Wealth inequality in the minority game

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To demonstrate the usefulness of physical approaches for the study of realistic economic systems, we investigate the inequality of players' wealth in one of the most extensively studied econophysical models, namely, the minority game (MG). We gauge the wealth inequality of players in the MG by a well-known measure in economics known as the modified Gini index. From our numerical results, we conclude that the wealth inequality in the MG is very severe near the point of maximum cooperation among players, where the diversity of the strategy space is approximately equal to the number of strategies at play. In other words, the optimal cooperation between players comes hand in hand with severe wealth inequality. We also show that our numerical results in the asymmetric phase of the MG can be reproduced semianalytically using a replica method.

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I. INTRODUCTION

Econophysics is the study of economic systems by employing methods and tools developed in physics. Up to now, many economists have been worrying that econophysicists are just reinventing the wheel, while many physicists are studying properties of toy economic models that are not directly relevant to economics [1]. In this paper, we investigate the inequality of wealth in a simple-minded econophysical model known as the minority game (MG) using the so-called replica trick [2–4]. By doing so, we hope to make a small step forward in the application of physical methods when studying real economic systems.

The MG is a simple-minded model of a complex adaptive system which captures the cooperative behavior of selfish players in a real market. In this game, N players have to choose one of the two possible alternatives in each turn based only on the minority sides in the previous M turns. The wealth of those who end up in the minority side will be increased by one while the wealth of the others will be reduced by one. To aid the players in making their choice, each of them is randomly and independently assigned S deterministic strategies once and for all when the game begins. Each deterministic strategy is nothing but a map from the set of all possible histories (a string of the minority side of the previous M turns) to the set of the two possible alternatives. All players make their choices according to their current best strategies [5,6]. In the MG, the complexity of the system is usually indicated by the control parameter $\alpha \equiv 2^{M+1}/NS$ which is the ratio of the size of the strategy space to the size of strategies at play [6–8].

Clearly, the mean attendance of either choice is $N/2$ as the game is symmetrical for both choices. In contrast, the variance of this probability, which is conventionally denoted by $\sigma^2(A)$, is highly nontrivial. It attains a very small value when $\alpha \approx 1$, indicating that the players are cooperating [7]. That is why previous studies of the MG and its variants [9–11] focus mainly on the study of $\sigma^2(A)$.

Since the strategies are assigned once and for all to each player, it is possible that some poorly-performing players are somehow forced to cooperate with some well-performing peers. Therefore, it makes sense to study the inequality of wealth in MG in detail.

In Sec. II, we introduce a common method that measures wealth inequality in economics known as the modified Gini index. We then study the Gini index in the MG numerically in Sec. III. Our numerical simulation shows that both the maximal cooperation point and the point of maximum wealth inequality occur around $2^{M+1} \approx NS$. This confirms our suspicion that the apparent cooperation of players shown in the $\sigma^2(A)$ does not tell us the complete story. In fact, we are able to explain the trend of a modified Gini index qualitatively using the crowd-anticrowd theory [12–14]. In particular, we find that the cooperation comes along with wealth inequality partially because poorly-performing players cannot change their strategies in the MG. In this way, we show that the crowd-anticrowd theory is not only able to explain $\sigma^2(A)$, but also explains other features of other quantities in the MG. In Sec. IV, we try to reproduce our numerically simulated Gini index in the so-called asymmetric phase using the replica method. we recall that one has to average over the disorder variables in the conventional replica method; the direct application of the replica trick cannot provide the wealth distribution of players and thus the Gini index of the MG. Fortunately, a careful semianalytic application of the replica method can be used to reproduce the Gini index qualitatively as a function of α. Finally, we wrap up by giving a brief summary of our work in Sec. V.

