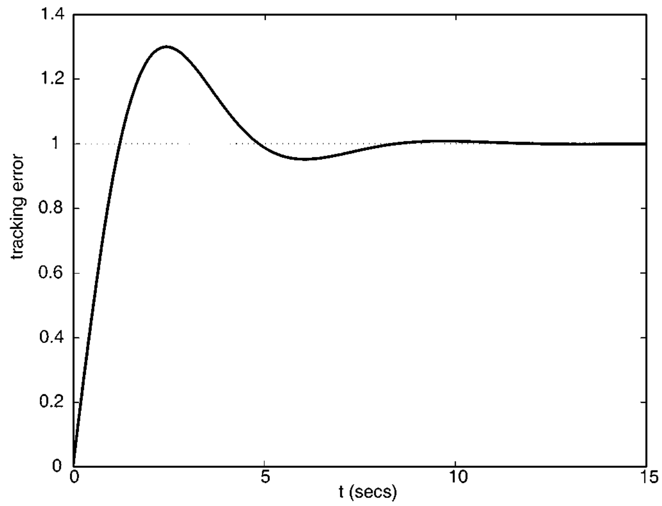

Fig. 3. Tracking-error e to a unit-ramp input for the reset control system.

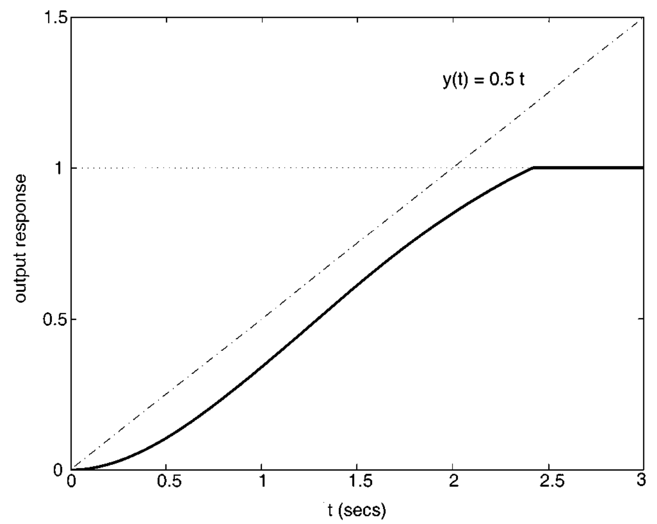

Fig. 4. Output response y to a unit-step input for the reset control system (dotted line shows rise time constraint).

track step inputs with zero steady-state error; see [7], [9], and [14] for more details.

IV. Conclusion

The main contribution of this note is an example of control specifications that can be achieved by reset control and not by linear feedback. This does not imply that reset control is superior; rather, that reset control has a different set of performance limitations. Such differences can be exploited in specific control applications as demonstrated in [4], [5], [8], and [11].

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On Quadratic Stability of Systems With Structured Uncertainty

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***Abstract*—This note considers the problem of stability robustness with respect to a class of nonlinear time-varying perturbations which are bounded in a component-wise rather than aggregated manner. A family of robustness bounds is parameterized in terms of a nonsingular symmetric matrix. It is shown that the problem of computing the largest robustness bound over the set of nonsingular symmetric matrices can be approximated by a smooth minimization problem over a compact set. A convergent algorithm for computing an optimal robustness bound is proposed in the form of a gradient flow.**

***Index Terms*—Linear systems, optimization, quadratic stability, stability robustness.**

I. Introduction

Linear systems are often subject to time-varying nonlinear perturbations including parametric uncertainties. The issue of stability robust-

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ness has been given a great deal of attention in recent decades. Many results have been obtained for determining the extent of uncertainty that the system can tolerate without becoming unstable [1]–[3]. However, it remains challenging to develop methods for finding less conservative robustness bounds in the presence of *structured* perturbations.

The notion of quadratic stability provides a convenient means of dealing with the stability robustness of a linear system to time-varying nonlinear perturbations. Quadratic stability is associated with the existence of a single independent Lyapunov function which guarantees stability for a set of time-varying nonlinear perturbations characterized by an upper bound. Obviously, quadratic stability implies robust stability defined in such a way that a Lyapunov function is allowed to be dependent on perturbations. Conversely, robust stability of a system with unstructured norm bounded parametric uncertainty implies quadratic stability even when the uncertainty is time-varying and nonlinear, see [4]. While this surprising result together with the availability of the maximum allowable bound on the norm of the uncertainty such that robust stability is still maintained drastically diminishes the significance of the robustness bounds recently given in [5] as well as some related bounds obtained later such as those in [6], it does not carried over to systems with structured uncertainty. It has been revealed through examples in [7] that robust stability is no longer equivalent to quadratic stability when the system is subject to at least two blocks of real parametric uncertainty. As such, those bounds obtained to measure the degree of robust stability for general structured uncertainty may not be applicable to quadratic stability even if the uncertainty is parametric. On the other hand, a robustness bound obtained with quadratic stability is always a bound for robust stability no matter whether the uncertainty is structured or unstructured. Interestingly, the real structured singular value used to express the sufficient condition for robust stability with respect to real structured uncertainty has been known to be NP-hard to compute [8].

The focus of this note is on quadratic stability of a linear system with time-varying nonlinear perturbations characterized by individual bounds on perturbation components. Our objective is twofold. First, we seek a condition on such bounds such that the perturbed system is quadratically stable; the derivation of the condition will be given in the next section. Second, the robustness index describing the condition will be optimized so as to obtain tight bounds for quadratic stability. The optimization problem to be discussed in Section III turns out to be a nonsmooth optimization problem which can be approximated by a smooth problem. A simple and effective algorithm for computing a minimum of the robustness index will be presented in Section III.

