (z'_1, \ldots, z'_n) \in \mathbb{C}^n$ be such that

$$[p_{\gamma_1,\alpha_1}(z_1),\ \ldots,\ p_{\gamma_n,\alpha_n}(z_n)] = [p'_{\gamma_1,\alpha_1}(z'_1),\ \ldots,\ p'_{\gamma_n,\alpha_n}(z'_n)] \in \mathcal{O}^\gamma.$$

Let t be any element in T satisfying (65). By Lemma 5.4,

$$\begin{aligned} t \cdot [p_{\gamma_1,\alpha_1}(z'_1),\ \ldots,\ p_{\gamma_n,\alpha_n}(z'_n)] &= [p'_{\gamma_1,\alpha_1}(z'_1),\ \ldots,\ p'_{\gamma_n,\alpha_n}(z'_n)] \\ &= [p_{\gamma_1,\alpha_1}(z_1),\ \ldots,\ p_{\gamma_n,\alpha_n}(z_n)]. \end{aligned}$$

It follows that

$$[p_{\gamma_1,\alpha_1}(z'_1),\ \ldots,\ p_{\gamma_n,\alpha_n}(z'_n)] = t^{-1} \cdot [p_{\gamma_1,\alpha_1}(z_1),\ \ldots,\ p_{\gamma_n,\alpha_n}(z_n)].$$

Denote by $(t^{-1})^* : \mathbb{C}[\mathcal{O}^\gamma] \to \mathbb{C}[\mathcal{O}^\gamma]$ the Poisson isomorphism on the algebra $\mathbb{C}[\mathcal{O}^\gamma]$ of regular functions on $\mathcal{O}^\gamma$ induced by the action of $t^{-1} \in T$. One then has $z_i' = (t^{-1})^* z_i$ for every $1 \leq i \leq n$, and hence for any $1 \leq i, k \leq n$,

$$\{z_i', \, z_k'\} = \{(t^{-1})^* z_i, \, (t^{-1})^* z_k\} = (t^{-1})^* \{z_i, \, z_k\} = (t^{-1})^* f_{i,k} = f_{i,k}(z_1', \ldots, z_n').$$

This finishes the proof of Proposition 5.1.

5.2. **The Poisson algebra $(\mathbb{C}[z_1, \ldots, z_n], \{\, , \,\}_\gamma)$ as an iterated T-Poisson Ore extension of $\mathbb{C}$.** Recall [18, 27, 37] that a Poisson polynomial algebra

$$A = (\mathbb{C}[z_1, \ldots, z_n], \; \{\, , \,\})$$

is said to be an *iterated Poisson Ore extension* (of $\mathbb{C}$) if the Poisson bracket $\{\, , \,\}$ satisfies

$$\{z_i, \mathbb{C}[z_{i+1}, \ldots, z_n]\} \subset z_i \mathbb{C}[z_{i+1}, \ldots, z_n] + \mathbb{C}[z_{i+1}, \ldots, z_n], \qquad 1 \leq i \leq n-1.$$

In such a case, define the derivations a_i and b_i on $\mathbb{C}[z_{i+1}, \ldots, z_n]$ by

$$\{z_i, f\} = z_i a_i(f) + b_i(f), \quad 1 \leq i \leq n-1, \;\; f \in \mathbb{C}[z_{i+1}, \ldots, z_n]. \tag{67}$$

Then [37] for each $1 \leq i \leq n-1$, a_i is a Poisson derivation, and b_i an a_i-Poisson derivation, of the Poisson subalgebra $\mathbb{C}[z_{i+1}, \ldots, z_n]$ of the Poisson algebra A, i.e.,

$$a_i\{f, \, g\} = \{a_i(f), \; g\} + \{f, \, a_i(g)\}, \tag{68}$$

$$b_i\{f, \, g\} = \{b_i(f), \; g\} + \{f, \, b_i(g)\} + a_i(f)b_i(g) - b_i(f)a_i(g) \tag{69}$$

for $f, g \in \mathbb{C}[z_{i+1}, \ldots, z_n]$. In this case, the Poisson algebra A is also denoted as

$$A = \mathbb{C}[z_n]\, [z_{n-1}; \; a_{n-1}, \; b_{n-1}] \cdots [z_2; \; a_2, \; b_2]\, [z_1; \; a_1, \; b_1]. \tag{70}$$

An iterated Poisson Ore extension as in (70) is said to be *nilpotent* [20, Definition 4] if b_i is a locally nilpotent derivation of $\mathbb{C}[z_{i+1}, \ldots, z_n]$ for each $1 \leq i \leq n-1$. The following Definition 5.5 follows [20, Definition 4] but emphasizes on the torus actions.

Definition 5.5. Let $A = (\mathbb{C}[z_1, \ldots, z_n], \{\, , \,\})$ be a polynomial Poisson algebra and $\mathbb{T}$ a complex algebraic torus with Lie algebra $\mathfrak{t}$ acting on A rationally [18] by Poisson algebra automorphisms. A is said to be an *iterated* $\mathbb{T}$*-Poisson Ore extension (of* $\mathbb{C}$*)* (with respect to the given $\mathbb{T}$-action) if each z_i, $1 \leq i \leq n$, is a weight vector for the $\mathbb{T}$-action with weight $\lambda_i \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^\times)$, and if

$$A = \mathbb{C}[z_n]\, [z_{n-1}; \; a_{n-1}, \; b_{n-1}] \cdots [z_2; \; a_2, \; b_2]\, [z_1; \; a_1, \; b_1]$$

is an iterated Poisson Ore extension such that there exist $h_1, \ldots, h_n \in \mathfrak{t}$ satisfying $\lambda_i(h_i) \neq 0$ and $a_i = h_i|_{\mathbb{C}[z_{i+1}, \ldots, z_n]}$ for each $1 \leq i \leq n-1$. Such an iterated $\mathbb{T}$-Poisson Ore extension is said to be *symmetric* if

$$b_i(z_k) \in \mathbb{C}[z_{i+1}, \ldots, z_{k-1}], \quad 1 \leq i < k \leq n,$$

and if, there exist $h_1', \ldots, h_n' \in \mathfrak{t}$ such that $\lambda_i(h_i') \neq 0$ for $2 \leq i \leq n$ and

$$\lambda_i(h_k') = \lambda_k(h_i), \qquad 1 \leq i < k \leq n. \tag{71}$$

Following [23] (see Remark 5.8), a polynomial Poisson algebra which is a symmetric iterated $\mathbb{T}$-Poisson Ore extension for some torus $\mathbb{T}$ is called a *symmetric Poisson CGL extension (of* $\mathbb{C}$*)*. ◇

Remark 5.6. For an iterated $\mathbb{T}$-Poisson Ore extension as in Definition 5.5, one has

$$\{z_i,\, z_k\} = a_i(z_k)z_i + b_i(z_k) = \lambda_k(h_i)z_iz_k + b_i(z_k) \in \lambda_k(h_i)z_iz_k + \mathbb{C}[z_{i+1}, \ldots, z_n]$$

for all $1 \leq i < k \leq n$, a property referred to as *semi-quadratic* in [20, Definition 4]. ◇

Remark 5.7. Let A be an iterated $\mathbb{T}$-Poisson Ore extension as in Definition 5.5. Then

$$[h|_{\mathbb{C}[z_{i+1},\ldots,z_n]},\, b_i] = \lambda_i(h)b_i, \qquad 1 \leq i \leq n-1, \;\; h \in \mathfrak{t}, \tag{72}$$

where the left hand side denotes the commutator bracket between the two derivations $h|_{\mathbb{C}[z_{i+1},\ldots,z_n]}$ and b_i of $\mathbb{C}[z_{i+1}, \ldots, z_n]$. In fact, (72) is equivalent to

$$[h|_{\mathbb{C}[z_{i+1},\ldots,z_n]},\, b_i](z_k) = \lambda_i(h)b_i(z_k), \quad 1 \leq i < k \leq n, \; h \in \mathfrak{t},$$

which, by the fact that z_j is a $\mathbb{T}$-weight vector with weight λ_j for each $1 \leq j \leq n$, is in turn equivalent to

$$h(\{z_i, z_k\}) = \{h(z_i), z_k\} + \{z_i, h(z_k)\}, \quad h \in \mathfrak{t}, \; 1 \leq i < k \leq n,$$

which follows from the assumption that $\mathbb{T}$ acts on A by Poisson automorphisms. In particular, one has