II. GINI INDEX WITH NEGATIVE WEALTH

In order to measure the inequality of wealth among players in the MG qualitatively, we follow our economics colleagues employing the so-called Gini index. In the original definition, the Gini index G_0 in a population is the mean of the absolute differences between the wealth all possible pairs of players [15]. That is to say,

$$G_0 = 1 - \frac{1}{N}\sum_{j=1}^{N}[1+2(N-j)]g_j. \quad (1)$$

In the above equation, N is the number of players in the population, g_j is the wealth earned by a player divided by the total wealth in the population. and the g_j's are ranked in ascending order, i.e., $g_1 \leqslant g_2 \leqslant \cdots \leqslant g_N$. Clearly, G_0 ranges from 0 to 1. The larger G_0, the more serious the wealth inequality. If $G_0=0$, the players' wealth is uniformly distributed. If $G_0 \simeq 1$, one of the players possesses the total wealth of the population and the wealth inequality is served. However, Eq. (1) is only applicable in two cases: (1) all players have positive wealth; or (2) all players have negative wealth. Since players in the MG may have positive or negative wealth, we cannot use G_0, in general, to measure wealth inequality. We employ an extension of G_0, introduced by Chen *et al.*, known as the modified Gini index G [16–18], is given by

$$G = \frac{\frac{2}{N}\sum_{j=1}^{N} j g_j - \frac{N+1}{N}}{1 + \frac{2}{N}\sum_{j=1}^{k} j g_j + \frac{1}{N}\sum_{j=1}^{k} g_j\left[\frac{\sum_{j=1}^{k} g_j}{g_{k+1}} - (1+2k)\right]}, \quad (2)$$

where k is defined in such a way that $\sum_{j=1}^{k} g_j < 0$ and $\sum_{j=1}^{k+1} g_j > 0$. For simplicity, we refer to the modified Gini index G as the Gini index from now on. Just like the original Gini index G_0, the modified Gini index G measures the normalized wealth inequality of players. Again, G ranges from 0 to 1. The larger the value of G, the more serious the wealth inequality. When all players are equally wealthy, i.e., $k=0$, the term $\sum_{j=1}^{k} j g_j$ vanishes and G becomes zero. In contrast, if the total wealth of the system is owned by a single player, i.e., $g_N=1$ and the term $\sum_{j=1}^{N-1} j g_j = 0$, then G attains a value of one as $N \to \infty$. Also, G is reduced to the original Gini index G_0 when all players have positive wealth or all players have negative wealth. Moreover, G is unchanged if the wealth of each player is multiplied by a nonzero constant.

III. NUMERICAL RESULTS AND QUALITATIVE EXPLANATIONS

In this section, we investigate the wealth inequality of the players in the MG. Since we are only interested in studying the generic properties of the Gini index, we average the Gini index over $N_r=500$ independent runs. Because G measures the normalized wealth distribution of players rather than simply the first and second moments of this distribution, the time of convergence of Gini index G is much longer than that of the variance of attendance and it differs for different initial configurations of the system. So we employ an adaptive scheme to check for system equilibration before taking any measurement. Specifically, in each run, we record the time series of G until the absolute difference of G between 10000 successive steps is less than 10^{-6}. Then, we obtain the equilibrated value of G by using finite size scaling. Finally, we take G to be the average over 50 measurements each separated by 1000 steps. G is a measure of the normalized wealth distribution. From our numerical simulation, G equilibrates logarithmically and slowly although the wealth of players is decreasing in each turn. We will explain the reason for convergence of G in detail at the end of this section. We have performed numerical simulations for the cases where players draw their strategies from full strategy space and reduced strategy space [5,8] respectively. The Gini indices obtained in these two cases are very similar. Since the analytical investigation performed in Sec. IV is simpler if we focus on reduced strategy space, we present the numerical results based on reduced strategy space here for consistency.