The following notation is adopted in the remainder of this note.

$\|x\|_1$, $\|x\|_2$, and $\|x\|_\infty$	denote the 1, 2, and ∞-norms of a vector x, respectively.
$\|X\|_2$ and $\|X\|_F$	denote the spectral and Frobenius norms of a matrix X, respectively.
$\mathbb{Q}$	denotes the set of invertible symmetric matrices.

II. Robustness Bounds

Consider a dynamic system described by

$$\dot{x} = Ax + f(x,t) \tag{2.1}$$

where $A \in \mathbb{R}^{n\times n}$ is a nominal matrix with eigenvalues in the open left-half plane and $f(x,t)$ is a possibly time-varying nonlinear uncertain function. A widely used constraint on $f(x,t)$ is of the form

$$\|f(x,t)\|_2 \le \mu \|x\|_2 \tag{2.2}$$

in which all components of $f(x,t)$ are weighted equally. In this note, we assume instead that $f(x,t)$ obeys the following constraints:

$$\begin{aligned} |f_i(x,t)| &\le \alpha_i \left|w_i^T x\right|, \quad i = 1,2,\ldots,r \\ f_i(x,t) &= 0, \quad i = r+1,\ldots,n \end{aligned} \tag{2.3}$$

where w_i is a constant weighting vector in $\mathbb{R}^n$, α_i is nonnegative, and r is the number of uncertain components. Of course, there will be no equality constraints when $r = n$. It is reasonable to expect that a smaller r tends to result in a larger robustness bound.

Quite obviously, this kind of constraints on $f(x,t)$ is different from the constraint of the form (2.2) and is capable of describing the structure of the uncertain term $f(x,t)$ more accurately in many practical situations. For example, consider the following system with norm bounded structured uncertainty:

$$\dot{x} = (A + B\Delta C)x \tag{2.4}$$

where $B \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{r\times n}$, and the uncertain term Δ is of the form

$$\Delta = \operatorname{diag}\{\delta_1,\ldots,\delta_r\}. \tag{2.5}$$

If B is of full-column rank, then a suitable similarity transformation of the form $x = Tz$ will convert the (2.4) into

$$\dot{z} = T^{-1}ATz + f(z,t) \tag{2.6}$$

where $f(z,t) = [\begin{smallmatrix} I \\ 0 \end{smallmatrix}]\Delta C T z$ obeys

$$|f_i(z,t)| = |\delta_i| \left|c_i^T T z\right|, \quad i = 1,2,\ldots,r \tag{2.7}$$

$$f_i(z,t) = 0, \quad i = r+1,\ldots,n \tag{2.8}$$

where c_i^T is the ith row of C.

The system (2.1) with (2.3) will be said to be quadratically stable if there is a common quadratic Lyapunov function for all the uncertain $f(x,t)$ obeying (2.3). The following lemma plays a pivotal role later on and may be of interest in its own right though its proof is simple.

Lemma 2.1: For any given matrix $M \in \mathbb{R}^{n\times m}$, there holds

$$\max_{\|x\|_2=1} \|Mx\|_\infty = \|M\|_{2,\infty}$$

where $\|M\|_{2,\infty}$ denotes the square root of the maximum diagonal element of MM^T.

Proof: Put

$$y_i = e_i^T Mx, \quad i = 1,2,\ldots,n$$

where e_i is the i-th column of the $n \times n$ identity matrix. Since

$$\max_{\|x\|_2=1} y_i^2 = \max_{\|x\|_2=1} x^T M^T e_i e_i^T Mx \tag{2.9}$$

$$= \left\|M^T e_i e_i^T M\right\|_2 = e_i^T MM^T e_i \tag{2.10}$$

it follows that:

$$\max_{\|x\|_2=1} \|Mx\|_\infty = \max_{\|x\|_2=1} \max_{1\le i\le n} |y_i| \tag{2.11}$$

$$= \max_{1\le i\le n} \max_{\|x\|_2=1} \sqrt{y_i^2} \tag{2.12}$$

$$= \max_{1\le i\le n} \sqrt{e_i^T MM^T e_i}. \tag{2.13}$$

This completes the proof. □

Remark 2.1: Note that $\|M\|_{2,\infty} = \|M\|_2$ when M is a row vector,

Theorem 2.1: The perturbed system (2.1) with (2.3) is quadratically stable if

$$\inf_{\mathbb{Q}} \sum_{i=1}^{r} \alpha_i \left\|w_i^T Q\right\|_2 \left\|p_i^T Q\right\|_2 < 1 \tag{2.14}$$

where p_i is the i-th column of the unique solution to the Lyapunov equation

$$PA + A^T P + 2Q^{-2} = 0. \tag{2.15}$$

Proof: Let $Q \in \mathbb{Q}$ be such that (2.14) holds and P be the symmetric positive definite solution to (2.15), and set the Lyapunov function

$$V(x) = x^T P x.$$

Then, there holds

$$\begin{aligned}
\dot{V}(x) &= x^T(PA + A^T P)x + 2f^T(x,t)Px && (2.16)\\
&= -2x^T Q^{-2} x + 2f^T(x,t)Px && (2.17)\\
&\le -2x^T Q^{-2} x + 2\sum_{i=1}^{r} \alpha_i \left|w_i^T x\right| \left|p_i^T x\right| && (2.18)\\
&= -2\|Q^{-1}x\|_2^2 \left(1 - \sum_{i=1}^{r} \alpha_i \frac{\left|w_i^T x\right|}{\|Q^{-1}x\|_2} \frac{\left|p_i^T x\right|}{\|Q^{-1}x\|_2}\right) && (2.19)\\
&\le -2\|Q^{-1}x\|_2^2 \left(1 - \sum_{i=1}^{r} \alpha_i \left\|w_i^T Q\right\|_2 \left\|p_i^T Q\right\|_2\right). && (2.20)
\end{aligned}$$

Because of (2.14), there results $\dot{V}(x) < 0$, which implies the asymptotic stability of the system (2.1). □

The above theorem enables one to test if the system (2.1) with (2.3) is quadratically stable when all the α_i are given. Moreover, one can derive a robustness bound on the vector

$$\alpha \triangleq [\,\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_r\,]^T \tag{2.21}$$

directly from the condition (2.14) for the system (2.1) with (2.3) to be quadratically stable.

