

## High-dimensional central limit theorems for a class of particle systems\*

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### Abstract

We consider a class of particle systems that generalizes the eigenvalues of a class of matrix-valued processes, of which the empirical measures converge to deterministic measures as the dimension goes to infinity. In this paper, we obtain central limit theorems (CLTs) to characterize the fluctuations of the empirical measures around the limit measures by using stochastic calculus. As applications, CLTs for Dyson’s Brownian motion and the eigenvalues of Wishart process are recovered under slightly more general initial conditions, and a CLT for the eigenvalues of a symmetric matrix-valued Ornstein-Uhlenbeck process is obtained.

**Keywords:** Dyson’s Brownian motion; Wishart process; particle system; squared Bessel particle system; central limit theorem; matrix-valued Ornstein-Uhlenbeck process.

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## 1 Introduction

Recently general stochastic differential equations (SDEs) on the group of symmetric matrices have attracted much interest. A prominent example is the process introduced in Graczyk and Małeckı (2013) as the solution of the following matrix-valued SDE,

$$dX_t^N = g_N(X_t^N)dB_t h_N(X_t^N) + h_N(X_t^N)dB_t^T g_N(X_t^N) + b_N(X_t^N)dt, \quad t \geq 0. \quad (1.1)$$

Here,  $B_t$  is a matrix-valued Brownian motion of dimension  $N \times N$ , and the continuous functions  $g_N, h_N, b_N : \mathbb{R} \rightarrow \mathbb{R}$  act on the spectrum of  $X_t^N$  (a function  $f$  acts on the

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spectrum of a symmetric matrix  $X = \sum_{j=1}^N \alpha_j u_j u_j^\top$  with eigenvalues  $(\alpha_j)$  and eigenvectors  $(u_j)$  if  $f(X) = \sum_{j=1}^N f(\alpha_j) u_j u_j^\top$ . The matrix-valued process (1.1) extends several well-known processes such as the celebrated symmetric matrix-valued Brownian motion (Dyson, 1962), Wishart process (Bru, 1991), and the symmetric matrix-valued Ornstein-Uhlenbeck processes (Chan, 1992).

Let  $\{\lambda_i^N(t)\}_{1 \leq i \leq N}$  be the eigenvalues of  $X_t^N$ . According to Theorem 3 in Graczyk and Małeckı (2013), if the initial eigenvalues  $\{\lambda_i^N(0)\}_{1 \leq i \leq N}$  are all distinct, then before the first collision time  $\tau_N = \inf\{t > 0 : \exists i \neq j, \lambda_i(t) = \lambda_j(t)\}$ , the eigenvalue processes satisfy the following system of SDEs: for  $1 \leq i \leq N$ ,

$$d\lambda_i^N(t) = 2g_N(\lambda_i^N(t))h_N(\lambda_i^N(t))dW_i(t) + \left( b_N(\lambda_i^N(t)) + \sum_{j:j \neq i} \frac{G_N(\lambda_i^N(t), \lambda_j^N(t))}{\lambda_i^N(t) - \lambda_j^N(t)} \right) dt, \tag{1.2}$$

where  $\{W_i, 1 \leq i \leq N\}$  are independent Brownian motions and

$$G_N(x, y) = g_N^2(x)h_N^2(y) + g_N^2(y)h_N^2(x). \tag{1.3}$$

In Graczyk and Małeckı (2013, 2014), some other conditions on the coefficient functions were imposed to ensure that (1.2) has a unique strong solution and the collision time  $\tau_N$  is infinite almost surely.

Let  $L_N(t)$  be the empirical measure of the eigenvalues  $\{\lambda_i^N(t)\}_{1 \leq i \leq N}$ , that is

$$L_N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i^N(t)}.$$

In connection with the theory of random matrices, it is of interest to investigate possible limits of these empirical measures  $\{L_N(t), t \in [0, T]\}$  when  $N$  grows to infinity (*high-dimensional limits*). An early result is the derivation of the Wigner semi-circle law as the only equilibrium point (with finite moments of all orders) of the equation satisfied by the limit of eigenvalue empirical measure processes in Chan (1992), where the entries of the symmetric matrix-valued processes are independent Ornstein-Uhlenbeck processes. The results were later generalized in Rogers and Shi (1993) to the following interacting particle system

$$dX_i = \sqrt{\frac{2\alpha}{N}} dB_i + \left( -\theta X_i + \frac{\alpha}{N} \sum_{j:j \neq i} \frac{1}{X_i - X_j} \right) dt, \quad 1 \leq i \leq N, \quad t \geq 0.$$

Cépa and Lépingle (1997) further generalized these SDEs to

$$dX_i = \sigma(X_i)dB_i + \left( b(X_i) + \sum_{j:j \neq i} \frac{\gamma}{X_i - X_j} \right) dt, \quad 1 \leq i \leq N, \quad t \geq 0,$$

with some coefficient functions  $b$ ,  $\sigma$  and constant  $\gamma$ . Another important case is the Marčenko-Pastur law for the eigenvalue empirical measure processes derived in Cabanal-Duvillard and Guionnet (2001). In the framework of free probability theory, the semi-circle law and Marčenko-Pastur law arise naturally as a consequence of the central limit theorem (CLT) for free random variables (Voiculescu (1991); Voiculescu et al. (1992)). We refer the interested reader to Hiai and Petz (2000); Mingo and Speicher (2017) and the references therein for this topic.

The eigenvalue SDEs (1.2) generalize the eigenvalue SDEs in Chan (1992) and Cabanal-Duvillard and Guionnet (2001), as well as the particle system in Rogers and

Shi (1993). High-dimensional limits for these eigenvalue SDEs appeared very recently in Song et al. (2020) and Małeckı and Pérez (2019). Particularly in the former article, it was proved that under proper conditions,  $\{L_N(t), t \in [0, T]\}_{N \in \mathbb{N}}$  is relatively compact in  $(C[0, T], M_1(\mathbb{R}))$  almost surely. Here  $M_1(\mathbb{R})$  is the set of probability measures on  $\mathbb{R}$  endowed with the topology induced by the weak convergence of measures.

Furthermore, any limit measure  $\{\mu_t, t \in [0, T]\}$  from a converging subsequence satisfies

$$\int \frac{\mu_t(dx)}{z-x} = \int \frac{\mu_0(dx)}{z-x} + \int_0^t \left[ \int \frac{b(x)}{(z-x)^2} \mu_s(dx) \right] ds + \int_0^t \left[ \iint \frac{G(x,y)}{(z-x)(z-y)^2} \mu_s(dx) \mu_s(dy) \right] ds, \quad \forall z \in \mathbb{C} \setminus \mathbb{R}, \quad (1.4)$$

with

$$b(x) = \lim_{N \rightarrow \infty} b_N(x) \quad \text{and} \quad G(x,y) = \lim_{N \rightarrow \infty} NG_N(x,y), \quad (1.5)$$

uniformly. To obtain the convergence of the relatively compact sequence of empirical measures  $\{L_N(t), t \in [0, T]\}_{n \in \mathbb{N}}$ , one would expect the solution of (1.4) to be unique. The uniqueness can be obtained under proper conditions for some symmetric matrix-valued processes, e.g., Dyson’s Brownian motion (Anderson et al. (2010); Chan (1992); Rogers and Shi (1993)), matrix-valued Ornstein-Uhlenbeck process (Chan (1992); Rogers and Shi (1993)), Wishart process (Cabanal-Duvillard and Guionnet (2001)), self-similar processes (Song et al. (2020)), and Dyson’s Brownian motion with a drift (Cépa and Lépingle (1997); Fontbona (2004); Li et al. (2020)). However, general conditions on the coefficient functions in (1.4) that guarantee the uniqueness of the solution are still unknown. In particular, Małeckı and Pérez (2019) provided examples of (1.4) which have at least two solutions.

We aim to study the fluctuations of  $\{L_N(t), t \in [0, T]\}$  of the eigenvalue SDE (1.2) around the limit  $\{\mu_t, t \in [0, T]\}$  as  $N \rightarrow \infty$ , up to considering a subsequence. Indeed, we shall consider the following general particle system introduced in Graczyk and Małeckı (2014) which includes (1.2) as a particular case: for  $1 \leq i \leq N$ ,

$$dx_i^N(t) = \sigma^N(x_i^N(t)) dW_i(t) + \left( b_N(x_i^N(t)) + \sum_{j:j \neq i} \frac{H_N(x_i^N(t), x_j^N(t))}{x_i^N(t) - x_j^N(t)} \right) dt, \quad t \geq 0, \quad (1.6)$$

with  $H_N(x,y)$  being a symmetric function. Under proper conditions, the existence and uniqueness of the non-colliding strong solution was obtained in Graczyk and Małeckı (2014), and it was shown in Song et al. (2020) that the family of empirical measures  $\{L_N(t), t \in [0, T]\}_{n \in \mathbb{N}}$  given by

$$L_N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N(t)} \quad (1.7)$$

is tight almost surely, and any limit  $\{\mu_t, t \in [0, T]\}$  satisfies

$$\int \frac{\mu_t(dx)}{z-x} = \int \frac{\mu_0(dx)}{z-x} + \int_0^t \left[ \int \frac{b(x)}{(z-x)^2} \mu_s(dx) \right] ds + \int_0^t \left[ \int \frac{\sigma(x)^2}{(z-x)^3} \mu_s(dx) \right] ds + \int_0^t \left[ \iint \frac{H(x,y)}{(z-x)(z-y)^2} \mu_s(dx) \mu_s(dy) \right] ds, \quad \forall z \in \mathbb{C} \setminus \mathbb{R}, \quad (1.8)$$

where the continuous functions  $b(x)$ ,  $\sigma(x)$  and  $H(x,y)$  are the uniform limits of  $b_N(x)$ ,  $\sigma^N(x)$  and  $NH_N(x,y)$ , respectively.

Up to considering a subsequence, we can assume, without loss of generality, the uniqueness of the limit process  $\{\mu_t, t \in [0, T]\}$ . Consider the random fluctuations,

$$\mathcal{L}_t^N(f) = N\langle f, L_N(t) - \mu_t \rangle = \sum_{i=1}^N f(\lambda_i^N(t)) - N\langle f, \mu_t \rangle, \quad N \in \mathbb{N}, \quad (1.9)$$

where  $f$  belongs to an appropriate space of test functions and

$$\langle f, \mu \rangle = \int f(x)\mu(dx),$$

for a measure  $\mu$  on  $\mathbb{R}$ .

The main purpose of the paper is to find, as  $N \rightarrow \infty$ , a Gaussian limit for the centered process

$$\begin{aligned} Q_t^N(f) &= \mathcal{L}_t^N(f) - \mathcal{L}_0^N(f) - \int_0^t \mathcal{L}_s^N(f'b)ds - \frac{1}{2} \int_0^t \int f''(x) (\tilde{\sigma}^2(x) - H(x, x)) \mu_s(dx) ds \\ &\quad - \int_0^t \mathcal{L}_s^N \left( \int \frac{f'(x) - f'(y)}{x - y} H(x, y) \mu_s(dx) \right) ds \\ &\quad - \frac{N}{2} \int_0^t \iint \frac{f'(x) - f'(y)}{x - y} H(x, y) [L_N(s)(dx) - \mu_s(dx)] [L_N(s)(dy) - \mu_s(dy)] ds, \end{aligned} \quad (1.10)$$

where  $\tilde{\sigma}(x)$  is the uniform limit of  $\sqrt{N}\sigma^N(x)$ .

To our best knowledge, the problem of fluctuations for empirical measures of eigenvalue processes was first studied in Cabanal-Duvillard (2001) and later extended in Anderson et al. (2010). More precisely, Cabanal-Duvillard (2001) considered Dyson's Brownian motion and Wishart process with null initial condition, and established CLT for polynomial test functions; (Anderson et al., 2010, Theorem 4.3.20) extended the result for Dyson's Brownian motion by allowing bounded initial condition. Similar results were also obtained in Pérez-Abreu and Tudor (2007) and Perez-Abreu and Tudor (2009).

The rest of this paper is organized as follows.

In Section 2, we establish CLTs for the empirical measures of the general particle system (1.6). The space  $\mathbb{F}$  of test functions is given by (2.1), which in general does not necessarily contain the set of all polynomials.

In Section 3, we apply the results in Section 2 to obtain CLTs for the eigenvalues of Wishart process in Section 3.2, for Dyson's Brownian motion in Section 3.3, and for the eigenvalues of symmetric matrix-valued Ornstein-Uhlenbeck process in Section 3.4, respectively. Note that in these three cases, one key ingredient is the following uniform bound for the moments of the eigenvalue processes

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \langle |x|^p, L_N(t) \rangle \right] \leq C^p, \quad (1.11)$$

which enables us to obtain more precise CLTs for a larger class of test functions (in comparison with  $\mathbb{F}$  given in (2.1)). For Dyson's Brownian motion, such bound was obtained in Anderson et al. (2010) by using the explicit joint density function of the eigenvalues (see Lemma 4.3.17 therein). This density approach is developed here in full detail for the Wishart case (Section 3.2). However, for more general particle systems that we consider in this paper, as the joint density functions are not available, some new tool is needed in order to derive the uniform moment bound (1.11). We thus establish in Section 3.1 a comparison principle for the particle system (1.6), which is also of interest

in itself. This comparison principle allows to obtain the uniform bound (1.11) and then extend the CLTs to a larger class of particle systems (Corollaries 3.7, 3.10 and 3.12).

Furthermore, due to the special structures of Wishart process, Dyson’s Brownian motion, and matrix-valued Ornstein-Uhlenbeck process, we are able to directly characterize the fluctuations  $\{\mathcal{L}_t(x^n), t \in [0, T]\}_{n \in \mathbb{N}}$ , where  $\mathcal{L}_t(x^n)$  is the limit of  $\mathcal{L}_t^N(x^n)$ , by recursive formulas (see Theorems 3.5, 3.8, 3.11 and the remarks thereafter). For Dyson’s Brownian motion and Wishart process, our CLTs provide extensions of the existing results in Cabanal-Duvillard (2001) and Anderson et al. (2010), while the CLT obtained in Section 3.4 for the eigenvalue processes of matrix-valued Ornstein-Uhlenbeck process seems to be new.

Finally, some useful lemmas are provided in Appendix A.

## 2 Central limit theorems

In this section, we derive a central limit theorem for the empirical measure (1.7) of the particle system (1.6).

Let  $C_b^k(\mathbb{R})$  be the space of bounded continuous real-valued functions with bounded continuous derivatives (up to the order  $k$ ). Recall that the continuous functions  $b(x)$ ,  $\sigma(x)$  and  $H(x, y)$  are the uniform limits of  $b_N(x)$ ,  $\sigma^N(x)$  and  $NH_N(x, y)$ , respectively, and  $Q_t^N(f)$  is defined in (1.10). We use the following space of test functions

$$\mathbb{F} = \left\{ f \in C_b^2(\mathbb{R}) : \|f'(x)b(x)\|_{L^\infty(\mathbb{R})} < \infty, \left\| \frac{f'(x) - f'(y)}{x - y} H(x, y) \right\|_{L^\infty(\mathbb{R}^2)} < \infty, \right. \\ \left. \|f'(x)\tilde{\sigma}(x)\|_{L^\infty(\mathbb{R})} < \infty, \|f''(x)\tilde{\sigma}^2(x)\|_{L^\infty(\mathbb{R})} < \infty \right\}, \tag{2.1}$$

where the continuous function  $\tilde{\sigma}(x)$  is the uniform limit of  $\sqrt{N}\sigma^N(x)$ .

**Remark 2.1.** For Dyson’s Brownian motion,  $\sigma^N(x) = \frac{\sqrt{2}}{\sqrt{N}}$ ,  $b_N(x) = 0$ ,  $H_N(x, y) = \frac{1}{N}$  (see also Section 3.3). Thus, the space  $\mathbb{F}$  of test functions is  $C_b^2(\mathbb{R})$ . For Wishart process,  $\sigma^N(x) = 2\frac{\sqrt{x}}{\sqrt{N}}$ ,  $b_N(x) = \frac{P}{N}$ ,  $H_N(x, y) = \frac{x+y}{N}$  (see also Section 3.2), the limits of the coefficient functions are  $\tilde{\sigma}(x) = 2\sqrt{x}$ ,  $b(x) = c$  and  $H(x, y) = x + y$ . In this case,  $\mathbb{F}$  is the set of functions  $f(x)$  in  $C_b^2(\mathbb{R})$  satisfying  $\|\sqrt{x}f'(x)\|_{L^\infty(\mathbb{R})} < \infty$  and  $\|xf''(x)\|_{L^\infty(\mathbb{R})} < \infty$ . For Ornstein-Uhlenbeck process,  $\sigma^N(x) = \frac{1}{\sqrt{N}}$ ,  $b_N(x) = -\frac{1}{2}x$ ,  $H_N(x, y) = \frac{1}{2N}$  (see also Section 3.4), and correspondingly  $\tilde{\sigma}(x) = 1$ ,  $b(x) = -\frac{1}{2}x$  and  $H(x, y) = \frac{1}{2}$ . In this case,  $\mathbb{F}$  is the set of functions  $f(x)$  in  $C_b^2(\mathbb{R})$  satisfying  $\|xf'(x)\|_{L^\infty(\mathbb{R})} < \infty$ . In Sections 3.2, 3.3 and 3.4, we will develop CLTs for a larger class of test functions which includes all polynomials for these three cases, respectively.