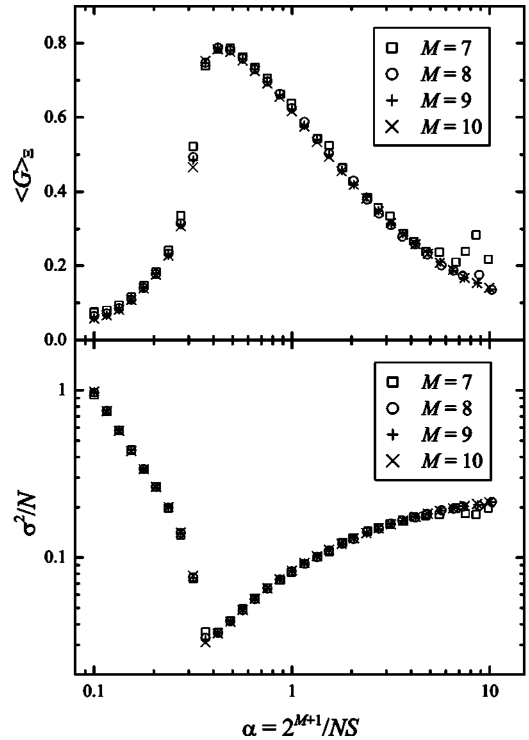

FIG. 1. The Gini index $\langle G\rangle_\Xi$ and the variance of attendance per player σ^2/N averaged over the initial configuration versus α in the MG with $S=2$ for different values of M. The error bar of $\langle G\rangle_\Xi$ is of order of at most 10^{-3}. The small bump around $\alpha=10$ for $M=7$ is due to finite size effect.

Let us study the Gini index averaged over the initial conditions $\langle G\rangle_\Xi$ versus the control parameter α as shown in Fig. 1. (Note that we use $\langle\cdot\rangle_\Xi$ to denote the average over the initial configuration of the system.) Our numerical results show that the curves of $\langle G\rangle_\Xi$ for different M coincide. which means that the Gini index $\langle G\rangle_\Xi$, just like the variance of attendance, depends only on the control parameter α in the MG.

We now move on to discuss the properties of the Gini index $\langle G\rangle_\Xi$ as a function of α in detail. Figure 1 shows that the Gini index $\langle G\rangle_\Xi$ is small when $\alpha \to 0$. In other words, the wealth of all players is roughly the same in such a case. In fact, the small value of $\langle G\rangle_\Xi$ can be explained by the crowd-anticrowd theory [12–14] as follows. In the small α regime, players are likely to have at least one high ranking strategy at each instance, as each player possesses a relatively large portion of strategies of the reduced strategy space. Thus, most of the players are using the crowd of high ranking strategies, i.e., those high ranking strategies are overcrowded. Due to the overcrowding of strategies, each strategy alternatively wins and loses one virtual score repeatedly when the same history appears, under the period-two dynamics [7,19]. That is to say, each strategy has approximately the same probability to win for any history. Therefore, all players have roughly the same amount of wealth and this leads to a small Gini index $\langle G \rangle_\Xi$.

As the control parameter α increases, the Gini index $\langle G \rangle_\Xi$ rises rapidly and subsequently attains its maximum value when the number of strategies at play is approximately equal to the reduced strategy space size. To explain this, we recall that the aim of each player in the MG is to maximize one's own wealth, which is achieved under the maximization of the global profit [8]. Subsequently, the attendance of each choice always tends to $\lfloor N/2 \rfloor$ upon equilibration for all values of α since the two alternatives are symmetric in the MG. That is to say, the system always "distributes" approximately the same amount of wealth to the population in each turn regardless of the value of α. Moreover, whenever $\alpha \approx \alpha_c$, unlike in the cases of symmetric and asymmetric phases of the MG, it is not uncommon for a player to hold only low ranking strategies since the number of strategies at play and the reduced strategy space size are of the same order. Consequently, a significant number of players are forced to use the crowd of low ranking strategies and keep on losing. On the other hand, those players picking the crowd of high ranking strategies have a higher winning probability and keep on using those strategies. Note that the ranking of the strategies is almost unchanged when $\alpha \approx \alpha_c$ [12–14]. As a result, the wealth distribution of players would become relatively diverse and the Gini index $\langle G \rangle_\Xi$ of the population attains its maximum value when $\alpha \to \alpha_c$.

