An Analytical Comparison of Partial Power-Feedback Designs for MIMO Block Fading Channels

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***Abstract*—It has been shown that with perfect feedback (CSIT), the optimal multiple input/multiple output (MIMO) transmission strategy is a cascade of channel encoder banks, *power control matrix*, and *eigen-beamforming* matrix. However, the feedback capacity requirement for perfect CSIT is $2n_T \times n_R$, which is not scalable with respect to n_T or n_R. In this letter, we shall compare the performance of two levels of partial power-feedback strategies, namely, the *scalar symmetric feedback* and the *vector feedback*, for MIMO block fading channels. Unlike quasi-static fading, variable rate encoding is not needed for block fading channels to achieve the optimal channel capacity.**

***Index Terms*—Multiple input/multiple output (MIMO) capacity with partial feedback, multiple input/multiple output (MIMO) feedback.**

I. Introduction

MULTIPLE-antenna technologies played an important role in 3G+ wireless systems such as high-speed data packet access (HSDPA) in universal mobile telcommunications systems (UMTS) because it has the potential of achieving extraordinary bit rates [1]. When there is no feedback, the optimal multiple input/multiple output (MIMO) transmission strategy is shown [2] to consist of a bank of n_T independent channel encoders, with uniform power allocation across the n_T antenna. On the other hand, the optimal MIMO transmission strategy with perfect CSIT is shown to be a cascade of channel encoder bank, *power control matrix*, as well as an *eigen-beamforming* matrix. However, the full feedback cost is quite high, and it is desirable to have some partial-feedback schemes to bridge the performance gap.

In this letter, we shall consider a MIMO link with partial feedback on the instantaneous characterization of the block fading channel where a coding frame spans over multiple realizations of fading blocks. Two cases of partial feedback, namely, the *scalar symmetric feedback* and the *vector feedback*, are considered. With scalar symmetric feedback, we assume the feedback channel carries a single scalar parameter. With vector feedback, we assume the feedback channel carries n_T parameters. Hence, the feedback loading of the latter case is n_T times that of the former case. The gain of the scalar feedback is contributed solely by temporal power waterfilling, while the gain of the vector feedback is contributed by both power waterfilling and power distribution across individual transmit antennas. Our focus is to investigate and compare the effectiveness of the partial feedback schemes in different n_T, n_R, and signal-to-noise ratio (SNR) regions.

This letter is organized as follows. In Section II, we shall outline the system model and the feedback model. In Section III, the optimal scalar feedback and the optimal vector feedback are derived. In Section IV, we present the numerical results and discuss the efficiency of feedback link capacity at various SNR regions, and various n_T and n_R. Finally, we conclude with a brief summary of results in Section V.

II. System Model

The received symbol $\mathbf{Y}$ (dimension $n_R \times 1$) is given by

$$\mathbf{Y} = \mathbf{S}\mathbf{X} + \mathbf{Z} \tag{1}$$

where $\mathbf{X}$ is the $n_T \times 1$ transmitted symbol, $\mathbf{Z}$ is the $n_R \times 1$ white Gaussian channel noise with covariance $\sigma_z^2 \mathbf{I}_{n_R}$, and $\mathbf{S}$ is the $n_R \times n_T$ channel matrix.

The channel capacity is shown to be

$$C_{\text{fwd}} = \max_{p(\mathbf{X}|\mathbf{U})} \mathcal{E}\left[I(\mathbf{X};\mathbf{Y}|\mathbf{S})\right] \tag{2}$$

where $\mathcal{E}[.]$ denotes expectation over the channel state $\mathbf{S}$, $p(\mathbf{X}|\mathbf{U})$ denotes the input distribution, and $\mathbf{U}$ denotes the feedback CSIT.