**Theorem 2.2.** *Suppose that*

$$\lim_{N \rightarrow \infty} N \|b_N(x) - b(x)\|_{L^\infty(\mathbb{R})} = 0; \tag{2.2}$$

$$\lim_{N \rightarrow \infty} N \|NH_N(x, y) - H(x, y)\|_{L^\infty(\mathbb{R}^2)} = 0.$$

*Also assume that (1.6) has a non-exploding and non-colliding strong solution, such that the sequence of the empirical measures  $\{L_N(t), t \in [0, T]\}_{N \in \mathbb{N}}$  converges weakly to  $\{\mu_t, t \in [0, T]\}$ .*

*Then, for any  $k \in \mathbb{N}$  and any  $f_1, \dots, f_k \in \mathbb{F}$ ,  $(Q_t^N(f_1), \dots, Q_t^N(f_k))_{t \in [0, T]}$  converges in distribution to a centered Gaussian process  $(G_t(f_1), \dots, G_t(f_k))_{t \in [0, T]}$  with covariance*

$$\mathbb{E}[G_t(f_i)G_s(f_j)] = \int_0^{t \wedge s} \langle f'_i(x)f'_j(x)\tilde{\sigma}^2(x), \mu_u \rangle du, \quad 1 \leq i, j \leq k. \tag{2.3}$$

*Proof.* By Itô's formula and (1.6), for  $f \in C^2(\mathbb{R})$ ,

$$df(x_i^N(t)) = f'(x_i^N(t))\sigma^N(x_i^N(t))dW_i(t) + f'(x_i^N(t))b_N(x_i^N(t))dt + \sum_{j:j \neq i} \frac{f'(x_i^N(t))H_N(x_i^N(t), x_j^N(t))}{x_i^N(t) - x_j^N(t)}dt + \frac{1}{2}f''(x_i^N(t))(\sigma^N(x_i^N(t)))^2dt.$$

Thus, we have, noting that  $H_N(x, y)$  is symmetric,

$$\begin{aligned} \langle f, L_N(t) \rangle &= \frac{1}{N} \sum_{i=1}^N f(x_i^N(t)) \\ &= \frac{1}{N} \sum_{i=1}^N f(x_i^N(0)) + \frac{1}{N} \sum_{i=1}^N \int_0^t f'(x_i^N(s))\sigma^N(x_i^N(s))dW_i(s) \\ &\quad + \frac{1}{N} \sum_{i=1}^N \int_0^t f'(x_i^N(s))b_N(x_i^N(s))ds + \frac{1}{N} \int_0^t \sum_{i \neq j} \frac{f'(x_i^N(s))H_N(x_i^N(s), x_j^N(s))}{x_i^N(s) - x_j^N(s)}ds \\ &\quad + \frac{1}{2N} \sum_{i=1}^N \int_0^t f''(x_i^N(s))(\sigma^N(x_i^N(s)))^2ds \\ &= \frac{1}{N} \sum_{i=1}^N f(x_i^N(0)) + \frac{1}{N} \sum_{i=1}^N \int_0^t f'(x_i^N(s))\sigma^N(x_i^N(s))dW_i(s) \\ &\quad + \frac{1}{N} \sum_{i=1}^N \int_0^t f'(x_i^N(s))b_N(x_i^N(s))ds + \frac{1}{2N} \sum_{i=1}^N \int_0^t f''(x_i^N(s))(\sigma^N(x_i^N(s)))^2ds \\ &\quad + \frac{1}{2N} \int_0^t \sum_{i \neq j} \frac{f'(x_i^N(s)) - f'(x_j^N(s))}{x_i^N(s) - x_j^N(s)}H_N(x_i^N(s), x_j^N(s))ds \\ &= \langle f, L_N(0) \rangle + M_f^N(t) + \int_0^t \langle f'b_N, L_N(s) \rangle ds + \frac{1}{2} \int_0^t \langle f''(\sigma^N)^2, L_N(s) \rangle ds \\ &\quad + \frac{1}{2N} \int_0^t \sum_{i,j=1}^N \frac{f'(x_i^N(s)) - f'(x_j^N(s))}{x_i^N(s) - x_j^N(s)}H_N(x_i^N(s), x_j^N(s))ds \\ &\quad - \frac{1}{2N} \int_0^t \sum_{i=1}^N f''(x_i^N(s))H_N(x_i^N(s), x_i^N(s))ds \\ &= \langle f, L_N(0) \rangle + M_f^N(t) + \int_0^t \langle f'b_N, L_N(s) \rangle ds \\ &\quad + \frac{1}{2} \int_0^t \int f''(x) ((\sigma^N(x))^2 - H_N(x, x)) L_N(s)(dx)ds \\ &\quad + \frac{N}{2} \int_0^t \iint \frac{f'(x) - f'(y)}{x - y} H_N(x, y) L_N(s)(dx) L_N(s)(dy)ds, \end{aligned} \tag{2.4}$$

where we use the convention  $\frac{f'(x) - f'(y)}{x - y} = f''(x)$  if  $x = y$ , and

$$M_f^N(t) = \frac{1}{N} \sum_{i=1}^N \int_0^t f'(x_i^N(s))\sigma^N(x_i^N(s))dW_i(s) \tag{2.5}$$

is a local martingale with quadratic variation

$$\langle M_f^N \rangle_t = \frac{1}{N^2} \sum_{i=1}^N \int_0^t (f'(x_i^N(s))\sigma^N(x_i^N(s)))^2 ds = \frac{1}{N} \int_0^t \langle |f'\sigma^N|^2, L_N(s) \rangle ds. \tag{2.6}$$

Recall that the limit measure  $\mu_t$  satisfies (1.8). For  $f \in \mathbb{F}$ , under condition (2.2), one may apply the approach used in the proof of Theorem 1 in Małeckı and Pérez (2019) (see also Theorem 2.2 in Song et al. (2020)) to get, noting that  $\sigma(x) \equiv 0$ ,

$$\langle f, \mu_t \rangle = \langle f, \mu_0 \rangle + \int_0^t \langle f' b, \mu_s \rangle ds + \frac{1}{2} \int_0^t \iint \frac{f'(x) - f'(y)}{x - y} H(x, y) \mu_s(dx) \mu_s(dy) ds. \quad (2.7)$$

Thus, (2.4) and (2.7) yield

$$\begin{aligned} \mathcal{L}_t^N(f) &= N \langle f, L_N(t) - \mu_t \rangle \\ &= N \langle f, L_N(0) - \mu_0 \rangle + N M_f^N(t) \\ &\quad + N \int_0^t \left( \langle f' b_N, L_N(s) \rangle - \langle f' b, \mu_s \rangle \right) ds \\ &\quad + \frac{N}{2} \int_0^t \int f''(x) \left( (\sigma^N(x))^2 - H_N(x, x) \right) L_N(s)(dx) ds \\ &\quad + \frac{N}{2} \int_0^t \iint \frac{f'(x) - f'(y)}{x - y} \left[ N H_N(x, y) L_N(s)(dx) L_N(s)(dy) - H(x, y) \mu_s(dx) \mu_s(dy) \right] ds. \end{aligned} \quad (2.8)$$

The third term on the right-hand side of (2.8) can be written as

$$\begin{aligned} &N \int_0^t \left( \langle f' b_N, L_N(s) \rangle - \langle f' b, \mu_s \rangle \right) ds \\ &= N \int_0^t \langle f' b_N - f' b, L_N(s) \rangle ds + N \int_0^t \langle f' b, L_N(s) - \mu_s \rangle ds \\ &= N \int_0^t \langle f' b_N - f' b, L_N(s) \rangle ds + \int_0^t \mathcal{L}_s^N(f' b) ds. \end{aligned}$$

Thus, by (2.2), we have

$$\begin{aligned} &\lim_{N \rightarrow \infty} \left| N \int_0^t \left( \langle f' b_N, L_N(s) \rangle - \langle f' b, \mu_s \rangle \right) ds - \int_0^t \mathcal{L}_s^N(f' b) ds \right| \\ &\leq \lim_{N \rightarrow \infty} N \int_0^t |\langle f' b_N - f' b, L_N(s) \rangle| ds \\ &\leq \lim_{N \rightarrow \infty} N T \|f'\|_{L^\infty(\mathbb{R})} \|b_N - b\|_{L^\infty(\mathbb{R})} = 0. \end{aligned} \quad (2.9)$$

For the fourth term on the right-hand side of (2.8),

$$\begin{aligned} &\frac{N}{2} \int_0^t \int f''(x) \left( (\sigma^N(x))^2 - H_N(x, x) \right) L_N(s)(dx) ds \\ &= \frac{1}{2} \int_0^t \int f''(x) \left( N(\sigma^N(x))^2 - \tilde{\sigma}^2(x) + H(x, x) - N H_N(x, x) \right) L_N(s)(dx) ds \\ &\quad + \frac{1}{2} \int_0^t \int f''(x) \left( \tilde{\sigma}^2(x) - H(x, x) \right) L_N(s)(dx) ds. \end{aligned}$$

Noting that both convergences  $\sqrt{N} \sigma^N \rightarrow \tilde{\sigma}$  and  $N H_N(x, x) \rightarrow H(x, x)$  are uniform as

$N \rightarrow \infty$ , we have

$$\begin{aligned}
 & \left| \frac{N}{2} \int_0^t f''(x) ((\sigma^N(x))^2 - H_N(x, x)) L_N(s)(dx) ds \right. \\
 & \quad \left. - \frac{1}{2} \int_0^t \int f''(x) (\tilde{\sigma}^2(x) - H(x, x)) \mu_s(dx) ds \right| \\
 & \leq \left| \frac{1}{2} \int_0^t \int f''(x) (N(\sigma^N(x))^2 - \tilde{\sigma}^2(x) + H(x, x) - NH_N(x, x)) L_N(s)(dx) ds \right| \\
 & \quad + \left| \frac{1}{2} \int_0^t \int f''(x) (\tilde{\sigma}^2(x) - H(x, x)) (L_N(s)(dx) - \mu_s(dx)) ds \right| \\
 & \leq \frac{T}{2} \|f''\|_{L^\infty(\mathbb{R})} \|N(\sigma^N(x))^2 - \tilde{\sigma}^2(x) + H(x, x) - NH_N(x, x)\|_{L^\infty(\mathbb{R})} \\
 & \quad + \frac{1}{2} \left| \int_0^t \langle f''(x) (\tilde{\sigma}^2(x) - H(x, x)), L_N(s)(dx) - \mu_s(dx) \rangle ds \right| \\
 & \rightarrow 0,
 \end{aligned} \tag{2.10}$$

as  $N \rightarrow \infty$ , where the last step follows from the weak convergence of  $\{L_N(t), t \in [0, T]\}_{N \in \mathbb{N}}$  and the continuity and boundedness of  $f''(x) (\tilde{\sigma}^2(x) - H(x, x))$  for  $f \in \mathbb{F}$ .

The fifth term on the right-hand side of (2.8) can be written as

$$\begin{aligned}
 & \frac{N}{2} \int_0^t \iint \frac{f'(x) - f'(y)}{x - y} [NH_N(x, y)L_N(s)(dx)L_N(s)(dy) - H(x, y)\mu_s(dx)\mu_s(dy)] ds \\
 & = \frac{N}{2} \int_0^t \iint \frac{f'(x) - f'(y)}{x - y} [NH_N(x, y) - H(x, y)] L_N(s)(dx)L_N(s)(dy) ds \\
 & \quad + \frac{N}{2} \int_0^t \iint \frac{f'(x) - f'(y)}{x - y} H(x, y) [L_N(s)(dx) - \mu_s(dx)] [L_N(s)(dy) - \mu_s(dy)] ds \\
 & \quad + \frac{N}{2} \int_0^t \iint \frac{f'(x) - f'(y)}{x - y} H(x, y)\mu_s(dx) [L_N(s)(dy) - \mu_s(dy)] ds \\
 & \quad + \frac{N}{2} \int_0^t \iint \frac{f'(x) - f'(y)}{x - y} H(x, y) [L_N(s)(dx) - \mu_s(dx)] \mu_s(dy) ds \\
 & = \frac{N}{2} \int_0^t \iint \frac{f'(x) - f'(y)}{x - y} [NH_N(x, y) - H(x, y)] L_N(s)(dx)L_N(s)(dy) ds \\
 & \quad + \frac{N}{2} \int_0^t \iint \frac{f'(x) - f'(y)}{x - y} H(x, y) [L_N(s)(dx) - \mu_s(dx)] [L_N(s)(dy) - \mu_s(dy)] ds \\
 & \quad + \int_0^t \mathcal{L}_s^N \left( \int \frac{f'(x) - f'(y)}{x - y} H(x, y)\mu_s(dx) \right) ds,
 \end{aligned} \tag{2.11}$$

where the last equality follows from the symmetry of  $\frac{f'(x) - f'(y)}{x - y} H(x, y)$ . For the first term on the right-hand side of (2.11), by (2.2), we have

$$\begin{aligned}
 & \lim_{N \rightarrow \infty} \left| \frac{N}{2} \int_0^t \iint \frac{f'(x) - f'(y)}{x - y} [NH_N(x, y) - H(x, y)] L_N(s)(dx)L_N(s)(dy) ds \right| \\
 & \leq \lim_{N \rightarrow \infty} \frac{NT}{2} \left\| \frac{f'(x) - f'(y)}{x - y} \right\|_{L^\infty(\mathbb{R}^2)} \|NH_N(x, y) - H(x, y)\|_{L^\infty(\mathbb{R}^2)} = 0.
 \end{aligned} \tag{2.12}$$

Therefore, by (1.10), (2.8) and the above estimations (2.9), (2.10), (2.11), and (2.12),



we have that the term

$$\begin{aligned}
 & Q_t^N(f) - NM_f^N(t) \\
 &= \mathcal{L}_t^N(f) - \mathcal{L}_0^N(f) - NM_f^N(t) - \int_0^t \mathcal{L}_s^N(f'b)ds \\
 &\quad - \frac{1}{2} \int_0^t \langle f''(x) (\tilde{\sigma}^2(x) - H(x, x)), \mu_s(dx) \rangle ds \\
 &\quad - \int_0^t \mathcal{L}_s^N \left( \int \frac{f'(x) - f'(y)}{x - y} H(x, y) \mu_s(dx) \right) ds \\
 &\quad - \frac{N}{2} \int_0^t \iint \frac{f'(x) - f'(y)}{x - y} H(x, y) [L_N(s)(dx) - \mu_s(dx)] [L_N(s)(dy) - \mu_s(dy)] ds \quad (2.13)
 \end{aligned}$$

converges to 0 almost surely as  $N \rightarrow \infty$ , uniformly in  $t \in [0, T]$ . Note that in (2.9), (2.10) and (2.12), the integrand function is bounded, and hence the convergence is also in  $L^p$  for all  $p \geq 1$ . Thus,  $Q_t^N(f) - NM_f^N(t)$  with  $f \in \mathbb{F}$  converges to 0 in  $L^p$  for all  $p \geq 1$  uniformly in  $t \in [0, T]$ .