Actually, the increase in the Gini index when $\alpha \to \alpha_c^+$ can be justified by the frozen probability of the MG. We recall that in the MG a player employs the virtual score system to determine which strategy to use in the next time step. In the asymmetric phase, the probability that a strategy assigned to a player has a virtual score asymptotically higher than all the other strategies assigned to the same player increases as α decreases. Some players end up using only one strategy after the system equilibrates, they are regard as frozen players. The frozen probability indicates the number of frozen players. A small frozen probability, i.e., most players in the game keep changing their best strategies, implies that only a few player will keep on winning or keep on losing all the time and the Gini index should be low. On the other hand, a high frozen probability may indicate that while some frozen players are using strategies that win most of time, the best performing strategies for the other frozen players are losing badly. Thus, there is a wide spread in wealth distribution of players. The Gini index should be high in this case. The frozen probability follows the same trend of Gini index as $\alpha \to \alpha_c^+$, which further supports the validity of the result of the Gini index. Moreover, when $\alpha \approx \alpha_c$, it is likely that those frozen players which form the majority of crowds and anticrowds in the game use anticorrelated strategy pairs, resulting in effective crowd-anticrowd cancellation between frozen players. Also, those frozen players who picked the anticorrelated strategy pairs keep winning or keep losing throughout the game.

After attaining the maximum value, the Gini index $\langle G \rangle_\Xi$ decreases and gradually tends to zero when the control parameter α further increases. According to crowd-anticrowd theory [12–14], it is because most of the strategies at play are uncorrelated to each other when the strategy space size becomes much larger than the number of strategies at play. Therefore, it is as if each player is making random choices in the game when α is large. Hence, the winning probability of all strategies is roughly the same. As a result, the Gini index $\langle G \rangle_\Xi$ of the population is small in this regime.

As we have reasoned above, the winning probability of each individual player is steady after equilibration of the system. Since the wealth distribution depends solely on the winning probabilities of individual players, the g_j's, and hence the Gini index G, converge over a sufficiently long time. Moreover, it is easy to check that the g_j's converge logarithmically. Therefore, the equilibration time for G is much longer than that of $\sigma^2(A)$.

IV. SEMIANALYTICAL STUDY OF THE GINI INDEX IN MG USING THE REPLICA TRICK

A. Methodology

In the previous section, we have explained the wealth inequality of the players in the MG qualitatively. In fact, the system of the MG can be described as a disorder spin system [2,3] since the dynamics of the MG indeed minimizes a global function related to market predictability. In this section, we calculate the Gini index G of the population in MG semianalytically by mapping the MG to a spin glass. As we shall see, this approach works well whenever $\alpha > \alpha_c$.

Let us start to link the MG, a repeated game with N players, to the spin glass. In this formalism, every player has to choose one out of two actions ± 1 corresponding to the two alternatives at each time step. We denote the action of the ith player at time t by $c_i(t)$. After all players have chosen their actions, those players choosing the minority action win and gain one unit of wealth while all the others lose one. In the MG, the only public information available to the players is the so-called history, which is the string of the minority action of the last M time steps. Namely, the history is a string $[\Pi(t-M),\ldots,\Pi(t-1)]$, where $\Pi(t)$ denotes the minority action at time t. For convenience, we label the history by an index μ as follows:

$$\mu(t) = \Pi(t-M) \times 2^{M-1} + \Pi(t-M-1) \times 2^{M-2} + \cdots + \Pi(t-1). \tag{3}$$

At the beginning of the game, each player picks once and for all S strategies randomly from the strategy space. In fact, a strategy specifies an action $a_{s,i}^\mu$ taken by the ith player for all possible histories $\mu=1,\ldots,2^M$. In the MG, agents make use of the virtual score, i.e., the hypothetical profit for using a strategy throughout the game, to evaluate the performance of a strategy. To guess the next global minority action, each player uses their own current best strategy which is the strategy with the highest virtual score at that moment. Assuming each player has $S=2$ strategies which are labeled by "+" and "−," we define the disorder variables $\{\omega_i^\mu, \xi_i^\mu\}$ as

$$w_i^\mu = \frac{a_{+,i}^\mu + a_{-,i}^\mu}{2}, \quad \xi_i^\mu = \frac{a_{+,i}^\mu - a_{-,i}^\mu}{2}. \tag{4}$$