The capacity-achieving distribution is well known to be circularly symmetric Gaussian with input covariance given by

$$\mathcal{E}[\mathbf{X}\mathbf{X}^*] = \mathbf{W}\mathbf{\Psi}_\rho\mathbf{W}^* \tag{3}$$

where $\mathbf{W}$ is the $n_T \times n_T$ unitary beamforming matrix, and

$$\mathbf{\Psi}_\rho = \begin{pmatrix} \rho_1 & 0 & 0 \\ 0 & . & 0 \\ & & \rho_{n_T} \end{pmatrix}$$

is the power control matrix. In general, both the beamforming matrix and the power control matrix are functions of CSIT $\mathbf{U}$. Hence, the transmitted symbol $\mathbf{X}$ could be expressed as

$$\mathbf{X} = \mathbf{W}\sqrt{\mathbf{\Psi}_\rho}\begin{pmatrix} \sqrt{\rho_1} & 0 & 0 \\ 0 & . & 0 \\ & & \sqrt{\rho_{n_T}} \end{pmatrix}\begin{pmatrix} T_1 \\ . \\ T_{n_T} \end{pmatrix} \tag{4}$$

where

$$\mathbf{T} = \begin{pmatrix} T_1 \\ . \\ T_{n_T} \end{pmatrix}$$

is the $n_T \times 1$ independent, identically distributed (i.i.d.) Gaussian inputs (with covariance $\mathbf{I}_{n_T}$) from the n_T bank of channel encoders.

Hence, the general MIMO transmission strategy with feedback is given by a bank of n_T independent channel encoders,

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cascaded with a power control matrix and a beamforming matrix. In other words, there is no need to do variable-rate encoding in block fading channels.

A. Partial Feedback Model

We constraint the transmission scheme to consist of an $n_T \times n_T$ adaptive power control matrix cascaded with a $n_T \times n_T$ fixed beamforming matrix. Hence, the power control matrix $\mathbf{\Psi}_\rho$ is a function of the CSIT U, while the beamforming matrix $\mathbf{W}$ is a constant. The problem is to derive the optimal partial-feedback schemes and the associated optimal forward-channel capacity.

Two cases of partial feedback are considered, namely, the scalar symmetric feedback and vector feedback. With scalar symmetric feedback, the CSIT $U(\mathbf{V})$ is a scalar analytical function symmetric with respect to the columns of $\mathbf{V}$. That is

$$U([\mathbf{V}_1, \ldots, \mathbf{V}_{n_T}]) = U\left(\left[\mathbf{V}_{\pi(1)}, \ldots, \mathbf{V}_{\pi(n_T)}\right]\right) \quad (5)$$

for any permutation $\pi(i)$.

With vector feedback, the CSIT $\mathbf{U}(\mathbf{V}) = [U_1(\mathbf{V}), \ldots, U_{n_T}(\mathbf{V})]$ is a $1 \times n_T$ vector function of CSIR $\mathbf{V}$.

III. Optimal Forward-Channel Capacity

From (2), the MIMO link capacity is given by

$$C_{\text{fwd}} = \mathcal{E}_{\mathbf{S}}\left[\log_2\left|\mathbf{I}_{n_R} + \frac{1}{\sigma_z^2}\mathbf{S}\mathbf{W}\mathbf{\Psi}_\rho\mathbf{W}^*\mathbf{S}^*\right|\right]. \quad (6)$$

A. Scalar Symmetric Feedback

When we have scalar symmetric feedback, all the n_T diagonal elements $\{\rho_1, \ldots, \rho_{n_T}\}$ of the power control matrix $\mathbf{\Psi}_\rho$ is a continuous function of the scalar feedback, U. We observe that for any scalar symmetric feedback, U, the optimal power control matrix has the general form given by

$$\mathbf{\Psi}_\rho = \frac{U}{n_T}\mathbf{I}_{n_T} \quad (7)$$

where $\{b_1, \ldots, b_{n_T}\}$ are constants independent of the feedback and $\sum_{t=1}^{n_T} b_t = 1$. Hence, the remaining optimization problem is given by the following.