$$[a_i,\, b_i] = \lambda_i(h_i)b_i, \qquad 1 \leq i \leq n-1.$$

Let $1 \leq i \leq n-1$ and consider the 2-dimensional Lie bialgebra $\mathfrak{b}_2 = \mathbb{C}x + \mathbb{C}y$ with Lie bracket $[x, y] = 2y$ and Lie co-bracket $\delta : \mathfrak{b}_2 \to \wedge^2\mathfrak{b}_2$ given by $\delta(x) = 0$ and $\delta(y) = -\frac{\lambda_i(h_i)}{2} x \wedge y$. Consider the Poisson subalgebra $A_{i+1} = \mathbb{C}[z_{i+1}, \ldots, z_n]$ of A and let $\mathrm{Der}_{\mathbb{C}}(A_{i+1})$ be the Lie algebra of derivations (for the commutative algebra structure) of A_{i+1}. Define the Lie algebra anti-homomorphism $\sigma : \mathfrak{b}_2 \to \mathrm{Der}_{\mathbb{C}}(A_{i+1})$ by

$$\sigma(x) = -\frac{2}{\lambda_i(h_i)}a_i, \quad \sigma(y) = \frac{1}{\lambda_i(h_i)}b_i.$$

Then (68) and (69) are equivalent to σ being a *left Poisson action of the Lie bialgebra* $(\mathfrak{b}_2, \delta)$ on the Poisson algebra A_{i+1} (see [30, §2]). Let $\mathfrak{b}_2^*$ be the dual vector space of $\mathfrak{b}_2$ with basis (x^*, y^*) dual to the basis (x, y) of $\mathfrak{b}_2$. Then the dual Lie bialgebra of $(\mathfrak{b}_2, \delta)$ is $\mathfrak{b}_2^*$ with Lie bracket $[x^*, y^*] = -\frac{\lambda_i(h_i)}{2}y^*$ and Lie co-bracket $x^* \mapsto 0$ and $y^* \mapsto 2x^* \wedge y^*$. Let $\rho : \mathfrak{b}_2^* \to \mathrm{Der}_{\mathbb{C}}\mathbb{C}[z_i]$ be the Lie algebra homomorphism given by

$$\rho(x^*) = \frac{\lambda_i(h_i)}{2}z_i\partial/\partial z_i, \quad \rho(y^*) = -\lambda_i(h_i)\partial/\partial z_i.$$

Then ρ is a *right Poisson action of the Lie bialgebra* $\mathfrak{b}_2^*$ on $\mathbb{C}[z_i]$ with the trivial Poisson bracket. The Poisson Ore extension $A_i := \mathbb{C}[z_i, z_{i+1}, \ldots, z_n]$ of A_{i+1} with the Poisson bracket given in (67) can now be interpreted as the *mixed product Poisson structure* on $A_i = \mathbb{C}[z_i] \otimes A_{i+1}$ defined by the pair (ρ, σ) of Poisson actions of Lie bialgebras introduced in [30]. ◇

Remark 5.8. A symmetric iterated $\mathbb{T}$-Poisson Ore extension is automatically nilpotent. Indeed, let $1 \leq i \leq n-1$ and let the notation be as in Definition 5.5. To show that b_i is locally nilpotent as a derivation of $\mathbb{C}[z_{i+1}, \ldots, z_n]$, observe first that for integers $m, N \geq 1$ and $f_1, f_2, \ldots, f_m \in \mathbb{C}[z_{i+1}, \ldots, z_n]$, $b_i^N(f_1f_2\cdots f_m)$ is a linear combination of terms of the form $b_i^{N_1}(f_1)b_i^{N_2}(f_1)\cdots b_i^{N_m}(f_m)$ with $N_1 + N_2 + \cdots + N_m = N$. Thus b_i is locally nilpotent if for each $i < k \leq n$, $b_i^{N_k}(z_k) = 0$ for some integer $N_k \geq 1$. As $b_i(z_{i+1}) \in \mathbb{C}$, one has $b_i^2(z_{i+1}) = 0$. Assume that there exist $N_j \geq 1$ such that $b_i^{N_j}(z_j) = 0$ for $i+1 \leq j \leq k-1$. As $b_i(z_k) \in \mathbb{C}[z_{i+1}, \ldots, z_{k-1}]$, the above observation shows that there is an integer $N_k \geq 1$ such that $b_i^{N_k}(z_k) = 0$. Induction on k now shows that b_i is locally nilpotent. Observe also that if A is a symmetric iterated $\mathbb{T}$-Poisson Ore extension, then for $1 \leq i < k \leq n$,

$$\{z_i,\, z_k\} = \lambda_k(h_i)z_iz_k + b_i(z_k) \in \lambda_k(h_i)z_iz_k + \mathbb{C}[z_{i+1}, \ldots, z_{k-1}] \subset \mathbb{C}[z_i, \ldots, z_k]. \tag{73}$$

Consequently, $\mathbb{C}[z_i, \ldots, z_k]$ is a Poisson subalgebra of A for all $1 \leq i < k \leq n$. ◇

Lemma 5.9. [23] *If $A = (\mathbb{C}[z_1, \ldots, z_n], \{\, ,\, \})$ is a symmetric iterated $\mathbb{T}$-Poisson Ore extension, then, with respect to the same $\mathbb{T}$-action, A is a $\mathbb{T}$-Poisson Ore extension in the reversed order of the variables. More precisely, in the notation of Definition 5.5, for each $2 \leq k \leq n$, $\mathbb{C}[z_1, \ldots, z_{k-1}]$ is a Poisson subalgebra of A, and*

$$\{f,\, z_k\} = a_k'(f)z_k + b_k'(f), \qquad f \in \mathbb{C}[z_1, \ldots, z_{k-1}], \tag{74}$$

where $a_k' = h_k'|_{\mathbb{C}[z_1,\ldots,z_{k-1}]}$ as a derivation of $\mathbb{C}[z_1, \ldots, z_{k-1}]$ and b_k' is the unique derivation of $\mathbb{C}[z_1, \ldots, z_{k-1}]$ such that $b_k'(z_i) = b_i(z_k) \in \mathbb{C}[z_{i+1}, \ldots, z_{k-1}]$ for $1 \leq i \leq k-1$. Moreover, for any $h \in \mathfrak{t}$, $[h|_{\mathbb{C}[z_1,\ldots,z_{k-1}]},\, b_k'] = \lambda_k(h)b_k'$ as derivations of $\mathbb{C}[z_1, \ldots, z_{k-1}]$.

Proof. It follows from (73) that $\mathbb{C}[z_1, \ldots, z_{k-1}]$ is a Poisson subalgebra of A for every $2 \leq k \leq n$. The assumption that $\lambda_i(h_k') = \lambda_k(h_i)$ for all $1 \leq i < k \leq n$ and the definition of the b_k''s imply that (74) holds for $f = z_i$ for each $i < k$, so it holds for all $f \in \mathbb{C}[z_1, \ldots, z_{k-1}]$. Let $h \in \mathfrak{t}$ and $2 \leq k \leq n$. Then for each $1 \leq i \leq k-1$, using (72), one has $h(b_i(z_k)) - b_i(h(z_k)) = \lambda_i(h)b_i(z_k)$, from which one has

$$h(b_i(z_k)) - \lambda_i(h)b_i(z_k) = b_i(h(z_k)) = \lambda_k(h)b_i(z_k),$$

and it follows that

$$h(b_k'(z_i)) - b_k'(h(z_i)) = h(b_i(z_k)) - \lambda_i(h)b_i(z_k) = \lambda_k(h)b_i(z_k) = \lambda_k(h)b_k'(z_i).$$

This proves that $[h|_{\mathbb{C}[z_1,\ldots,z_{k-1}]},\, b_k'] = \lambda_k(h)b_k'$ as derivations of $\mathbb{C}[z_1, \ldots, z_{k-1}]$.

Q.E.D.

Notation 5.10. In the context of Lemma 5.9, one also writes

$$A = \mathbb{C}[z_1]\,[z_2;\, a_2',\, b_2'] \cdots [z_{n-1};\, a_{n-1}',\, b_{n-1}']\,[z_n;\, a_n',\, b_n']. \tag{75}$$

◇

We now return to the Bott-Samelson variety $Z_{\mathbf{u}}$ with the Poisson structure π_n, where $\mathbf{u} = (s_1, \ldots, s_n) = (s_{\alpha_1}, \ldots, s_{\alpha_n})$, and choose again a set $\{e_\alpha : \alpha \in \Gamma\}$ of root vectors for the simple roots, so that one has the parametrization $\Phi^\gamma : \mathbb{C}^n \to \mathcal{O}^\gamma$ for each $\gamma \in \Upsilon_{\mathbf{u}}$.

