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Residual Julia Sets of Meromorphic Functions

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Dedicated to the memory of Prof. I.N. Baker

Abstract

In this paper, we study the residual Julia sets of meromorphic functions. In fact, we prove that if a meromorphic function f belongs to the class $\mathbf{S}$ and its Julia set is locally connected, then the residual Julia set of f is empty if and only if its Fatou set $F(f)$ has a completely invariant component or consists of only two components. We also show that if f is a meromorphic function which is not of the form $\alpha + (z-\alpha)^{-k}e^{g(z)}$, where k is a natural number, α is a complex number and g is an entire function, then f has buried components provided that f has no completely invariant components and its Julia set $J(f)$ is disconnected. Moreover, if $F(f)$ has an infinitely connected component, then the singleton buried components are dense in $J(f)$. This generalizes a result of Baker and Domínguez. Finally, we give some examples of meromorphic functions with buried points but without any buried components.

1. *Introduction*

Let $f\colon \mathbb{C} \to \mathbb{C}_\infty$ be a meromorphic function, where $\mathbb{C}_\infty$ is the Riemann sphere. Throughout this paper, we assume that f is neither constant nor a Möbius transformation. We shall denote the nth iterate of f by f^n. Then f^n is defined for all z in $\mathbb{C}$ except a countable set which consists of the poles of $f, f^2, \ldots, f^{n-1}$. If f is

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rational, then f has a meromorphic extension to $\mathbb{C}_\infty$ and we denote the extension again by f. In this case f is defined and meromorphic in $\mathbb{C}_\infty$. The Fatou set, $F(f)$ consists of point z in $\mathbb{C}_\infty$ at which the sequence $\{f^n\}_{n\in\mathbb{N}}$ is defined and normal in some neighbourhood of z. The Julia set, $J(f)$ is defined as the complement of $F(f)$ in $\mathbb{C}_\infty$. Note that if f is transcendental, then by definition ∞ is always in the Julia set of f. Clearly $F(f)$ is open and $J(f)$ is closed. It is also known that $J(f)$ is perfect (and therefore $J(f)$ is uncountable, and no point of $J(f)$ is isolated). A set E is said to be forward invariant (under f) if $f(E) \subset E$, and backward invariant (under f) if $f^{-1}(E) \subset E$. E is completely invariant if it is both forward and backward invariant. It is easy to check that both $F(f)$ and $J(f)$ are completely invariant. Moreover, it is well known that $J(f)$ is the smallest closed completely invariant set which contains at least three points. For the details of the general theory of iterations of meromorphic functions, we refer the reader to a series of papers by Baker, Kotus and Lü [**7**–**10**], the survey article [**13**], as well as the book [**26**]. For the iteration theory of rational functions, we refer to the books [**12**], [**27**] or [**37**].

In [**38**], Sullivan drew the attention to the dictionary of correspondence between the theory of Kleinian groups and iteration theory of rational functions. For example if Γ is a Kleinian group, $\Omega(\Gamma)$ and $\Lambda(\Gamma) = \mathbb{C}_\infty - \Omega(\Gamma)$ are its ordinary set and limit set respectively, then the limit set $\Lambda(\Gamma)$ is the smallest closed Γ invariant set with at least three points. The details of the dictionary can be found in [**32**, chapter 5] or [**24**, section 10].

It is known that $\partial\Omega_i \subset \Lambda(\Gamma)$ for any component Ω_i of the ordinary set $\Omega(\Gamma)$. Hence, $\bigcup_i \partial\Omega_i \subset \Lambda(\Gamma)$. In [**1**] and [**2**], Abikoff defined the residual limit set $\Lambda_r(\Gamma)$ to be

