Fault Tolerant Decentralized H_∞ Control for Symmetric Composite Systems

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***Abstract*—This note discusses a class of large-scale systems composed of symmetrically interconnected identical subsystems. We consider the decentralized H_∞ control design problem and study the fault tolerance of the resulting system. By exploiting the special structure of the systems, a sufficient condition for the existence of a decentralized H_∞ controller is derived. Moreover, for the nominal case as well as for contingent situations characterized by control channel failures, the poles and the H_∞-norm of the closed-loop system can be calculated easily based on certain systems of reduced dimensions. Consequently, the tolerance to actuator failure can be easily tested.**

***Index Terms*— Decentralized control, fault tolerance, H_∞ control, large-scale systems.**

I. Introduction

In the last decade, a great deal of attention has been paid to the H_∞ control of dynamic systems, and some important design procedures have been established (e.g., [1]–[3]). Unfortunately, these control designs may result in unsatisfactory performance or even unexpected instability in the event of control component failures (e.g., actuator failures and sensor failures). Since failures of control components do occur in real world applications, they should be taken into account when a practical control system is designed.

Recently, Veillette *et al.* [4] studied the design of reliable control systems. The resulting control systems provide guaranteed stability and satisfy an H_∞-norm disturbance attenuation bound not only when all control components are operational, but also in case of actuator or sensor outages in the systems. The reliable control using redundant controllers was studied in [5].

This note considers a special kind of large-scale system—symmetric composite systems. Symmetric composite systems are composed of identical subsystems which are symmetrically interconnected. These systems are encountered in electric power systems, industrial manipulators, computer networks, etc. (see [6]–[8] for other examples and references). Many analyzes and design problems for symmetric composite systems can be simplified because of the special structure of the system. For example, Lunze [6] discussed the stability, controllability, and observability for such systems. The output regulation problem is investigated in [9]. Hovd and Skogestad [7] studied the H_2 and H_∞ control problems using centralized controllers. Lam and Yang [10] studied the balanced model reduction of such systems. Yang *et al.* [11] considered the primary contingency case of reliable H_∞ controller design problem.

For the decentralized control of symmetric composite systems, Lunze [6] proved that the system has no decentralized fixed modes if and only if it is completely controllable and observable. Sundareshan and Elbanna [8] presented a sufficient condition for such systems to be decentralized stabilized, but they did not consider the performance of the closed-loop systems.

This note is concerned with the fault tolerant decentralized H_∞ control for symmetric composite systems. Differing from [4], we only study the tolerance to actuator failure. Moreover, the method used here is distinct from that of [4]. In [4], the method was to design directly a controller which is reliable in case outages occur within a prespecified subset of control components. In this work, the controller is first designed and tested against its tolerance to actuator failure exactly by calculating the poles and the H_∞-norm of the closed-loop system. It will be shown that the effort of these computations can be significantly reduced by exploiting the special structure of the system. The note is organized as follows. Section II gives the state-space model of the system and the problem statement. In Section III, a sufficient condition for the existence of a decentralized H_∞ controller is derived. In Section IV, a new methodology to test the tolerance to actuator failure is presented. In order to clearly demonstrate the methodology proposed, a possible design procedure and an example are given in Section V. Finally, a conclusion is given in Section VI.

II. System Description and Problem Statement

Consider a large-scale system composed of N subsystems, the ith subsystem is given by

$$\begin{aligned} \dot{x}_i &= A_1 x_i + \sum_{k=1,\, k\neq i}^{N} A_2 x_k + B_1 u_i + G_1 w_i + \sum_{k=1,\, k\neq i}^{N} G_2 w_k \\ z_i &= C_1 x_i + D_1 u_i \end{aligned}$$

where $i = 1, 2, \cdots, N$ and $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $w_i \in \mathbb{R}^r$, $z_i \in \mathbb{R}^s$ $(i = 1, \cdots, N)$ are the n-, m-, r-, and s-dimensional state, control input, exogenous input, and penalty, respectively. $A_1, A_2 \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m}$, $G_1, G_2 \in \mathbb{R}^{n\times r}$, $C_1 \in \mathbb{R}^{s\times n}$, $D_1 \in \mathbb{R}^{s\times m}$. Then the state-space model of the overall system is

$$\begin{aligned} \dot{x} &= Ax + Bu + Gw \\ z &= Cx + Du \end{aligned} \tag{1}$$

where $x = (x_1^T, \cdots, x_N^T)^T$, $u = (u_1^T, \cdots, u_N^T)^T$, $w = (w_1^T, \cdots, w_N^T)^T$, $z = (z_1^T, \cdots, z_N^T)^T$, and $A \in \mathbb{R}^{Nn\times Nn}$, $B \in \mathbb{R}^{Nn\times Nm}$, $G \in \mathbb{R}^{Nn\times Nr}$, $C \in \mathbb{R}^{Ns\times Nn}$, $D \in \mathbb{R}^{Ns\times Nm}$ have the structure

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_2 \\ A_2 & A_1 & \cdots & A_2 \\ \vdots & \vdots & & \vdots \\ A_2 & A_2 & \cdots & A_1 \end{bmatrix}, \qquad G = \begin{bmatrix} G_1 & G_2 & \cdots & G_2 \\ G_2 & G_1 & \cdots & G_2 \\ \vdots & \vdots & & \vdots \\ G_2 & G_2 & \cdots & G_1 \end{bmatrix}$$

$$B = \mathrm{diag}[B_1, \cdots, B_1], \qquad C = \mathrm{diag}[C_1, \cdots, C_1]$$
$$D = \mathrm{diag}[D_1, \cdots, D_1].$$

Remark 1: Just as in [6] and [9], we shall hereafter refer system (1) to as a *symmetric composite system*. In [7], Hovd and Skogestad called a system with this structure a parallel system, whereas in Sundareshan and Elbanna [8], it was a symmetrically interconnected system.