Corollary 2.1: The perturbed system (2.1) with (2.3) is quadratically stable if

$$\|\alpha\|_1 < \frac{1}{\|WQ\|_{2,\infty}\|P_r Q\|_{2,\infty}} \tag{2.22}$$

for some $Q \in \mathbb{Q}$, where P_r is the $r \times n$ upper submatrix of the unique solution to the Lyapunov equation (2.15) and W is given by

$$W = \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_r^T \end{bmatrix}.$$

Apparently, the robustness bound given above can be optimized by minimizing the function $J(Q) : \mathbb{Q} \mapsto \mathbb{R}$ defined as

$$J(Q) = \|WQ\|_{2,\infty}\|PQ\|_{2,\infty} \tag{2.23}$$

over the set $\mathbb{Q}$. As such, $J(Q)$ will be termed the *robustness index* in the sequel. There are two main difficulties with the problem of minimizing this index. First, $J(Q)$ generally is not a differentiable or convex function in $\mathbb{Q}$. Second, $J(Q)$ may not have a minimum in $\mathbb{Q}$ despite the existence of the infimum.

We end this section by pointing out that all the techniques to be developed in the next two sections for computing the infimum of $J(Q)$ as defined in (2.23) are equally applicable to the problem of computing the infimum on the left-hand side of (2.14) in Theorem 2.1.

III. Optimization of Robustness Index

In this section, we will introduce a smooth auxiliary cost function which has a global minimum and approximates the robustness index $J(Q)$.

Let Z be a fixed constant matrix such that $\begin{bmatrix} W \\ Z \end{bmatrix}$ is of full-column rank, and let

$$W_k = \begin{bmatrix} W \\ k^{-1}Z \end{bmatrix} \quad \text{and} \quad U_k = \begin{bmatrix} I_r & 0 \\ 0 & k^{-1}I_{n-r} \end{bmatrix}$$

where k is a positive integer. Further, let $\mathcal{D}_k(\cdot)$ denote the operator defined by

$$\mathcal{D}_k(X) = \operatorname{diag}\left\{x_{11}^k, \ldots, x_{nn}^k\right\}$$

for $X = (x_{ij})_{n \times n}$. Now, for each given k, define a function $J_k(Q) : \mathbb{Q} \mapsto \mathbb{R}$ as

$$J_k(Q) \triangleq \left\{\operatorname{trace}\left[\mathcal{D}_k\left(W_k Q^2 W_k^T\right)\right] \times \operatorname{trace}\left[\mathcal{D}_k\left(U_k P Q^2 P U_k^T\right)\right]\right\}^{\frac{1}{2k}} \tag{3.1}$$

where P is the solution to the Lyapunov equation (2.15).

Theorem 3.1: For any given positive integer k, the function $J_k(Q)$ defined as in (3.1) is smooth and has a global minimum in $\mathbb{Q}$.

Proof: That $J_k(Q)$ is smooth in $\mathbb{Q}$ is obvious from the definition. Also, it is easy to verify that $J_k(Q)$ satisfies

$$J_k(\alpha Q) = J_k(Q), \quad \forall Q \in \mathbb{Q}, \quad \alpha \in \mathbb{R}, \quad \alpha \ne 0$$

which implies that $J_k(Q)$ has the same infimum in the set $\{Q \in \mathbb{Q}; \|Q\|_F = 1\}$ as in $\mathbb{Q}$. As such, it suffices to prove that the set

$$\Pi = \{Q \in \mathbb{Q}; \|Q\|_F = 1 \text{ and } J_k(Q) \le a\}$$

is closed for any given number $a > 0$. To this end, fix $a > 0$ and let $Q \in \Pi$. Since it is true that $W_k Q = 0$ if and only if $Q = 0$, there exists a constant $\gamma > 0$ such that

$$\operatorname{trace}\left[\mathcal{D}_k\left(W_k Q^2 W_k^T\right)\right] \ge \gamma^{2k}, \quad \forall Q \in \Pi. \tag{3.2}$$

Therefore, it follows that:

$$\begin{aligned}
\|P\|_2 &\le \|PQ\|_2\|Q^{-1}\|_2 && (3.3)\\
&\le \|U_k PQ\|_2 \left\|U_k^{-1}\right\|_2 \|Q^{-1}\|_2 && (3.4)\\
&\le \sqrt{n}\|U_k PQ\|_{2,\infty} \left\|U_k^{-1}\right\|_2 \|Q^{-1}\|_2 && (3.5)\\
&\le \sqrt{n}\{\operatorname{trace}[\mathcal{D}_k(U_k PQ^2 PU_k)]\}^{\frac{1}{2k}} \left\|U_k^{-1}\right\|_2 \|Q^{-1}\|_2 && (3.6)\\
&= \frac{\sqrt{n}J_k(Q)}{\{\operatorname{trace}[\mathcal{D}_k(W_k Q^2 W_k^T)]\}^{\frac{1}{2k}}} \left\|U_k^{-1}\right\|_2 \|Q^{-1}\|_2 && (3.7)\\
&\le \frac{a\sqrt{n}}{\gamma} \left\|U_k^{-1}\right\|_2 \|Q^{-1}\|_2. && (3.8)
\end{aligned}$$