Definition 5.11. For $\gamma \in \Upsilon_{\mathbf{u}}$, to emphasize on the dependence of γ, let $\{\, ,\, \}_\gamma$ denote the Poisson structure on the polynomial algebra $\mathbb{C}[z_1, \ldots, z_n]$ defined by the Poisson structure π_n on $\mathcal{O}^\gamma$ via the parametrization $\Phi^\gamma : \mathbb{C}^n \to \mathcal{O}^\gamma$. ◇

Fix $\gamma \in \Upsilon_{\mathbf{u}}$. Recall that the maximal torus T acts on $\mathcal{O}^\gamma$ by (23), which gives rise to a rational action of T on $(\mathbb{C}[z_1, \ldots, z_n], \{\, ,\, \}_\gamma)$ by Poisson automorphisms. More precisely,

$$t \cdot z_i = t^{-\gamma^i(\alpha_i)} z_i, \qquad 1 \leq i \leq n. \tag{76}$$

For $h \in \mathfrak{h} = \mathrm{Lie}(T)$, let h also denote the Poisson derivation of $(\mathbb{C}[z_1, \ldots, z_n], \{\,,\,\}_\gamma)$ generating the T-action in the direction of h, i.e,

$$h(z_i) = -\gamma^i(\alpha_i)(h)z_i, \quad 1 \leq i \leq n, \; h \in \mathfrak{h}. \tag{77}$$

Note that both the T-action and the derivations h on $\mathbb{C}[z_1, \ldots, z_n]$ for $h \in \mathfrak{h}$ depend on γ, but for notational simplicity we do not include the dependence on γ in the notation. For $1 \leq i \leq n-1$, recall from Notation 3.1 the vector field σ_i on the Bott-Samelson variety $Z_{(s_{i+1},\ldots,s_n)}$ and that for $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$, elements in $\mathbb{C}[z_{i+1}, \ldots, z_n]$ are regarded as regular functions on both $\mathcal{O}^{(\gamma_{i+1},\ldots,\gamma_n)} \subset Z_{(s_{i+1},\ldots,s_n)}$ and $\mathcal{O}^\gamma$.

Theorem 5.12. *For each $\gamma \in \Upsilon_{\mathbf{u}}$, $(\mathbb{C}[z_1, \ldots, z_n], \{\,,\,\}_\gamma)$ is an iterated T-Poisson Ore extension of $\mathbb{C}$ with respect to the T-action on given in* (76). *More explicitly,*

$$(\mathbb{C}[z_1, \ldots, z_n], \{\,,\,\}_\gamma) = \mathbb{C}[z_n]\,[z_{n-1};\, a_{n-1},\, b_{n-1}] \cdots [z_2;\, a_2,\, b_2]\,[z_1;\, a_1,\, b_1], \tag{78}$$

where for $1 \leq i \leq n-1$,

$$a_i = -\frac{\langle \alpha_i, \alpha_i \rangle}{2} \gamma^{i-1}(h_{\alpha_i})|_{\mathbb{C}[z_{i+1},\ldots,z_n]}, \quad b_i = \begin{cases} 0, & \text{if } \gamma_i = e, \\ -\langle \alpha_i, \alpha_i \rangle \sigma_i, & \text{if } \gamma_i = s_i. \end{cases} \tag{79}$$

When $\gamma = \mathbf{u}$, the extension is symmetric. More explicitly, for $\gamma = \mathbf{u}$, one also has

$$A = \mathbb{C}[z_1]\,[z_2;\, a_2',\, b_2'] \cdots [z_{n-1};\, a_{n-1}',\, b_{n-1}']\,[z_n;\, a_n',\, b_n'], \tag{80}$$

where for $2 \leq k \leq n$, $a_k' = -\frac{\langle \alpha_k, \alpha_k \rangle}{2} \gamma^{k-1}(h_{\alpha_k})|_{\mathbb{C}[z_1,\ldots,z_{k-1}]}$, and b_k' is the unique derivation of $\mathbb{C}[z_1, \ldots, z_{k-1}]$ such that $b_k'(z_i) = -\langle \alpha_i, \alpha_i \rangle \sigma_i(z_k)$ for $1 \leq i \leq k-1$.

Proof. Let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$ and let $\lambda_i = -\gamma^i(\alpha_i)$ for $1 \leq i \leq n$. By (76), z_i is a weight vector for the T-action on $\mathbb{C}[z_1, \ldots, z_n]$ with weight λ_i. For $1 \leq i \leq n$, define $h_i \in \mathfrak{h} = \mathrm{Lie}(T)$ by

$$h_i = -\frac{\langle \alpha_i, \alpha_i \rangle}{2} \gamma^{i-1}(h_{\alpha_i}) = \begin{cases} -\frac{\langle \alpha_i, \alpha_i \rangle}{2} \gamma^i(h_{\alpha_i}), & \text{if } \gamma_i = e, \\ \frac{\langle \alpha_i, \alpha_i \rangle}{2} \gamma^i(h_{\alpha_i}), & \text{if } \gamma_i = s_i. \end{cases} \tag{81}$$

Then for $1 \leq i < k \leq n$,

$$h_i(z_k) = \lambda_k(h_i) z_k = -\gamma^k(\alpha_k)(h_i) z_k = \langle \gamma^{i-1}(\alpha_i),\, \gamma^k(\alpha_k) \rangle z_k.$$

It now follows from Theorem 4.14 that (78) holds with the a_i's and b_i's given by (79). Moreover, for each $1 \leq i \leq n$, $\lambda_i(h_i) \neq 0$, as

$$\lambda_i(h_i) = \langle \gamma^{i-1}(\alpha_i),\, \gamma^i(\alpha_i) \rangle = \langle \alpha_i,\, \gamma_i(\alpha_i) \rangle = \begin{cases} \langle \alpha_i,\, \alpha_i \rangle, & \gamma_i = e, \\ -\langle \alpha_i,\, \alpha_i \rangle, & \gamma_i = s_i. \end{cases} \tag{82}$$

Thus $(\mathbb{C}[z_1, \ldots, z_n], \{\,,\,\}_\gamma)$ is an iterated T-Poisson Ore extension of $\mathbb{C}$.

Assume now that $\gamma = \mathbf{u}$ is the full subexpression of $\mathbf{u}$. In this case, let

$$h_i = -\frac{\langle \alpha_i,\, \alpha_i \rangle}{2} \gamma^{i-1}(h_{\alpha_i}) = -\frac{\langle \alpha_i,\, \alpha_i \rangle}{2} s_1 s_2 \cdots s_{i-1}(h_{\alpha_i}) \in \mathfrak{h}, \quad 1 \leq i \leq n,$$

and let $h_k' = h_k$ for $2 \leq k \leq n$. With $\lambda_i = s_1 s_2 \cdots s_{i-1}(\alpha_i)$, one has, for $1 \leq i < k \leq n$,

$$\lambda_i(h_k') = -\langle \gamma^i(\alpha_i),\, \gamma^k(\alpha_k) \rangle = -\langle s_1 s_2 \cdots s_{i-1}(\alpha_i),\, s_1 s_2 \cdots s_{k-1}(\alpha_k) \rangle = \lambda_k(h_i).$$

By Theorem 4.14, one also has $b_i(z_k) \in \mathbb{C}[z_{i+1}, \ldots, z_{k-1}]$ for $1 \leq i < k \leq n$. This shows that $(\mathbb{C}[z_1, \ldots, z_n], \{\,,\,\}_{\mathbf{u}})$, as an iterated T-Poisson Ore extension of $\mathbb{C}$ with respect to the T-action given in (76), is symmetric. By Lemma 5.9, (80) holds.

Q.E.D.