Manuscript received December 18, 1997. Recommended by Associate Editor, P. G. Voulgaris. This work was supported in part by the National Natural Science Foundation of China under Grant 69774005, the University Doctoral Foundation of Chinese State Education Commission under Grant 97014508, and the University of Hong Kong CRCG under Grant 10202003/301.

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Publisher Item Identifier S 0018-9286(99)08611-0.

For a symmetric composite system (1), the decentralized H_∞ control problem under consideration is to design a decentralized state feedback control law

$$u_i = K_1 x_i, \qquad i = 1, \cdots, N \tag{2}$$

such that

S1) $\operatorname{spec}(A+BK) \subset C^-$, ($K = \operatorname{diag}[K_1, \cdots, K_1]$), where C^- denotes the open left-half plane.

S2) The transfer matrix $T(s)$ of the closed-loop system

$$\begin{aligned} \dot{x} &= (A+BK)x + Gw \\ z &= (C+DK)x \end{aligned} \tag{3}$$

satisfies $\|T\|_\infty \le \alpha$ for some prescribed $\alpha > 0$.

Remark 2: Since all the subsystems in system (1) are identical, it is an intuitive idea to use decentralized controller of the form (2). Although it had been pointed out in [7], [12], and [13] that decentralized control with identical local controllers is not optimal for all cases, we may still prefer the decentralized controller of the form (2) because of practical reasons, such as easier maintenance and tuning [12]. □

III. Decentralized H_∞ Control

In the rest of this note, we denote

$$\begin{aligned} A_s &= A_1 - A_2, \qquad A_o = A_1 + (N-1)A_2 \\ G_s &= G_1 - G_2, \qquad G_o = G_1 + (N-1)G_2. \end{aligned}$$

The following theorem gives a sufficient condition for the existence of a decentralized H_∞ controller of the form (2).

Theorem 1: Suppose that system (1) satisfies the following two assumptions:

H1) $R_1 = D_1^T D_1$ is nonsingular;

H2) for every real number ω;

$$\begin{aligned} &\operatorname{rank}\begin{bmatrix} A_s - j\omega I & B_1 \\ C_1 & D_1 \end{bmatrix} = n+m \\ &\operatorname{rank}\begin{bmatrix} A_o - j\omega I & B_1 \\ C_1 & D_1 \end{bmatrix} = n+m \qquad (j = \sqrt{-1}). \end{aligned}$$

Let α be a positive constant. Suppose that there exists a symmetric definite positive matrix P_1 such that the following two Riccati algebraic inequalities hold:

$$\begin{aligned} &\left(A_s - B_1 R_1^{-1} D_1^T C_1\right)^T P_1 + P_1\left(A_s - B_1 R_1^{-1} D_1^T C_1\right) \\ &\quad + P_1\left(\frac{1}{\alpha^2} G_s G_s^T - B_1 R_1^{-1} B_1^T\right) P_1 \\ &\quad + C_1^T\left(I - D_1 R_1^{-1} D_1^T\right) C_1 < 0 \end{aligned} \tag{4}$$

$$\begin{aligned} &\left(A_o - B_1 R_1^{-1} D_1^T C_1\right)^T P_1 + P_1\left(A_o - B_1 R_1^{-1} D_1^T C_1\right) \\ &\quad + P_1\left(\frac{1}{\alpha^2} G_o G_o^T - B_1 R_1^{-1} B_1^T\right) P_1 \\ &\quad + C_1^T\left(I - D_1 R_1^{-1} D_1^T\right) C_1 < 0 \end{aligned} \tag{5}$$

then the decentralized state feedback control law

$$u_i = K_1 x_i = -R_1^{-1}\left(B_1^T P_1 + D_1^T C_1\right) x_i, \qquad i = 1, \cdots, N$$

stabilizes system (1), and the closed-loop transfer matrix satisfies $\|T\|_\infty \le \alpha$.

The proof of Theorem 1 is based on the following notations and results.

For a positive integer p, we denote

$$m_k = \begin{bmatrix} 1 & v_k & v_k^2 & \cdots & v_k^{p-1} \end{bmatrix}^T, \qquad k = 1, 2, \cdots, p$$

where $v_k = \exp(2\pi(k-1)j/p)$, $k = 1, 2, \cdots, p$, i.e., v_k is a root of the equation $v^p = 1$.

Let $t = (p+1)/2$ if p is odd, $t = p/2$ if p is even. Denote $r_1 = m_1 = [1 \ 1 \ \cdots \ 1]^T$, $r_{(p/2)+1} = m_{(p/2)+1}$ if p is an even number, $r_i = (1/\sqrt{2})(m_i + m_{p+2-i})$, $r_{p+2-i} = (j/\sqrt{2})(m_i - m_{p+2-i})$ ($i = 2, 3, \cdots, t$). Define

$$R_p = \frac{1}{\sqrt{p}}[r_1 \quad r_2 \quad \cdots \quad r_p]. \tag{6}$$

Then from the results in [7], R_p is a real orthogonal matrix, and the following lemma holds.