Therefore, to prove the desired result, it suffices to show that, for any  $k \in \mathbb{N}$  and  $f_1, \dots, f_k \in \mathbb{F}$ , the vector-valued process  $(NM_{f_1}^N(t), NM_{f_2}^N(t), \dots, NM_{f_k}^N(t))_{t \in [0, T]}$  converges in distribution to a centered Gaussian process  $(G_t(f_1), G_t(f_2), \dots, G_t(f_k))_{t \in [0, T]}$  with covariance given by (2.3). To this end, by Lemma A.1 it suffices to prove that  $\{NM_f^N(t), t \in [0, T]\}_{N \in \mathbb{N}}$  are martingales for  $f \in \mathbb{F}$  such that the following limit holds in  $L^1(\Omega)$ ,

$$\lim_{N \rightarrow \infty} \langle NM_{f_1}^N, NM_{f_2}^N \rangle_t = \int_0^t \langle f'_1 f'_2 \tilde{\sigma}^2, \mu_s \rangle ds, \quad \forall f_1, f_2 \in \mathbb{F}.$$

By the uniform convergence of  $\sqrt{N}\sigma^N$  towards  $\tilde{\sigma}$ , the boundedness of  $f'(x)\tilde{\sigma}(x)$  and (2.6), one can show that  $\{NM_f^N(t), t \in [0, T]\}_{N \in \mathbb{N}}$  are martingales. It follows from (2.5) that, for  $f_1, f_2 \in \mathbb{F}$ ,

$$\begin{aligned}
 \langle NM_{f_1}^N, NM_{f_2}^N \rangle_t &= \sum_{i=1}^N \int_0^t f'_1(x_i^N(s)) f'_2(x_i^N(s)) (\sigma^N(x_i^N(s)))^2 ds \\
 &= N \int_0^t \langle f'_1 f'_2 (\sigma^N)^2, L_N(s) \rangle ds \\
 &= \int_0^t \langle f'_1 f'_2 (N(\sigma^N)^2 - \tilde{\sigma}^2), L_N(s) \rangle ds + \int_0^t \langle f'_1 f'_2 \tilde{\sigma}^2, L_N(s) \rangle ds.
 \end{aligned}$$

The term  $\int_0^t \langle f'_1 f'_2 (N(\sigma^N)^2 - \tilde{\sigma}^2), L_N(s) \rangle ds$  converges to 0 a.s. and in  $L^p$  for all  $p \geq 1$  due to the boundedness of  $f'_1(x)$  and  $f'_2(x)$  and the uniform convergence of  $\sqrt{N}\sigma^N$  towards  $\tilde{\sigma}$ . Furthermore, the following convergence

$$\lim_{N \rightarrow \infty} \int_0^t \langle f'_1 f'_2 \tilde{\sigma}^2, L_N(s) \rangle ds = \int_0^t \langle f'_1 f'_2 \tilde{\sigma}^2, \mu_s \rangle ds,$$

holds a.s. and in  $L^p$  for all  $p \geq 1$ , because of the weak convergence of  $\{L_N(t), t \in [0, T]\}_{N \in \mathbb{N}}$  to  $\{\mu_t, t \in [0, T]\}$  and the boundedness of  $f'_1 f'_2 \tilde{\sigma}^2$ . Therefore,  $\langle NM_{f_1}^N, NM_{f_2}^N \rangle_t$  converges to  $\int_0^t \langle f'_1 f'_2 \tilde{\sigma}^2, \mu_s \rangle ds$  a.s. and in  $L^p$  for all  $p \geq 1$ .

The proof is concluded. □

If the particles in (1.6) are bounded, the test function space  $\mathbb{F}$  can be enlarged by removing the boundedness condition in Theorem 2.2.

**Corollary 2.3.** *Assume the same conditions as in Theorem 2.2. Moreover, for  $T < \infty$ , assume that*

$$\limsup_{N \rightarrow \infty} \sup_{t \in [0, T]} \max_{1 \leq i \leq N} |x_i^N(t)| \leq C(T), \tag{2.14}$$

*almost surely for some constant  $C(T)$  depending on  $T$ . Then Theorem 2.2 still holds if the set  $\mathbb{F}$  of test function is replaced by  $C^2(\mathbb{R})$ .*

*Proof.* Note that by (2.14), all but finitely many terms in  $\{\sup_{t \in [0, T]} \max_{1 \leq i \leq N} |x_i^N(t)|\}_{N \in \mathbb{N}}$  are bounded by  $A(T) = C(T) + 1$  a.s.. Thus, there exists a measurable set  $A \subset \Omega$  with  $\mathbb{P}(A) = 1$  and a random variable  $N_0(\omega) \in \mathbb{N}$ , such that for  $\omega \in A$ , the empirical measures  $\{L_N(t)(\omega), t \in [0, T]\}$  is supported in  $[-A(T), A(T)]$  for all  $N \geq N_0(\omega)$ . Hence the limit  $\{\mu_t, t \in [0, T]\}$  also has the same support. By (Rudin, 1991, 1.46), there exists a cut-off function  $\eta(x) \in C^\infty(\mathbb{R})$  equal to 1 on  $[-A(T), A(T)]$ , of which the support is  $[-2A(T), 2A(T)]$ . If we replace  $f$  by  $f\eta$ , noting that  $f\eta \in \mathbb{F}$  for  $f \in C^2(\mathbb{R})$  and that  $f\eta = f$  on  $[-A(T), A(T)]$ , we can show that the term  $Q_t^N(f) - NM_f^N(t)$  in (2.13) converges to 0 a.s. using the same argument as in the proof of Theorem 2.2. Then following the rest part of the proof, it is easy to get the result of Theorem 2.2.  $\square$

**Remark 2.4.** Under the conditions in Theorem 2.2, (2.14) yields almost sure convergence of  $Q_t^N(f) - NM_f^N(t)$  towards 0 for  $f \in C^2(\mathbb{R})$ . However, it is not clear whether the convergence also holds in  $L^p$  for  $p \geq 1$ .

The following corollary provides a sufficient condition for  $L^p$ -convergence for  $p \geq 1$ .

**Corollary 2.5.** *Assume the same conditions as in Theorem 2.2. For  $T < \infty$  and all  $p \geq 1$  and all  $N \geq cp$  for some positive constant  $c$ , assume that*

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \langle |x|^p, L_N(t) \rangle \right] \leq C(T)^p, \tag{2.15}$$

*for some positive constant  $C(T)$  which depends only on  $T$ . Furthermore, assume that  $(\tilde{\sigma}(x)^2 - H(x, x))f''(x)$  and its derivative has at most polynomial growth. Then for  $f \in C^3(\mathbb{R})$  of which the derivatives have at most polynomial growth,  $Q_t^N(f) - NM_f^N(t)$  converges to 0 in  $L^p$  for all  $p \geq 1$  uniformly in  $t \in [0, T]$ .*

*Proof.* By the analysis in the proof of Theorem 2.2, it suffices to show

$$\limsup_{N \rightarrow \infty} \mathbb{E} \left[ \sup_{t \in [0, T]} |\langle g, L_N(t) \rangle - \langle g, \mu_t \rangle|^p \right] = 0, \tag{2.16}$$

for  $p \geq 1$  and  $g \in C^1(\mathbb{R})$  of which the derivative has at most polynomial growth. More precisely, one can check that under conditions (2.15) and (2.16), the convergences to 0 in (2.9), (2.10) and (2.12) are uniform in  $L^p$ , and hence  $Q_t^N(f) - NM_f^N(t)$  in (2.13) converges to 0 in  $L^p$  uniformly.

By Markov inequality and (2.15),

$$\begin{aligned} \mathbb{P} \left( \sup_{t \in [0, T]} \max_{1 \leq i \leq N} |x_i^N(t)| > C(T) + 1 \right) &\leq (C(T) + 1)^{-p} \mathbb{E} \left[ \sup_{t \in [0, T]} \max_{1 \leq i \leq N} |x_i^N(t)|^p \right] \\ &\leq (C(T) + 1)^{-p} N \mathbb{E} \left[ \sup_{t \in [0, T]} \langle |x|^p, L_N(t) \rangle \right] \\ &\leq N \left( \frac{C(T)}{C(T) + 1} \right)^p. \end{aligned}$$

Choosing  $p = \ln^2 N$ , we have

$$\begin{aligned} \sum_{N=1}^{\infty} \mathbb{P} \left( \sup_{t \in [0, T]} \max_{1 \leq i \leq N} |x_i^N(t)| > C(T) + 1 \right) &\leq \sum_{N=1}^{\infty} N \left( \frac{C(T)}{C(T) + 1} \right)^p \\ &= \sum_{N=1}^{\infty} N^{1 + \ln N \ln \frac{C(T)}{C(T)+1}} \\ &< \infty. \end{aligned}$$

By Borel-Cantelli lemma, we get that almost surely,

$$\limsup_{N \rightarrow \infty} \sup_{t \in [0, T]} \max_{1 \leq i \leq N} |x_i^N(t)| \leq C(T) + 1.$$

By the proof of Corollary 2.3, the limiting measure  $\{\mu_t, t \in [0, T]\}$  is supported in  $[-C(T) - 1, C(T) + 1]$ .

For  $g \in C^1(\mathbb{R})$  with  $|g'(x)| \leq a|x|^{n-1} + b$  for some  $a, b \in \mathbb{R}$ ,  $n \in \mathbb{N}_+$ , define

$$g_\delta(x) = g \left( \frac{x}{1 + \delta x^2} \right)$$

for  $\delta > 0$ . Then  $g_\delta(x)$  is a bounded continuous function, and hence

$$\lim_{N \rightarrow \infty} \sup_{t \in [0, T]} |\langle g_\delta, L_N(t) \rangle - \langle g_\delta, \mu_t \rangle| = 0,$$

almost surely. By dominated convergence theorem,

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ \sup_{t \in [0, T]} |\langle g_\delta, L_N(t) \rangle - \langle g_\delta, \mu_t \rangle|^p \right] = 0. \tag{2.17}$$

Note that  $g'(x)$  grows no faster than polynomials of degree  $n - 1$ , by the mean value theorem, it is not difficult to show  $|g(x) - g_\delta(x)| \leq C\delta(|x|^{n+2} + |x|^3)$ , which implies that  $g_\delta$  converges to  $g$  uniformly in any compact interval as  $\delta \rightarrow 0^+$ . Thus,

$$\lim_{\delta \rightarrow 0^+} \sup_{t \in [0, T]} |\langle g, \mu_t \rangle - \langle g_\delta, \mu_t \rangle| = 0. \tag{2.18}$$

Finally, by Jensen's inequality and (2.15), we obtain that, as  $\delta \rightarrow 0^+$ ,

$$\begin{aligned} &\mathbb{E} \left[ \sup_{t \in [0, T]} |\langle g, L_N(t) \rangle - \langle g_\delta, L_N(t) \rangle|^p \right] \\ &\leq C^p \delta^p \mathbb{E} \left[ \sup_{t \in [0, T]} |\langle |x|^{n+2} + |x|^3, L_N(t) \rangle|^p \right] \\ &\leq C^p \delta^p \mathbb{E} \left[ \sup_{t \in [0, T]} |\langle (|x|^{n+2} + |x|^3)^p, L_N(t) \rangle| \right] \\ &\leq 2^p C^p (C(T)^{(n+2)p} + C(T)^{3p}) \delta^p \rightarrow 0 \end{aligned} \tag{2.19}$$

uniformly in  $N \in \mathbb{N}_+$ .

By (2.17), (2.18), (2.19) and the triangle inequality, we can obtain (2.16), and the proof is concluded.  $\square$

**Remark 2.6.** The proof of Corollary 2.5 clearly shows that condition (2.15) implies condition (2.14). Therefore, under (2.15), for the test function  $f$  satisfying conditions in Corollary 2.5,  $Q_t^N(f) - NM_f^N(t)$  converges to 0 a.s. and in  $L^p$  for all  $p \geq 1$  uniformly in  $t \in [0, T]$ . As a consequence, Theorem 2.2 holds for such test functions.

To end this section, we provide the following linear property for the Gaussian family  $\{G_t(f), f \in \mathbb{F}\}$  obtained in Theorem 2.2.

**Proposition 2.7.** Consider the centered Gaussian family  $\{G_t(f), f \in \mathbb{F}\}$  with covariance

$$\mathbb{E}[G_t(f)G_t(g)] = \int_0^t \langle f'(x)g'(x)\tilde{\sigma}^2(x), \mu_u \rangle du, \quad \forall f, g \in \mathbb{F}.$$

For  $f_1, f_2 \in \mathbb{F}$  and  $\alpha_1, \alpha_2 \in \mathbb{R}$ , we have

$$G_t(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 G_t(f_1) + \alpha_2 G_t(f_2), \quad \forall t \in [0, T], \tag{2.20}$$

almost surely.

*Proof.* For  $f_1, f_2 \in \mathbb{F}$  and  $\alpha_1, \alpha_2 \in \mathbb{R}$ , it is easy to check that  $\alpha_1 f_1 + \alpha_2 f_2 \in \mathbb{F}$ . By the proof of Theorem 2.2, the random vector  $(NM_{f_1}^N(t), NM_{f_2}^N(t), NM_{\alpha_1 f_1 + \alpha_2 f_2}^N(t))_{t \in [0, T]}$  converges in distribution to  $(G_t(f_1), G_t(f_2), G_t(\alpha_1 f_1 + \alpha_2 f_2))_{t \in [0, T]}$ . Hence, the linear combination  $(\alpha_1 NM_{f_1}^N(t) + \alpha_2 NM_{f_2}^N(t) - NM_{\alpha_1 f_1 + \alpha_2 f_2}^N(t))_{t \in [0, T]}$  converges in distribution to  $(\alpha_1 G_t(f_1) + \alpha_2 G_t(f_2) - G_t(\alpha_1 f_1 + \alpha_2 f_2))_{t \in [0, T]}$ .

By (2.5), we can see that the martingale  $M_f^N(t)$  is linear with respect to the function  $f$ , so  $\alpha_1 NM_{f_1}^N(t) + \alpha_2 NM_{f_2}^N(t) = NM_{\alpha_1 f_1 + \alpha_2 f_2}^N(t)$  for all  $t \in [0, T]$  and all  $N \in \mathbb{N}$ , which implies that the process  $(\alpha_1 NM_{f_1}^N(t) + \alpha_2 NM_{f_2}^N(t) - NM_{\alpha_1 f_1 + \alpha_2 f_2}^N(t))_{t \in [0, T]}$  is actually a zero process. Thus, as the limit of the convergence in distribution,  $(\alpha_1 G_t(f_1) + \alpha_2 G_t(f_2) - G_t(\alpha_1 f_1 + \alpha_2 f_2))_{t \in [0, T]}$  is also a zero process, which implies (2.20).  $\square$

### 3 Applications

In this section, we first provide a comparison principle in Section 3.1, and then we apply our main results obtained in Section 2 to the eigenvalues of Wishart process (Section 3.2), Dyson’s Brownian motion (Section 3.3) and the eigenvalues of symmetric matrix-valued Ornstein-Uhlenbeck process (Section 3.4).

For these three cases, we are able to obtain the boundedness of the moments for the empirical measures assuming proper initial conditions (see Lemma 3.4 for Wishart process, Eq. (3.30) for Dyson’s Brownian motion, and Eq. (3.45) for Ornstein-Uhlenbeck process). This enables us to apply Corollary 2.5 and Remark 2.6. As a consequence, the CLT in Theorem 2.2 holds for a larger space of test functions consisting of all the functions in  $C^3(\mathbb{R})$  of which the derivatives have at most polynomial growth. In particular, if we choose the space  $\mathbb{R}[x]$  of polynomial functions as the space of test functions, we are able to obtain recursive formulas for the basis  $\{\mathcal{L}_t(x^n), t \in [0, T]\}_{n \in \mathbb{N}}$  of  $\{\mathcal{L}_t(f), t \in [0, T]\}_{f \in \mathbb{R}[x]}$ . Note that these results are more precise than the general results in Section 2, where we study the centered processes  $\{Q_t^N(f)\}_{N \in \mathbb{N}}$  for more restricted test functions  $f \in \mathbb{F} \subset C_b^2(\mathbb{R})$ , recalling that  $\mathbb{F}$  is given in (2.1).

#### 3.1 Comparison principle

In this subsection, we provide comparison principles for SDE (1.2) and particle system (1.6), which allow us to obtain the boundedness of the eigenvalues/particles under more general initial conditions in Sections 3.2, 3.3 and 3.4.