Remark 5.13. We already know from Remark 5.7 that for $h \in \mathfrak{t}$ and $1 \leq i \leq n-1$, the two derivations $a_h := h|_{\mathbb{C}[z_{i+1},\ldots,z_n]}$ and b_i on $\mathbb{C}[z_{i+1}, \ldots, z_n]$ in Theorem 5.12 satisfy $[a_h, b_i] = \lambda_i(h)b_i$. This can also be checked directly: it clearly holds when $\gamma_i = e$. Assume that $\gamma_i = s_i$. In the notation of (46) and by Lemma 2.2, one has $a_h = \sigma^{(i+1)}_{(\gamma^i)^{-1}(h)}$ and $b_i = -\langle \alpha_i, \alpha_i \rangle \sigma^{(i+1)}_{e_{\alpha_i}}$. Thus

$$[a_h,\, b_i] = -\langle \alpha_i, \alpha_i \rangle \left[\sigma^{(i+1)}_{(\gamma^i)^{-1}(h)},\, \sigma^{(i+1)}_{e_{\alpha_i}}\right] = \langle \alpha_i, \alpha_i \rangle \sigma^{(i+1)}_{[(\gamma^i)^{-1}(h), e_{\alpha_i}]} = \lambda_i(h) b_i.$$

$\diamond$

Remark 5.14. For an arbitrary $\gamma \in \Upsilon_{\mathbf{u}}$, $(\mathbb{C}[z_1, \ldots, z_n], \{\, ,\, \}_\gamma)$ expressed as an iterated T-Poisson Ore extension as in (78) is not necessarily a Poisson CGL extension in the sense of [23], as the definition in [23] requires the derivations b_i be locally nilpotent (recall Remark 5.8). In Example 3.3 for $\gamma = (s_{\alpha_1}, e, e)$, the derivation b_1 on $\mathbb{C}[z_2, z_3]$ is given by $b_1(z_2) = 0$ and $b_1(z_3) = 2z_3^2$ which is not locally nilpotent. $\diamond$

5.3. **The Poisson structure π_n in $\mathcal{O}^{\mathbf{u}}$.** We now look in more detail at the Poisson polynomial algebra $(\mathbb{C}[z_1, \ldots, z_n], \{\, ,\, \}_{\mathbf{u}})$. In this case, T acts on $\mathbb{C}[z_1, \ldots, z_n]$ by

$$t \cdot z_i = t^{s_1 s_2 \cdots s_{i-1}(\alpha_i)} z_i, \qquad t \in T,\ 1 \leq i \leq n. \tag{83}$$

Due to its connection with the Levendorskii-Soibelman straightening law explained in §1.3 and applications to the standard Poisson structures on generalized Bruhat cells explained in §1.4, we now extract from Theorem 4.14 and (59) the details of the explicit formula for the Poisson bracket $\{\, ,\, \}_{\mathbf{u}}$.

Theorem 5.15. *In the coordinates $(z_1, \ldots, z_n)$ on $\mathcal{O}^{\mathbf{u}}$ given by the parametrization $\Phi^{\mathbf{u}} : \mathbb{C}^n \to \mathcal{O}^{\mathbf{u}}$, one has*

$$\{z_i,\, z_k\}_{\mathbf{u}} = c_{i,k} z_i z_k + b_i(z_k)$$

for $1 \leq i < k \leq n$, where

$$c_{i,k} = -\langle s_1 s_2 \cdots s_{i-1}(\alpha_i),\ s_1 s_2 \cdots s_{k-1}(\alpha_k) \rangle, \tag{84}$$

and $b_i(z_k) \in \mathbb{C}[z_{i+1}, \ldots, z_{k-1}]$ is given as follows:

1) If $k = i+1$, one has $b_i(z_{i+1}) = 0$ if $\alpha_{i+1} \neq \alpha_i$, and $b_i(z_{i+1}) = -\langle \alpha_i, \alpha_i \rangle$ if $\alpha_{i+1} = \alpha_i$;

2) Assume that $k > i+1$. For $(j_{i+1}, \ldots, j_{k-1}) \in \mathbb{N}^{k-i-1}$ and $i+1 \leq l \leq k-1$, let

$$\begin{aligned}\beta_{(j_{i+1},\ldots,j_l)} &= s_l s_{l-1} \cdots s_{i+2} s_{i+1}(\alpha_i) - j_{i+1} s_l s_{l-1} \cdots s_{i+2}(\alpha_{i+1}) - \cdots - j_{l-1} s_l(\alpha_{l-1}) - j_l \alpha_l \\ &= s_l(\beta_{(j_{i+1},\ldots,j_{l-1})}) - j_l \alpha_l \in \mathfrak{h}^*,\end{aligned}$$

where $\beta_{(j_{i+1},\ldots,j_{l-1})} = \alpha_i$ if $l = i+1$. Let $J_{i,k} \subset \mathbb{N}^{k-i-1}$ be given by

$$\begin{aligned}J_{i,k} = \{(j_{i+1}, \ldots, j_{k-1}) \in \mathbb{N}^{k-i-1} :\ & \beta_{(j_{i+1},\ldots,j_l)} \in \Delta_+ \ \forall\, i+1 \leq l \leq k-1 \text{ and} \\ & \beta_{(j_{i+1},\ldots,j_{k-1})} = \alpha_k\}.\end{aligned}$$

If $J_{i,k} = \emptyset$, then $b_i(z_k) = 0$. Otherwise,

$$b_i(z_k) = -\langle \alpha_i,\, \alpha_i \rangle \sum_{(j_{i+1},\ldots,j_{k-1}) \in J_{i,k}} c_{j_{i+1},\ldots,j_{k-1}} z_{i+1}^{j_{i+1}} \cdots z_{k-1}^{j_{k-1}}, \tag{85}$$

where for $(j_{i+1}, \ldots, j_{k-1}) \in J_{i,k}$,

$$c_{j_{i+1},\ldots,j_{k-1}} = c^{s_{i+1}, j_{i+1}}_{\alpha_{i+1}, \alpha_i}\, c^{s_{i+2}, j_{i+2}}_{\alpha_{i+2}, \beta_{(j_{i+1})}} \cdots c^{s_{k-1}, j_{k-1}}_{\alpha_{k-1}, \beta_{(j_{i+1},\ldots,j_{k-2})}} \neq 0,$$

and for $i+1 \leq l \leq k-1$, $c^{s_l, j_l}_{\alpha_l, \beta_{(j_{i+1}, \ldots, j_{l-1})}}$ *is a certain binomial coefficient with plus or minus sign, with the binomial coefficient being determined by the* α_l*-string of roots through* $\beta_{(j_{i+1}, \ldots, j_{l-1})}$ *and the plus or minus sign determined by the signs of the structure constants of* $\mathfrak{g}$ *in the chosen Chevalley basis, as in* (42)*,* (50) *and* (52)*.*

By Theorem 5.12, for $2 \leq k \leq n$, one also has the derivation b'_k of $\mathbb{C}[z_1, \ldots, z_{k-1}]$ such that $b'_k(z_i) = b_i(z_k)$ for $1 \leq i \leq k-1$. In the rest of §5.3, we give the geometric meaning of the derivation b'_k, similarly to that of the derivation b_i on $\mathbb{C}[z_{i+1}, \ldots, z_n]$ given in Theorem 4.14. To this end, consider the quotient manifold

$$F'_{-n} = B_- \backslash G \times_{B_-} G \times \cdots \times_{B_-} G$$

of G^n by $(B_-)^n$, where $(B_-)^n$ acts on G^n from the left by

$$(86)\quad (b_1, b_2, \ldots, b_n) \cdot (g_1, g_2, \ldots, g_n) = (b_1 g_1 b_2^{-1},\, b_2 g_2 b_3^{-1}, \ldots, b_n g_n), \qquad b_j \in B_-,\ g_j \in G.$$

Let $\rho_- : G^n \to F'_{-n}$ be the natural projection. Similar to the case of the quotient manifold F_n in (8), the product Poisson structure $\pi^n_{\rm st}$ on G^n projects by ρ_- to a well-defined Poisson structure on F'_{-n}, which will be denoted by π'_{-n}. Let $P_{-s_i} = B_- \cup B_- s_i B_-$ for $1 \leq i \leq n$. As each P_{-s_i} is a Poisson submanifold of $(G, \pi_{\rm st})$, the closed submanifold

$$Z'_{-\mathbf{u}} = B_- \backslash P_{-s_1} \times_{B_-} P_{-s_2} \times \cdots \times_{B_-} P_{-s_n}$$

of F'_{-n} is a Poisson submanifold with respect to π'_{-n}. We will also call $Z'_{-\mathbf{u}}$ a Bott-Samelson variety. Note that for each $1 \leq i \leq n$, one has

$$u_{\alpha_i}(z) \dot{s}_i = \dot{s}_i u_{-\alpha_i}(-z), \qquad z \in \mathbb{C}.$$