Lemma 1 [7]: For a positive integer $p \ge 2$, let

$$D_p = \begin{bmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & & \vdots \\ b & b & \cdots & a \end{bmatrix} \in \mathbb{R}^{p \times p}$$

where a, b are two arbitrarily given numbers. Then we have

$$R_p^{-1} D_p R_p = \operatorname{diag}[a + (p-1)b, a-b, \cdots, a-b] \in \mathbb{R}^{p \times p}.$$

In this note, we further denote

$$T_{pi} = R_p \otimes I_i \tag{7}$$

where I_i is the $i \times i$ identity matrix and $\otimes$ denotes the Kronecker product. Then from Lemma 1 we have

$$\begin{aligned} T_{Nn}^{-1} A T_{Nn} &= \operatorname{diag}[A_o, A_s, \cdots, A_s] \\ T_{Nn}^{-1} G T_{Nr} &= \operatorname{diag}[G_o, G_s, \cdots, G_s] \\ T_{Nn}^{-1} B T_{Nm} &= \operatorname{diag}[B_1, \cdots, B_1] \\ T_{Ns}^{-1} C T_{Nn} &= \operatorname{diag}[C_1, \cdots, C_1] \\ T_{Ns}^{-1} D T_{Nm} &= \operatorname{diag}[D_1, \cdots, D_1]. \end{aligned} \tag{8}$$

From [2, Theorem 2.4.1], the following lemma holds.

Lemma 2 [2]: Consider system (1); suppose

i) $R = D^T D$ is nonsingular;

ii) for every real number ω;

$$\operatorname{rank}\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix} = Nn + Nm.$$

Let α be a given positive constant. If there exists a symmetric definite positive matrix P such that the following Riccati algebraic inequality holds:

$$\begin{aligned} &(A - BR^{-1}D^T C)^T P + P(A - BR^{-1}D^T C) \\ &\quad + P\left(\frac{1}{\alpha^2} GG^T - BR^{-1}B^T\right) P \\ &\quad + C^T(I - DR^{-1}D^T)C < 0 \end{aligned} \tag{9}$$

then the state feedback control law $u = Kx = -R^{-1}(B^T P + D^T C)x$ stabilizes the system (1) and the closed-loop transfer matrix satisfies $\|T\|_\infty \le \alpha$.

Proof of Theorem 1: From H1), $R = D^T D$ is nonsingular. From H2) and (8), we have

$$\begin{aligned}
&\mathrm{rank}\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix} \\
&\quad = \mathrm{rank}\left\{ \begin{bmatrix} T_{Nn}^{-1} & 0 \\ 0 & T_{Ns}^{-1} \end{bmatrix} \begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix} \begin{bmatrix} T_{Nn} & 0 \\ 0 & T_{Nm} \end{bmatrix} \right\} \\
&\quad = \mathrm{rank}\begin{bmatrix} T_{Nn}^{-1}(A - j\omega I)T_{Nn} & T_{Nn}^{-1} B T_{Nm} \\ T_{Ns}^{-1} C T_{Nn} & T_{Ns}^{-1} D T_{Nm} \end{bmatrix} \\
&\quad = \mathrm{rank}\begin{bmatrix} A_o - j\omega I & B_1 \\ C_1 & D_1 \end{bmatrix} + (N-1)\,\mathrm{rank}\begin{bmatrix} A_s - j\omega I & B_1 \\ C_1 & D_1 \end{bmatrix} \\
&\quad = (n+m) + (N-1)(n+m) \\
&\quad = Nn + Nm.
\end{aligned}$$

Thus, i) and ii) in Lemma 2 hold.

Suppose (4) and (5) hold; let

$$P = \mathrm{diag}[P_1, \cdots, P_1].$$

Noting that $T_{Nn}^{-1} = T_{Nn}^T$, from (8), we have

$$\begin{aligned}
&T_{Nn}^T \Big[(A - BR^{-1}D^T C)^T P + P(A - BR^{-1}D^T C) \\
&\quad + P\left(\frac{1}{\alpha^2} GG^T - BR^{-1}B^T\right)P \\
&\quad + C^T(I - DR^{-1}D^T)C \Big] T_{Nn} \\
&\quad = T_{Nn}^T (A - BR^{-1}D^T C)^T T_{Nn} T_{Nn}^{-1} P T_{Nn} \\
&\qquad + T_{Nn}^{-1} P T_{Nn} T_{Nn}^{-1} (A - BR^{-1}D^T C) T_{Nn} \\
&\qquad + \frac{1}{\alpha^2} T_{Nn}^{-1} P T_{Nn} T_{Nn}^{-1} G T_{Nr} T_{Nr}^{-1} G^T T_{Nn} T_{Nn}^{-1} P T_{Nn} \\
&\qquad - T_{Nn}^{-1} P B R^{-1} B^T P T_{Nn} + T_{Nn}^{-1} C^T (I - DR^{-1}D^T) C T_{Nn} \\
&\quad = \mathrm{diag}\Big[\left(A_o - B_1 R_1^{-1} D_1^T C_1\right)^T, \left(A_s - B_1 R_1^{-1} D_1^T C_1\right)^T, \\
&\qquad \cdots, \left(A_s - B_1 R_1^{-1} D_1^T C_1\right)^T \Big] \mathrm{diag}[P_1, P_1, \cdots, P_1] \\
&\qquad + \mathrm{diag}[P_1, P_1, \cdots, P_1]\, \mathrm{diag}\Big[\left(A_o - B_1 R_1^{-1} D_1^T C_1\right), \\
&\qquad \left(A_s - B_1 R_1^{-1} D_1^T C_1\right), \cdots, \left(A_s - B_1 R_1^{-1} D_1^T C_1\right) \Big] \\
&\qquad + \mathrm{diag}[P_1, \cdots, P_1]\, \mathrm{diag}\Big[\frac{1}{\alpha^2} G_o G_o^T - B_1 R_1^{-1} B_1^T, \\
&\qquad \frac{1}{\alpha^2} G_s G_s^T - B_1 R_1^{-1} B_1^T, \cdots, \frac{1}{\alpha^2} G_s G_s^T - B_1 R_1^{-1} B_1^T \Big] \\
&\qquad \times \mathrm{diag}[P_1, \cdots, P_1] + \mathrm{diag}\Big[C_1^T \left(I - D_1 R_1^{-1} D_1^T\right) C_1, \\
&\qquad C_1^T \left(I - D_1 R_1^{-1} D_1^T\right) C_1, \cdots, C_1^T \left(I - D_1 R_1^{-1} D_1^T\right) C_1 \Big].
\end{aligned}$$