$$\Lambda_r(\Gamma) = \Lambda(\Gamma) - \bigcup_i \partial\Omega_i.$$

In other words, the residual limit set $\Lambda_r(\Gamma)$ is the subset of those points of the limit set $\Lambda(\Gamma)$ which do not lie on the boundary of any component of the ordinary set $\Omega(\Gamma)$. Abikoff also gave examples to show that $\Lambda_r(\Gamma)$ can be nonempty. In [**2**], Abikoff, obtained a complete classification of the limit sets of finitely generated Kleinian groups, namely the limit set is a union of quasi-circles, limit sets of degenerate groups, the residual limit set and a discrete set. He also proved that $\Lambda_r(\Gamma)$ is empty if and only if Γ is a function group (i.e. having an invariant component of the set of discontinuity) or has a subgroup of index 2 which is quasi-Fuchsian.

As an analogue of the residual limit set of a Kleinian group, Morosawa [**29**] defined the residual limit set $J_r(f)$ to be the subset of those points of the Julia set $J(f)$ which do not lie on the boundary of any component of the Fatou set $F(f)$. A point in $J_r(f)$ is called a buried point. A component of $J(f)$ which consists of buried points only is called a buried component. Buried points and buried components were first considered by McMullen in [**28**] and he gave the first example of a rational function with buried components. In [**33**], Qiao gave examples of transcendental entire function with nonempty Fatou set and residual Julia set.

In Section 2, we introduce the Makienko's conjecture on the residual Julia sets and discuss some results of Beardon [**11**], Morosawa [**29**, **31**] and Qiao [**34**, **35**] on conditions for a rational function to have buried points or buried components. Some of these results are extended to certain classes of the meromorphic functions (see Theorems 2·1 and 2·2). In particular, Theorem 2·2 is a generalization of a result obtained by Baker and Domínguez in [**4**]. In order to prove Theorems 2·1 and 2·2, it is useful to consider a more general class of functions, namely the meromorphic functions outside small sets. We briefly recall the basic facts of this class of functions in Section 3. We then prove Theorem 2·1 in Section 4 and Theorem 2·2 in Section 5. In Section 6, we show that the Newton's map of the polynomial $z^n - 1$ has buried points when $n \geqslant 3$. Finally, we consider some meromorphic functions whose Julia sets are Sierpinski curves in Section 7. We show that for these functions, their residual Julia sets are nonempty.

2. *Motivations and results*

By analogy of Abikoff's theorem on residual limit sets mentioned in the last section, it is natural to pose the following conjecture which is often referred as

Makienko's conjecture. Let f be a rational function with degree at least two. Then $J_r(f)$ is empty if and only if $F(f)$ has a completely invariant component or consists of only two components.

Notice that the original Makienko's conjecture was first mentioned in [**24**, p. 578] and in the original formulation, the possibility for $F(f)$ to have exactly two components was not mentioned. In [**35**], Qiao stated the correct conjecture in a way slightly different from the above formulation. Makienko's conjecture was proved to be true by Morosawa [**29**] for hyperbolic rational functions first and then extended to subhyperbolic rational functions in [**31**]. Note that a rational function f is hyperbolic if every critical point of f has a forward orbit that accumulates at an attracting cycle of f. While a rational function is subhyperbolic if each critical point of f has a forward orbit that is finite or accumulates at an attracting cycle of f. It is known that if a rational functions f is hyperbolic or subhyperbolic and $J(f)$ is connected, then $J(f)$ is locally connected (see [**27**], p. 191). The same is also true for geometrically finite rational functions. Here a rational function is geometrically finite if the orbit of every critical point in its Julia set is eventually periodic.