On the other hand, it is seen from (2.15) that

$$\|2Q^{-2}\|_2 = \|PA + A^T P\|_2 \le 2\|A\|_2\|P\|_2$$

i.e.,

$$\|Q^{-1}\|_2^2 \le \|A\|_2\|P\|_2. \tag{3.9}$$

A combination of (3.8) and (3.9) yields

$$\|Q^{-1}\|_2 \le \frac{a\sqrt{n}}{\gamma} \left\|U_k^{-1}\right\|_2 \|A\|_2. \tag{3.10}$$

Since Q is arbitrary, (3.10) implies that any limit point of Π is nonsingular and thus belongs to Π due to the fact that both $\|Q\|_F$ and $J_k(Q)$

are continuous with respect to Q. In this way, the set Π is shown to be closed in$\mathbb{Q}$. $\square$

As a result of the following lemma, the robustness index $J(Q)$ as defined in (2.23) can be approximated by the smooth function $J_k(Q)$ when k is large.

Lemma 3.1: Given $X \in \mathbb{R}^{p\times q}$, there holds

$$\|X\|_{2,\infty} \leq [\operatorname{trace} \mathcal{D}_k(XX^T)]^{\frac{1}{2k}} < p^{\frac{1}{2k}}\|X\|_{2,\infty}, \quad k = 1, 2, \ldots.$$

Proof: Let the diagonal entries of XX^T be

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p.$$

Then it is easily seen that

$$\|X\|_{2,\infty}^2 = \sigma_1$$
$$[\operatorname{trace} \mathcal{D}_k(X^TX)^k]^{\frac{1}{k}} = \left[\sum_{i=1}^{p} \sigma_i^k\right]^{\frac{1}{k}} \leq p^{\frac{1}{k}}\sigma_1$$

from which the lemma is concluded. $\square$

Remark 3.1: It is seen that Lemma 3.1 is closely related to the well-known formula

$$\|x\|_\infty = \max_i |x_i| = \lim_{p\to\infty} \|x\|_p = \lim_{p\to\infty} \left(\sum_i |x_i|^p\right)^{1/p}$$

for the $\mathcal{L}_p$ norm.

We move on to show that $J(Q)$ can be minimized by minimizing the smooth function $J_k(Q)$ as defined in (3.1) as integer k tends to infinity.

Theorem 3.2: Let the function $J_k(Q) : \mathbb{Q} \mapsto \mathbb{R}$ be defined as in (3.1) for $k = 1, 2, \ldots$ and the robustness index $J(Q) : \mathbb{Q} \mapsto \mathbb{R}$ be as defined in (2.23). There holds

$$\inf_{\mathbb{Q}} J(Q) = \lim_{k\to\infty} \min_{\mathbb{Q}} J_k(Q); \tag{3.11}$$

moreover, if $J_k(Q)$ assumes its minimum at $Q_k \in \mathbb{Q}$, then there holds

$$\inf_{\mathbb{Q}} J(Q) = \lim_{k\to\infty} J(Q_k). \tag{3.12}$$

Proof: First, it is seen from Lemma 3.1 that

$$\begin{aligned} J(Q) &\leq \|W_kQ\|_{2,\infty}\|P_kQ\|_{2,\infty} \\ &\leq J_k(Q) \leq (np)^{\frac{1}{2k}}\|W_kQ\|_{2,\infty}\|P_kQ\|_{2,\infty}, \\ &\qquad \forall k \geq 1, \quad Q \in \mathbb{Q} \end{aligned} \tag{3.13}$$

where p is the number of rows of W_k. Since

$$\lim_{k\to\infty} (np)^{\frac{1}{2k}}\|W_kQ\|_{2,\infty}\|P_kQ\|_{2,\infty} = \|WQ\|_{2,\infty}\|PQ\|_{2,\infty}$$

there results

$$J(Q) = \lim_{k\to\infty} J_k(Q), \quad \forall Q \in \mathbb{Q}. \tag{3.14}$$

Now take an arbitrary number $\epsilon > 0$. Then there exists Q_ϵ such that

$$J(Q_\epsilon) - \inf_{\mathbb{Q}} J(Q) \leq \epsilon$$

leading to

$$\begin{aligned} \min_{\mathbb{Q}} J_k(Q) - \inf_{\mathbb{Q}} J(Q) &= \min_{\mathbb{Q}} J_k(Q) - J_k(Q_\epsilon) + J_k(Q_\epsilon) \\ &\quad - J(Q_\epsilon) + J(Q_\epsilon) - \inf_{\mathbb{Q}} J(Q) \\ &\leq \epsilon + J_k(Q_\epsilon) - J(Q_\epsilon). \end{aligned} \tag{3.15}$$

On the other hand, it follows from (3.13) that:

$$\min_{\mathbb{Q}} J_k(Q) - \inf_{\mathbb{Q}} J(Q) \geq 0.$$

Consequently, it is deduced that

$$0 \leq \min_{\mathbb{Q}} J_k(Q) - \inf_{\mathbb{Q}} J(Q) \leq \epsilon + J_k(Q_\epsilon) - J(Q_\epsilon).$$