Here we use the spin variable $s_i(t)=\pm 1$ to denote the strategy used by the ith player at time t. Thus the action of this player is given by

$$c_i(t) = \omega_i^{\mu(t)} + s_i(t)\xi_i^{\mu(t)}. \tag{5}$$

With the above formalism, we can employ a statistical tool called the replica trick [3,4] to study the stationary state properties of the MG by solving the ground state of the Hamiltonian H:

$$H\{\vec{m}\} = \overline{\Omega^2} + 2\sum_{i=1}^{N} \overline{\Omega\xi_i} m_i + \sum_{i,j}^{N} \overline{\xi_i\xi_j} m_i m_j, \tag{6}$$

where $m_i \equiv \langle s_i(t)\rangle$ and $\Omega^\mu = \Sigma_{i=1}^N \omega_i^\mu$. Note that $\overline{O}$ denotes the average over history μ and $\langle\cdot\rangle$ denotes the average over time t. In other words, our aim is to find the minimum of $H\{\vec{m}\}$ defined by

$$\min_{\vec{m}\in[-1,1]^N} H\{\vec{m}\} = -\lim_{\beta\to\infty} \frac{1}{\beta}\langle \ln Z(\beta)\rangle_\Xi, \tag{7}$$

where the partition function

$$Z(\beta) = \mathrm{Tr}_{\vec{m}} e^{-\beta H\{\vec{m}\}}. \tag{8}$$

Here, $\mathrm{Tr}_{\vec{m}}$ denotes the integral of $\vec{m}$ on $[-1,1]^N$, $\langle\cdot\rangle_\Xi$ denotes the average over the disorder variables $a_{s,i}^\mu$ (i.e., the quenched disorder Ξ of the system) and β stands for the inverse temperature. In fact, the ground state solution of H depends on the disorder variables. However, in the thermodynamic limit, the ground state of H has a unique solution for all quenched disorder. Thus, in the replica calculation, we seek for ground state solution of the Hamiltonian H on average of the quenched disorder. In order to evaluate $\langle \ln Z\rangle_\Xi$, we construct the partition function Z^n by studying n (a non-negative integer) replicas of the system with identical disorder variables $\{a_{s,i}^\mu\}$. Then, we perform a semianalytical continuation to extend this function for non-integer n. In this way, the average of $\ln Z$ over $\{a_{s,i}^\mu\}$ is reduced to

$$\langle \ln Z\rangle_\Xi = \lim_{n\to 0} \frac{1}{n} \ln\langle Z^n\rangle_\Xi. \tag{9}$$

We also define the free energy density $F_\beta(\hat{Q},\hat{r})$ by

$$\langle Z^n\rangle_\Xi = \int d\hat{r} \int d\hat{Q}\, \exp[-\beta n N F_\beta(\hat{Q},\hat{r})], \tag{10}$$

where $Q_{a,b}=(1/N)\Sigma_i^N m_i^a m_i^b$ is the overlap matrix and $r_{a,b}$ are the associated Lagrange multipliers. Hence, we can find the stationary state solution of H in the thermodynamic limit $N\to\infty$ by finding the minima of $F_\beta(\hat{Q},\hat{r})$ as

$$\lim_{N\to\infty} \min_{\vec{m}\in[1,-1]^N} \frac{H\{m\}}{N} \approx \lim_{\beta\to\infty}\lim_{n\to 0} \min F_\beta(\hat{Q},\hat{r}). \tag{11}$$

In fact, we can find the minima in the replica symmetric (RS) ansatz by solving the saddle point equations [4,10]:

$$\frac{\partial F_\beta}{\partial r_{a,b}} = 0 \text{ and } \frac{\partial F_\beta}{\partial Q_{a,b}} = 0 \ \forall a,b. \tag{12}$$