From Morosawa's results, one may expect that Makienko's conjecture is true if the Julia set is not too complicated. In fact, Qiao proved in [**35**] that the conjecture is true if the Julia set $J(f)$ is locally connected. However, Qiao's proof is quite complicated and we find it difficult to follow his argument. Theorem 2·1 stated below covers Qiao's result and the proof of it is very different from that of Qiao. In Theorem 2·1, we consider the class **S** of meromorphic functions which consists of meromorphic functions with finitely many critical and asymptotic values. It is not difficult to show that rational functions, rational functions of $\exp{(az)}$, $a \in \mathbb{C}$, and elliptic functions are class **S** functions. Class **S** functions are considered as a natural generalization of rational functions. Many results on iterations of rational functions have been generalized to functions in class **S**. For example, Sullivan's well-known theorem that a rational function has no wandering domains has been extended to transcendental meromorphic functions in class **S** (see [**10**]). In this paper, we extend Qiao's result to functions in class **S** by proving the following:

THEOREM 2·1. *Let f be a meromorphic function in the class* **S**. *Suppose that $J(f)\backslash\{\infty\}$ is locally connected, then $J_r(f)$ is empty if and only if $F(f)$ has a completely invariant component or consists of only two components.*

Examples of functions with locally connected Julia sets can be found in [**15**], [**25**] and [**30**]. In particular, in [**30**] and [**25**], examples of class **S** functions are constructed in such a way that their Julia sets are Sierpinski curves. In Section 7, we show that these functions must have buried points.

The existence of buried components of meromorphic functions is also an interesting topic to consider. We would like to find some necessary and sufficient conditions on the existence of buried components of meromorphic functions. It is clear that the Julia set must be disconnected if it has some buried components. Therefore, we focus on disconnected Julia sets. In [**11**], Beardon proved that the Julia set $J(f)$ of a rational function with degree at least two has a buried component if $J(f)$ is disconnected and every Fatou component is of finite connectivity. He obtained his result by first proving that the number of components of a Julia set is uncountable if the Julia set is disconnected and noting that the number of Fatou components is always countable, and then under the assumption that every Fatou component is of finite connectivity, it follows easily that the Julia set must have some buried components. Qiao [**34**] then extended Beardon's result and proved that a rational function f has buried components if and only if $J(f)$ is disconnected and $F(f)$ has no completely invariant components.

In [**4**], Baker and Domínguez considered the case of transcendental meromorphic functions and obtained the following result.

Theorem A ([**4**]). Let f be a transcendental meromorphic function with no wandering domains. Then $J(f)$ has buried components if f has no completely invariant components and its Julia set is disconnected in the following way. Either $F(f)$ has a component of connectivity at least three, or $F(f)$ has three doubly-connected components $U_i, 1 \leqslant i \leqslant 3$, such that one of the following conditions holds.

(a) Each U_i lies in the unbounded component of the complement of the other two.

(b) Two components U_1, U_2 lie in the bounded component of $\mathbb{C}_\infty \backslash U_3$ but U_1 lies in the unbounded component of $\mathbb{C}_\infty \backslash U_2$ and U_2 lies in the unbounded component of $\mathbb{C}_\infty \backslash U_1$.

We shall extend this result to the following:

Theorem 2·2. *Let f be a meromorphic function which is not of the form $\alpha + (z-\alpha)^{-k}e^{g(z)}$, where k is a natural number, α is a complex number and g is an entire function. Then $J(f)$ has buried components if f has no completely invariant components and its Julia set is disconnected. Moreover, if $F(f)$ has an infinitely connected component, then the singleton buried components are dense in $J(f)$.*

If f is a transcendental meromorphic function of the above form $\alpha + (z-\alpha)^{-k}e^{g(z)}$, then it was proved by Baker in [**3**] that f can have at most one multiply-connected Fatou component and the connectivity of such component is 2. Therefore, such f cannot satisfy the assumptions of Theorem A and hence Theorem 2·2 is really a generalization of Theorem A.

It is possible to have a rational function with buried points and does not have any buried components. For example, it was shown by Morosawa in [**29**] that the rational function

$$\frac{-2z+1}{(z-1)^2}$$

has buried points while its Julia set $J(f)(\neq \mathbb{C}_\infty)$ is connected and hence f has no buried components. A more detailed study of the dynamics of this rational function can be found in Baker and Domínguez [**4**]. We give more examples of meromorphic functions with buried points and without any buried components in Sections 6 and Section 7. For other results concerning buried points and buried components, we refer the reader to [**20**, **22**, **39**, **40**] and the recent paper [**23**].