This implies that

$$\limsup_{k\to\infty} \left[\min_{\mathbb{Q}} J_k(Q) - \inf_{\mathbb{Q}} J(Q)\right] - \liminf_{k\to\infty} \left[\min_{\mathbb{Q}} J_k(Q) - \inf_{\mathbb{Q}} J(Q)\right] \leq \epsilon$$

due to

$$\lim_{k\to\infty} [J_k(Q_\epsilon) - J(Q_\epsilon)] = 0.$$

As ϵ is arbitrary, one obtains

$$\lim_{k\to\infty} \left[\min_{\mathbb{Q}} J_k(Q) - \inf_{\mathbb{Q}} J(Q)\right] = 0$$

i.e., (3.11). To prove (3.12), note from (3.13) that

$$\inf_{\mathbb{Q}} J(Q) \leq J(Q_k) \leq J_k(Q_k) \leq J_k(Q_\epsilon).$$

So there holds

$$0 \leq J(Q_k) - \inf_{\mathbb{Q}} J(Q) \leq J_k(Q_\epsilon) - \inf_{\mathbb{Q}} J(Q).$$

In this way, (3.12) is concluded from

$$\lim_{k\to\infty} [J_k(Q_\epsilon) - \inf_{\mathbb{Q}} J(Q)] = J(Q_\epsilon) - \inf_{\mathbb{Q}} J(Q) \leq \epsilon$$

and the fact that ϵ is arbitrary. $\square$

IV. Computational Algorithm

Recall from Section II that the robustness bound given in (2.22) can be maximized by finding the infimum of $J(Q)$ over the set of invertible symmetric matrices. Owing to Theorem 3.2, this infimum can be arbitrarily approximated by the minimum of the auxiliary cost function $J_k(Q)$ with a sufficiently large k. The purpose of this section is to develop a method for performing the minimization of $J_k(Q)$. The way to achieve this purpose is through the use of differential techniques.

Introduce the following notation:

$$F_k = \frac{J_k(Q)}{2\operatorname{trace}[\mathcal{D}_k(W_kQ^2W_k^T)]}\mathcal{D}_{k-1}\left(W_kQ^2W_k^T\right) \tag{4.1}$$

$$G_k = \frac{J_k(Q)}{2\operatorname{trace}[\mathcal{D}_k(U_kPQ^2PU_k^T)]}\mathcal{D}_{k-1}\left(U_kPQ^2PU_k^T\right) \tag{4.2}$$

with $\Sigma_k \in \mathbb{Q}$ satisfying the Lyapunov equation

$$\Sigma_kA^T + A\Sigma_k + Q^2PU_k^TG_kU_k + U_k^TG_kU_kPQ^2 = 0. \tag{4.3}$$

In what follows, an algorithm for minimizing $J_k(Q)$ will be presented in the form of a matrix differential equation, which can be easily integrated using an appropriate numerical routine, e.g., in Matlab on a digital computer. Recently, analog computing has gained renewed interest in view of advances in neural networks which allow massively parallel processing. As a result, it becomes increasingly acceptable to make use of differential equations for solving various problems such as optimization and linear algebra problems, see e.g., [9]–[11] and the references therein.

Theorem 4.1: Consider the function $J_k(Q) : \mathbb{Q} \mapsto \mathbb{R}$ as defined in (3.1).

1) The gradient of $J_k(Q)$ is given by

$$\nabla J_k(Q) = R_k + R_k^T \tag{4.4}$$

where

$$R_k \triangleq F_kW_kQ + PU_k^TG_kU_kPQ - 2Q^{-1}\Sigma_kQ^{-2}. \tag{4.5}$$

2) The differential equation

$$\boxed{\dot{Q}(t) = -\nabla J_k(Q(t)), \quad Q(0) = Q_0} \tag{4.6}$$

has no finite escape time in $\mathbb{Q}$ for any $Q_0 \in \mathbb{Q}$ with

$$\|Q(t)\|_F = \|Q_0\|_F, \quad \forall t \geq 0. \tag{4.7}$$

3) The cost function $J_k(Q)$ is monotonically decreasing along the solution $Q(t)$ for $t > 0$ and the gradient $\nabla J_k(Q)$ converges to zero along $Q(t)$ as $t \to \infty$.

Proof:

1) Denote the Fréchet derivatives of $J_k(Q)$ and P as a function of Q at $Q \in \mathbb{Q}$ by dJ_k and dP, respectively. By definition, dJ_k and dP are simply linear operators on $\mathbb{R}^{n\times n}$. Straightforward calculations give

$$\begin{aligned}
dJ_k(X) & \\
&= \frac{1}{2}\left\{\mathrm{trace}\left[\mathcal{D}_k\left(W_kQ^2W_k^T\right)\right]\right\}^{\frac{1}{2k}-1} \\
&\quad \times \left\{\mathrm{trace}\left[\mathcal{D}_k\left(U_kPQ^2PU_k^T\right)\right]\right\}^{\frac{1}{2k}} \\
&\quad \times \mathrm{trace}\left[\mathcal{D}_{k-1}\left(W_kQ^2W_k^T\right)(W_k(QX+XQ)W_k^T\right] \\
&\quad + \frac{1}{2}\left\{\mathrm{trace}\left[\mathcal{D}_k\left(W_kQ^2W_k^T\right)\right]\right\}^{\frac{1}{2k}} \\
&\quad \times \left\{\mathrm{trace}\left[\mathcal{D}_k\left(U_kPQ^2PU_k^T\right)\right]\right\}^{\frac{1}{2k}-1} \\
&\quad \times \mathrm{trace}\left[\mathcal{D}_{k-1}\left(U_kPQ^2PU_k^T\right)\right. \\
&\quad \times U_k(dP(X)Q^2P + PQ^2dP(X) \\
&\quad \left. + PQXP + PXQP)U_k^T\right] \qquad (4.8)\\
&= \mathrm{trace}\left[F_k(W_k(QX+XQ)W_k^T\right] \\
&\quad + \mathrm{trace}\left[(G_kU_k(dP(X)Q^2P + PQ^2dP(X)\right. \\
&\quad \left. + PQXP + PXQP)U_k^T\right] \qquad (4.9)\\
&= \mathrm{trace}\left[\left(F_kW_kQ + QW_k^TF_k\right)X\right] \\
&\quad + \mathrm{trace}\left[\left(PU_k^TG_kU_kPQ + QPU_k^TG_kU_kP\right)X\right] \\
&\quad + \mathrm{trace}\left[\left(Q^2PU_k^TG_kU_k + U_k^TG_kU_kPQ^2\right)dP(X)\right].
\end{aligned} \tag{4.10}$$