Throughout this subsection, the dimension  $N$  is fixed and thus subscripts/superscripts are removed. Precisely, consider the following two particle systems: for  $1 \leq i \leq N, t \geq 0$ ,

$$\begin{cases} dx_i(t) = \sigma_i(x_i(t))dW_i(t) + \left( b_i(x_i(t)) + \sum_{j:j \neq i} \frac{H_{ij}(x_i(t), x_j(t))}{x_i(t) - x_j(t)} \right) dt, \\ x_1(t) \leq \dots \leq x_N(t), \end{cases} \tag{3.1}$$

and

$$\begin{cases} dy_i(t) = \sigma_i(y_i(t))dW_i(t) + \left( \tilde{b}_i(y_i(t)) + \sum_{j:j \neq i} \frac{H_{ij}(y_i(t), y_j(t))}{y_i(t) - y_j(t)} \right) dt, \\ y_1(t) \leq \dots \leq y_N(t), \end{cases} \quad (3.2)$$

with non-colliding initial values  $x(0) = (x_1(0), \dots, x_N(0))$  and  $y(0) = (y_1(0), \dots, y_N(0))$ , respectively. Here, the functions  $\sigma_i(x)$ ,  $b_i(x)$  and  $\tilde{b}_i(x)$  for  $1 \leq i \leq N$  are continuous, and  $H_{ij}(x, y)$  with  $i \neq j$  is a continuous, non-negative and symmetric function satisfying the condition (Graczyk and Małeck, 2014, (A1)):

$$\frac{H_{ij}(w, z)}{z - w} \leq \frac{H_{ij}(x, y)}{y - x}, \quad \forall w < x < y < z, \quad 1 \leq i \neq j \leq N. \quad (3.3)$$

Note that conditions for the existence and uniqueness of a non-colliding and non-exploding strong solution to (3.1) (or (3.2)) were obtained in Graczyk and Małeck (2014). In particular, under conditions (A2) – (A5) therein, the particles will separate from each other immediately after starting from a colliding initial state, and will not collide forever.

**Theorem 3.1.** *Suppose  $x(t) = (x_1(t), \dots, x_N(t))$  and  $y(t) = (y_1(t), \dots, y_N(t))$  are the non-exploding and non-colliding unique strong solutions to (3.1) and (3.2), respectively. Assume that there exists a strictly increasing function  $\rho : [0, \infty) \rightarrow [0, \infty)$  with  $\rho(0) = 0$  and*

$$\int_{0+} \rho^{-2}(u)du = \infty,$$

such that

$$|\sigma_i(u) - \sigma_i(v)| \leq \rho(|u - v|), \quad \forall u, v \in \mathbb{R}, \quad 1 \leq i \leq N.$$

If we further assume that  $b_i(u) \leq \tilde{b}_i(u)$  for all  $u \in \mathbb{R}$ , and  $x_i(0) \leq y_i(0)$  a.s.,  $1 \leq i \leq N$ , then

$$\mathbb{P}(x_i(t) \leq y_i(t), \forall t \geq 0, 1 \leq i \leq N) = 1.$$

*Proof.* The continuity of the functions  $H_{ij}$  and the condition (3.3) imply that for all  $1 \leq i \neq j \leq N$ ,

$$\frac{H_{ij}(x, z)}{x - z} \geq \frac{H_{ij}(x, y)}{x - y}, \quad \forall x < y \leq z,$$

and

$$\frac{H_{ij}(w, y)}{y - w} \leq \frac{H_{ij}(x, y)}{y - x}, \quad \forall w \leq x < y.$$

Hence, the drift functions

$$F(u) = \left( b_i(u_i) + \sum_{j:j \neq i} \frac{H_{ij}(u_i, u_j)}{u_i - u_j} \right)_{1 \leq i \leq N}, \quad \tilde{F}(u) = \left( \tilde{b}_i(u_i) + \sum_{j:j \neq i} \frac{H_{ij}(u_i, u_j)}{u_i - u_j} \right)_{1 \leq i \leq N},$$

satisfy the quasi-monotonously increasing condition in Lemma A.2.

In order to apply Lemma A.2 to get the desired result, we use an approximation argument to remove the singularities of the drift functions  $F$  and  $\tilde{F}$ . For  $\epsilon > 0$ , let

$$\Delta_\epsilon = \{u = (u_1, \dots, u_N) \in \mathbb{R}^N : \forall 1 \leq i \leq N - 1, u_{i+1} - u_i > \epsilon\}$$

and define the stopping time

$$\tau_\epsilon = \inf_{t>0} \left\{ \min_{1 \leq i \leq N-1} (x_{i+1}(t) - x_i(t)) \wedge (y_{i+1}(t) - y_i(t)) \leq \epsilon \right\}.$$

One can find continuous quasi-monotonously increasing functions  $F_\epsilon$  and  $\tilde{F}_\epsilon$ , such that they coincide with  $F$  and  $\tilde{F}$  in  $\Delta_\epsilon$ , respectively. Before time  $\tau_\epsilon$ , both  $x$ -particles and  $y$ -particles stay in  $\Delta_\epsilon$  and thus satisfy (3.1) and (3.2) with drift functions  $\tilde{F}_\epsilon$  and  $\tilde{F}_\epsilon$ , respectively.

Applying Lemma A.2 to the processes  $x^\epsilon$  and  $y^\epsilon$ , we have

$$\mathbb{P}(x_i^\epsilon(t) \leq y_i^\epsilon(t), \forall t \geq 0, 1 \leq i \leq N) = 1,$$

which implies

$$\mathbb{P}(x_i(t) \leq y_i(t), \forall t \in [0, \tau_\epsilon], 1 \leq i \leq N) = 1.$$

The desired result now follows from the non-colliding property  $\lim_{\epsilon \rightarrow 0^+} \tau_\epsilon = \infty$ . □

As a corollary of Theorem 3.1, we have the following comparison principle for SDE (1.2) of eigenvalue processes. Note that the existence and uniqueness of the non-colliding and non-exploding strong solution was obtained under proper conditions in Graczyk and Małeckı (2013).

**Corollary 3.2.** *Suppose that the following systems of eigenvalue SDEs*

$$d\lambda_i(t) = 2g_N(\lambda_i(t))h_N(\lambda_i^N(t))dW_i(t) + \left( b_N(\lambda_i^N(t)) + \sum_{j:j \neq i} \frac{G_N(\lambda_i(t), \lambda_j(t))}{\lambda_i(t) - \lambda_j(t)} \right) dt, \quad 1 \leq i \leq N,$$

$$\lambda_1(t) \leq \dots \leq \lambda_N(t), \quad t \geq 0,$$

and

$$d\theta_i(t) = 2g_N(\theta_i(t))h_N(\theta_i^N(t))dW_i(t) + \left( \tilde{b}_N(\theta_i^N(t)) + \sum_{j:j \neq i} \frac{G_N(\theta_i(t), \theta_j(t))}{\theta_i(t) - \theta_j(t)} \right) dt, \quad 1 \leq i \leq N,$$

$$\theta_1(t) \leq \dots \leq \theta_N(t), \quad t \geq 0,$$

with non-colliding initial values  $\lambda(0) = (\lambda_1(0), \dots, \lambda_N(0))$  and  $\theta(0) = (\theta_1(0), \dots, \theta_N(0))$ , respectively, have non-exploding and non-colliding unique strong solutions  $\lambda(t) = (\lambda_1(t), \dots, \lambda_N(t))$  and  $\theta(t) = (\theta_1(t), \dots, \theta_N(t))$ , respectively. Here,  $g_N(x)$ ,  $h_N(x)$ ,  $b_N(x)$  and  $\tilde{b}_N(x)$  are continuous functions, and  $G_N(x, y) = g_N^2(x)h_N^2(y) + g_N^2(y)h_N^2(x)$  satisfies

$$\frac{G_N(w, z)}{z - w} \leq \frac{G_N(x, y)}{y - x}, \quad \forall w < x < y < z. \tag{3.4}$$

Assume that there exists a strictly increasing function  $\rho : [0, \infty) \rightarrow [0, \infty)$  with  $\rho(0) = 0$  and

$$\int_{0^+} \rho^{-2}(u)du = \infty,$$

such that

$$|g_N(u)h_N(u) - g_N(v)h_N(v)| \leq \rho(|u - v|), \quad \forall u, v \in \mathbb{R}.$$

Furthermore, we assume that  $b_N(u) \leq \tilde{b}_N(u)$  for all  $u \in \mathbb{R}$ . If  $\lambda_i(0) \leq \theta_i(0)$  for all  $1 \leq i \leq N$  almost surely, then

$$\mathbb{P}(\lambda_i(t) \leq \theta_i(t), \forall t \geq 0, 1 \leq i \leq N) = 1.$$

### 3.2 Application to eigenvalues of Wishart process

In this subsection, we discuss the limit theorem for Wishart process. As illustrated in Graczyk and Małeckı (2013) and Song et al. (2020), the scaled Wishart process  $X_t^N = \tilde{B}^\top(t)\tilde{B}(t)/N$ , where  $\tilde{B}(t)$  is a  $P \times N$  matrix-valued Brownian motion with  $P > N - 1$ , is the solution to (1.1) with the coefficient functions

$$g_N(x)h_N(y) = \frac{\sqrt{x}}{\sqrt{N}}, \quad b_N(x) = \frac{P}{N}.$$

The eigenvalue processes now satisfy

$$d\lambda_i^N(t) = 2\frac{\sqrt{\lambda_i^N(t)}}{\sqrt{N}}dW_i(t) + \left(\frac{P}{N} + \frac{1}{N} \sum_{j:j \neq i} \frac{\lambda_i^N(t) + \lambda_j^N(t)}{\lambda_i^N(t) - \lambda_j^N(t)}\right) dt, \quad 1 \leq i \leq N, \quad t \geq 0. \quad (3.5)$$

Hence, the eigenvalue processes are the particles in (1.6) with

$$\sigma^N(x) = 2\frac{\sqrt{x}}{\sqrt{N}}, \quad b_N(x) = \frac{P}{N}, \quad H_N(x, y) = \frac{x + y}{N},$$

and thus,

$$NH_N(x, y) = H(x, y) = x + y, \quad b(x) = \lim_{N \rightarrow \infty} \frac{P}{N} = c \geq 1, \quad \sqrt{N}\sigma^N(x) = \tilde{\sigma}(x) = 2\sqrt{x}. \quad (3.6)$$

By (Graczyk and Małeckı, 2019, Theorem 3), all the components of the solution to (3.5) are non-negative if all the components of the initial value are non-negative. Let  $\mathbb{P}^N$  be the distribution on  $\Delta_N = \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : 0 < x_1 < \dots < x_N\}$  with density

$$p(x) = C_{N,P} \exp\left(-\frac{N}{2} \sum_{i=1}^N x_i\right) \prod_{i=1}^N x_i^{(P-N-1)/2} \prod_{1 \leq j < i \leq N} (x_i - x_j), \quad (3.7)$$

where  $C_{N,P} > 0$  is a normalization constant. Then we have the following estimation on the eigenvalues.

**Lemma 3.3.** *Let  $\xi^N = (\xi_1^N, \dots, \xi_N^N)$  be a random vector independent of  $(W_1, \dots, W_N)$  with (3.7) as its joint probability density function. Assume that  $(\lambda_1^N(0), \dots, \lambda_N^N(0))$  is independent of  $(W_1, \dots, W_N)$  and that there exists a constant  $a > 0$ , such that  $\lambda_i^N(0) \leq a\xi_i^N$  for  $1 \leq i \leq N$  almost surely. Then there exists a stationary stochastic process  $u^N(t)$  with initial value  $u^N(0) = \xi^N$  satisfying, for  $1 \leq i \leq N$  and  $t \geq 0$ ,*

$$\lambda_i^N(t) \leq v_i^N(t) = (t + a)u_i^N(t).$$

*Proof.* Consider the following system of SDEs, for  $1 \leq i \leq N$ ,

$$du_i^N(t) = 2\frac{\sqrt{u_i^N(t)}}{\sqrt{N(t+a)}}dW_i(t) + \frac{1}{t+a} \left(\frac{P}{N} - u_i^N(t) + \frac{1}{N} \sum_{j:j \neq i} \frac{u_i^N(t) + u_j^N(t)}{u_i^N(t) - u_j^N(t)}\right) dt, \quad t \geq 0, \quad (3.8)$$

with initial value  $u_i^N(0) = \xi_i^N(0)$  distributed according to  $\mathbb{P}^N$  and  $u_1^N(t) \leq \dots \leq u_N^N(t)$ .

Note that the pathwise uniqueness proved in (Graczyk and Małeckı, 2013, Theorem 2) is still valid if the coefficient functions depend on the time  $t$  and the corresponding conditions therein hold uniformly in  $t$ . Furthermore, the boundedness estimation and the McKean’s argument in (Graczyk and Małeckı, 2013, Theorem 5) is also valid when  $t \geq 0$ . Therefore, the system of SDEs (3.8) has a unique non-colliding strong solution.

If at any time  $t$ ,  $u^N(t)$  has the distribution  $\mathbb{P}^N$ , then Lemma A.3 yields that  $\frac{d}{dt}\mathbb{E}[f(u^N(t))]$  vanishes for  $f \in C_b^2(\mathbb{R})$ . Since  $u^N(0)$  is distributed according to  $\mathbb{P}^N$ , we can conclude that  $(u^N(t))_{t \geq 0}$  is a stationary process with marginal distribution  $\mathbb{P}^N$ .

Now let  $v_i^N(t) = (t+a)u_i^N(t)$  for  $1 \leq i \leq N$  and  $v^N(t) = (v_1^N(t), \dots, v_N^N(t))$ . Then the Itô formula shows that  $v^N(t)$  is a solution to (3.5) with initial value  $v^N(0) = au^N(0) = a\xi^N$ . Noting that the solution of (3.5) is non-negative and that  $G_N(x, y) = (x+y)/N$  with non-negative variables satisfies condition (3.3), we can apply the comparison principle in Theorem 3.1 to obtain

$$\lambda_i^N(t) \leq v_i^N(t) = (t+a)u_i^N(t).$$

The proof is concluded. □

**Lemma 3.4.** *Assume the same conditions as in Lemma 3.3. Then for any  $T < \infty$ , there exists a positive constant  $C(a, T)$  depending only on  $(a, T)$ , such that for all  $p \geq 1$ ,*

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \langle |x|^p, L_N(t) \rangle \right] \leq C(a, T)^p,$$

almost surely for  $N \geq (2p-1)/\alpha$  for some positive constant  $\alpha$ .

*Proof.* Noting that the probability density of  $u^N(t)$  considered in Lemma 3.3 is (3.7) for all  $t$ , we can obtain the following tail probability estimation with  $\alpha$  being a positive constant independent of  $N$ ,

$$\mathbb{P}(u_N^N(t) \geq x) = \mathbb{P}^N(x_N \geq x) \leq \exp(-\alpha Nx), \text{ for } t \geq 0. \tag{3.9}$$

By Lemma 3.3 and (3.9), we have for  $t \geq 0$ ,

$$\begin{aligned} \mathbb{E}[\lambda_N^N(t)^k] &\leq (t+a)^k \mathbb{E}[u_N^N(t)^k] = k(t+a)^k \int_0^\infty x^{k-1} \mathbb{P}(u_N^N(t) \geq x) dx \\ &\leq k(t+a)^k \int_0^\infty x^{k-1} \exp(-\alpha Nx) dx = \frac{\Gamma(k+1)}{(\alpha N)^k} (t+a)^k \\ &\leq (t+a)^k, \end{aligned} \tag{3.10}$$

for  $k \in [0, \alpha N]$ , where  $\Gamma(x)$  is the gamma function.