From (4) and (5),

$$\begin{aligned}
&T_{Nn}^T \Big[(A - BR^{-1}D^T C)^T P + P(A - BR^{-1}D^T C) \\
&\quad + P\left(\frac{1}{\alpha^2} GG^T - BR^{-1}B^T\right)P \\
&\quad + C^T(I - DR^{-1}D^T)C \Big] T_{Nn} < 0.
\end{aligned}$$

Thus (9) holds.

From Lemma 2, the state feedback control law $u = Kx = -R^{-1}(B^T P + D^T C)x$ stabilizes system (1) and the closed-loop transfer matrix satisfies $\|T\|_\infty \le \alpha$. Since $P = \mathrm{diag}[P_1, \cdots, P_1]$, hence $K = \mathrm{diag}[K_1, \cdots, K_1]$ where $K_1 = -R_1^{-1}(B_1^T P_1 + D_1^T C_1)$. The proof is completed. □

The following theorem shows that the poles and the H_∞-norm of the closed-loop system (3) can be calculated easily.

Theorem 2: The set of poles of the closed-loop system (3) is

$$\mathrm{spec}(A_c) = \mathrm{spec}(A_o + B_1 K_1) \cup \mathrm{spec}(A_s + B_1 K_1). \tag{10}$$

The H_∞-norm of the closed-loop transfer matrix is

$$\|T\|_\infty = \max\{\|T_{oc}\|_\infty, \|T_{sc}\|_\infty\} \tag{11}$$

where

$$\begin{aligned}
T_{oc}(s) &= (C_1 + D_1 K_1)[sI - (A_o + B_1 K_1)]^{-1} G_o \\
T_{sc}(s) &= (C_1 + D_1 K_1)[sI - (A_s + B_1 K_1)]^{-1} G_s.
\end{aligned} \tag{12}$$

Proof: Noting that

$$\begin{aligned}
A_c &= A + BK = A + \mathrm{diag}[B_1 K_1, \cdots, B_1 K_1] \\
T(s) &= (C + DK)[sI - (A + BK)]^{-1} G
\end{aligned}$$

and

$$\mathrm{spec}(A_c) = \mathrm{spec}\big(T_{Nn}^{-1} A_c T_{Nn}\big), \qquad \|T\|_\infty = \big\|T_{Ns}^{-1} T(s) T_{Nr}\big\|_\infty$$

from (8), we can easily prove this theorem. □

Remark 3: Sundareshan and Elbanna [8] also proved (10), but they did not consider the H_∞-norm disturbance attenuation of the closed-loop system. □

Remark 4: Theorem 2 shows that the design of a decentralized H_∞ controller of the form (2) for system (1) is equivalent to finding a gain matrix K_1 that provides stability and H_∞ attenuation for the systems

$$\begin{aligned}
\dot{x} &= (A_o + B_1 K_1)x + G_o w \\
z &= (C_1 + D_1 K_1)x
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
\dot{x} &= (A_s + B_1 K_1)x + G_s w \\
z &= (C_1 + D_1 K_1)x
\end{aligned} \tag{14}$$

simultaneously. If H_∞ disturbance attenuation is not considered, the methods for simultaneous control design (e.g., [14]) can be employed to solve this problem. But if H_∞ disturbance attenuation is considered, there is no systematic simultaneous control design method to apply. □

Since Theorem 1 gives only a sufficient condition, when inequalities (4) and (5) do not hold simultaneously, it does not imply the nonexistence of the controller of the form (2) to guarantee stability and satisfy the H_∞ disturbance attenuation condition $\|T\|_\infty \le \alpha$. However, from Theorem 2, for any given K_1, the poles and the H_∞-norm of the closed-loop system can be determined easily, thus allowing the designer to know whether the controller $u_i = K_1 x_i$ satisfies the specifications or not. In other words, Theorem 2 is also very useful for designing the decentralized controller.

In this section, we studied the design of decentralized H_∞ controller of the form (2). However, when actuator failures occur in the closed-loop system, the resulting system may become unstable. In the next section, we will study the tolerance to actuator failure of the decentralized controller (2).

IV. Tolerance to Actuator Failure

This section studies the tolerance to actuator failure of the decentralized controller (2). For a given $\gamma > 0$, we want to find the integer l_0 which corresponds to the smallest number of failures that make the closed-loop system unstable or cause the closed-loop system to violate the disturbance attenuation bound γ. It will be shown that l_0 can be obtained easily as a result of the special structure of system (1). The main results of this section are given by the following theorems.