Problem 1: Find U such that the function is maximized

$$f(U) = \mathcal{E}\left[\log_2\left|\mathbf{I}_{n_R} + \frac{1}{n_T\left(\sigma_z^2\right)}U\mathbf{S}\mathbf{S}^*\right|\right] - \lambda\mathcal{E}[U]. \quad (8)$$

Express $\mathbf{S}\mathbf{S}^*$ as $\mathbf{\Gamma}\mathbf{S}_v\mathbf{\Gamma}^*$ by singular value decomposition, where $\mathbf{\Gamma}$ is the eigenvector matrix and $\mathbf{S}_v$ is the diagonal eigenvalue matrix. $f(U)$ could be simplified as

$$f(U) = \mathcal{E}\left[\sum_{r=1}^{n_R}\log_2\left(1 + \frac{1}{n_T\left(\sigma_z^2\right)}U\delta_r\right)\right] - \lambda\mathcal{E}[U] \quad (9)$$

where δ_r is the rth eigenvalue of $\mathbf{S}\mathbf{S}^*$.

The solution of the above optimization is given by $df/dU = 0$, which reduces to

$$\sum_{r=1}^{n_R}\frac{\dfrac{\delta_r}{\left(n_T\left(\sigma_z^2\right)\right.}}{\dfrac{1+\delta_r}{\left(n_T\left(\sigma_z^2\right)\right.}} = \lambda. \quad (10)$$

B. Vector Feedback

Without loss of generality, we could assume the tth diagonal element of $\mathbf{\Psi}_\rho$ is associated with the tth feedback CSIT, U_t. The channel capacity is given by

$$C_{\text{fwd}} = \mathcal{E}\left[\max_{\rho_1,\ldots,\rho_{n_T}}\log_2\left|\mathbf{I}_{n_R} + \frac{1}{\sigma_z^2}\mathbf{S}'\mathbf{\Psi}_\rho\mathbf{S}'^*\right|\right] \quad (11)$$

where $\mathbf{S}' = \mathbf{S}\mathbf{W}$ has the same distribution as $\mathbf{S}$ because $\mathbf{W}$ is a fixed and unitary matrix. Hence, without loss of generality, we could assume the fixed beamforming matrix to be identity matrix.

Including the average transmit power constraint, $\mathcal{E}[tr\mathbf{\Psi}_\rho] = \mathcal{E}[\sum_{t=1}^{n_T}\rho_t] \leq P$, the optimization problem is given by the following.

Problem 2: Given the CSIR, $\mathbf{S}$, find the optimal power allocation $\{\rho_1, \ldots, \rho_{n_T}\}$ so that $f(\rho_1, \ldots, \rho_{n_T}) = \log_2|\mathbf{I}_{n_R} + (1/\sigma_z^2)\mathbf{S}\mathbf{\Psi}_\rho\mathbf{S}^*| - \lambda tr\mathbf{\Psi}_\rho$ is maximized.

Observe that

$$df = tr\left\{\left[\mathbf{S}^*\left(\mathbf{I}_{n_R} + \frac{1}{\sigma_z^2}\mathbf{S}\mathbf{\Psi}_\rho\mathbf{S}^*\right)^{-1}\mathbf{S} - \lambda\mathbf{I}_{n_T}\right]d\mathbf{\Psi}_\rho\right\}. \quad (12)$$

The necessary condition for the optimal point $\mathbf{\Psi}_\rho$ that optimizes f is given by $df = 0$ for all $d\mathbf{\Psi}_\rho$. This is equivalent to

$$\text{Diag}\left[\mathbf{S}^*\left(\mathbf{I}_{n_R} + \frac{1}{\sigma_z^2}\mathbf{S}\mathbf{\Psi}_\rho\mathbf{S}^*\right)^{-1}\mathbf{S} - \lambda\mathbf{I}_{n_T}\right] = \mathbf{0}. \quad (13)$$

IV. Results and Discussions

A. Performance of MIMO Link With $n_T = n_R$

We consider the case of $n_T = n_R$ first. Fig. 1(a) and (b) illustrates the 2×2 and 4×4 forward MIMO channel capacity versus the average forward SNR (-10 to 10 dB) with no feedback, scalar symmetric feedback, vector feedback, and full feedback. In the low SNR region (SNR $= -2$ dB), there is a significant SNR gain of around 3 dB in channel capacity between full feedback and no feedback for both 2×2 and 4×4 systems. The gains of scalar feedback and vector feedback versus no feedback for the 2×2 system are given by 1 and 2 dB, respectively. The corresponding gains for the 4×4 system is given by 0.5 and 1 dB, respectively. In other words, the vector feedback realized about 67% of the ideal feedback gain, while scalar feedback realized about 33% of the ideal feedback gain in the low SNR region.