By differentiating both sides of the Lyapunov (2.15) with respect to Q, one obtains

$$dP(X)A + A^TdP(X) - 2\left(Q^{-2}XQ^{-1} + Q^{-1}XQ^{-2}\right) = 0$$

from which it follows that:

$$\begin{aligned}
&\mathrm{trace}\left[\left(Q^2PU_k^TG_kU_k + U_k^TG_kU_kPQ^2\right)dP(X)\right] \\
&\quad = -\mathrm{trace}\left[2\left(Q^{-2}XQ^{-1} + Q^{-1}XQ^{-2}\right)\Sigma_k\right] \\
&\quad = -\mathrm{trace}\left[2\left(Q^{-1}\Sigma_kQ^{-2} + Q^{-2}\Sigma_kQ^{-1}\right)X\right].
\end{aligned} \tag{4.11}$$

Combining this with (4.10) immediately yields

$$dJ_k(X) = \mathrm{trace}\left[\left(R_k + R_k^T\right)X\right] \tag{4.12}$$

which implies (4.4) by definition.

2) Recall from the Proof of Theorem 3.1 that

$$J_k(\alpha Q) = J_k(Q), \quad \forall Q \in \mathbb{Q}, \quad \alpha \in \mathbb{R}, \quad \alpha \neq 0$$

which implies that

$$\mathrm{trace}[Q^T\nabla J_k(Q)] = 0, \quad \forall Q \in \mathbb{Q}. \tag{4.13}$$

Therefore, it follows that:

$$\|Q(t)\|_F = \|Q_0\|_F, \quad \forall t \geq 0. \tag{4.14}$$

By employing an argument similar to that used in the Proof of Theorem 3.1, it can be shown that $\|Q^{-1}(t)\|_F$ is bounded by a constant for all $t \geq 0$. It is thus concluded that the differential equation (4.6) has no finite escape time.

3) The proof is omitted due to its simplicity.

□

Remark 4.1: In the case where there are isolated minimum points in the set $\{Q \in \mathbb{Q}; \|Q\|_F = 1\}$, the solution to the Eq. (4.6) is bound to converge to one of the minimum points of $J_k(Q)$.

Now it is appropriate to recap briefly the main results developed so far before a numerical algorithm is presented.

- Theorem 2.1 gives a sufficient condition for quadratic stability of the uncertain system (2.1)–(2.3) in terms of the inequality

$$\inf_{\mathbb{Q}} \sum_{i=1}^{r} \alpha_i \left\|w_i^TQ\right\|_2 \left\|p_i^TQ\right\|_2 < 1$$

  where α_i characterizes the uncertainty of the i-th dynamic equation together with a weighting vector w_i via (2.3) and r is the number of uncertain equations. This sufficient condition is verifiable only when the uncertainty parameters α_i are given.
- By Corollary 2.1, the largest robustness bound on the 1-norm of the uncertainty vector in the sense of (2.22) is the reciprocal of the infimum of $J(Q)$ over the set of invertible symmetric matrices where

$$J(Q) \triangleq \|WQ\|_{2,\infty}\|PQ\|_{2,\infty}.$$

- By Theorem 3.2, the infimum of $J(Q)$ is equal to the limit of a sequence of global minima of a smooth function $J_k(Q)$ over the compact set of invertible symmetric matrices with Frobenius norm equal to 1, where

$$J_k(Q) \triangleq \left\{\mathrm{trace}\left[\mathcal{D}_k\left(W_kQ^2W_k^T\right)\right] \times \mathrm{trace}\left[\mathcal{D}_k\left(U_kPQ^2PU_k^T\right)\right]\right\}^{\frac{1}{2k}}.$$

- By Theorem 4.1, the solution to the ODE

$$\dot{Q}(t) = -R_k - R_k^T, \quad Q(0) = Q_0$$

  converges to a local minimum of $J_k(Q)$ for any initial invertible symmetric matrix Q with $\|Q\|_F = 1$, where R_k is a function of Q as defined by (4.5).

The summarized theoretic results naturally give rise to the following numerical procedure for computing a suboptimal robustness bound.

Algorithm 1:

Step 1) Choose an initial index k and a starting point $Q_0 \in \mathbb{Q}$.

Step 2) Seek a minimum point $\bar{Q}$ of the cost function $J_k(Q)$ by finding a limiting solution to the ODE (4.6) with the initial condition $Q(0) = Q_0$.

Step 3) If $|J(Q_0) - J(\bar{Q})|$ is less than a preset tolerance, stop; otherwise, go back to Step 2 with a larger k and $Q_0 = \bar{Q}$.

Finally, a remark concerning the practical implementation of Algorithm 1 is in order.