In this ansatz, the matrices $\hat{r}$, $\hat{Q}$ corresponding to $\min F_\beta$, are assumed to be in the following form:

$$Q_{a,b} = \frac{1}{N}\sum_i^N m_i^a m_i^b = \begin{cases} q & \text{for } a\neq b, \\ Q & \text{for } a=b, \end{cases} \tag{13}$$

and

$$r_{a,b} = \begin{cases} 2r & \text{for } a\neq b, \\ R & \text{for } a=b, \end{cases} \tag{14}$$

for all $a,b=1,2,\ldots,n$. Therefore, using the *RS* ansatz, the minimum value of F_β in the $n\to 0$ limit is given by

$$\begin{aligned} F^{(\mathrm{RS})} &= \lim_{n\to 0}\min F_\beta(\hat{Q},\hat{r}) = \frac{\alpha}{2\beta}\ln\left[1+\frac{\beta}{\alpha}(Q-q)\right] \\ &\quad + \frac{\alpha(1+q)}{2[\alpha+\beta(Q-q)]} - \frac{1}{\beta}\int d\Phi(\lambda)\ln\left[\int_{-1}^{1} dm \right. \\ &\quad \left. \times\exp(-\beta V(m|\lambda))\right] + \frac{\alpha\beta}{2}(RQ-rq), \end{aligned} \tag{15}$$

where $\Phi(\lambda)$ is the normal distribution and the potential $V(m|\lambda)=-\sqrt{\alpha r}\lambda m+(\alpha\beta/2)(r-R)m^2$.

Using the saddle point equations, we arrive at [4,10]

$$\frac{\alpha}{\rho^2} = 2 - \sqrt{\frac{2}{\pi}}\frac{1}{\rho}e^{-\rho^2/2} - \left(1-\frac{1}{\rho^2}\right)\mathrm{erf}\left(\frac{\rho}{\sqrt{2}}\right), \tag{16}$$

where ρ is a disorder variable and depends on the control parameter α. The probability distribution of the "average action" of a player, m, is then given by [3]

$$P(m) = \frac{\phi(\rho)}{2}[\delta(m-1)+\delta(m+1)] + \frac{\rho}{\sqrt{2\pi}}e^{-(\rho m)^2/2}, \tag{17}$$

where $\phi(\rho)=1-\mathrm{erf}(\rho/\sqrt{2})$, $\delta(0)=1$ and $\delta(x)=0$ whenever $x\neq 0$. Note that Eqs. (16) and (17) are only valid for $\alpha>\alpha_c$. For $\alpha<\alpha_c$, the replica calculation cannot give correct predictions for the probability distribution of spin variable m because it is unable to reproduce the period-two dynamics of the system [4].

Our aim is to calculate the Gini index of the players in the MG using the replica trick. At first glance, one might argue that the distribution of g_i can be reproduced analytically using the replica trick. However, the replica trick can only generate the average gain of a group of players rather than the wealth of an individual player. This is why Challet did not compute the theoretical gain of individual players analytically by using the replica trick for the MG. In fact, he computes the gain semianalytically using the disorder spin variable m_i measured in the simulations instead [10,20]. To reproduce the wealth distribution of players, we need to know the actions of each individual player $s_i(t)$ at time t for each individual particular quenched disorder. However, $s_i(t)$ cannot be found by the replica trick. So we approximate $s_i(t)$ by the disorder spin variable m_i generated stochastically from the distribution $P(m)$ which is found by the replica trick. Then, the Gini index $G(\Xi)$ can be calculated from the wealth distribution of the players for that quenched disorder. As we are only interested in the generic properties of the Gini index, we calculate the Gini index averaged over quenched disorder $\langle G\rangle_\Xi$. This should be done by calculating the Gini index of each individual quenched disorder $G(\Xi)$ first and then taking average over all quenched disorders.