3. *Functions meromorphic outside a small set*

We first notice that if we iterate a meromorphic function f, we are also iterating f^p for each natural number p, which in general has countably many essential singularities in $\mathbb{C}$. Since it is often very convenient in proofs to replace a function by its higher order iterates, it is natural to extend our iteration theory to include this kind of functions. This has been done by Baker, Domínguez and Herring in [**5**]. Here, we first recall the basic definitions and some results of this extended theory. Then we state and prove some lemmata we are going to use later. The presentation here follows closely to that of [**5**].

Let E be any compact totally disconnected set in $\mathbb{C}_\infty$ and f be a function meromorphic in $E^c = \mathbb{C}_\infty \backslash E$. For any $z_0 \in E$, the cluster set $C(f, E^c, z_0)$ is defined as $\{w\colon w = \lim_{n\to\infty} f(z_n)$ for some $z_n \in E^c$ with $z_n \to z_0\}$. Denote by $\mathbf{M}$ the class of functions f such that f is meromorphic outside some compact totally disconnected set $E = E(f) \subset \mathbb{C}_\infty$ and $C(f, E^c, z_0) = \mathbb{C}_\infty$ for all $z_0 \in E$. In case $E = \varnothing$ we further assume that f is neither constant nor a Möbius transformation. A related class $\mathbf{K} \subset \mathbf{M}$ consists of functions f with the property that there is a compact *countable* set $E = E(f) \subset \mathbb{C}_\infty$ such that f is non-constant meromorphic in $\mathbb{C}_\infty \backslash E$ but in no proper superset. The class $\mathbf{K}$ is studied in [**17**] and [**18**]. Let $f \in \mathbf{K}$ and $e \in E(f)$. It is known that for any open neighbourhood U of e, f takes in $U \backslash E(f)$ every value in $\mathbb{C}_\infty$ with at most 2 exceptions. We call such exceptional values the Picard values of f.

One reason to consider the class $\mathbf{M}$ is that it is closed under functional composition. In fact, if f and g belong to $\mathbf{M}$, then so is $f \circ g$ and $E(f \circ g) = E(g) \cup g^{-1}(E(f))$ which is still a compact totally disconnected set ([**5**, lemma 2]). In particular, for any $f \in \mathbf{M}$ and $n \in \mathbb{N}$, $f^n \in \mathbf{M}$ and $E_n = E(f^n) = \bigcup_{i=0}^{n-1} f^{-i}(E(f))$, where f^0 is defined to be the identity function on $\mathbb{C}_\infty$. Clearly, if we set $J_1(f) = \overline{\bigcup_{n=1}^{\infty} E_n}$ and $F_1(f) = \mathbb{C}_\infty \backslash J_1(f)$, then $F_1(f)$ is the largest open set in which all f^n are defined. Furthermore $f(F_1) \subset F_1$. We obtain an extension of the standard Fatou and Julia theory by defining the Fatou set $F(f)$ to be the largest open set in which (i) all $f^n, n \in \mathbb{N}$, are meromorphic and (ii) $\{f^n\}$ is a normal family. The Julia set is defined to be $J(f) = \mathbb{C}_\infty \backslash F(f)$. If $J_1(f)$ is either empty or contains one point (in this case, assume without loss of generality that $J_1(f) = E_1(f) = \{\infty\}$) or two points (may assume that $J_1(f) = \{0, \infty\}$), then we are dealing with maps which are rational or (conjugate to) entire functions or analytic maps of the punctured plane $\mathbb{C}^*$ respectively. In this case the condition (i) above is trivial and the Fatou and Julia sets are determined by (ii). In the other cases Montel's theorem shows us that $F(f) = F_1(f)$ and $J(f) = J_1(f)$ as defined above.