Setting $\rho_-(g_1, g_2, \ldots, g_n) = [g_1, g_2, \ldots, g_n]_- \in F'_{-n}$ for $(g_1, g_2, \ldots, g_n) \in G^n$, it follows that one has the open affine chart

$$\mathcal{O}'^{,\mathbf{u}}_- := B_- \backslash (B_- s_1 B_-) \times_{B_-} (B_- s_2 B_-) \times \cdots \times_{B_-} (B_- s_n B_-)$$

of $Z'_{-\mathbf{u}}$, with the parametrization by $\mathbb{C}^n$ via

$$(87)\qquad \mathbb{C}^n \ni (z_1, z_2, \ldots, z_n) \longmapsto [u_{\alpha_1}(z_1)\dot{s}_{\alpha_1},\ u_{\alpha_2}(z_2)\dot{s}_{\alpha_2},\ \ldots,\ u_{\alpha_n}(z_n)\dot{s}_{\alpha_n}]_- \in \mathcal{O}'^{,\mathbf{u}}_-.$$

The restriction of the Poisson structure π'_{-n} to $\mathcal{O}'^{,\mathbf{u}}_-$ will also be denoted by π'_{-n}.

Proposition 5.16. *The map* $I : (\mathcal{O}^{\mathbf{u}}, \pi_n) \to (\mathcal{O}'^{,\mathbf{u}}_-, \pi'_{-n})$ *given by*

$$[u_{\alpha_1}(z_1)\dot{s}_{\alpha_1},\ u_{\alpha_2}(z_2)\dot{s}_{\alpha_2},\ \ldots,\ u_{\alpha_n}(z_n)\dot{s}_{\alpha_n}] \longmapsto [u_{\alpha_1}(z_1)\dot{s}_{\alpha_1},\ u_{\alpha_2}(z_2)\dot{s}_{\alpha_2},\ \ldots,\ u_{\alpha_n}(z_n)\dot{s}_{\alpha_n}]_-,$$

where $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$*, is a Poisson anti-isomorphism.*

Proof. Let $\rho : G^n \to F_n$ be the natural projection, so that $\pi_n = \rho(\pi^n_{\rm st})$. It is proved in [30, §8] that the pair

$$\rho : \ (G^n, \pi^n_{\rm st}) \longrightarrow (F_n, \pi_n) \quad \text{and} \quad \rho_- : \ (G^n, \pi^n_{\rm st}) \longrightarrow \left(F'_{-n},\ \pi'_{-n}\right)$$

of Poisson submersions is a Poisson pair (see §A the Appendix), i.e., the map

$$(\rho,\ \rho_-) : \ (G^n, \pi^n_{\rm st}) \longrightarrow (F_n \times F'_{-n},\ \pi_n \times \pi'_{-n}),\ \ g \longmapsto (\rho(g),\ \rho_-(g)), \qquad g \in G^n,$$

is Poisson. For $\alpha \in \Gamma$, let Σ_α be the symplectic leaf of $\pi_{\rm st}$ in G through the point $\dot{s}_\alpha \in G$. To describe the two-dimensional symplectic manifold $(\Sigma_\alpha, \pi_{\rm st}|_{\Sigma_\alpha})$, consider the surface

$$\Sigma = \{(p, q, t) \in \mathbb{C}^3 : \ t^2(1 - pq) = 1\}$$

in $\mathbb{C}^3$ and equip Σ with the Poisson structure π given by

$$\{p,q\} = 2(1-pq), \quad \{p,t\} = pt, \quad \{q,t\} = -qt. \tag{88}$$

A calculation in $SL(2,\mathbb{C})$ shows that the embedding

$$J: \ \Sigma \longrightarrow SL(2,\mathbb{C}), \quad (p,q,t) \longmapsto \begin{pmatrix} pt & -t \\ t & -qt \end{pmatrix}, \quad (p,q,t) \in \Sigma,$$

identifies (Σ, π) as the symplectic leaf through $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2,\mathbb{C})$ of the Poisson structure $\pi_{SL(2,\mathbb{C})}$ on $SL(2,\mathbb{C})$ in (14). Using the Poisson homomorphism θ_α in (15), one sees [26] that

$$\Sigma_\alpha = \{g_\alpha(p,q,t): \ (p,q,t) \in \Sigma\},$$

and $\pi_{\rm st}|_{\Sigma_\alpha} = \frac{\langle \alpha,\alpha\rangle}{2}(\theta_\alpha \circ J)(\pi)$, where for $(p,q,t) \in \Sigma$,

$$g_\alpha(p,q,t) = \theta_\alpha \begin{pmatrix} pt & -t \\ t & -qt \end{pmatrix} = u_\alpha(p)\dot{s}_\alpha \alpha^\vee(t) u_\alpha(-q) = u_{-\alpha}(q)\alpha^\vee(t)\dot{s}_\alpha u_{-\alpha}(-p). \tag{89}$$

Consider now the product manifold $\Sigma_{\mathbf{u}} = \Sigma_{\alpha_1} \times \Sigma_{\alpha_2} \times \cdots \times \Sigma_{\alpha_n}$ and denote the restriction of the product Poisson structure $\pi_{\rm st}^n$ to $\Sigma_{\mathbf{u}}$ still by $\pi_{\rm st}^n$. It follows from (89) that

$$\rho(\Sigma_{\mathbf{u}}) = \mathcal{O}^{\mathbf{u}} \quad \text{and} \quad \rho_-(\Sigma_{\mathbf{u}}) = \mathcal{O}_-^{\prime,\mathbf{u}},$$

and, denoting again by ρ (resp. ρ_-) the induced map from $\Sigma_{\mathbf{u}}$ to $\mathcal{O}^{\mathbf{u}}$ (resp. to $\mathcal{O}_-^{\prime,\mathbf{u}}$),

$$\rho: \ (\Sigma_{\mathbf{u}}, \ \pi_{\rm st}^n) \longrightarrow (\mathcal{O}^{\mathbf{u}}, \ \pi_n) \quad \text{and} \quad \rho_-: \ (\Sigma_{\mathbf{u}}, \ \pi_{\rm st}^n) \longrightarrow (\mathcal{O}_-^{\prime,\mathbf{u}}, \ \pi_{-n}^\prime) \tag{90}$$

are Poisson submersions and form a Poisson pair. Moreover, the submanifold

$$L := \{(u_{\alpha_1}(z_1)\dot{s}_{\alpha_1}, \ u_{\alpha_2}(z_2)\dot{s}_{\alpha_2}, \ \ldots, \ u_{\alpha_n}(z_n)\dot{s}_{\alpha_n}): \ (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n\}$$

of $\Sigma_{\mathbf{u}}$ is Lagrangian with respect to $\pi_{\rm st}^n$, and it is clear that $\rho|_L: L \to \mathcal{O}^{\mathbf{u}}$ is a diffeomorphism. It now follows from Lemma A.1 in the Appendix that $I = \rho_- \circ (\rho|_L)^{-1}$: $(\mathcal{O}^{\mathbf{u}}, \ \pi_n) \to (\mathcal{O}_-^{\prime,\mathbf{u}}, \ \pi_{-n}^\prime)$ is a Poisson anti-isomorphism.

Q.E.D.

We now prove a fact similar to that in Lemma 2.2: let (X, π_X) be a Poisson manifold with a right Poisson action by the Poisson Lie group $(B_-, \pi_{\rm st})$, let α be a simple root, and consider the quotient manifold $Z = X \times_{B_-} P_{-s_\alpha}$ (see notation in §2.2) equipped with Poisson structure π_Z which is the projection to Z of the product Poisson structure $\pi_X \times \pi_{\rm st}$ on $X \times P_{-s_\alpha}$. Denote by $[x,p]$ the image of $(x,p) \in X \times P_{-s_\alpha}$ in Z. Fix any $\mathfrak{sl}(2,\mathbb{C})$-triple $\{e_\alpha, e_{-\alpha}, h_\alpha\}$ and consider

$$\phi: \ X \times \mathbb{C} \longrightarrow Z_0, \ (x,z) \longmapsto [x, \ u_\alpha(z)\dot{s}_\alpha], \quad x \in X, \ z \in \mathbb{C}.$$

Then ϕ is an embedding, and we regard ϕ as a diffeomorphism from $X \times \mathbb{C}$ to $Z_0 = \phi(X \times \mathbb{C})$. For $\xi \in \mathfrak{b}_-$, let $\sigma_\xi^\prime$ be the vector field on X defined by

$$\sigma_\xi^\prime(x) = \frac{d}{dt}|_{t=0}(x \exp(t\xi)), \quad x \in X.$$

Using the second part of (17), the proof of the following Lemma 5.17 is similar to that of Lemma 2.2 and is omitted.