Now we apply (2.4) and (2.6) with  $f(x) = x^{n+2}$  for  $n \geq -1$  to obtain

$$\begin{aligned} \langle x^{n+2}, L_N(t) \rangle &= \langle x^{n+2}, L_N(0) \rangle + M_{x^{n+2}}^N(t) + \frac{(P+n+1)(n+2)}{N} \int_0^t \langle x^{n+1}, L_N(s) \rangle ds \\ &\quad + \frac{n+2}{2} \int_0^t \int \sum_{k=0}^n x^k y^{n-k} (x+y) L_N(s)(dx) L_N(s)(dy) ds. \end{aligned} \tag{3.11}$$

where the martingale term  $M_{x^{n+2}}^N(t)$  has the quadratic variation

$$\langle M_{x^{n+2}}^N \rangle_t = \frac{4(n+2)^2}{N^2} \int_0^t \langle x^{2n+3}, L_N(s) \rangle ds.$$

By the Cauchy-Schwarz inequality, Burkholder-Davis-Gundy inequality, Hölder inequality and the estimation (3.10), for  $(2n+3)q \leq \alpha N$ ,  $q \in \mathbb{N}$ , and  $\Lambda_q$  being a positive constant



depending only on  $q$ ,

$$\begin{aligned}
 & \mathbb{E} \left[ \left| \sup_{u \in [0, t]} M_{x^{n+2}}^N(u) \right|^q \right] \leq \left\{ \mathbb{E} \left[ \sup_{u \in [0, t]} M_{x^{n+2}}^N(u)^{2q} \right] \right\}^{1/2} \\
 & \leq \sqrt{\Lambda_q} \left\{ \mathbb{E} [\langle M_{x^{n+2}}^N \rangle_t^q] \right\}^{1/2} \leq \frac{2^q(n+2)^q \sqrt{\Lambda_q}}{N^q} \left\{ \mathbb{E} \left[ \int_0^t \langle x^{2n+3}, L_N(s) \rangle ds \right]^q \right\}^{1/2} \\
 & = \frac{2^q(n+2)^q \sqrt{\Lambda_q}}{N^q} \left\{ \mathbb{E} \left[ \int_0^t \frac{1}{N} \sum_{i=1}^N \lambda_i^N(s)^{2n+3} ds \right]^q \right\}^{1/2} \\
 & \leq \frac{2^q(n+2)^q \sqrt{\Lambda_q}}{N^q} \left\{ \mathbb{E} \left[ \int_0^t \lambda_N^N(s)^{2n+3} ds \right]^q \right\}^{1/2} \\
 & \leq \frac{2^q(n+2)^q \sqrt{\Lambda_q}}{N^q} \left\{ \mathbb{E} \left[ t^{q-1} \int_0^t \lambda_N^N(s)^{(2n+3)q} ds \right] \right\}^{1/2} \\
 & \leq \frac{2^q(n+2)^q \sqrt{\Lambda_q}}{N^q} \left\{ t^{q-1} \int_0^t (s+a)^{(2n+3)q} ds \right\}^{1/2} \\
 & \leq \frac{2^q(n+2)^q \sqrt{\Lambda_q} t^q (t+a)^{(2n+3)q}}{N^q}.
 \end{aligned} \tag{3.12}$$

Defining, for  $k \geq 1$ ,

$$E_t^N(k) = \mathbb{E} \left[ \sup_{u \in [0, t]} \langle x^k, L_N(u) \rangle \right],$$

it follows from (3.11) that for  $n \geq -1$ ,

$$\begin{aligned}
 & E_t^N(n+2) \\
 & \leq E_0^N(n+2) + \mathbb{E} \left[ \sup_{u \in [0, t]} M_{x^{n+2}}^N(u) \right] + \frac{(P+n+1)(n+2)}{N} \mathbb{E} \left[ \sup_{u \in [0, t]} \int_0^u \langle x^{n+1}, L_N(s) \rangle ds \right] \\
 & \quad + \frac{n+2}{2} \mathbb{E} \left[ \sup_{u \in [0, t]} \int_0^u \iint \sum_{k=0}^n x^k y^{n-k} (x+y) L_N(s)(dx) L_N(s)(dy) ds \right].
 \end{aligned} \tag{3.13}$$

For the third and the fourth terms on the right-hand side of (3.13), we have by (3.10),

$$\begin{aligned}
 & \frac{(P+n+1)(n+2)}{N} \mathbb{E} \left[ \sup_{u \in [0, t]} \int_0^u \langle x^{n+1}, L_N(s) \rangle ds \right] \\
 & \leq \frac{(P+n+1)(n+2)}{N} \mathbb{E} \left[ \int_0^t |\lambda_N^N(s)|^{n+1} ds \right] \\
 & \leq \frac{(P+n+1)(n+2)}{N} \int_0^t (s+a)^{n+1} ds \\
 & \leq \frac{(P+n+1)(n+2)t(t+a)^{n+1}}{N},
 \end{aligned}$$

and

$$\begin{aligned} & \frac{n+2}{2} \mathbb{E} \left[ \sup_{u \in [0, t]} \int_0^u \iint \sum_{k=0}^n x^k y^{n-k} (x+y) L_N(s)(dx) L_N(s)(dy) ds \right] \\ &= \frac{n+2}{2} \sum_{k=0}^n \mathbb{E} \left[ \sup_{u \in [0, t]} \int_0^u (\langle x^{k+1}, L_N(s) \rangle \langle y^{n-k}, L_N(s) \rangle + \langle x^k, L_N(s) \rangle \langle y^{n+1-k}, L_N(s) \rangle) ds \right] \\ &\leq \frac{(n+2)}{2} \sum_{k=0}^n \mathbb{E} \left[ \sup_{u \in [0, t]} \int_0^u (|\lambda_N^N(s)|^{k+1} |\lambda_N^N(s)|^{n-k} + |\lambda_N^N(s)|^k |\lambda_N^N(s)|^{n+1-k}) ds \right] \\ &\leq (n+2)(n+1) \mathbb{E} \left[ \int_0^t |\lambda_N^N(s)|^{n+1} ds \right] \\ &\leq (n+2)(n+1)t(t+a)^{n+1} \end{aligned}$$

for  $n+1 \leq \alpha N$ . Hence, by (3.12), (3.13), and the above two estimations, for  $n \geq -1$  such that  $2n+3 \leq \alpha N$  and  $t \in [0, T]$ , we have

$$\begin{aligned} E_t^N(n+2) &\leq E_0^N(n+2) + \frac{2(n+2)\sqrt{\Lambda_1 t(t+a)^{2n+3}}}{N} \\ &\quad + \frac{(P+n+1)(n+2)t(t+a)^{n+1}}{N} + (n+2)(n+1)t(t+a)^{n+1}. \end{aligned}$$

Thus, for all  $-1 \leq n \leq \frac{\alpha N - 3}{2}$ , noting that  $E_0^N(n+2) \leq \mathbb{E}[\lambda_N^N(0)^{n+2}] \leq a^{n+2}$  by (3.10), we have

$$E_T^N(n+2) \leq C_{a,T}^{n+2},$$

for some positive constant  $C_{a,T}$  depending on  $(a, T)$  only.

The proof is concluded. □

Under the conditions in Lemma 3.3, Lemma 3.4 implies  $\limsup_{N \rightarrow \infty} \sup_N \lambda_N^N(0) < \infty$  almost surely. One can check that the conditions (A) – (D) in Song et al. (2020) are satisfied, hence  $\{L_N(t), t \in [0, T]\}_{n \in \mathbb{N}}$  is tight (see also (Song et al., 2020, Remark 3.3)), and we know that it converges to  $\{\mu_t, t \in [0, T]\}$ , where  $\mu_t$  is a scaled Marchenko-Pastur law. Recall that  $c = \lim_{N \rightarrow \infty} P/N$  and that  $\mathcal{L}_t^N(f)$  is defined by (1.9) in Theorem 2.2.

**Theorem 3.5.** *Assume that  $\lim_{N \rightarrow \infty} |P - cN| = 0$ , and that for any polynomial  $f(x) \in \mathbb{R}[x]$ , the initial value  $\mathcal{L}_0^N(f)$  converges in probability to a random variable  $\mathcal{L}_0(f)$ . Besides, assume the same condition on  $\{\lambda_i^N(0), i = 1, 2, \dots, N\}$  as in Lemma 3.3 for all  $N \in \mathbb{N}$ . Furthermore, assume that for all  $n \in \mathbb{N}$ ,*

$$\sup_N \mathbb{E}[|N(\langle x^n, L_N(0) \rangle - \langle x^n, \mu_0 \rangle)|^q] < \infty, \tag{3.14}$$

for all  $q \geq 1$ . Then for any  $0 < T < \infty$ , there exists a family of processes  $\{\mathcal{L}_t(f), t \in [0, T]\}_{f \in \mathbb{R}[x]}$ , such that for any  $n \in \mathbb{N}$  and any polynomials  $P_1, \dots, P_n \in \mathbb{R}[x]$ , the vector-valued process  $(\mathcal{L}_t^N(P_1), \dots, \mathcal{L}_t^N(P_n))_{t \in [0, T]}$  converges to  $(\mathcal{L}_t(P_1), \dots, \mathcal{L}_t(P_n))_{t \in [0, T]}$  in distribution, as  $N \rightarrow \infty$ .

The limit processes  $\{\mathcal{L}_t(f), t \in [0, T]\}_{f \in \mathbb{R}[x]}$  are characterized by the following properties.

1. For  $P_1, P_2 \in \mathbb{R}[x]$ ,  $\alpha_1, \alpha_2 \in \mathbb{R}$ ,  $t \in [0, T]$ ,

$$\mathcal{L}_t(\alpha_1 P_1 + \alpha_2 P_2) = \alpha_1 \mathcal{L}_t(P_1) + \alpha_2 \mathcal{L}_t(P_2).$$

2. The basis  $\{\mathcal{L}_t(x^n), t \in [0, T]\}_{n \in \mathbb{N}}$  of  $\{\mathcal{L}_t(P), t \in [0, T]\}_{P \in \mathbb{R}[x]}$  satisfies

$$\mathcal{L}_t(1) = 0, \quad \mathcal{L}_t(x) = \mathcal{L}_0(x) + G_t(x),$$

and for  $n \geq 0$ ,

$$\begin{aligned} \mathcal{L}_t(x^{n+2}) &= \mathcal{L}_0(x^{n+2}) + c(n+2) \int_0^t \mathcal{L}_s(x^{n+1}) ds + (n+2)(n+1) \int_0^t \langle x^{n+1}, \mu_s \rangle ds \\ &\quad + (n+2) \sum_{k=0}^n \int_0^t (\mathcal{L}_s(x^{n-k}) \langle x^{k+1}, \mu_s \rangle + \mathcal{L}_s(x^{n+1-k}) \langle x^k, \mu_s \rangle) ds + G_t(x^{n+2}), \end{aligned} \tag{3.15}$$

where  $\{G_t(x^n), t \in [0, T]\}_{n \in \mathbb{N}}$  is a family of centered Gaussian processes with covariance

$$\mathbb{E}[G_t(x^n)G_s(x^m)] = 4mn \int_0^{t \wedge s} \langle x^{n+m-1}, \mu_u \rangle du, \quad n, m \geq 1. \tag{3.16}$$

*Proof.* First, note that by Lemma 3.4 and Corollary 2.5,  $Q_t^N(x^n)$  defined by (1.10) converges in distribution to a centered Gaussian family  $\{G_t(x^n), t \in [0, T]\}_{n \in \mathbb{N}}$  with covariance given by (3.16). Furthermore, by (1.9), (1.10) and (3.6), for  $n \geq -1$ , we have

$$\begin{aligned} &Q_t^N(x^{n+2}) \\ &= \mathcal{L}_t^N(x^{n+2}) - \mathcal{L}_0^N(x^{n+2}) - c(n+2) \int_0^t \mathcal{L}_s^N(x^{n+1}) ds - (n+2)(n+1) \int_0^t \langle x^{n+1}, \mu_s \rangle ds \\ &\quad - (n+2) \int_0^t \mathcal{L}_s^N \left( \int \sum_{k=0}^n x^k y^{n-k} (x+y) \mu_s(dx) \right) ds \\ &\quad - \frac{N(n+2)}{2} \int_0^t \iint \sum_{k=0}^n x^k y^{n-k} (x+y) [L_N(s)(dx) - \mu_s(dx)] [L_N(s)(dy) - \mu_s(dy)] ds \\ &= \mathcal{L}_t^N(x^{n+2}) - \mathcal{L}_0^N(x^{n+2}) - c(n+2) \int_0^t \mathcal{L}_s^N(x^{n+1}) ds - (n+2)(n+1) \int_0^t \langle x^{n+1}, \mu_s \rangle ds \\ &\quad - (n+2) \sum_{k=0}^n \int_0^t (\mathcal{L}_s^N(x^{n-k}) \langle x^{k+1}, \mu_s \rangle + \mathcal{L}_s^N(x^{n+1-k}) \langle x^k, \mu_s \rangle) ds \\ &\quad - \frac{(n+2)}{2N} \sum_{k=0}^n \int_0^t (\mathcal{L}_s^N(x^{n-k}) \mathcal{L}_s^N(x^{k+1}) + \mathcal{L}_s^N(x^{n+1-k}) \mathcal{L}_s^N(x^k)) ds. \end{aligned} \tag{3.17}$$

In Corollary 2.3 and Corollary 2.5, we have shown  $Q_t^N(x^{n+2}) - NM_{x^{n+2}}^N$  converges to 0 almost surely and in  $L^q$  for all  $q \geq 1$  as  $N \rightarrow \infty$ , uniformly in  $t \in [0, T]$ . Thus, by (3.12), (3.17), and the condition (3.14), it is not difficult to show

$$\sup_{N \in \mathbb{N}} \mathbb{E} \left[ \sup_{t \in [0, T]} |\mathcal{L}_t^N(x^n)|^q \right] < \infty,$$

for  $q \geq 1$  and  $n \in \mathbb{N}$  by using an induction argument on  $n$ .

To estimate the last term on the right-hand side of (3.17), we apply the Cauchy-

Schwarz inequality to obtain, for  $0 \leq k \leq n$ ,

$$\begin{aligned} & \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \frac{n+2}{2N} \int_0^t \mathcal{L}_s^N(x^{n-k}) \mathcal{L}_s^N(x^{k+1}) ds \right|^q \right] \\ & \leq \frac{(n+2)^q T^q}{2^q N^q} \mathbb{E} \left[ \sup_{t \in [0, T]} |\mathcal{L}_t^N(x^{n-k})|^q \sup_{t \in [0, T]} |\mathcal{L}_t^N(x^{k+1})|^q \right] \\ & \leq \frac{(n+2)^q T^q}{2^q N^q} \left\{ \mathbb{E} \left[ \sup_{t \in [0, T]} |\mathcal{L}_t^N(x^{n-k})|^{2q} \right] \mathbb{E} \left[ \sup_{t \in [0, T]} |\mathcal{L}_t^N(x^{k+1})|^{2q} \right] \right\}^{1/2} \\ & \leq C(n, T, q) N^{-q}, \end{aligned}$$

for some constant  $C(n, T, q)$ . Thus, the last term on the right-hand side of (3.17) converges to 0 in  $L^q$  for  $q > 1$ , as  $N$  tends to infinity. By Markov inequality and Borel-Cantelli Lemma, one can also obtain the almost sure convergence.

If we define

$$\begin{aligned} \tilde{Q}_t^N(x^{n+2}) &= \mathcal{L}_t^N(x^{n+2}) - \mathcal{L}_0^N(x^{n+2}) - c(n+2) \int_0^t \mathcal{L}_s^N(x^{n+1}) ds \\ &\quad - (n+2)(n+1) \int_0^t \langle x^{n+1}, \mu_s \rangle ds \\ &\quad - (n+2) \sum_{k=0}^n \int_0^t (\mathcal{L}_s^N(x^{n-k}) \langle x^{k+1}, \mu_s \rangle + \mathcal{L}_s^N(x^{n+1-k}) \langle x^k, \mu_s \rangle) ds, \end{aligned} \quad (3.18)$$

for  $n \geq -1$ , then the difference  $|\tilde{Q}_t^N(x^{n+2}) - Q_t^N(x^{n+2})|$  converges to 0 almost surely and in  $L^q$  for  $q > 1$ . Thus, Corollary 2.3 implies that  $(\tilde{Q}_t^N(x^k), \tilde{Q}_t^N(x^{k-1}), \dots, \tilde{Q}_t^N(x))_{t \in [0, T]}$  converges in distribution to  $(G_t(x^k), G_t(x^{k-1}), \dots, G_t(x))_{t \in [0, T]}$  with covariance (3.16).

Now we deduce the convergence in distribution of  $(\mathcal{L}_t^N(x^k))_{t \in [0, T]}$  for  $k \in \mathbb{N}$ . First of all, we have  $\mathcal{L}_t^N(1) = 0$  and  $\mathcal{L}_t^N(x) = \mathcal{L}_0^N(x) + \tilde{Q}_t^N(x)$  converges in distribution since the initial value converges in probability. By induction, if we assume  $(\mathcal{L}_t^N(x^k), \dots, \mathcal{L}_t^N(x))_{t \in [0, T]}$  convergence in distribution to  $(\mathcal{L}_t(x^k), \dots, \mathcal{L}_t(x))_{t \in [0, T]}$ , then the convergence in distribution of  $(\tilde{Q}_t^N(x^{k+1}), \tilde{Q}_t^N(x^k), \dots, \tilde{Q}_t^N(x))_{t \in [0, T]}$  implies the convergence in distribution of  $(\tilde{Q}_t^N(x^{k+1}), \mathcal{L}_t^N(x^k), \dots, \mathcal{L}_t^N(x))_{t \in [0, T]}$ , and hence  $(\mathcal{L}_t^N(x^{k+1}), \dots, \mathcal{L}_t^N(x))_{t \in [0, T]}$  converges in distribution.