Many properties of $F(f)$ and $J(f)$ are similar for $f \in \mathbf{M}$ to those for rational, entire or analytic maps of the punctured plane but different proofs are often needed. For example, we still have $F(f) = F(f^n)$ and $J(f) = J(f^n)$. If U is a component of $F(f)$ then for each $k \in \mathbb{N}$, $f^k(U)$ is contained in a unique component U_k of $F(f)$. If $U_k \neq U_n$ whenever $k \neq n$, then U is called a wandering domain; otherwise we call U eventually periodic. Attracting domains, Parabolic domains, Siegel disks, Herman rings and Baker domains are defined for $f \in \mathbf{M}$ just as for rational functions, transcendental entire functions and analytic self-maps of the punctured plane, with little change in the discussion. In particular, we still have the classification theorem of periodic Fatou components (see [**5**, theorem C]). Moreover, the No Wandering Domains Theorem for the class $\mathbf{S}$ meromorphic functions has also been extended to the class $\mathbf{MS} \cap \mathbf{K} \subset \mathbf{M}$ (see [**5**, p. 660]). Here we say that $f \in \mathbf{M}$ belongs to the class $\mathbf{MS}$ if $sing(f^{-1})$, the set of singularities of f^{-1} is finite. This set of singularities consists of the critical values, asymptotic values of f as well as their limit points. Here, we say that w_1 is an asymptotic value of f at $e \in E(f)$ if $f(z)$ tends to w_1 as z tends to $e \in E(f)$ along some path γ. The precise definition of singularities of f^{-1} for $f \in \mathbf{M}$ can be found in [**5**, p. 659]. Notice that when $f \in \mathbf{MS}$ is meromorphic in $\mathbb{C}$, the singularities of f^{-1} are the usual critical values and asymptotic values of f and in this case f belongs to the class $\mathbf{S}$. It is known that if $f, g \in \mathbf{MS}$, then so is $f \circ g$. Also note that if $f \in \mathbf{MS}$, then f has no Baker domains ([**5**, p. 664]). The periodic domains are closely related to the set of singularities of f^{-1} as can be seen from the following theorem.

Theorem B ([**5**, lemma 10]).

(i) If $f \in \mathbf{M}$ and $G_1, \ldots, G_p$ is a periodic cycle of Fatou components in which the iterates converge to either an attracting or parabolic periodic cycle of points, then $G_1 \cup \cdots \cup G_p$ contains the forward orbit of some singularity of f^{-1}.

(ii) If $G_1, \ldots, G_p$ is a cycle of Siegel discs or Herman rings (i.e. each G_i is a Siegel disc or Herman ring of f^p), then each point of $\bigcup_{i=1}^{p} \partial G_i$ is a limit of points in the forward orbit of singularities of f^{-1}.

We shall also need the following results on the completely invariant components of $f \in \mathbf{MS}$.

Theorem C ([**6**]). Suppose that $f \in \mathbf{MS}$. If $E(f)$ has an isolated point, then f has at most two completely invariant Fatou components.

By proving that if $f \in \mathbf{S}$, then $f^p \in \mathbf{MS}$ and $E(f^p)$ has an isolated point, we obtain the following lemma.

Lemma 3·1. *Suppose that $f \in \mathbf{S}$, then for any natural number p, f^p has at most two completely invariant Fatou components.*