Lemma 5.17. *With the notation as above, one has*

$$\phi^{-1}(\pi_Z)(x,z) = \pi_X(x) + \frac{\langle \alpha,\alpha\rangle}{2}\frac{d}{dz} \wedge \left(z\sigma_{h_\alpha}^\prime(x) + 2\sigma_{e_{-\alpha}}^\prime(x)\right).$$

Returning now to the Bott-Samelson variety $Z'_{-\mathbf{u}}$ for $\mathbf{u} = (s_1, \ldots, s_n) = (s_{\alpha_1}, \ldots, s_{\alpha_n})$, let $2 \leq k \leq n$, and consider

$$Z'_{-(s_1,\ldots,s_{k-1})} = B_-\backslash P_{-s_1} \times_{B_-} P_{-s_2} \times \cdots \times_{B_-} P_{-s_{k-1}}.$$

Denote again by $[p_1, \ldots, p_{k-1}]_-$ the image of $(p_1, \ldots, p_{k-1}) \in P_{-s_1} \times \cdots \times P_{-s_{k-1}}$ in $Z'_{-(s_1,\ldots,s_{k-1})}$, and let B_- act on $Z'_{-(s_1,\ldots,s_{k-1})}$ from the right by

$$[p_1, \ldots, p_{k-2}, p_{k-1}]_- \cdot b_- = [p_1, \ldots, p_{k-2}, p_{k-1}b_-], \quad b_- \in B_-, \, p_i \in P_{-s_i}, \, 1 \leq i \leq k-1.$$

For $\xi \in \mathfrak{b}_-$, denote by $\sigma_\xi^{\prime,(k-1)}$ the vector field on $Z'_{-(s_1,\ldots,s_{k-1})}$ given by

$$\sigma_\xi^{\prime,(k-1)}([p_1, \ldots, p_{k-2}, p_{k-1}]) = \frac{d}{dt}|_{t=0}[p_1, \ldots, p_{k-2}, p_{k-1} \exp(t\xi)]_-, \tag{91}$$

where $p_i \in P_{-s_i}$ for $1 \leq i \leq k-1$, so $\sigma_\xi^{\prime,(k-1)}$ generates the action of B_- on $Z'_{-(s_1,\ldots,s_{k-1})}$ in the direction of ξ. Let

$$\sigma'_k = \sigma_{e_{-\alpha}}^{\prime,(k-1)}. \tag{92}$$

Consider the coordinates $(z_1, z_2, \ldots, z_n)$ on $\mathcal{O}_-^{\prime,\mathbf{u}}$ given in (87). Then $(z_1, \ldots, z_{k-1})$ can be considered as coordinates on the open submanifold

$$\begin{aligned} \mathcal{O}_-^{\prime,(s_1,\ldots,s_{k-1})} &= B_-\backslash(B_-s_1B_-) \times_{B_-} (B_-s_2B_-) \times \cdots \times_{B_-} (B_-s_{k-1}B_-) \\ &= \{[u_{\alpha_1}(z_1)\dot{s}_{\alpha_1}, \, \ldots, \, u_{\alpha_{k-1}}(z_{k-1})\dot{s}_{\alpha_{k-1}}]_- : \, (z_1, \ldots, z_{k-1}) \in \mathbb{C}^{k-1}\} \end{aligned}$$

of $Z'_{-(s_1,\ldots,s_{k-1})}$, and σ'_k can be regarded as a derivation on $\mathbb{C}[z_1, \ldots, z_{k-1}]$.

Lemma 5.18. *In the coordinates $(z_1, z_2, \ldots, z_n)$ on $\mathcal{O}_-^{\prime,\mathbf{u}}$ given in (87), the Poisson structure π'_{-n} is given by*

$$\{z_i, \, z_k\} = -c_{i,k}z_iz_k - \langle\alpha_k, \, \alpha_k\rangle\sigma'_k(z_i), \quad 1 \leq i < k \leq n, \tag{93}$$

where for $1 \leq i, k \leq n$, $c_{i,k}$ is given in (84).

Proof. By repeatedly applying Lemma 5.17 to the Poisson manifold $(\mathcal{O}_-^{\prime,\mathbf{u}}, \pi'_{-n})$, one sees that π'_{-n} is given in the coordinates $(z_1, z_2, \ldots, z_n)$ on $\mathcal{O}_-^{\prime,\mathbf{u}}$ by (see notation in (91))

$$\{z_i, \, z_k\} = -\frac{\langle\alpha_k, \, \alpha_k\rangle}{2} z_k\sigma_{h_{\alpha_k}}^{\prime,(k-1)}(z_i) - \langle\alpha_k, \, \alpha_k\rangle\sigma'_k(z_i), \quad 1 \leq i < k \leq n,$$

For $h \in \mathfrak{t}$, one checks directly from the definition of the vector field $\sigma_h^{\prime,(k-1)}$ that

$$\sigma_h^{\prime,(k-1)}(z_i) = (s_{k-1}s_{k-2} \cdots s_{i+1}(\alpha_i)(h))z_i, \quad 1 \leq i \leq k-1. \tag{94}$$

Identity (93) now follows from

$$\begin{aligned} \frac{\langle\alpha_k, \, \alpha_k\rangle}{2}\sigma_{h_{\alpha_k}}^{\prime,(k-1)}(z_i) &= \langle s_{k-1}s_{k-2} \cdots s_{i+1}(\alpha_i), \, \alpha_k\rangle z_i \\ &= -\langle s_1s_2 \cdots s_{i-1}(\alpha_i), \, s_1s_2 \cdots s_{k-1}(\alpha_k)\rangle z_i \\ &= c_{i,k}z_i. \end{aligned}$$

Q.E.D.

Corollary 5.19. *In the notation in Theorem 5.12 for the case of $\gamma = \mathbf{u}$, one has*

$$b'_k = \langle\alpha_k, \alpha_k\rangle\sigma'_k, \quad 2 \leq k \leq n.$$

Proof. By Proposition 5.16 and Lemma 5.18, the Poisson structure π_n is given in the coordinates $(z_1, \ldots, z_n)$ on the affine chart $\mathcal{O}^{\mathbf{u}}$ by

$$\{z_i,\, z_k\} = c_{i,k} z_i z_k + \langle \alpha_k,\, \alpha_k\rangle \sigma_k'(z_i), \qquad 1 \leq i < k \leq n.$$

It follows from the definition of b_k' that $b_k' = \langle \alpha_k, \alpha_k\rangle \sigma_k'$ for $2 \leq k \leq n$.

Q.E.D.

Remark 5.20. We already know from Lemma 5.9 that for any $h \in \mathfrak{t}$ and $2 \leq k \leq n$, $[a_h',\, b_k'] = \lambda_k(h) b_k'$, as derivations of $\mathbb{C}[z_1, \ldots, z_{k-1}]$, where $a_h' = h|_{\mathbb{C}[z_1,\ldots,z_{k-1}]}$ and $\lambda_k = s_1 s_2 \cdots s_{k-1}(\alpha_k)$. This fact can also be checked directly from Corollary 5.19. Indeed, that in the notation of (91), it follows from (94) that $a_h' = -\sigma'^{,(k-1)}_{s_{k-1}\cdots s_2 s_1(h)}$ and $b_k' = \langle \alpha_k, \alpha_k\rangle \sigma'^{,(k-1)}_{e_{-\alpha_k}}$, so

$$[a_h',\, b_k'] = -\langle \alpha_k, \alpha_k\rangle \left[\sigma'^{,(k-1)}_{s_{k-1}\cdots s_2 s_1(h)},\; \sigma'^{,(k-1)}_{e_{-\alpha_k}}\right] = \lambda_k(h) b_k'.$$

$\diamond$

5.4. **The polynomial rings** $(\mathbb{Z}[z_1, \ldots, z_n], \{\,,\,\}_\gamma)$**.** Recall from §2 that once the Borel subgroup B and the maximal torus $T \subset B$ of G are fixed, the definition of the Poisson structure π_n on $Z_{\mathbf{u}}$ depends only on the choice of a symmetric non-degenerate invariant bilinear form $\langle\,,\,\rangle$ on $\mathfrak{g}$ and not on the choices of root vectors e_α for $\alpha \in \Delta$. Although a choice of the set $\{e_\alpha : \alpha \in \Gamma\}$ of root vectors for the simple roots is needed to define the coordinates $(z_1, \ldots, z_n)$ on $\mathcal{O}^\gamma$ for $\gamma \in \Upsilon_{\mathbf{u}}$, we proved in Proposition 5.1 that the polynomials $f_{i,k} := \{z_i, z_k\}_\gamma \in \mathbb{C}[z_1, \ldots, z_n]$ for $1 \leq i, k \leq n$ are independent on the choices of the root vectors for the simple roots. For each $\gamma \in \Upsilon_{\mathbf{u}}$, one thus has a well-defined Poisson polynomial algebra $(\mathbb{C}[z_1, \ldots, z_n], \{\,,\,\}_\gamma)$.