Thus, by (3.18) we have

$$\begin{aligned} G_t(x^{n+2}) &\stackrel{d}{=} \mathcal{L}_t(x^{n+2}) - \mathcal{L}_0(x^{n+2}) \\ &\quad - c(n+2) \int_0^t \mathcal{L}_s(x^{n+1}) ds - (n+2)(n+1) \int_0^t \langle x^{n+1}, \mu_s \rangle ds \\ &\quad - (n+2) \sum_{k=0}^n \int_0^t (\mathcal{L}_s(x^{n-k}) \langle x^{k+1}, \mu_s \rangle + \mathcal{L}_s(x^{n+1-k}) \langle x^k, \mu_s \rangle) ds, \end{aligned}$$

where “ $\stackrel{d}{=}$ ” means equality in distribution. The proof is concluded.  $\square$

**Remark 3.6.** By the self-similarity of Brownian motion, when  $X_0^N = 0$ , we have  $X_t^N \stackrel{d}{=} tX_1^N$ . Thus,  $(\lambda_1^N(t), \dots, \lambda_N^N(t)) \stackrel{d}{=} (t\lambda_1^N(1), \dots, t\lambda_N^N(1))$ . Therefore,

$$\langle f(x), L_N(t) \rangle = \frac{1}{N} \sum_{i=1}^N f(\lambda_i^N(t)) \stackrel{d}{=} \frac{1}{N} \sum_{i=1}^N f(t\lambda_i^N(1)) = \langle f(tx), L_N(1) \rangle,$$

and

$$\langle f(x), \mu_t \rangle \stackrel{d}{=} \langle f(tx), \mu_1 \rangle.$$

Hence,  $\mathcal{L}_t^N(f(x)) \stackrel{d}{=} \mathcal{L}_1^N(f(tx))$ , and thus,  $\mathcal{L}_t(f(x)) \stackrel{d}{=} \mathcal{L}_1(f(tx))$ . With these identities and the linearity of  $\mathcal{L}_t(\cdot)$ , (3.15) can be simplified as, for  $n \geq 0$ ,

$$\begin{aligned} \mathcal{L}_1(x^{n+2}) &= c\mathcal{L}_1(x^{n+1}) + (n+1)\langle x^{n+1}, \mu_1 \rangle \\ &\quad + \sum_{k=0}^n (\mathcal{L}_1(x^{n-k})\langle x^{k+1}, \mu_1 \rangle + \mathcal{L}_1(x^{n+1-k})\langle x^k, \mu_1 \rangle) + \frac{1}{t^{n+2}}G_t(x^{n+2}), \quad t > 0, \end{aligned} \tag{3.19}$$

where the Gaussian family  $\{G_t(x^n), t \in [0, T]\}_{n \in \mathbb{N}}$  has the covariance functions

$$\mathbb{E}[G_t(x^n)G_s(x^m)] = \frac{4mn}{m+n}(t \wedge s)^{n+m}\langle x^{n+m-1}, \mu_1 \rangle, \quad n, m \geq 1.$$

Note that the case  $t = 1$  corresponds to the classical Wishart matrix, and  $\mu_1$  is the Marchenko–Pastur law. More precisely, recalling that  $\mathcal{L}_1(1) = 0$  and  $\mathcal{L}_1(x) = G_1(x)$ , we get by (3.19)  $\mathcal{L}_1(x^2) = \langle x, \mu_1 \rangle + (c+1)G_1(x) + G_1(x^2)$ , for  $m \geq 3$ , and more generally  $\mathcal{L}_1(x^m) = c_{m,0} + c_{m,1}G_1(x) + \dots + c_{m,m}G_1(x^m)$  for some coefficients  $(c_{m,j})_{0 \leq j \leq m}$  which are determined recursively by (3.19).

We now study a more general particle system:

$$d\lambda_i^N(t) = 2\frac{\sqrt{\lambda_i^N(t)}}{\sqrt{N}}dW_i(t) + \left( b_N(\lambda_i^N(t)) + \frac{1}{N} \sum_{j:j \neq i} \frac{\lambda_i^N(t) + \lambda_j^N(t)}{\lambda_i^N(t) - \lambda_j^N(t)} \right) dt, \quad 1 \leq i \leq N, \quad t \geq 0. \tag{3.20}$$

Compared to (3.5), the constant  $P/N$  is replaced by a function  $b_N(x)$  that will be assumed to converge to a constant  $c$  in Corollary 3.7 below. Despite the extension being small, the system (3.20) may not have an explicit joint density function or/and stationary distribution, and hence cannot be treated in the same way as for the eigenvalues of Wishart process.

**Corollary 3.7.** Consider the SDEs (3.20), where  $b_N(x)$  satisfies, for some constant  $c \geq 1$ ,

$$\lim_{N \rightarrow \infty} N\|b_N(x) - c\|_{L^\infty(\mathbb{R})} = 0. \tag{3.21}$$

Assume the same initial conditions as in Theorem 3.5. Then the conclusion of Theorem 3.5 still holds.

*Proof.* Let  $p_1 = N(c - \|b_N(x) - c\|_{L^\infty})$  and  $p_2 = N(c + \|b_N(x) - c\|_{L^\infty})$  be two constants depending on  $N$ . Then (3.21) implies  $p_2 \geq p_1 > N - 1$  when  $N$  is large. Clearly,  $p_1 \leq N\|b_N(x)\|_{L^\infty(\mathbb{R})} \leq p_2$ . Consider the following two systems of SDEs:

$$dx_i^N(t) = 2\frac{\sqrt{x_i^N(t)}}{\sqrt{N}}dW_i(t) + \left( \frac{p_1}{N} + \frac{1}{N} \sum_{j:j \neq i} \frac{x_i^N(t) + x_j^N(t)}{x_i^N(t) - x_j^N(t)} \right) dt, \quad 1 \leq i \leq N, \quad t \geq 0, \tag{3.22}$$

and

$$dy_i^N(t) = 2\frac{\sqrt{y_i^N(t)}}{\sqrt{N}}dW_i(t) + \left( \frac{p_2}{N} + \frac{1}{N} \sum_{j:j \neq i} \frac{y_i^N(t) + y_j^N(t)}{y_i^N(t) - y_j^N(t)} \right) dt, \quad 1 \leq i \leq N, \quad t \geq 0, \tag{3.23}$$

with the initial conditions  $x_i^N(0) = y_i^N(0) = \lambda_i^N(0)$ . By the comparison principle in Corollary 3.2, we have

$$\mathbb{P}(x_i^N(t) \leq \lambda_i^N(t) \leq y_i^N(t), \forall t \geq 0, \forall 1 \leq i \leq N) = 1.$$

Thus, almost surely,

$$\begin{aligned} \sup_{t \in [0, T]} \langle |x|^p, L_N(t) \rangle &= \sup_{t \in [0, T]} \frac{1}{N} \sum_{i=1}^N |\lambda_i^N(t)|^p \\ &\leq \sup_{t \in [0, T]} \frac{1}{N} \sum_{i=1}^N (|x_i^N(t)|^p + |y_i^N(t)|^p) \\ &\leq \sup_{t \in [0, T]} \langle |x|^p, L_N^{(x)}(t) \rangle + \sup_{t \in [0, T]} \langle |x|^p, L_N^{(y)}(t) \rangle, \end{aligned} \tag{3.24}$$

where  $\{L_N^{(x)}(t), t \in [0, T]\}_{N \in \mathbb{N}}$  and  $\{L_N^{(y)}(t), t \in [0, T]\}_{N \in \mathbb{N}}$  are the empirical measures of the two particle systems  $(x_i^N(t))_{1 \leq i \leq N}$  and  $(y_i^N(t))_{1 \leq i \leq N}$ , respectively.

Noting that  $p_1/N$  and  $p_2/N$  converge to  $c$  as  $N \rightarrow \infty$  by (3.21), we have that Lemma 3.4 holds for the two systems (3.22) and (3.23), and thus also holds for (3.20) by (3.24). Furthermore, condition (3.21) also yields that  $b_N(x) \rightarrow c$  uniformly as  $N \rightarrow \infty$ , and hence (3.17) still holds. Then the rest of the proof follows that of Theorem 3.5.  $\square$

### 3.3 Application to Dyson’s Brownian motion

In this subsection, we discuss the CLT for Dyson’s Brownian motion. It was shown in Anderson et al. (2010); Graczyk and Malecki (2014); Song et al. (2020), the scaled symmetric matrix-valued Brownian motion  $X_i^N = (\tilde{B}^\top(t) + \tilde{B}(t))/\sqrt{2N}$ , where  $\tilde{B}(t)$  is a  $N \times N$  matrix-valued Brownian motion, is the solution of the matrix SDE (1.1) with the coefficient functions

$$g_N(x)h_N(y) = \frac{1}{\sqrt{2N}}, \quad b_N(x) = 0.$$

The system of SDEs of the eigenvalue processes, that is, Dyson’s Brownian motion, is

$$d\lambda_i^N(t) = \frac{\sqrt{2}}{\sqrt{N}} dW_i(t) + \frac{1}{N} \sum_{j:j \neq i} \frac{1}{\lambda_i^N(t) - \lambda_j^N(t)} dt, \quad 1 \leq i \leq N, \quad t \geq 0. \tag{3.25}$$

Hence, the eigenvalue processes are the particles in (1.6) with

$$\sigma^N(x) = \frac{\sqrt{2}}{\sqrt{N}}, \quad b_N(x) = 0, \quad H_N(x, y) = \frac{1}{N},$$

and thus,

$$NH_N(x, y) = H(x, y) = 1, \quad b(x) = 0, \quad \sqrt{N}\sigma^N(x) = \tilde{\sigma}(x) = \sqrt{2}. \tag{3.26}$$

Here, we consider the distribution  $\mathbb{Q}^N$  on  $\Delta'_N = \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_1 < \dots < x_N\}$  with the density function

$$C_N \exp\left(-\frac{N}{4} \sum_{i=1}^N x_i^2\right) \prod_{1 \leq j < i \leq N} |x_i - x_j|, \tag{3.27}$$

where  $C_N$  is a normalization constant.

Similar to Wishart process, we can obtain the following central limit theorem.

**Theorem 3.8.** Let  $\xi^N = (\xi_1^N, \dots, \xi_N^N)$  be a random vector independent of  $(W_1, \dots, W_N)$  with (3.27) as its joint probability density function. Assume that  $(\lambda_1^N(0), \dots, \lambda_N^N(0))$  is independent of  $(W_1, \dots, W_N)$  and that there exist constants  $a, b \geq 0$ , such that

$$\sqrt{a}\xi_i^N - b \leq \lambda_i^N(0) \leq \sqrt{a}\xi_i^N + b \tag{3.28}$$

for  $1 \leq i \leq N$  almost surely. Besides, assume that for any polynomial  $f(x) \in \mathbb{R}[x]$ , the initial value  $\mathcal{L}_0^N(f)$  converges in probability to a random variable  $\mathcal{L}_0(f)$ . Furthermore, assume that for all  $n \in \mathbb{N}$ ,

$$\sup_N \mathbb{E}[|N(\langle x^n, L_N(0) \rangle - \langle x^n, \mu_0 \rangle)|^p] < \infty,$$

for all  $p \geq 1$ .

Then for any  $0 < T < \infty$ , there exists a family of processes  $\{\mathcal{L}_t(f), t \in [0, T]\}_{f \in \mathbb{R}[x]}$ , such that for any  $n \in \mathbb{N}$  and any polynomials  $P_1, \dots, P_n \in \mathbb{R}[x]$ , the vector-valued process  $(\mathcal{L}_t^N(P_1), \dots, \mathcal{L}_t^N(P_n))_{t \in [0, T]}$  converges to  $(\mathcal{L}_t(P_1), \dots, \mathcal{L}_t(P_n))_{t \in [0, T]}$  in distribution.

The limit processes  $\{\mathcal{L}_t(f), t \in [0, T]\}_{f \in \mathbb{R}[x]}$  are characterized by the following properties.

1. For  $P_1, P_2 \in \mathbb{R}[x]$ ,  $\alpha_1, \alpha_2 \in \mathbb{R}$ ,  $t \in [0, T]$ ,

$$\mathcal{L}_t(\alpha_1 P_1 + \alpha_2 P_2) = \alpha_1 \mathcal{L}_t(P_1) + \alpha_2 \mathcal{L}_t(P_2).$$

2. The basis  $\{\mathcal{L}_t(x^n), t \in [0, T]\}_{n \in \mathbb{N}}$  of  $\{\mathcal{L}_t(f), t \in [0, T]\}_{f \in \mathbb{R}[x]}$  satisfies

$$\mathcal{L}_t(1) = 0, \quad \mathcal{L}_t(x) = \mathcal{L}_0(x) + G_t(x),$$

and for  $n \geq 0$ ,

$$\begin{aligned} \mathcal{L}_t(x^{n+2}) = & \mathcal{L}_0(x^{n+2}) + \frac{(n+2)(n+1)}{2} \int_0^t \langle x^n, \mu_s \rangle ds \\ & + (n+2) \sum_{k=0}^n \int_0^t \mathcal{L}_s(x^{n-k}) \langle x^k, \mu_s \rangle ds + G_t(x^{n+2}), \end{aligned} \tag{3.29}$$

where  $\{G_t(x^n), t \in [0, T]\}_{n \in \mathbb{N}}$  is a centered Gaussian family with the covariance

$$\mathbb{E}[G_t(x^n)G_s(x^m)] = 2mn \int_0^{t \wedge s} \langle x^{n+m-2}, \mu_u \rangle du, \quad n, m \geq 1.$$

*Proof.* The proof is similar to the proofs of the Wishart case (Lemma 3.3, Lemma 3.4 and Theorem 3.5), which is sketched below.

Consider the following SDE, for  $1 \leq i \leq N$ ,

$$du_i^N(t) = \frac{\sqrt{2}}{\sqrt{N(t+a)}} dW_i(t) + \frac{1}{t+a} \left( -\frac{1}{2}u_i^N(t) + \frac{1}{N} \sum_{j:j \neq i} \frac{1}{u_i^N(t) - u_j^N(t)} \right) dt, \quad t \geq 0.$$

Then  $\frac{d}{dt} \mathbb{E}[f(u^N(t))]$  vanishes for any  $f \in C_b^2(\mathbb{R})$  if  $u^N(t)$  has the distribution  $\mathbb{Q}^N$  given in (3.27), and hence the process  $u^N(t)$  with initial value  $u^N(0) = \xi^N$  is stationary (see (Anderson et al., 2010, Lemma 4.3.17)). Let  $v_i^N(t) = \sqrt{t+au_i^N(t)} + b$  for  $1 \leq i \leq N$ . Then  $v^N(t)$  and  $\lambda^N(t)$  solve the same SDEs (3.25), and by the comparison principle in Corollary 3.2, we have

$$\lambda_i^N(t) \leq v_i^N(t) = \sqrt{t+au_i^N(t)} + b.$$

A similar argument leads to

$$-\lambda_i^N(t) \leq -\sqrt{t+a}u_i^N(t) + b.$$

Therefore,

$$|\lambda_i^N(t)| \leq \sqrt{t+a}|u_i^N(t)| + b.$$

Using the tail probability estimation based on the density function (3.27) of  $u_i^N(t)$ ,

$$\mathbb{P}(|u_i^N(t)| \geq x) \leq \mathbb{P}(u_i^N(t) \geq x) + \mathbb{P}(u_i^N(t) \leq -x) \leq 2\mathbb{P}(u_i^N(t) \geq x) \leq \exp(-\alpha Nx),$$

where  $\alpha$  is positive constant independent of  $N$ , we obtain

$$\begin{aligned} \mathbb{E}[|\lambda_i^N(t)|^k] &\leq \mathbb{E}\left[\left(\sqrt{t+a}|u_i^N(t)| + b\right)^k\right] \\ &\leq 2^k \sqrt{t+a}^k \mathbb{E}[|u_i^N(t)|^k] + 2^k b^k \\ &= 2^k \sqrt{t+a}^k k \int_0^\infty x^{k-1} \mathbb{P}(|u_i^N(t)| \geq x) dx + 2^k b^k \\ &\leq 2^k \sqrt{t+a}^k k \int_0^\infty x^{k-1} \exp(-\alpha Nx) dx + 2^k b^k \\ &= 2^k \sqrt{t+a}^k \frac{\Gamma(k+1)}{(\alpha N)^k} + 2^k b^k \\ &\leq 2^k \sqrt{t+a}^k + 2^k b^k \\ &\leq 2(4b^2 + 4(t+a))^{k/2} \end{aligned}$$

for  $k \in [0, \alpha N]$ . Then a similar argument in the proof of Lemma 3.4 leads to

$$\mathbb{E}\left[\sup_{t \in [0, T]} \langle |x|^p, L_N(t) \rangle\right] \leq C(a, b, T)^p \tag{3.30}$$

for some positive constant  $C(a, b, T)$  depending only on  $(a, b, T)$  and all  $p \geq 0$ ,  $N \geq \alpha p$  for some positive constant  $\alpha$ .