Proof. Note that $f \in \mathbf{S}$ is the same as $f \in \mathbf{MS}$ and $E(f) = \{\infty\}$. Therefore, we have $f^p \in \mathbf{MS}$ and $E(f^p) = \bigcup_{i=0}^{p-1} f^{-i}(\{\infty\})$. We claim that any $z_0 \in E(f^p)\backslash\{\infty\}$ is an isolated point of $E(f^p)$. Assume to the contrary that there exists a sequence $\{z_n\}_1^{\infty} \subset E(f^p)$ such that $\lim_{n\to\infty} z_n = z_0$. Since $E(f^p) = \bigcup_{i=0}^{p-1} f^{-i}(\{\infty\})$, there exists some $1 \leqslant j \leqslant p-1$ such that infinitely many z_n belongs to $f^{-j}(\{\infty\})$. Hence $z_0 (\neq \infty)$ is an accumulation point of the set $f^{-j}(\{\infty\})$ which contradicts the fact that the solution set of the equation $f^j(z) = \infty$ has no finite accumulation points for the non-constant meromorphic function f. Now apply Theorem C and we conclude that f^p has at most two completely invariant Fatou components.

Using Theorem C, Lemma 3·1 and the result that any $f \in \mathbf{MS} \cap \mathbf{K}$ has no wandering domains, one can obtain the following lemma easily.

LEMMA 3·2. *Suppose that* $f \in \mathbf{MS} \cap \mathbf{K}$. *If* $E(f)$ *has an isolated point, then the number of Fatou components is either* $0, 1, 2$ *or* ∞. *In particular, this is true for any* f^p *with* $f \in \mathbf{S}$.

The following result of Bergweiler and Eremenko takes care of what happens when $f \in \mathbf{MS}$ has precisely two completely invariant components. They only prove the result for the class $\mathbf{S}$, but their proof also works for the class $\mathbf{MS}$.

THEOREM D ([**14**]). Suppose that $f \in \mathbf{MS}$ and f has two completely invariant Fatou components D_1 and D_2. Then the set of singularities of f^{-1} is contained in $D_1 \cup D_2$.

By using the above results, we have the following lemma which is a generalization of Theorem 1 in [**19**].

LEMMA 3·3. *Suppose that* $f \in \mathbf{MS} \cap \mathbf{K}$ *and* f *has two completely invariant Fatou components* D_1 *and* D_2, *then* f *has exactly two Fatou components* D_1 *and* D_2.

Proof. Let D_3 be a Fatou component of f other than D_1 and D_2. This D_3 must be eventually periodic as any function in $\mathbf{MS} \cap \mathbf{K}$ has no wandering domains. Consider the forward orbit of D_3, that is, $D_3, f(D_3), f^2(D_3), \ldots$. Then this orbit will eventually be attracted to a cycle of attracting domains or parabolic domains or Siegel discs or Herman rings as $f \in \mathbf{MS}$ has no Baker domains. Let this cycle be $G_1, \ldots, G_p$. By Theorem D and the fact that D_1 and D_2 are forward invariant, all the forward orbits of the singularities of f^{-1} must lie inside $D_1 \cup D_2$. Hence it follows from Theorem B that the cycle $G_1, \ldots, G_p$ cannot be a cycle of Siegel discs or Herman rings. Therefore it must be a cycle of attracting domains or parabolic domains which will contain the forward orbit of some singularity of f^{-1}. Since all such forward orbits are contained in the completely invariant components D_1 and D_2, we have $p = 1$ and $G_1 = D_1$ or D_2. Now $G_1 = D_1$ or D_2 implies that G_1 is backward invariant and therefore $D_3 \subset G_1$ which is a contradiction.

THEOREM E ([**4**, corollary 1]). Suppose that $f \in \mathbf{K}$ and has no wandering domains. If D is a multiply-connected periodic Fatou component such that $\partial D = J(f)$, then D is completely invariant.

Baker and Domínguez only proved the above result for f meromorphic in $\mathbb{C}$. However, their proof also works for any $f \in \mathbf{K}$. Finally, using this result, we prove the following lemma which is crucial for the proof of Theorem 2·1.

LEMMA 3·4. *Let* $f \in \mathbf{MS} \cap \mathbf{K}$ *and* $E(f)$ *has an isolated point. Assume that* $J(f)\backslash\{\infty\}$ *is locally connected. If* D *is a forward invariant Fatou component of* f *such that* $\partial D = J(f)$, *then* D *is completely invariant.*

Proof. We shall first consider the case that $J(f)$ is disconnected. Since $J(f) = \partial D$ and $J(f)$ is disconnected, we know that D must be multiply-connected. Note that f has no wandering domains and therefore we can apply Theorem E to conclude that D is completely invariant.