Theorem 5.21. *Suppose that the symmetric non-degenerate invariant bilinear form $\langle\,,\,\rangle$ on $\mathfrak{g}$ is chosen such that $\frac{1}{2}\langle \alpha,\, \alpha\rangle \in \mathbb{Z}$ for each $\alpha \in \Delta$. Then for any $\gamma \in \Upsilon_{\mathbf{u}}$, the Poisson structure $\{\,,\,\}_\gamma$ on $\mathbb{C}[z_1, \ldots, z_n]$ has the property that $\{z_i, z_k\} \in \mathbb{Z}[z_i, \ldots, z_k] \subset \mathbb{Z}[z_1, \ldots, z_n]$ for all $1 \leq i < k \leq n$.*

Proof. Choose any set $\{e_\alpha : \alpha \in \Gamma\}$ of root vectors for the simple roots and extend it to a Chevalley basis of $\mathfrak{g}$. Theorem 5.21 now follows from Remark 4.5 and the fact that for any $\alpha, \beta \in \Delta$, $\langle \alpha,\, \beta\rangle = \dfrac{2\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle} \dfrac{\langle \alpha, \alpha\rangle}{2} \in \mathbb{Z}$.

Q.E.D.

Note that a canonical choice of the bilinear form $\langle\,,\,\rangle$ on $\mathfrak{g}$ is such that $\langle \alpha, \alpha\rangle = 2$ for the short roots for each of the simple factors of $\mathfrak{g}$.

Remark 5.22. By Theorem 5.21, each $\gamma \in \Upsilon_{\mathbf{u}}$ gives rise to a Poisson algebra

$$(\mathbf{k}[z_1, \ldots, z_n],\; \{\,,\}_\gamma)$$

over any field $\mathbf{k}$ of arbitrary characteristic. In particular, it follows from (60) in Theorem 4.14 that the Poisson structure $\{\,,\,\}_\gamma$ on $\mathbf{k}[z_1, \ldots, z_n]$ is log-canonical for every $\gamma \in \Upsilon_{\mathbf{u}}$ if $\mathrm{char}(\mathbf{k}) = 2$. On the other hand, suppose that the bilinear form $\langle\,,\,\rangle$ on $\mathfrak{g}$ is such that $\langle \alpha, \alpha\rangle = 2$ for all the short roots. Then $\langle \alpha, \alpha\rangle \in \{2, 4, 6\}$ for all $\alpha \in \Gamma$. It follows from (82) that $(\mathbf{k}[z_1, \ldots, z_n], \{\,,\,\}_{\mathbf{u}})$ is a symmetric Poisson CGL extension of any field $\mathbf{k}$ with $\mathrm{char}(\mathbf{k}) \neq 2, 3$. $\diamond$

5.5. **Examples.** Assume that $\mathfrak{g}$ is simple and let $\langle\, ,\, \rangle$ be such that $\langle \alpha, \alpha\rangle = 2$ for the short roots of $\mathfrak{g}$. Based on Theorem 4.14, the first author has written a computer program in the GAP language [15] which computes the Poisson bracket $\{\, ,\, \}_\gamma$ on $\mathbb{Z}[z_1, \ldots, z_n]$ for any $\mathbf{u} = (s_1, \ldots, s_n)$ and any $\gamma \in \Upsilon_{\mathbf{u}}$. We given some examples.

Example 5.23. Consider G_2 with the two simple roots α_1 and α_2 satisfying

$$\langle \alpha_2, \alpha_2\rangle = 3\langle \alpha_1, \alpha_1\rangle = 6.$$

Let $\mathbf{u} = (s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_1}, s_{\alpha_2})$ and note that $s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}$ is the longest element in the Weyl group of G_2. For $\gamma = \mathbf{u}$, one has

$$\begin{aligned}
&\{z_1, z_2\} = -3z_1z_2, \quad \{z_1, z_3\} = -z_1z_3 - 2z_2, \quad \{z_1, z_4\} = -6z_3^2,\\
&\{z_1, z_5\} = z_1z_5 - 4z_3, \quad \{z_1, z_6\} = 3z_1z_6 - 6z_5, \quad \{z_2, z_3\} = -3z_2z_3\\
&\{z_2, z_4\} = -6z_3^3 - 3z_2z_4, \quad \{z_2, z_5\} = -6z_3^2, \quad \{z_2, z_6\} = 3z_2z_6 - 18z_3z_5 + 6z_4\\
&\{z_3, z_4\} = -3z_3z_4, \quad \{z_3, z_5\} = -z_3z_5 - 2z_4, \quad \{z_3, z_6\} = -6z_5^2\\
&\{z_4, z_5\} = -3z_4z_5, \quad \{z_4, z_6\} = -6z_5^3 - 3z_4z_6, \quad \{z_5, z_6\} = -3z_5z_6.
\end{aligned}$$

For the same $\mathbf{u}$ but $\gamma = (s_{\alpha_1}, s_{\alpha_2}, e, e, s_{\alpha_1}, e)$, one has

$$\begin{aligned}
&\{z_1, z_2\} = -3z_1z_2, \quad \{z_1, z_3\} = 2z_2z_3^2 + z_1z_3, \quad \{z_1, z_4\} = -6z_2z_3z_4 + 6z_3z_4^2 - 3z_1z_4,\\
&\{z_1, z_5\} = -4z_2z_3z_5 + 6z_3z_4z_5 - z_1z_5 - 2z_2 + 2z_4,\\
&\{z_1, z_6\} = 6z_3z_5^3z_6^2 + 6z_5^2z_6^2 + 6z_2z_3z_6 - 6z_3z_4z_6, \quad \{z_2, z_3\} = 3z_2z_3,\\
&\{z_2, z_4\} = -6z_2z_4 + 6z_4^2, \quad \{z_2, z_5\} = -3z_2z_5 + 6z_4z_5,\\
&\{z_2, z_6\} = 6z_5^3z_6^2 + 3z_2z_6 - 6z_4z_6, \quad \{z_3, z_4\} = -3z_3z_4, \quad \{z_3, z_5\} = -2z_3z_5,\\
&\{z_3, z_6\} = 3z_3z_6, \quad \{z_4, z_5\} = 3z_4z_5, \quad \{z_4, z_6\} = -3z_4z_6, \quad \{z_5, z_6\} = 3z_5z_6.
\end{aligned}$$

◇

Example 5.24. Consider $G = SL(2)$ with the only simple root denoted by α and $s = s_\alpha$ and $\langle \alpha,\, \alpha\rangle = 2$. Let $\mathbf{u} = (s, s, s, s, s)$. For $\gamma = \mathbf{u}$, one has

$$\begin{aligned}
&\{z_1, z_2\} = 2z_1z_2 - 2, \quad \{z_1, z_3\} = -2z_1z_3, \quad \{z_1, z_4\} = 2z_1z_4, \quad \{z_1, z_5\} = -2z_1z_5,\\
&\{z_2, z_3\} = 2z_2z_3 - 2, \quad \{z_2, z_4\} = -2z_2z_4, \quad \{z_2, z_5\} = 2z_2z_5, \quad \{z_3, z_4\} = 2z_3z_4 - 2,\\
&\{z_3, z_5\} = -2z_3z_5, \quad \{z_4, z_5\} = 2z_4z_5 - 2.
\end{aligned}$$