On the other hand, the effectiveness of both the scalar feedback and the vector feedback is reduced at the high SNR region (6 dB). The results demonstrate that temporal power waterfilling gain is more effective across low SNR regions. This is reasonable, because when the average SNR is large, the penalty of transmitting power less efficiently is small, compared with the case when the average SNR is small. Furthermore, both scalar feedback and vector feedback offer similar SNR gains relative to the performance with no feedback.

B. Performance of MIMO Link With $n_T > n_R$

We consider $n_T = 4$ and $n_R = 1, 2$. Fig. 2(a) and (b) illustrate the forward-channel capacity versus the average forward SNR with no feedback, scalar feedback, vector feedback, and

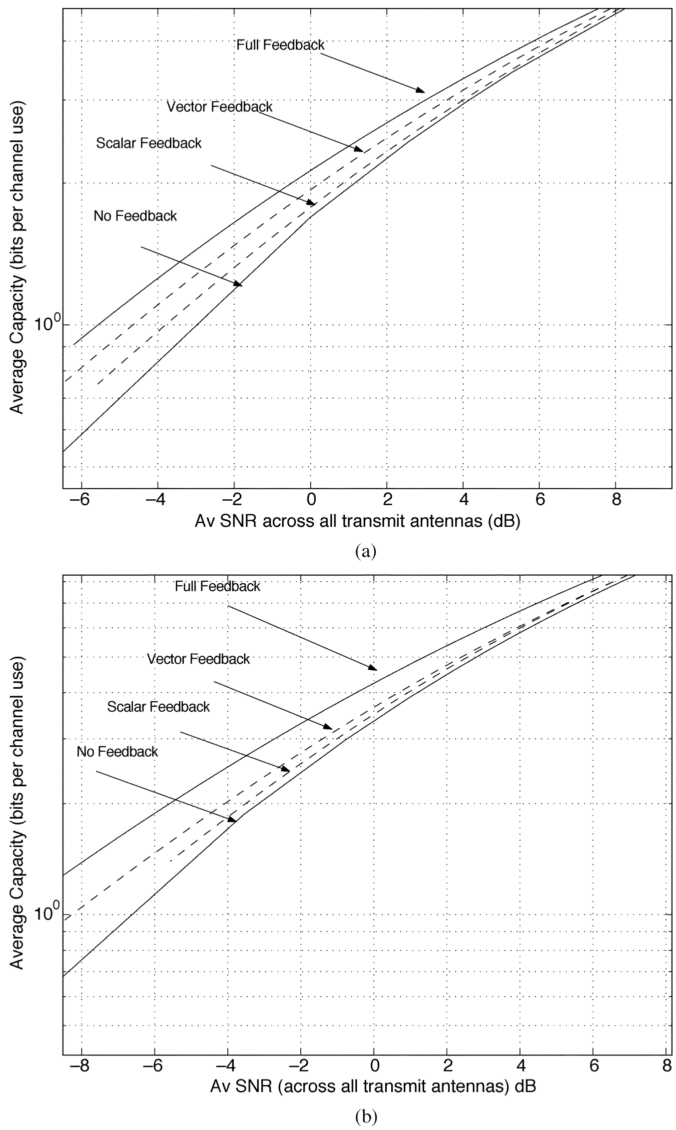

Fig. 1. Partial feedback performance of MIMO systems, illustration of waterfilling gain. (a) 2 × 2 performance. (b) 4 × 4 performance.