In practice, we perform the stochastic simulation to generate the wealth distribution of population in the MG for an individual quenched disorder in the following way. Before starting the simulation, the quenched disorder Ξ is formed by allowing each player to pick two strategies randomly from the reduced strategy space. Next, each player draws the spin variable m from the distribution $P(m)$ as shown in Eq. (17). Those players with $m=\pm 1$ are called frozen players because they keep on using a strategy throughout the game. Then, in each step of the game, players choose one of their own strategies according to their own spin variable m to guess the next global minority side. In practice, the strategy used by the ith player at time t, $\sigma_i(t)$, is chosen by calling a uniform random variate ζ on $[-1,1]$. Then we set $\sigma_i(t)=1$ if $m_i \geqslant \zeta$ and $\sigma_i(t)=-1$ otherwise. Therefore, for the history $\nu(t)$, the action of the ith player at time t can be written as

$$\chi_i(t) = \omega_i^{\nu(t)} + \sigma_i(t)\xi_i^{\nu(t)}. \tag{18}$$

Note that the history $\nu(t)$ is generated randomly at each time step t in our simulation. In addition, the difference in the numbers of players choosing the two alternatives at time t is given by

$$X(t) = \sum_{i=1}^{N} \chi_i(t). \tag{19}$$

So we obtain the minority side at time t

$$\Theta(t) = -\,\mathrm{sgn}(X(t)). \tag{20}$$

After determining the minority side, the wealth of the ith players, $w_i(t)$, is updated by

$$w_i(t+1) = w_i(t) + 2\delta(\chi_i(t) - \Theta(t)) - 1. \tag{21}$$

We repeat the above algorithm N_s times for the system to equilibrate. After the equilibration, we measure on the Gini index of the population for the quenched disorder Ξ using Eq. (2). Then we calculate the average Gini index for 500 independent runs. We denote the Gini index calculated by this algorithm with averaging over the quenched disorder by $\langle G\rangle_\Xi^{\mathrm{R}}$. In fact, we find that the average Gini index $\langle G\rangle_\Xi$ converges after $N_s=500P$ iterations, where $P=2^M$ is the number of possible histories.

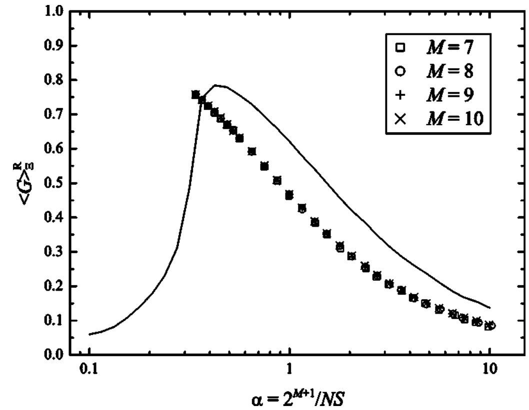

FIG. 2. The average Gini index found in stochastic simulation using random history $\langle G\rangle_\Xi^{\mathrm{R}}$ versus the control parameter α in the asymmetric phase of the MG with $N_s=500P$ and $S=2$ for different M. For comparison purpose, the solid line indicates the corresponding numerical results in the MG with $M=9$.

B. Semianalytical results using stochastic simulation

Figure 2 gives the Gini index obtained from semianalytical calculation of $\langle G\rangle_\Xi^{\mathrm{R}}$ versus the control parameter α for MG with $\alpha > \alpha_c$. We find that the trend of the curves of $\langle G\rangle_\Xi^{\mathrm{R}}$ agrees with the numerical findings. This implies that we have successfully reproduced the numerical results of the Gini index in the asymmetric phase of the MG by using the replica method. However, the curves of $\langle G\rangle_\Xi^{\mathrm{R}}$ are systematically lower than those from numerical simulation. This is because the coupling between the actions of players and the dynamics of the system is completely ignored in our stochastic simulation as the actions of the players depend only on the spin variable m. Consequently, the global cooperation among the players is suppressed in our semianalytical calculation. Hence, the wealth distribution of players is less diverse which results in underestimation of the Gini index in the MG.