Theorem 3: Consider the closed-loop system (3), when only one of the subsystem controllers fails, the set of poles of the resulting closed-loop system is

$$\begin{aligned}\text{spec}(A_{c1}) &= \text{spec}(A_s + B_1K_1)\\ &\cup \text{spec}\left\{\begin{bmatrix} A_1 & \sqrt{N-1}A_2 \\ \sqrt{N-1}A_2 & A_1 + (N-2)A_2 + B_1K_1\end{bmatrix}\right\}.\end{aligned}$$

Moreover, in this case, the H_∞-norm of the resulting closed-loop transfer matrix is

$$\|T_1\|_\infty = \max\{\|T_{1c}\|_\infty, \|T_{sc}\|_\infty\}$$

where

$$\begin{aligned}T_{1c}(s) = &\begin{bmatrix} C_1 & 0 \\ 0 & C_1 + D_1K_1\end{bmatrix}\\ &\times \left\{sI - \begin{bmatrix} A_1 & \sqrt{N-1}A_2 \\ \sqrt{N-1}A_2 & A_1 + (N-2)A_2 + B_1K_1\end{bmatrix}\right\}^{-1}\\ &\times \begin{bmatrix} G_1 & \sqrt{N-1}G_2 \\ \sqrt{N-1}G_2 & G_1 + (N-2)G_2\end{bmatrix}\end{aligned}$$

and $T_{sc}(s)$ is defined in (12).

Theorem 4: Consider the closed-loop system (3), for positive integer l $(2 \le l \le N-2)$, when l of the subsystem controllers fail, the set of poles of the resulting closed-loop system is

$$\begin{aligned}\text{spec}(A_{cl}) &= \text{spec}(A_s) \cup \text{spec}(A_s + B_1K_1)\\ &\cup \text{spec}\left\{\begin{bmatrix} A_1 + (l-1)A_2 & \sqrt{l(N-l)}A_2 \\ \sqrt{l(N-l)}A_2 & A_1 + (N-l-1)A_2 + B_1K_1\end{bmatrix}\right\}.\end{aligned} \tag{15}$$

Moreover, in this case, the H_∞-norm of the resulting closed-loop transfer matrix is

$$\|T_l\|_\infty = \max\{\|T_{lc}\|_\infty, \|T_s\|_\infty, \|T_{sc}\|_\infty\} \tag{16}$$

where as shown in (17) at the bottom of the page, and

$$T_s(s) = C_1(sI - A_s)^{-1}G_s \tag{18}$$

and $T_{sc}(s)$ is defined in (12).

Remark 5: When l $(2 \le l \le N-2)$ of the subsystem controllers fail, the resulting closed-loop system can be regarded as composed of two symmetric composite systems: one is an ln-dimensional "open-loop system" (with no K_1 in it), another is an $(N-l)n$-dimensional "closed-loop system" (with K_1 in every subsystems). In (15), spec(A_s) is part of the poles of the "open-loop system," spec$(A_s + B_1K_1)$ is part of the poles of the "closed-loop system," and

$$\text{spec}\left\{\begin{bmatrix} A_1 + (l-1)A_2 & \sqrt{l(N-l)}A_2 \\ \sqrt{l(N-l)}A_2 & A_1 + (N-l-1)A_2 + B_1K_1\end{bmatrix}\right\}$$

is the rest of the poles. The H_∞-norm result can be explained similarly. □

Theorem 5: Consider the closed-loop system (3), when $N-1$ of the subsystem controllers fail, the set of poles of the resulting closed-loop system is

$$\begin{aligned}\text{spec}(A_{c(N-1)}) &= \text{spec}(A_s) \cup \text{spec}\left\{\begin{bmatrix} A_1 + (N-2)A_2 & \sqrt{N-1}A_2 \\ \sqrt{N-1}A_2 & A_1 + B_1K_1\end{bmatrix}\right\}.\end{aligned}$$

Moreover, in this case, the H_∞-norm of the resulting closed-loop transfer matrix is

$$\|T_{N-1}\|_\infty = \max\{\|T_{(N-1)c}\|_\infty, \|T_s\|_\infty\}$$

where

$$\begin{aligned}T_{(N-1)c}(s) = &\begin{bmatrix} C_1 & 0 \\ 0 & C_1 + D_1K_1\end{bmatrix}\\ &\times \left\{sI - \begin{bmatrix} A_1 + (N-2)A_2 & \sqrt{N-1}A_2 \\ \sqrt{N-1}A_2 & A_1 + B_1K_1\end{bmatrix}\right\}^{-1}\\ &\times \begin{bmatrix} G_1 + (N-2)G_2 & \sqrt{N-1}G_2 \\ \sqrt{N-1}G_2 & G_1\end{bmatrix}\end{aligned}$$

and $T_s(s)$ is defined in (18).

Remark 6: Theorems 4 and 5 show that the part of the poles of the open-loop system (1), given by spec(A_s), cannot be changed when more than two controllers failures occur. Hence, spec$(A_s) \subset C^-$ is a necessary condition for the closed-loop system to tolerate more than two controllers failure. □

The proofs of Theorems 3–5 require the following lemma.