For $\gamma = (s, e, e, e, s)$, one has

$$\begin{aligned}
&\{z_1, z_2\} = -2z_1z_2 + 2z_2^2, \quad \{z_1, z_3\} = -2z_1z_3 + 4z_2z_3 + 2z_3^2,\\
&\{z_1, z_4\} = -2z_1z_4 + 4z_2z_4 + 4z_3z_4 + 2z_4^2,\\
&\{z_1, z_5\} = 2z_1z_5 - 4z_2z_5 - 4z_3z_5 - 4z_4z_5 - 2,\\
&\{z_2, z_3\} = 2z_2z_3, \quad \{z_2, z_4\} = 2z_2z_4, \quad \{z_2, z_5\} = -2z_2z_5,\\
&\{z_3, z_4\} = 2z_3z_4, \quad \{z_3, z_5\} = -2z_3z_5, \quad \{z_4, z_5\} = -2z_4z_5.
\end{aligned}$$

In general, it is easy to see from Theorem 4.14 that for the sequence $\mathbf{u} = (s, s, \ldots, s)$ of length n, and $\gamma = \mathbf{u}$, the Poisson bracket $\{\, ,\, \}_\gamma$ on $\mathbb{Z}[z_1, \ldots, z_n]$ is given by

$$\begin{aligned}
&\{z_i, z_{i+1}\} = 2z_iz_{i+1} - 2, \quad 1 \leq i \leq n-1,\\
&\{z_i, z_k\} = 2(-1)^{k-i+1}z_iz_k, \quad 1 \leq i < k \leq n,\ k - i \geq 2.
\end{aligned}$$

The coefficient 2 in all the Poisson brackets results from that fact that $\langle \alpha, \alpha \rangle = 2$. $\diamond$

Appendix A. Poisson pairs

In [30, §8.5], a *Poisson pair* is defined to be a pair of Poisson maps

$$\rho_Y : \ (X, \pi_X) \longrightarrow (Y, \pi_Y) \quad \text{and} \quad \rho_Z : \ (X, \pi_X) \longrightarrow (Z, \pi_Z) \tag{95}$$

between Poisson manifolds such that the map

$$(\rho_Y, \rho_Z): \ (X, \pi_X) \longrightarrow (Y \times Z, \ \pi_Y \times \pi_Z), \quad x \longmapsto (\rho_Y(x), \ \rho_Z(x)), \quad x \in X,$$

is Poisson. If (Y, π_Y) and (Z, π_Z) are two Poisson manifolds, the projections from the product Poisson manifold $(Y \times Z, \ \pi_Y \times \pi_Z)$ to the two factors clearly form a Poisson pair. Moreover, for a differentiable map $\phi : Y \to Z$, it is well-known [39] that $\phi : (Y, \pi_Y) \to (Z, \pi_Z)$ is anti-Poisson if and only if $\text{Graph}(\phi) = \{(y, \ \phi(y) \ : \ y \in Y\}$ is a coisotropic submanifold of $(Y \times Z, \ \pi_Y \times \pi_Z)$. The following Lemma A.1 is a (partial) generalization of this fact to the case of Poisson pairs.

Lemma A.1. *Let* (ρ_Y, ρ_Z) *be a Poisson pair as in* (95). *Suppose that* X' *is a coisotropic submanifold of* (X, π_X) *such that* $\rho_Y|_{X'} : X' \to Y$ *is a diffeomorphism. Then*

$$\phi = \rho_Z \circ (\rho_Y|_{X'})^{-1} : \ (Y, \pi_Y) \longrightarrow (Z, \pi_Z)$$

is an anti-Poisson map.

Proof. Fix $x \in X'$ and let $\rho_Y(x) = y$ and $z = \rho_Z(x) \in Z$. Let

$$\rho_{Y,x} : \ T_xX \longrightarrow T_yY \quad \text{and} \quad \rho_{Z,x} : \ T_xX \longrightarrow T_zZ$$

be respectively the differentials of ρ_Y and ρ_Z at x. Lemma A.1 now follows from the following Lemma A.2 by taking $(V, \pi) = (T_xX, \pi_X(x))$, $V_1 = \ker \rho_{Y,x}$, $V_2 = \ker \rho_{Z,x}$, and $U = T_xX'$. Indeed, in the notation stated below for Lemma A.2, the assumption $\pi_X(x)(V_1^0, V_2^0) = 0$ is the same

$$\pi_X(x)(\rho_{Y,x}^*(T_y^*Y), \ \rho_{Z,x}^*(T_z^*Z)) = 0$$

which is satisfied because $(\rho_Y, \rho_Z) : (X, \ \pi_X) \to (Y \times Z, \ \pi_Y \times \pi_Z)$ is a Poisson map.

Q.E.D.

In the following Lemma A.2, for a finite dimensional vector space V and a subspace $U_1 \subset V$, set $U_1^0 = \{\xi \in V^* : \xi|_{U_1} = 0\} \subset V^*$, and U_1 is said to be coisotropic with respect to $\pi \in \wedge^2 V$ if $\pi \in U_1 \wedge V$, where for any subspace U_2 of V,

$$U_1 \wedge U_2 = (\wedge^2 V) \cap (U_1 \otimes U_2 + U_2 \otimes U_1) \subset \wedge^2 V.$$

Lemma A.2. *Let* V *be a finite dimensional vector space, let* $\pi \in \wedge^2 V$, *and let* V_1 *and* V_2 *be two vector subspaces of* V *such that* $\pi(V_1^0, V_2^0) = 0$. *For* $j = 1, 2$, *let* $\rho_j : V \to V/V_j$ *be the projections so that* $\rho_j(\pi) \in \wedge^2(V/V_j)$. *Assume that* U *is a coisotropic subspace of* V *and that* $\rho_1|_U : U \to V/V_1$ *is an isomorphism. Let* $\psi = \rho_2 \circ (\rho_1|_U)^{-1} : V/V_1 \to V/V_2$. *Then* $\psi(\rho_1(\pi)) = -\rho_2(\pi)$.

Proof. For $\pi' = \sum_j v_j \wedge v'_j \in \wedge^2 V$ and $\xi \in V^*$, let $\xi \rfloor \pi' = \sum_j (\langle \xi, v_j\rangle v'_j - \langle \xi, v'_j\rangle v_j)$, where $\langle\, ,\, \rangle$ denotes the pairing between V and V^*. Then the condition $\pi(V_1^0, V_2^0) = 0$ is equivalent to $\xi \rfloor \pi \in V_2$ for all $\xi \in V_1^0$. By assumption, $V = U + V_1$ is a direct sum. As U is coisotropic with respect to π, one can uniquely write $\pi = \pi_U + \pi_1$, where $\pi_U \in \wedge^2 U$ and $\pi_1 \in U \wedge V_1$. Let $\{u_1, \ldots, u_m\}$ be a basis of U and let $\xi_i \in V_1^0$, $1 \leq i \leq m$, be such that $\langle u_i, \xi_j\rangle = \delta_{i,j}$ for $1 \leq i, j \leq m$. Then

$$\pi_U = \frac{1}{2}\sum_{i=1}^m u_i \wedge (\xi_i \rfloor \pi_U) \quad \text{and} \quad \pi_1 = \sum_{i=1}^m u_i \wedge (\xi_i \rfloor \pi_1).$$

For $1 \leq i \leq m$, let $x_i = \xi_i \rfloor \pi = \xi_i \rfloor (\pi_U + \pi_1)$. Then

$$\begin{aligned}\pi &= \frac{1}{2}\sum_{i=1}^m u_i \wedge (\xi_i \rfloor \pi_U) + \sum_{i=1}^m u_i \wedge (\xi_i \rfloor (\pi_U + \pi_1)) - \sum_{i=1}^m u_i \wedge (\xi_i \rfloor \pi_U) \\ &= -\frac{1}{2}\sum_{i=1}^m u_i \wedge (\xi_i \rfloor \pi_U) + \sum_{i=1}^m u_i \wedge x_i = -\pi_U + \sum_{i=1}^m u_i \wedge x_i.\end{aligned}$$

As $x_i \in V_2$ for each $1 \leq i \leq m$, $\rho_2(\sum_{i=1}^m u_i \wedge x_i) = 0$, so $\psi(\rho_1(\pi)) = \rho_2(\pi_U) = -\rho_2(\pi)$.

Q.E.D.

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