Then applying Corollary 2.5 and following the approach in the proof of Theorem 3.5, we may get the desired result.  $\square$

**Remark 3.9.** A CLT was obtained in (Anderson et al., 2010, Theorem 4.3.20) for Dyson’s Brownian motion with *bounded* initial values. (We would like to point out that there should be a constant factor  $2/\beta$  in the covariance function which equals to 2 in the real case and equals to 1 in the complex case in Anderson et al. (2010).) Thanks to the comparison principle Corollary 3.2, the CLT also applies to Dyson’s Brownian motion with possibly *unbounded* initial values satisfying (3.28), as stated in Theorem 3.8. For example, the CLT is valid for Dyson’s Brownian motion with unbounded initial value  $\xi^N$ . Note that  $\xi^N$  has the joint density (3.27), which is also the joint density of  $\{\lambda_i(1)\}_{1 \leq i \leq N}$ , assuming  $\{\lambda_i(t)\}_{1 \leq i \leq N}$  is a Dyson’s Brownian motion starting at 0.

Similar to the Wishart case, the self-similarity of the Brownian motion implies  $\mathcal{L}_t(f(x)) \stackrel{d}{=} \mathcal{L}_1(f(\sqrt{t}x))$  and  $\langle f(x), \mu_t \rangle = \langle f(\sqrt{t}x), \mu_1 \rangle$  when the initial value  $X_0^N = 0$ . Thus, (3.29) can be simplified as, for  $n \geq 0$ ,

$$\mathcal{L}_1(x^{n+2}) = (n+1)\langle x^n, \mu_1 \rangle + 2 \sum_{k=0}^n \mathcal{L}_1(x^{n-k}) \mu_1(x^k) + \frac{1}{t^{\frac{n+2}{2}}} G_t(x^{n+2}), \quad t > 0, \tag{3.31}$$



with covariance functions

$$\mathbb{E}[G_t(x^n)G_s(x^m)] = \frac{4mn}{m+n}(t \wedge s)^{\frac{m+n}{2}} \langle x^{m+n-2}, \mu_1 \rangle, \quad n, m \geq 1.$$

The case  $t = 1$  corresponds to the classical GOE matrix, and  $\mu_1$  is the semicircle law. Some beginning terms are  $\mathcal{L}_1(1) = 0, \mathcal{L}_1(x) = G_1(x)$  and  $\mathcal{L}_1(x^2) = 1 + G_1(x^2)$ . By (3.31), for  $m \geq 2, \mathcal{L}_1(x^m)$  has the distribution of a linear combination of central Gaussian variables  $\{G_1(x^j), 1 \leq j \leq m\}$ .

The following Corollary extends the result of Theorem 3.8 by allowing asymptotical constant drift coefficient functions.

**Corollary 3.10.** *Consider the following SDEs*

$$d\lambda_i^N(t) = \frac{\sqrt{2}}{\sqrt{N}}dW_i(t) + \left( b_N(\lambda_i^N(t)) + \frac{1}{N} \sum_{j:j \neq i} \frac{1}{\lambda_i^N(t) - \lambda_j^N(t)} \right) dt, \quad 1 \leq i \leq N, \quad t \geq 0, \tag{3.32}$$

where  $b_N(x)$  satisfies, for some constant  $c \in \mathbb{R}$ ,

$$\lim_{N \rightarrow \infty} N \|b_N(x) - c\|_{L^\infty(\mathbb{R})} = 0. \tag{3.33}$$

Furthermore, assume the same initial conditions as in Theorem 3.8. Then the conclusion of Theorem 3.8 still holds with (3.29) replaced by

$$\begin{aligned} \mathcal{L}_t(x^{n+2}) = & \mathcal{L}_0(x^{n+2}) + c(n+2) \int_0^t \mathcal{L}_s(x^{n+1})ds + \frac{(n+2)(n+1)}{2} \int_0^t \langle x^n, \mu_s \rangle ds \\ & + (n+2) \sum_{k=0}^n \int_0^t \mathcal{L}_s(x^{n-k})\mu_s(x^k)ds + G_t(x^{n+2}), \end{aligned} \tag{3.34}$$

for  $n \geq -1$ .

*Proof.* Set  $c_1 = c - 1$  and  $c_2 = c + 1$ . Then by (3.33), there exists  $N_0 \in \mathbb{N}$  such that for  $N \geq N_0, c_1 \leq \|b_N(x)\|_{L^\infty(\mathbb{R})} \leq c_2$ . Without loss of generality, we assume  $c_1 \leq \|b_N(x)\|_{L^\infty(\mathbb{R})} \leq c_2$  for all  $N \geq 1$ .

Consider the following two systems of SDEs:

$$dx_i^N(t) = \frac{\sqrt{2}}{\sqrt{N}}dW_i(t) + \left( c_1 + \frac{1}{N} \sum_{j:j \neq i} \frac{1}{x_i^N(t) - x_j^N(t)} \right) dt, \quad 1 \leq i \leq N, \quad t \geq 0. \tag{3.35}$$

and

$$dy_i^N(t) = \frac{\sqrt{2}}{\sqrt{N}}dW_i(t) + \left( c_2 + \frac{1}{N} \sum_{j:j \neq i} \frac{1}{y_i^N(t) - y_j^N(t)} \right) dt, \quad 1 \leq i \leq N, \quad t \geq 0, \tag{3.36}$$

with the initial conditions  $x_i^N(0) = y_i^N(0) = \lambda_i^N(0)$  for  $1 \leq i \leq N$ . By the comparison principle Theorem 3.1, we have

$$\mathbb{P}(x_i^N(t) \leq \lambda_i^N(t) \leq y_i^N(t), \forall 1 \leq i \leq N, \forall t > 0) = 1.$$

Thus, for  $p \geq 1$ , we have

$$\begin{aligned} \sup_{t \in [0, T]} \langle |x|^p, L_N(t) \rangle &= \sup_{t \in [0, T]} \frac{1}{N} \sum_{i=1}^N |\lambda_i^N(t)|^p \leq \sup_{t \in [0, T]} \frac{1}{N} \sum_{i=1}^N (|x_i^N(t)|^p + |y_i^N(t)|^p) \\ &\leq \sup_{t \in [0, T]} \frac{1}{N} \sum_{i=1}^N 2^p (|x_i^N(t) - c_1 t|^p + (c_1 t)^p + |y_i^N(t) - c_2 t|^p + (c_2 t)^p) \\ &\leq 2^p \left( \sup_{t \in [0, T]} \langle |x|^p, L_N^{(x)}(t) \rangle + \sup_{t \in [0, T]} \langle |x|^p, L_N^{(y)}(t) \rangle + (c_1 T)^p + (c_2 T)^p \right), \end{aligned} \tag{3.37}$$

almost surely, where  $\{L_N^{(x)}(t), t \in [0, T]\}_{N \in \mathbb{N}}$  and  $\{L_N^{(y)}(t), t \in [0, T]\}_{N \in \mathbb{N}}$  are the empirical measures of the two particle systems  $(x_i^N(t) - c_1 t)_{1 \leq i \leq N}$  and  $(y_i^N(t) - c_2 t)_{1 \leq i \leq N}$ , respectively.

It is easy to verify that both  $(x_i^N(t) - c_1 t)_{1 \leq i \leq N}$  and  $(y_i^N(t) - c_2 t)_{1 \leq i \leq N}$  solve Dyson's SDEs (3.25). By (3.30) in the proof Theorem 3.8, we have

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \langle |x|^p, L_N^{(x)}(t) \rangle \right] \leq C(a, b, T)^p \quad \text{and} \quad \mathbb{E} \left[ \sup_{t \in [0, T]} \langle |x|^p, L_N^{(y)}(t) \rangle \right] \leq C(a, b, T)^p,$$

and consequently, by (3.37)

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \langle |x|^p, L_N(t) \rangle \right] \leq C(a, b, T)^p,$$

for some positive constant  $C(a, b, T)$  depending only on  $(a, b, T)$  and all  $p \geq 1$ ,  $N \geq \alpha p$  for some positive constant  $\alpha$ .

Note that (3.33) also implies that  $b_N(x)$  converges to the constant  $c$  uniformly as  $N \rightarrow \infty$ . Then applying Corollary 2.5 and following the approach in the proof of Theorem 3.5, we get the desired result.  $\square$

### 3.4 Application to eigenvalues of symmetric matrix-valued OU matrix

In this subsection, we discuss the CLT for the eigenvalues of a symmetric matrix-valued Ornstein-Uhlenbeck process. It was shown in Chan (1992), the symmetric  $N \times N$  matrix  $X^N(t)$ , whose entries  $\{X_{ij}^N(t), i \leq j\}$  are independent Ornstein-Uhlenbeck processes with invariant distribution  $N(0, (1 + \delta_{ij})/(2N))$ , where  $\delta_{ij}$  is the Kronecker delta function, is the solution of the matrix SDE (1.1) with the coefficient functions

$$g_N(x)h_N(y) = \frac{1}{2\sqrt{N}}, \quad b_N(x) = -\frac{1}{2}x.$$

The SDEs of the eigenvalue processes are

$$d\lambda_i^N(t) = \frac{1}{\sqrt{N}} dW_i(t) + \left( -\frac{1}{2}\lambda_i^N(t) + \frac{1}{2N} \sum_{j:j \neq i} \frac{1}{\lambda_i^N(t) - \lambda_j^N(t)} \right) dt, \quad 1 \leq i \leq N, \quad t \geq 0. \tag{3.38}$$

Hence, the eigenvalue processes are the particles in (1.6) with

$$\sigma^N(x) = \frac{1}{\sqrt{N}}, \quad b_N(x) = -\frac{1}{2}x, \quad H_N(x, y) = \frac{1}{2N},$$

and thus,

$$NH_N(x, y) = H(x, y) = \frac{1}{2}, \quad b(x) = -\frac{1}{2}x, \quad \sqrt{N}\sigma^N(x) = \tilde{\sigma}(x) = 1.$$

Similar to the eigenvalues of Wishart process and Dyson’s Brownian motion, we have the following CLT.

**Theorem 3.11.** *Let  $\xi^N = (\xi_1^N, \dots, \xi_N^N)$  be a random vector that is independent of  $(W_1, \dots, W_N)$  and has (3.27) as its joint probability density function. Assume that  $(\lambda_1^N(0), \dots, \lambda_N^N(0))$  is independent of  $(W_1, \dots, W_N)$  and that there exist constants  $a, b \geq 0$ , such that*

$$\sqrt{a}\xi_i^N - b \leq \lambda_i^N(0) \leq \sqrt{a}\xi_i^N + b$$

for  $1 \leq i \leq N$  almost surely.

Besides, assume that for any polynomial  $f(x) \in \mathbb{R}[x]$ , the initial value  $\mathcal{L}_0^N(f)$  converges in probability to a random variable  $\mathcal{L}_0(f)$ . Furthermore, assume that for all  $n \in \mathbb{N}$ ,

$$\sup_N \mathbb{E}[|N(\langle x^n, L_N(0) \rangle - \langle x^n, \mu_0 \rangle)|^p] < \infty,$$

for all  $p \geq 1$ .

Then for any  $0 < T < \infty$ , there exists a family of processes  $\{\mathcal{L}_t(f), t \in [0, T]\}_{f \in \mathbb{R}[x]}$ , such that for any  $n \in \mathbb{N}$  and any polynomials  $P_1, \dots, P_n \in \mathbb{R}[x]$ , the vector-valued process  $(\mathcal{L}_t^n(P_1), \dots, \mathcal{L}_t^n(P_n))_{t \in [0, T]}$  converges to  $(\mathcal{L}_t(P_1), \dots, \mathcal{L}_t(P_n))_{t \in [0, T]}$  in distribution.

The limit processes  $\{\mathcal{L}_t(f), t \in [0, T]\}_{f \in \mathbb{R}[x]}$  are characterized by the following properties.

1. For  $P_1, P_2 \in \mathbb{R}[x]$ ,  $\alpha_1, \alpha_2 \in \mathbb{R}$ ,  $t \in [0, T]$ ,

$$\mathcal{L}_t(\alpha_1 P_1 + \alpha_2 P_2) = \alpha_1 \mathcal{L}_t(P_1) + \alpha_2 \mathcal{L}_t(P_2).$$

2. The basis  $\{\mathcal{L}_t(x^n), t \in [0, T]\}_{n \in \mathbb{N}}$  of  $\{\mathcal{L}_t(f), t \in [0, T]\}_{f \in \mathbb{R}[x]}$  satisfies

$$\mathcal{L}_t(1) = 0, \quad \mathcal{L}_t(x) = -\mathcal{L}_0(x) + G_t(x) - \frac{1}{2}e^{-t/2} \int_0^t e^{s/2} (G_s(x) - \mathcal{L}_0(x)) ds,$$

and for  $n \geq 0$ ,

$$\begin{aligned} \mathcal{L}_t(x^{n+2}) &= e^{-\frac{n+2}{2}t} \mathcal{L}_0(x^{n+2}) + R_t(n) + G_t(x^{n+2}) \\ &\quad - \frac{n+2}{2} e^{-\frac{n+2}{2}t} \int_0^t e^{\frac{n+2}{2}s} (R_s(n) + G_s(x^{n+2})) ds. \end{aligned} \quad (3.39)$$

where

$$R_t(n) = \frac{(n+2)(n+1)}{4} \int_0^t \langle x^n, \mu_s \rangle ds + \frac{n+2}{2} \sum_{k=0}^n \int_0^t \mathcal{L}_s(x^{n-k}) \langle x^k, \mu_s \rangle ds \quad (3.40)$$

and  $\{G_t(x^n), t \in [0, T]\}_{n \in \mathbb{N}}$  is a centered Gaussian family with the covariance

$$\mathbb{E}[G_t(x^n)G_s(x^m)] = mn \int_0^{t \wedge s} \langle x^{n+m-2}, \mu_u \rangle du, \quad n, m \geq 1. \quad (3.41)$$

*Proof.* Consider the symmetric matrix-valued OU matrix  $X_t^N$ , of which the entries  $\{X_{ij}^N(t)\}$  satisfy

$$dX_{ij}^N(t) = -\frac{1}{2}X_{ij}^N(t)dt + \frac{2\delta_{ij} + \sqrt{2}(1 - \delta_{ij})}{2\sqrt{N}}dB_{ij}(t), \quad 1 \leq i \leq j \leq N, \quad t \geq 0, \quad (3.42)$$

where  $\{B_{ij}(t), i \leq j\}$  is a family of independent Brownian motions. Denoting by

$$\sigma_{ij} = \frac{2\delta_{ij} + \sqrt{2}(1 - \delta_{ij})}{2\sqrt{N}},$$

the solution to (3.42) is given by

$$X_{ij}^N(t) = X_{ij}^N(0)e^{-t/2} + \sigma_{ij}e^{-t/2} \int_0^t e^{s/2} dB_{ij}(s).$$

The stochastic integral is a martingale with quadratic variation

$$\left\langle \int_0^t e^{s/2} dB_{ij}(s) \right\rangle_t = e^t - 1.$$

By Knight's Theorem, there exists a family of independent standard one-dimensional Brownian motions  $\{\tilde{B}_{ij}(t), i \leq j\}$ , such that

$$\int_0^t e^{s/2} dB_{ij}(s) = \tilde{B}_{ij}(e^t - 1).$$

Thus, we have

$$X_{ij}^N(t) = e^{-t/2} \left( X_{ij}^N(0) + \sigma_{ij} \tilde{B}_{ij}(e^t - 1) \right). \tag{3.43}$$

Let  $Y_t^N$  be a symmetric matrix-valued stochastic process whose entries  $\{Y_{ij}^N(t), i \leq j\}$  are given by

$$Y_{ij}^N(t) = Y_{ij}^N(0) + \sqrt{2}\sigma_{ij}\tilde{B}_{ij}(t), \tag{3.44}$$

with  $Y_{ij}^N(0) = \sqrt{2}X_{ij}^N(0), 1 \leq i \leq j \leq N$ . Then  $Y^N$  is the scaled symmetric matrix-valued Brownian motion introduced in section 3.3. By (3.43) and (3.44),

$$\sqrt{2}e^{t/2}X_{ij}^N(t) = Y_{ij}^N(e^t - 1), \quad 1 \leq i \leq j \leq N,$$

and hence

$$\sqrt{2}e^{t/2}\lambda_i^N(t) = \tilde{\lambda}_i^N(e^t - 1), \quad 1 \leq i \leq N,$$

where  $\{\lambda_i^N(t)\}_{1 \leq i \leq N}$  and  $\{\tilde{\lambda}_i^N(t)\}_{1 \leq i \leq N}$  are the eigenvalues of  $X^N(t)$  and  $Y^N(t)$ , respectively.