Lemma 3: For positive integers $p \ge 2$ and $q \ge 2$, let

$$E_{pq} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1\end{bmatrix} \in \mathbb{R}^{p\times q}.$$

$$T_{lc}(s) = \begin{bmatrix} C_1 & 0 \\ 0 & C_1 + D_1K_1\end{bmatrix} \left\{sI - \begin{bmatrix} A_1 + (l-1)A_2 & \sqrt{l(N-l)}A_2 \\ \sqrt{l(N-l)}A_2 & A_1 + (N-l-1)A_2 + B_1K_1\end{bmatrix}\right\}^{-1} \begin{bmatrix} G_1 + (l-1)G_2 & \sqrt{l(N-l)}G_2 \\ \sqrt{l(N-l)}G_2 & G_1 + (N-l-1)G_2\end{bmatrix} \tag{17}$$

Then the following equality holds:

$$R_p^{-1} E_{pq} R_q = \begin{bmatrix} \sqrt{pq} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{p\times q}$$

where R_p and R_q are defined by (6).

Proof: The lemma can be established through straightforward algebraic manipulations. □

For Theorems 3–5, we only prove Theorem 4. The proofs of Theorems 3 and 5 are similar and thus omitted.

Proof of Theorem 4: Consider the closed-loop system (3), since the subsystems of system (1) are symmetrically interconnected, without loss of generality, we can assume that the first l of the subsystem controllers fail. In this case, the decentralized controller becomes

$$\begin{aligned} u_i &= 0, && i = 1, \cdots, l \\ u_i &= K_1 x_i, && i = l+1, \cdots, N. \end{aligned}$$

Thus, the resulting closed-loop system matrix becomes as shown in (18a) at the bottom of the page.

Denote $W_1 = A_1 + (l-1)A_2$, $W_2 = \sqrt{l(N-l)}A_2$, $W_3 = A_1 + (N-l-1)A_2 + B_1K_1$. Then from Lemmas 1 and 3, we have

$$\text{spec}(A_{cl}) = \text{spec}\left\{\begin{bmatrix} T_{ln}^{-1} & 0 \\ 0 & T_{(N-l)n}^{-1}\end{bmatrix} A_{cl} \begin{bmatrix} T_{ln} & 0 \\ 0 & T_{(N-l)n}\end{bmatrix}\right\}$$

where T_{ln} and $T_{(N-l)n}$ are defined in (7) as shown in (18b) at the bottom of the page. Thus (15) holds.

Moreover, the resulting closed-loop transfer matrix becomes

$$\begin{aligned} T_l(s) = {} & \text{diag}[C_1, \cdots, C_1, C_1 + D_1K_1, \cdots, C_1 + D_1K_1] \\ & \times (sI - A_{cl})^{-1} G. \end{aligned}$$

Since premultiplication or postmultiplication of $T_l(s)$ by orthogonal matrices will leave the H_∞-norm unchanged, hence we have

$$\begin{aligned} \|T_l\|_\infty = {} & \left\| \begin{bmatrix} T_{ls}^{-1} & 0 \\ 0 & T_{(N-l)s}^{-1} \end{bmatrix} T_l(s) \begin{bmatrix} T_{lr} & 0 \\ 0 & T_{(N-l)r} \end{bmatrix} \right\|_\infty \\ = {} & \Bigg\| \text{diag}[C_1,\, C_1 + D_1K_1,\, C_1,\, \cdots,\, C_1,\, C_1 + D_1K_1, \\ & \cdots, C_1 + D_1K_1]\, \text{diag}\Bigg[\left(sI - \begin{bmatrix} W_1 & W_2 \\ W_2 & W_3 \end{bmatrix}\right)^{-1}, \\ & (sI - A_s)^{-1}, \cdots, (sI - A_s)^{-1}, \\ & [sI - (A_s + B_1K_1)]^{-1}, \cdots, [sI - (A_s + B_1K_1)]^{-1}\Bigg] \\ & \times \text{diag}\Bigg[\begin{bmatrix} G_1 + (l-1)G_2 & \sqrt{l(N-l)}G_2 \\ \sqrt{l(N-l)}G_2 & G_1 + (N-l-1)G_2 \end{bmatrix}, \\ & [G_s, \cdots, G_s]\Bigg] \Bigg\|_\infty \\ = {} & \max\{\|T_{lc}\|_\infty, \|T_s\|_\infty, \|T_{sc}\|_\infty\}. \end{aligned}$$

Thus (16) holds. □

From Theorems 3–5, the poles and the H_∞-norm of the resulting closed-loop system can be easily computed when arbitrary controller failures occur. Thus, after decentralized controller (2) (the gain matrix K_1) is obtained, the fault tolerance of the controller (l_0) can be

$$A_{cl} = \begin{bmatrix} A_1 & A_2 & \cdots & A_2 & A_2 & A_2 & \cdots & A_2 \\ A_2 & A_1 & \cdots & A_2 & A_2 & A_2 & \cdots & A_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ A_2 & A_2 & \cdots & A_1 & A_2 & A_2 & \cdots & A_2 \\ A_2 & A_2 & \cdots & A_2 & A_1 + B_1K_1 & A_2 & \cdots & A_2 \\ A_2 & A_2 & \cdots & A_2 & A_2 & A_1 + B_1K_1 & \cdots & A_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ A_2 & A_2 & \cdots & A_2 & A_2 & A_2 & \cdots & A_1 + B_1K_1 \end{bmatrix} \tag{18a}$$

$$\begin{aligned} & = \text{spec}\left\{\begin{bmatrix} W_1 & & & & W_2 & & & \\ & A_s & & & & & & \\ & & \ddots & & & & & \\ & & & A_s & & & & \\ W_2 & & & & W_3 & & & \\ & & & & & A_s + B_1K_1 & & \\ & & & & & & \ddots & \\ & & & & & & & A_s + B_1K_1 \end{bmatrix}\right\} \\ & = \text{spec}\left\{\text{diag}\left[\begin{bmatrix} W_1 & W_2 \\ W_2 & W_3 \end{bmatrix}, A_s, \cdots, A_s, A_s + B_1K_1, \cdots, A_s + B_1K_1\right]\right\} \\ & = \text{spec}(A_s) \cup \text{spec}(A_s + B_1K_1) \cup \text{spec}\left\{\begin{bmatrix} W_1 & W_2 \\ W_2 & W_3 \end{bmatrix}\right\}. \end{aligned} \tag{18b}$$

assessed by computing the poles and the H_∞-norm of different actuator failure cases. In next section, we shall provide a possible design procedure and an example to illustrate the details.