Remark 4.2: To implement the above algorithm, it is often adequate and convenient to set the initial point Q_0 to the identity matrix in light of the fact that $J_k(Q)$ assumes the same minimum in $\{Q \in \mathbb{Q}; \|Q\|_F = 1\}$ as in $\mathbb{Q}$. The proposed algorithm is not guaranteed to generate a sequence convergent to the infimum of $J(Q)$ since the limiting solution to the ODE associated with $J_k(Q)$ obtained in Step 2) is not necessarily a global minimum of $J_k(Q)$. It is worthwhile

$$Q = \begin{bmatrix} 1.0418 & -0.2348 & 0.2956 & -0.0628 & 0.0526 \\ -0.2348 & 0.7136 & -0.2550 & 0.4125 & -0.1699 \\ 0.2956 & -0.2550 & 1.0788 & -0.0691 & 0.0298 \\ -0.0628 & 0.4125 & -0.0691 & 1.0286 & -0.1445 \\ 0.0526 & -0.1699 & 0.0298 & -0.1445 & 0.5508 \end{bmatrix}$$

TABLE I
ROBUSTNESS BOUNDS VERSUS THE NUMBER OF CONSTRAINTS

Number of Inequality Constraints (r)	Robustness Bound
1	0.2867
2	0.2013
3	0.1490
4	0.1490
5	0.1490

to mention that general purpose optimization algorithms could also be used to find a local minimum of $J_k(Q)$.

V. AN EXAMPLE

In this section, we consider the system

$$\dot{x} = Ax + f(x,t) \tag{5.1}$$

with

$$A = \begin{bmatrix} -0.201 & 0.755 & 0.351 & -0.075 & 0.033 \\ -0.149 & -0.696 & -0.160 & 0.110 & -0.048 \\ 0.081 & 0.004 & -0.189 & -0.003 & 0.001 \\ -0.173 & 0.802 & 0.251 & -0.804 & 0.056 \\ 0.092 & -0.467 & -0.127 & 0.075 & -1.162 \end{bmatrix}$$

and

$$|f_i(x,t)| \le \alpha_i \|x\|_\infty, \quad i = 1, \ldots, r \tag{5.2}$$

$$|f_i(x,t)| = 0, \quad i = r+1, \ldots, 5. \tag{5.3}$$

This system was discussed in [1] and [5] when subject to the unstructured perturbation of the form

$$\|f(x,t)\|_2 \le \mu \|x\|_2. \tag{5.4}$$

It is also known from [4] that under the same type of perturbation, the largest μ for quadratic stability is $1/(\|(sI-A)^{-1}\|_\infty)$, i.e., 0.1116, which is well greater than the two previously obtained bounds 0.0774 and 0.0929 in [1], [5].

We are interested in computing an optimal bound on $\sum_{i=1}^r \alpha_i$ for every r from 1 to 5 such that the perturbed system is asymptotically stable. Table I shows the robustness bounds obtained by implementing the proposed algorithm in the previous section with $Q_0 = I$ and $k = 1, 50, 100$ in Matlab. In particular, this implies that the system (5.1) is asymptotically stable for any perturbation $f(x,t)$ obeying

$$|f_i(x,t)| \le \alpha_i \|x\|_\infty, \quad i = 1, \ldots, 5 \tag{5.5}$$

with

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 0.1490.$$

It is worth pointing out that this inequality characterizes a large class of perturbations which are not covered by the unstructured constraint (5.4) with the optimal $\mu = 0.1116$. It is also interesting to note that the algorithm gives the same robustness bound for $r = 3, 4, 5$ and that α_1 is allowed to be much larger without violating the stability when there are less than 3 inequality constraints on the perturbation, i.e., r equals 1 or 2.

The final Q obtained with the algorithm is shown in the equation at the top of the page, at which the robustness bound $1/J(Q)$ equals 0.1490.

Remark 5.1: To demonstrate the usefulness of the obtained robustness bounds given in Table I, let us consider a simple case where all components of $f(x,t)$ except $f_1(x,t)$ are known to be identically zero, i.e., $r = 1$. In this case, the structured constraint (5.2) with $\alpha_1 = 0.2867$ becomes

$$|f_1(x,t)| \le 0.2867 \|x\|_\infty \tag{5.6}$$

while the unstructured constraint (5.4) with $\mu = 0.1116$ becomes

$$|f_1(x,t)| \le 0.1116 \|x\|_2. \tag{5.7}$$

It is seen that the latter inequality strictly implies the former because of

$$0.1116\|x\|_2 \le 0.1116 \times \sqrt{5}\|x\|_\infty = 0.2495\|x\|_\infty.$$

In other words, the new robustness bound is capable of describing a larger set of structured uncertainties against which the system is quadratically stable.

VI. CONCLUSION

A sufficient condition has been derived for quadratic stability of a linear system with time-varying nonlinear perturbations whose components are individually bounded. The problem of finding an optimal robustness bound based on the condition has been treated with an effective numerical algorithm proposed. An extension of the present method to the case of delayed perturbations can be envisaged in view of the work in [3].

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Design of Fault Diagnosis Filters and Fault-Tolerant Control for a Class of Nonlinear Systems

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***Abstract*—This note presents a set of algorithms for fault diagnosis and fault tolerant control strategy for affine nonlinear systems subjected to an unknown time-varying fault vector. At first, the design of fault diagnosis filter is performed using nonlinear observer techniques, where the system is decoupled through a nonlinear transformation and an observer is used to generate the required residual signal. By introducing an extra input to the observer, a direct estimation of the time-varying fault is obtained when the residual is controlled, by this extra input, to zero. The stability analysis of this observer is proved and some relevant sufficient conditions are obtained. Using the estimated fault vector, a fault tolerant controller is established which guarantees the stability of the closed loop system. The proposed algorithm is applied to a combined pH and consistency control system of a pilot paper machine, where simulations are performed to show the effectiveness of the proposed approach.**