To make our semianalytical calculation more "realistic," we allow the history $\nu(t)$ to be updated sequentially by

$$\nu(t) = [2\nu(t-1) + \Theta(t)] \bmod P, \tag{22}$$

and we denote the Gini index averaged over the quenched disorder calculated in this approach by $\langle G\rangle_\Xi^{\mathrm{S}}$. Note that $\langle G\rangle_\Xi^{\mathrm{R}}$ and $\langle G\rangle_\Xi^{\mathrm{S}}$ are calculated using the same algorithm except that the history is updated in a different way. Figure 3 shows the Gini index $\langle G\rangle_\Xi^{\mathrm{S}}$ versus the control parameter α in the MG. We observe that the values of $\langle G\rangle_\Xi^{\mathrm{S}}$ agree well with the numerical results when α is large. According to crowd-anticrowd theory, if α is large, most strategies of the players are uncorrelated to each other due to the undersampling of

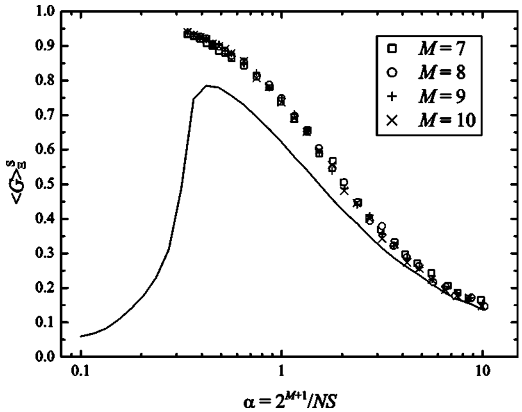

FIG. 3. The average Gini index found in stochastic simulation using sequential history $\langle G\rangle^{S}_{\Xi}$ versus the control parameter α in the asymmetric phase of MG with $N_s=500P$ and $S=2$ for different M. For comparison purpose, the solid line indicates the corresponding numerical results in MG with $M=9$.

the strategy space. Moreover, most of the strategies are used by either one or none of the players in the MG whenever $\alpha \to \infty$. Therefore, the cooperation between the players can be neglected for $\alpha \to \infty$. In addition, the probability of the occurrence of different histories is not the same in the MG when $\alpha \to \infty$ [21]. Indeed, these two conditions are satisfied in our stochastic simulation using the sequential history. So, the values of $\langle G\rangle^{S}_{\Xi}$ match the numerical estimates when α is large.

On the other hand, when α approaches α_c^+, the values of $\langle G\rangle^{S}_{\Xi}$ become larger than the numerical results. This discrepancy can be explained as follows. As mentioned in Sec. III, since there is effective crowd-anticrowd cancellation, the history in the MG becomes more uniform as α approaches α_c^+ [21]. In contrast, although players still have the same chance to pick anticorrelated pairs separately at the beginning of the game in our sequential simulation, the strategy actually used by each player at each turn is not determined by its virtual score, but a randomly assigned disorder spin variable m_i instead. Consequently, two players are less likely to be frozen on an anticorrelated strategy pair. This makes the crowd-anticrowd cancellation less effective among frozen players in our sequential simulation. So, the actions among these frozen players may give a strong bias in the output, especially for $\alpha \approx \alpha_c$, where frozen probability is highest. In turn, the history becomes much more nonuniform. This greatly increases the Gini index as some players have more chance to stay at the winning (or losing) side.

Finally, we remark that both $\langle G\rangle^{R}_{\Xi}$ and $\langle G\rangle^{S}_{\Xi}$ calculated by the stochastic simulation are independent of M. This is expected, since the results of the replica calculation do not depend explicitly on M.

V. CONCLUSION

In summary, we have investigated the inequality of wealth among players in the MG using the well-known measure in economics called the Gini index. In particular, our numerical findings show that the wealth inequality of players is very severe near the point of maximum global cooperation α_c. That is to say, in the minority game, global cooperation comes hand in hand with uneven distribution of players' wealth. Specifically, a significant number of players are forced to use the low ranking strategies and cooperate with those players using the high ranking strategies since the number of strategies at play and the reduced strategy space size are of the same order whenever $\alpha \to \alpha_c$. In this respect, we have showed that the crowd-anticrowd theory offers a simple and effective platform to study the wealth inequality in the MG.

In addition, we have studied the Gini index semianalytically by mapping the system of the MG to a spin glass. With this formalism, we semianalytically reproduce our numerically simulated Gini index in the asymmetric phase of MG by investigating the stationary state properties of MG using the replica trick.

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