V. A Possible Design Procedure and Example

Using Theorems 1–5, for a given $\gamma > 0$, a design scheme for a decentralized H_∞ controller is suggested and its fault tolerance properties are tested for the symmetric composite system (1) as follows.

A. Possible DesignProcedure

Step 1) Select α and ϵ, ($0 < \alpha < \gamma$, $\epsilon > 0$, for example, $\alpha = \gamma/2$) solve Riccati equations

$$\begin{aligned}&\left(A_s - B_1R_1^{-1}D_1^TC_1\right)^T P_s + P_s\left(A_s - B_1R_1^{-1}D_1^TC_1\right)\\&\quad + P_s\left(\frac{1}{\alpha^2}G_sG_s^T - B_1R_1^{-1}B_1^T\right)P_s\\&\quad + C_1^T\left(I - D_1R_1^{-1}D_1^T\right)C_1 + \epsilon I = 0\end{aligned}\tag{19}$$

and

$$\begin{aligned}&\left(A_o - B_1R_1^{-1}D_1^TC_1\right)^T P_o + P_o\left(A_o - B_1R_1^{-1}D_1^TC_1\right)\\&\quad + P_o\left(\frac{1}{\alpha^2}G_oG_o^T - B_1R_1^{-1}B_1^T\right)P_o\\&\quad + C_1^T\left(I - D_1R_1^{-1}D_1^T\right)C_1 + \epsilon I = 0\end{aligned}\tag{20}$$

to obtain P_s and P_o.

Step 2) Test Riccati inequality

$$\begin{aligned}&\left(A_s - B_1R_1^{-1}D_1^TC_1\right)^T P_o + P_o\left(A_s - B_1R_1^{-1}D_1^TC_1\right)\\&\quad + P_o\left(\frac{1}{\alpha^2}G_sG_s^T - B_1R_1^{-1}B_1^T\right)P_o\\&\quad + C_1^T\left(I - D_1R_1^{-1}D_1^T\right)C_1 < 0.\end{aligned}\tag{21}$$

If (21) holds, then let $P_1 = P_o$, go to Step 7.

Step 3) Test Riccati inequality

$$\begin{aligned}&\left(A_o - B_1R_1^{-1}D_1^TC_1\right)^T P_s + P_s\left(A_o - B_1R_1^{-1}D_1^TC_1\right)\\&\quad + P_s\left(\frac{1}{\alpha^2}G_oG_o^T - B_1R_1^{-1}B_1^T\right)P_s\\&\quad + C_1^T\left(I - D_1R_1^{-1}D_1^T\right)C_1 < 0.\end{aligned}\tag{22}$$

If (22) holds, then let $P_1 = P_s$, go to Step 7.

Step 4) Let $K_1 = -R_1^{-1}(B_1^TP_o + D_1^TC_1)$ (or let $K_1 = -R_1^{-1}(B_1^TP_s + D_1^TC_1)$).

Step 5) Compute spec(A_c) and $\|T\|_\infty$ (using Theorem 2). If spec$(A_c) \subset C^-$ and $\|T\|_\infty \le \gamma$, then go to Step 8.

Step 6) Go back to Step 1, select α and ϵ again (decrease ϵ and/or increase α).

Step 7) Let $K_1 = -R_1^{-1}(B_1^TP_1 + D_1^TC_1)$.

Step 8) The decentralized H_∞ control law can be chosen as $u_i = K_1x_i$, $i = 1, \cdots, N$.

Step 9) Let $l = 1$.

Step 10) Compute spec(A_{cl}) and $\|T_l\|_\infty$ (using Theorems 3–5).

Step 11) If spec$(A_{cl}) \subset C^-$ and $\|T_l\|_\infty \le \gamma$, then let $l = l + 1$, go back to Step 10.

Step 12) Let $l_0 = l$, and one can conclude that the closed-loop system will maintain its stability with $\|T\|_\infty \le \gamma$ when less than l_0 of the subsystem controllers fail.

Remark 7: If for some α and ϵ, (21) or (22) holds, then the above algorithm will converge, and we can obtain both the decentralized H_∞ controller and its tolerance level to actuator failure. If (21) and (22) do not hold, we suggest choosing K_1 as in Step 4 and using Step 5 to test its stabilization and disturbance attenuation properties. This choice very often works in our numerical examples. Up till now, a systematic method for choosing K_1 to ensure spec$(A_c) \subset C^-$ and $\|T\|_\infty \le \gamma$ is not available.