***Index Terms*—Fault detection, fault estimation, fault tolerant, feedback, nonlinear observers, nonlinear systems.**

I. Introduction

In fault detection and diagnosis (FDD), the residual generator [3] takes the input and the output of the process and delivers a signal which indicates the system healthy status. The analysis of the nonzero residual signals can help to determine which fault has occurred [4]. Indeed, residual generation for linear systems have been well documented in the literature [10], [11]. However, few results exist for nonlinear systems [5], [1], where the identification of faults for nonlinear systems is not considered in most of the existing approaches. Only the methods based on parameter estimation techniques [2] can give the identification of multiplicative faults and provide some fault tolerant control using an adaptive control framework. This is because in most cases it is difficult to use residuals alone to determine the size of the fault. One way for fault estimation could be to use the system inversion techniques in order to estimate the fault which affects the residual signal [6]. However, such an approach is not always robust with respect to measurement noises. As such, it is necessary to develop effective fault identification and fault tolerant control algorithms for nonlinear systems. This forms the main purpose of this paper where the contributions are to 1) reformulate the problem of residual generation so as to incorporate fault identification for nonlinear systems; 2) provide the estimation of the fault by characterizing "fault estimability" through the concept of input observability and input detectability for a class of nonlinear systems; 3) develop a simple method which can link the residual generation based fault diagnosis techniques to the design of fault tolerant control; and 4) establish a fault tolerant control using directly the diagnosis information so as to stabilize the closed loop system.

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II. Residual Generation and Fault Reconstruction

Consider the following class of known nonlinear systems:

$$\begin{cases} \dot{x}(t) = g_0(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_i + \sum_{i=1}^{n_f} e_i(x(t))f_i \\ y(t) = h(x(t)) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ is the output vector, $u = (u_1, u_2, \ldots, u_m) \in U \subset \mathbb{R}^m$ is the bounded input vector of the system and U is the set of admissible inputs. The vector fields g_i $(i = 0, \ldots, m), e_i$ $(i = 0, \ldots, n_f)$ and h are assumed to be smooth with respect to their arguments. The $f_i \in \mathbb{R}^{n_{fi}}$ are unknown but bounded fault vectors. The purpose here is to use $\{u_i(t), y(t)\}$ to estimate f_i and then construct a fault tolerant control algorithm. For this purpose, the reformulation of the fundamental problem of the residual generation (FPRG) [5], [10] is made to incorporate the fault estimation task into the design of residuals.

Definition 1: The purpose of solving the problem of fault detection and identification (PFDI) with respect to f_i is to find a dynamical system of the form shown in the (2) at the bottom of the page, where $z \in \mathbb{R}^{\bar{n}}, r \in \mathbb{R}^{\bar{p}}$ is the residual, and $\hat{f}_i(t)$ are the estimate of f_i, such that 1) r only depends on f_i; 2) if $f_i = 0$, then $\lim_{t\to\infty} r(t) = 0$; 3) if $\lim_{t\to\infty} r(t) = 0$ then $\lim_{t\to\infty}(\hat{f}_i - f_i) = 0$, holds for $\forall x(0), z(0)$ and $\forall u \in U$. In particular, if $\lim_{t\to\infty} r(t) = 0$ and $\lim_{t\to\infty} \hat{f}_i = 0$ then $f_i = 0$.

These conditions summarize an asymptotic property of input observability [6]. If system (2) can be constructed so that the residual $r(t)$ realizes 1), 2), and 3), then $\hat{f}_i$ can be regarded as the estimate of f_i. Using the procedures in [8], system (1) can be decoupled into an interconnected system whose jth subsystem is expressed by

$$\begin{cases} \dot{\xi}_j = A_j\xi_j + G_{0j}(\xi) + \sum_{l=1}^{m} u_l G_{lj}(\xi) + E_{ij}(\xi)f_i \\ \bar{y}_j = \varphi_j(y) = C_j\xi_j, \quad j = 1, \ldots, \bar{p} \le p \end{cases} \tag{3}$$

where $\xi_j = (\xi_{j,1}, \ldots, \xi_{j,n_j})^\top \in \mathbb{R}^{n_j}$, $\sum_{j=1}^{\bar{p}} n_j = \tilde{n} \le n, A_j = [a_{l,s}]_{1\le l,s\le n_j}$ is a $(n_j \times n_j)$ matrix, $a_{l,l+1} = 1$ and $a_{l,s} = 0$ for $s \ne l+1$. Also, $G_{0j} = G_{0j,n_j}(\xi)B_j, G_{0j,n_j} \in \mathbb{R}$. $B_j^\top = (0 \ \cdots \ 0 \ 1)$ and $C_j = [1 \ 0 \ \cdots \ 0]$ are $(n_j \times 1)$ and $(1 \times n_j)$ matrices, respectively. Moreover, we require that the output function $\varphi : \mathbb{R}^p \to \mathbb{R}^{\bar{p}}, \bar{p} \le p$, and a state transformation $\xi = (\xi_1, \ldots, \xi_{\bar{p}})^\top$ defined on an open set V_0 of $\mathbb{R}^n$ satisfy

i)

$$G_{lj}(\xi) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ G_{lj,k_j}(\xi_{1,1}, \ldots, \xi_{1,k_j}, \ldots, \xi_{\bar{p},1}, \ldots, \xi_{\bar{p},k_j}) \\ G_{lj,k_j+1}(\xi_{1,1}, \ldots, \xi_{1,k_j+1}, \ldots, \xi_{\bar{p},1}, \ldots, \xi_{\bar{p},k_j+1}) \\ \vdots \\ G_{lj,n_j}(\xi) \end{pmatrix}, \quad l = 1, \ldots, m.$$