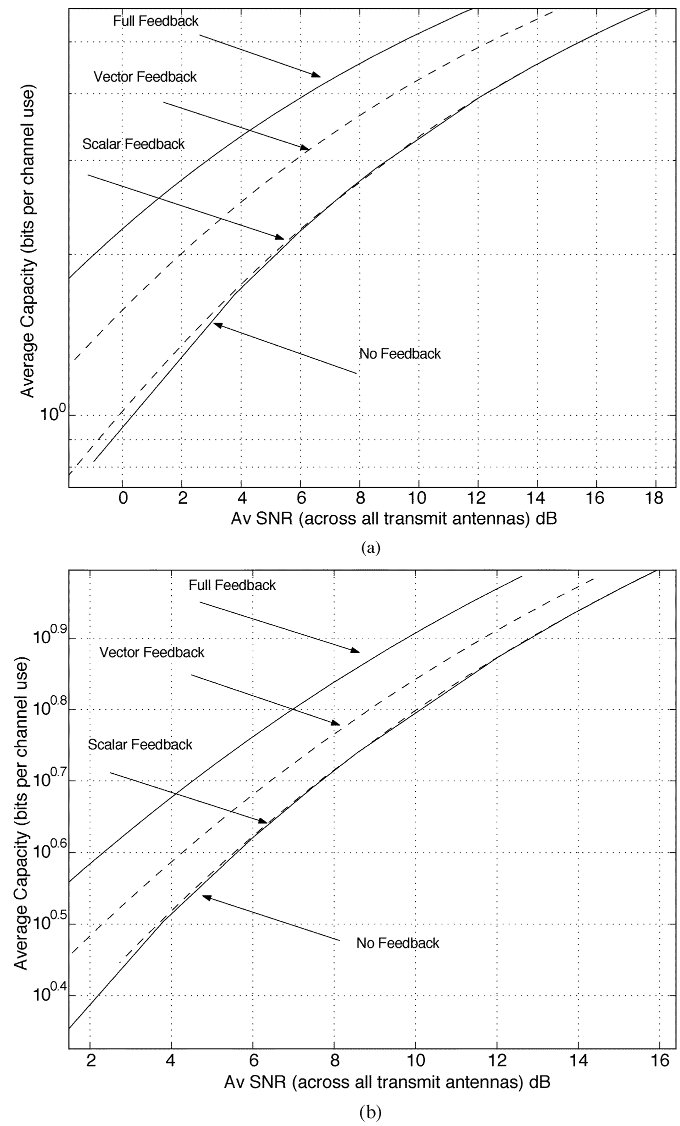

Fig. 2. Partial feedback performance for 4 × 1 and 4 × 2 systems, illustration of power distribution gain. (a) 4 × 1 system. (b) 4 × 2 system.

full feedback for 4 × 1 and 4 × 2 systems, respectively. We observe a significant gain in all SNR regions. For example, when SNR = 12 dB, there are significant SNR gains of 6 and 3 dB between the full feedback and no feedback for the 4 × 1 and 4 × 2 systems, respectively. These results demonstrate that the feedback performance as a result of transmission power distribution across active eigenchannels is very effective across all SNR regions.

Note that there are also significant SNR gains for vector feedback of 3 and 1 dB at SNR = 12 dB for the 4 × 1 and 4 × 2 systems, respectively. The corresponding SNR gains of scalar feedback are both less than 0.1 dB for the 4 × 1 and 4 × 2 systems, which are very insignificant. This is because with symmetric scalar feedback, no gain in power distribution across eigenchannels could be realized, and therefore, the performance gain is entirely contributed by power temporal waterfilling, which is insignificant at large SNR.

V. Conclusion

In this letter, we compare the performance of two partial power-feedback schemes, namely, the scalar symmetric feedback and the vector feedback. In general, the performance of feedback is contributed by: 1) temporal power waterfilling; and 2) power distribution across active eigenchannels. The former factor enhances the forward-channel capacity more effectively in the low SNR region, compared with the high SNR region. The latter factor enhances the forward-channel capacity effectively in all SNR regions when $n_T > n_R$. Finally, we have shown that the gain of scalar feedback vanishes at the large SNR region.

References

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