Remark 8: Before starting the design procedure, we should first compute spec$(A) =$ spec$(A_o) \cup$ spec(A_s) and the H_∞-norm of the open-loop transfer matrix

$$\|T\|_\infty = \max\{\|T_o\|_\infty, \|T_s\|_\infty\}$$

where $T_o(s) = C_1(sI - A_o)^{-1}G_o$ and $T_s(s)$ is defined in (18). If spec$(A) \subset C^-$ and $\|T\|_\infty \le \gamma$, then we do not need to design the controller. On the other hand, if we need to design the controller, this computation will also simplify the computation of spec(A_{cl}) and $\|T_l\|_\infty$ in Step 10. □

In the following, we use an example to illustrate the design procedure stated above. All H_∞-computations in the example are performed with the μ-Analysis and Synthesis Toolbox for MATLAB.

Example: Consider the voltage/reactive power behavior of a multimachine power system, the overall system consists of several synchronous machines including their PI-voltage controller, which feed the load through a distribution net [6]. The system can be modeled by

$$\begin{aligned}\dot{x}_i &= \begin{bmatrix}-2.51 & -0.16\\ 2.55 & 0\end{bmatrix}x_i + \sum_{k=1,\,k\neq i}^{N}\begin{bmatrix}-0.065 & 0\\ -0.0027 & 0\end{bmatrix}x_k\\&\quad + \begin{bmatrix}0.9\\ -1\end{bmatrix}u_i + \begin{bmatrix}0.2\\ 0.1\end{bmatrix}w_i + \sum_{k=1,\,k\neq i}^{N}\begin{bmatrix}0.1\\ 0.1\end{bmatrix}w_k\\ z_i &= [2.54 \quad 0]x_i + u_i, \qquad i = 1, 2, \cdots, N.\end{aligned}$$

Suppose $N = 20$, computing directly, we have

$$\begin{aligned}A_s &= \begin{bmatrix}-2.445 & -0.16\\ 2.5527 & 0\end{bmatrix}\\ A_o &= \begin{bmatrix}-3.745 & -0.16\\ 2.4987 & 0\end{bmatrix}\\ G_s &= \begin{bmatrix}0.1\\ 0\end{bmatrix}\\ G_o &= \begin{bmatrix}2.1\\ 2\end{bmatrix}.\end{aligned}$$

Suppose $\gamma = 0.8$, we choose $\alpha = 0.4, \epsilon = 0.0002$, solving the Riccati equations (19) and (20), we have

$$\begin{aligned}P_o &= \begin{bmatrix}0.001 & 0.0011\\ 0.0011 & 0.0014\end{bmatrix}\\ P_s &= \begin{bmatrix}0.000694 & 0.000625\\ 0.000625 & 0.0006025\end{bmatrix}.\end{aligned}$$

TABLE I
SUMMARY OF RESULTS

l	$\mathrm{spec}(A_{cl})$	$\|T_l\|_\infty$
1	$-4.552,\ -0.179,\ -5.8517,\ -2.3089,\ -0.1802,\ -0.1352$	0.4501
2	$-4.552,\ -0.179,\ -2.2646,\ -0.1804,\ -5.8082,\ -2.3541,\ -0.1801,\ -0.1336$	0.6368
3	$-4.552,\ -0.179,\ -2.2646,\ -0.1804,\ -5.7635,\ -2.4005,\ -0.18,\ -0.1321$	0.7803
4	$-4.552,\ -0.179,\ -2.2646,\ -0.1804,\ -5.7176,\ -2.448,\ -0.1799,\ -0.1305$	0.9014

(open-loop system $\|T\|_\infty = 2.0331$)

By testing, we know that inequalities (21) and (22) do not hold, we try by choosing

$$K_1 = -R_1^{-1}\left(B_1^T P_o + D_1^T C_1\right)$$

and obtain $K_1 = [-2.5397, 0.0003]$. From Theorem 2, we get $\mathrm{spec}(A_c) = \{-4.552, -0.179, -5.8942, -0.1368\} \subset C^-$ and $\|T\|_\infty = 0.0083 < \gamma$. Thus the decentralized H_∞ control law can be chosen as

$$u_i = K_1 x_i = [-2.5397, 0.0003]x_i, \qquad i = 1, \cdots, N.$$

For $l = 1, 2, 3, 4$, Theorems 3 and 4 are used to compute $\mathrm{spec}(A_{cl})$ and $\|T_l\|_\infty$. The results are summarized in Table I.

Since for $l = 1, 2, 3$, $\mathrm{spec}(A_{cl}) \subset C^-$ and $\|T_l\|_\infty < \gamma$, but $\|T_4\|_\infty > \gamma$, hence $l_0 = 4$. As a result, the closed-loop system will maintain its stability and the transfer matrix will satisfy $\|T\|_\infty \leq \gamma$ when less than four subsystem controllers fail.

VI. CONCLUSION

In this note, we studied the state feedback decentralized H_∞ control for symmetric composite systems. First, we gave a sufficient condition for the existence of a decentralized H_∞ controller. Second, we proved that the poles and the H_∞-norm of the closed-loop system can be computed easily, even when some actuator faults eliminate the state feedback in some of the subsystems. Using these results, we then know the tolerance to actuator failure as soon as the decentralized state feedback controller is designed.

Since only a sufficient condition for the existence of a state feedback decentralized H_∞ controller is obtained, further work is still needed before a complete design framework can be established. Moreover, the fault tolerant decentralized H_∞ control for symmetric composite systems via output feedback is also a further research problem.

It should be noted that the special structure of symmetric composite systems allows us to use the methodology presented in this note. The methodology is not suitable for general large-scale systems, since the computation of the poles and the H_∞-norm is computationally more demanding.

ACKNOWLEDGMENT

The authors are greatly indebted to the reviewers for useful suggestions and corrections on the initial manuscript of this work.

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