Thus, almost surely, we have

$$\begin{aligned} \langle |x|^p, L_N(t) \rangle &= \frac{1}{N} \sum_{i=1}^N |\lambda_i^N(t)|^p \\ &= 2^{-p/2} e^{-pt/2} \frac{1}{N} \sum_{i=1}^N |\tilde{\lambda}_i^N(e^t - 1)|^p \\ &= 2^{-p/2} e^{-pt/2} \langle |x|^p, \tilde{L}_N(e^t - 1) \rangle, \quad \forall t > 0, \end{aligned}$$

where  $L_N(t)$  and  $\tilde{L}_N(t)$  are the empirical measures of  $\{\lambda_i^N(t)\}_{1 \leq i \leq N}$  and  $\{\tilde{\lambda}_i^N(t)\}_{1 \leq i \leq N}$ , respectively. Note that  $\tilde{\lambda}_i^N(0) = \sqrt{2}\lambda_i^N(0)$  satisfies condition (3.28) in Theorem 3.8 with the constants  $a$  and  $b$  replaced by  $2a$  and  $\sqrt{2}b$ . By the estimation (3.30), for all  $p \geq 1$  and  $N \geq \alpha p$  for some positive constant  $\alpha$ , we have

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \in [0, T]} \langle |x|^p, L_N(t) \rangle \right] &\leq 2^{-p/2} \mathbb{E} \left[ \sup_{t \in [0, e^T - 1]} \langle |x|^p, \tilde{L}_N(t) \rangle \right] \\ &\leq 2^{-p/2} C(2a, \sqrt{2}b, e^T - 1)^p \\ &= C'(a, b, T)^p, \end{aligned} \tag{3.45}$$

where  $C'(a, b, T)$  is positive constant depending only on  $(a, b, T)$ .

Thus, by Lemma 3.4 and Corollary 2.5,  $Q_t^N(x^n)$  defined by (1.10) converges in distribution to a centered Gaussian family  $\{G_t(x^n), t \in [0, T]\}_{n \in \mathbb{N}}$  with covariance given by (3.41). Similar to (3.17), for  $n \geq -1$ , we have

$$Q_t^N(x^{n+2}) = \mathcal{L}_t^N(x^{n+2}) - \mathcal{L}_0^N(x^{n+2}) + \frac{n+2}{2} \int_0^t \mathcal{L}_s^N(x^{n+2}) ds - \frac{(n+2)(n+1)}{4} \int_0^t \langle x^n, \mu_s \rangle ds - \frac{n+2}{2} \sum_{k=0}^n \int_0^t \mathcal{L}_s^N(x^{n-k}) \langle x^k, \mu_s \rangle ds - \frac{(n+2)}{4N} \sum_{k=0}^n \int_0^t \mathcal{L}_s^N(x^{n-k}) \mathcal{L}_s^N(x^k) ds.$$

Letting  $N \rightarrow \infty$ , we have

$$G_t(x^{n+2}) \stackrel{d}{=} \mathcal{L}_t(x^{n+2}) - \mathcal{L}_0(x^{n+2}) + \frac{n+2}{2} \int_0^t \mathcal{L}_s(x^{n+2}) ds - \frac{(n+2)(n+1)}{4} \int_0^t \langle x^n, \mu_s \rangle ds - \frac{n+2}{2} \sum_{k=0}^n \int_0^t \mathcal{L}_s(x^{n-k}) \langle x^k, \mu_s \rangle ds = \mathcal{L}_t(x^{n+2}) - \mathcal{L}_0(x^{n+2}) + \frac{n+2}{2} \int_0^t \mathcal{L}_s(x^{n+2}) ds - R_t(n)$$

where  $R_t(n)$  is given in (3.40). Without loss of generality, we may replace “ $\stackrel{d}{=}$ ” by “ $=$ ” in the above equation. Thus we have

$$\mathcal{L}_t(x^{n+2}) + \frac{n+2}{2} \int_0^t \mathcal{L}_s(x^{n+2}) ds = \mathcal{L}_0(x^{n+2}) + G_t(x^{n+2}) + R_t(n),$$

whose solution is given by (3.39).

The proof is concluded. □

Now we extend the result of Theorem 3.11 to a generalized system of (3.38).

**Corollary 3.12.** Consider the following SDEs

$$d\lambda_i^N(t) = \frac{1}{\sqrt{N}} dW_i(t) + \left( b_N(\lambda_i^N(t)) + \frac{1}{2N} \sum_{j:j \neq i} \frac{1}{\lambda_i^N(t) - \lambda_j^N(t)} \right) dt, \quad 1 \leq i \leq N, \quad t \geq 0, \tag{3.46}$$

where  $b_N(x)$  satisfies, for some constant  $c \in \mathbb{R}$ ,

$$\lim_{N \rightarrow \infty} N \left\| b_N(x) + \frac{1}{2}x - c \right\|_{L^\infty(\mathbb{R})} = 0. \tag{3.47}$$

Furthermore, assume the same initial conditions as in Theorem 3.11. Then the conclusion of Theorem 3.11 still holds with  $R_t(n)$  in (3.40) replaced by

$$R_t(n) = c(n+2) \int_0^t \mathcal{L}_s^N(x^{n+1}) ds + \frac{(n+2)(n+1)}{4} \int_0^t \langle x^n, \mu_s \rangle ds + \frac{n+2}{2} \sum_{k=0}^n \int_0^t \mathcal{L}_s(x^{n-k}) \langle x^k, \mu_s \rangle ds.$$

*Proof.* The proof is similar to the proof of Corollary 3.10, which is sketched below.

By (3.47), without loss of generality, we assume

$$-\frac{1}{2}x + c - 1 \leq b_N(x) \leq -\frac{1}{2}x + c + 1,$$

for all  $N \geq 1$ . Then we have

$$\mathbb{P}(x_i^N(t) \leq \lambda_i^N(t) \leq y_i^N(t), \forall 1 \leq i \leq N, \forall t > 0) = 1, \tag{3.48}$$

where the processes  $(x_i^N(t))_{1 \leq i \leq N}$  and  $(y_i^N(t))_{1 \leq i \leq N}$  are the solutions of the following systems of SDEs respectively:

$$dx_i^N(t) = \frac{1}{\sqrt{N}} dW_i(t) + \left( -\frac{1}{2}x_i^N(t) + c - 1 + \frac{1}{2N} \sum_{j:j \neq i} \frac{1}{x_i^N(t) - x_j^N(t)} \right) dt, \quad 1 \leq i \leq N, \quad t \geq 0,$$

and

$$dy_i^N(t) = \frac{1}{\sqrt{N}} dW_i(t) + \left( -\frac{1}{2}y_i^N(t) + c + 1 + \frac{1}{2N} \sum_{j:j \neq i} \frac{1}{y_i^N(t) - y_j^N(t)} \right) dt, \quad 1 \leq i \leq N, \quad t \geq 0,$$

with the initial conditions  $x_i^N(0) = y_i^N(0) = \lambda_i^N(0)$  for  $1 \leq i \leq N$ . Noting that  $(x_i^N(t) - 2c + 2)_{1 \leq i \leq N}$  and  $(y_i^N(t) - 2c - 2)_{1 \leq i \leq N}$  solve the SDEs (3.38), by (3.45) and (3.48), we get that the uniform  $L^p$  bound (2.15) holds for system (3.46).

Then applying Corollary 2.5 and following the approach in the proof of Theorem 3.5, we get the desired result.  $\square$

## A Some lemmas

In this section, we provide some results that were used in the preceding sections.

The following CLT for martingales was used in the proof of Theorem 2.2.

**Lemma A.1** (Rebolledo’s Theorem). *Let  $n \in \mathbb{N}$ , and let  $\{M_N\}_{N \in \mathbb{N}}$  be a sequence of continuous centered martingales with values in  $\mathbb{R}^n$ . If the quadratic variation  $\langle M_N \rangle_t$  converges in  $L^1(\Omega)$  to a continuous deterministic function  $\phi(t)$  for all  $t > 0$ , then for any  $T > 0$ , as a continuous process from  $[0, T]$  to  $\mathbb{R}^n$ ,  $(M_N(t), t \in [0, T])$  converges in law to a Gaussian process  $G$  with mean 0 and covariance*

$$\mathbb{E}[G(s)G(t)^\top] = \phi(t \wedge s).$$

Section 3.1 was based on the following comparison principle for multi-dimensional SDEs which is a direct consequence of (Geiß and Manthey, 1994, Theorem 1.1 and Theorem 1.2).

**Lemma A.2.** *On a certain complete probability space equipped with a filtration that satisfies the usual conditions ((Karatzas and Shreve, 1991, Definition 2.25)), consider the following SDEs*

$$\begin{aligned} Y(t) &= Y(0) + \int_0^t b^{(1)}(s, Y(s)) ds + \int_0^t \sigma(s, Y(s)) dW(s), \\ Z(t) &= Z(0) + \int_0^t b^{(2)}(s, Z(s)) ds + \int_0^t \sigma(s, Z(s)) dW(s), \end{aligned} \tag{A.1}$$

where  $\{W(t), t \geq 0\}$  is a  $d$ -dimensional Brownian motion. Assume the solutions to SDEs (A.1) are pathwisely unique and non-exploding. If the following conditions are satisfied,

1. the drift functions  $b^{(1)}(t, x)$  and  $b^{(2)}(t, x)$  are continuous mappings from  $[0, \infty) \times \mathbb{R}^n$  to  $\mathbb{R}^n$ . Besides, they are quasi-monotonously increasing in the sense that for  $1 \leq i \leq n$  and  $j = 1, 2$ ,  $b_i^{(j)}(t, x) \leq b_i^{(j)}(t, y)$ , whenever  $x_i = y_i$  and  $x_l \leq y_l$  for  $l \in \{1, \dots, n\} \setminus \{i\}$ ;

2. the dispersion matrix  $\sigma(t, x)$  is a continuous mapping from  $[0, \infty) \times \mathbb{R}^n$  to  $\mathbb{R}^{n \times d}$  that satisfies the following condition

$$\sum_{j=1}^d |\sigma_{ij}(t, x) - \sigma_{ij}(t, y)| \leq \rho(|x_i - y_i|)$$

for all  $t \geq 0$  and  $x = (x_1, \dots, x_n)^\top, y = (y_1, \dots, y_n)^\top \in \mathbb{R}^n$ , where  $\rho : [0, \infty) \rightarrow [0, \infty)$  is a strictly increasing function with  $\rho(0) = 0$  and

$$\int_{0+} \rho^{-2}(u) du = \infty;$$

3.  $b_i^{(1)}(t, x) \leq b_i^{(2)}(t, x)$  for all  $1 \leq i \leq n, t \geq 0, x \in \mathbb{R}^n$ ;

4. for  $1 \leq i \leq n, Y_i(0) \leq Z_i(0)$  almost surely,

then we have

$$\mathbb{P}(Y_i(t) \leq Z_i(t), \forall t \geq 0, 1 \leq i \leq n) = 1.$$

The following lemma was employed in the proof of Lemma 3.3.

**Lemma A.3.** Let  $u^N(t)$  be the strong solution to (3.8). If  $u^N(t)$  is distributed according to  $\mathbb{P}^N$  in (3.7), then for  $f \in C_b^2(\mathbb{R}^N)$ ,

$$\frac{d}{dt} \mathbb{E}[f(u^N(t))] = 0.$$

*Proof.* For  $f \in C_b^2(\mathbb{R}^N)$ , applying Itô's formula to (3.8), we have

$$\begin{aligned} f(u^N(t)) &= f(u^N(t_0)) + \sum_{i=1}^N \int_0^t \partial_i f(u^N(s)) \cdot 2 \frac{\sqrt{u_i^N(s)}}{\sqrt{N(s+a)}} dW_i(s) \\ &\quad + \sum_{i=1}^N \int_0^t \partial_i f(u^N(s)) \cdot \frac{1}{s+a} \left( \frac{P}{N} - u_i^N(s) + \frac{1}{N} \sum_{j:j \neq i} \frac{u_i^N(s) + u_j^N(s)}{u_i^N(s) - u_j^N(s)} \right) ds \\ &\quad + \frac{1}{2} \sum_{i=1}^N \int_0^t \partial_i^2 f(u^N(s)) \cdot 4 \frac{u_i^N(s)}{N(s+a)} ds. \end{aligned}$$

Here,  $\partial_i$  is the partial derivative with respect to the  $i$ -th component  $x_i$ . Therefore, for  $t \geq 0$ ,

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[f(u^N(t))] &= \mathbb{E} \left[ \frac{1}{t+a} \sum_{i=1}^N \partial_i f(u^N(t)) \cdot \left( \frac{P}{N} - u_i^N(t) \right) \right] \\ &\quad + \mathbb{E} \left[ \frac{1}{N(t+a)} \sum_{i \neq j} \partial_i f(u^N(t)) \cdot \frac{u_i^N(t) + u_j^N(t)}{u_i^N(t) - u_j^N(t)} \right] \\ &\quad + \mathbb{E} \left[ \frac{2}{N(t+a)} \sum_{i=1}^N \partial_i^2 f(u^N(t)) u_i^N(t) \right]. \end{aligned}$$

Thus it suffices to show, with the density function  $p(x)$  in (3.7),

$$\begin{aligned} &\sum_{i=1}^N \int_{\Delta_N} \partial_i f(x) \cdot \left( \frac{P}{N} - x_i \right) p(x) dx + \frac{1}{N} \sum_{i \neq j} \int_{\Delta_N} \partial_i f(x) \cdot \frac{x_i + x_j}{x_i - x_j} p(x) dx \\ &+ \frac{2}{N} \sum_{i=1}^N \int_{\Delta_N} \partial_i^2 f(x) x_i p(x) dx = 0, \end{aligned} \tag{A.2}$$

where  $\Delta_N = \{x \in \mathbb{R}^N : 0 < x_1 < \dots < x_N\}$  is the support of  $\mathbb{P}^N$ . Noting that  $p(x)$  vanishes on  $\partial\Delta_N$ , we have by the integration by parts formula,

$$\begin{aligned} \int_{\Delta_N} \partial_i^2 f(x) x_i p(x) dx &= \int_{\partial\Delta_N} \partial_i f(x) x_i p(x) dS - \int_{\Delta_N} \partial_i f(x) \partial_i (x_i p(x)) dx \\ &= - \int_{\Delta_N} \partial_i f(x) (p(x) + x_i \partial_i p(x)) dx. \end{aligned}$$

Hence, to show (A.2), it is sufficient to verify

$$\sum_{i=1}^N \left( \frac{P}{N} - x_i \right) p(x) + \frac{1}{N} \sum_{i \neq j} \frac{x_i + x_j}{x_i - x_j} p(x) - \frac{2}{N} \sum_{i=1}^N (p(x) + x_i \partial_i p(x)) = 0.$$

By the chain rule,

$$\partial_i p(x) = -\frac{N}{2} p(x) + \frac{P - N - 1}{2} \frac{1}{x_i} p(x) + \sum_{j: j \neq i} \frac{1}{x_i - x_j} p(x).$$

Hence,

$$\begin{aligned} \frac{2}{N} \sum_{i=1}^N x_i \partial_i p(x) &= - \sum_{i=1}^N x_i p(x) + (P - N - 1) p(x) + \frac{2}{N} \sum_{i \neq j} \frac{x_i}{x_i - x_j} p(x) \\ &= - \sum_{i=1}^N x_i p(x) + (P - N - 1) p(x) + \frac{1}{N} \sum_{i \neq j} \left( \frac{x_i + x_j}{x_i - x_j} + 1 \right) p(x) \\ &= - \sum_{i=1}^N x_i p(x) + (P - 2) p(x) + \frac{1}{N} \sum_{i \neq j} \frac{x_i + x_j}{x_i - x_j} p(x), \end{aligned}$$

which gives the desired result. □

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