Now it remains to consider the case that $J(f)$ is connected. Suppose the number of Fatou components is finite. Then by Lemma 3·2, the number of Fatou components must be one or two. For the first case, obviously D is completely invariant. If the number of Fatou components is two, then let D_1 be another Fatou component. As D is forward invariant, D_1 must be backward invariant and hence D_1 is forward invariant. Thus, D must be backward invariant and therefore D is completely invariant.

From now on we shall assume that the number of Fatou components is infinite. As $\overline{D} = J(f) \bigcup D$, $\mathbb{C}_\infty - \overline{D} = \bigcup_i D_i$, where D_i are all the Fatou components other than D. Since $\overline{D}$ is connected, each of the Fatou component $D_i (\neq D)$ is simply connected (see [**12**, proposition 5·1·3]). Since $J(f) = \partial D$ is connected, D is also simply connected.

Take a Fatou component E other than D such that $\overline{E}$ contains no singular values (this is possible as there are infinitely many Fatou components and only finitely many singular values). Then $f^{-1}(E)$ is a union of distinct Fatou components $E_1, \ldots, E_d, \ldots$ such that each $f_i = f_{|E_i}\colon E_i \to E$ is a conformal map with $f_i^{-1}(\partial E) = \partial E_i$.

As D is forward invariant, none of E_i is equal to D. Suppose D is not backward invariant, then $f^{-1}(D)$ has a component D_1 other than D. Let $f_0 = f_{|\overline{D}_1}\colon \overline{D}_1 \to \overline{D}$. Then $f_0^{-1}(\partial D \backslash E_f) = \partial D_1$, where E_f is the set of Picard exceptional values of f. Note that $\partial E \subset J(f) = \partial D$. Since every Picard exceptional value is an asymptotic value (the proof is similar to the usual one for meromorphic functions in $\mathbb{C}$), we have $\partial E \subset \partial D \backslash E_f$ and hence $f_0^{-1}(\partial E) \subset \partial D_1$. On the other hand, $f_0^{-1}(\partial E)$ is the union of some of the $f_i^{-1}(\partial E) = \partial E_i$. Therefore each of these ∂E_i is a subset of ∂D_1 and we may for example assume that $\partial E_1 \subset \partial D_1 \subset \partial D = J(f)$.

Recall that a boundary point a of a domain G is said to be accessible if there exists an arc $\gamma\colon [0, 1) \to G$ such that $\lim_{t \to 1} \gamma(t) = a$. It is known that the accessible boundary points of a domain G form a dense subset of ∂G (see [**37**, p. 9, exercise 1]). Therefore we can take two finite accessible points a and b in ∂E_1 such that a and b are connected by some Jordan path γ so that except the two endpoints a and b, the whole γ is lying inside E_1.

Note that a and b are also boundary points of D. Since $\partial D = J(f)$ is locally connected at any finite point, by lemma 17·17 of [**27**], ∂D is also locally path connected at any finite point. Hence, there exist path connected neighbourhoods of both a and b in ∂D, say A and B respectively. Since a and b are also boundary points of D_1 and the accessible boundary points of D_1 form a dense subset of ∂D_1, there exist two accessible points of D_1, a_1 and b_1 in the two neighbourhoods $A \bigcap \partial D_1$ and $B \bigcap \partial D_1$ of ∂D_1 respectively so that $a_1, b_1 \in \partial D_1$ can be connected by a Jordan path γ_1 lying inside D_1 (except the two endpoints a_1, b_1). Let $\sigma_1 \subset A, \sigma_2 \subset B$ be a path joining $a, a_1 \in A$ and $b, b_1 \in B$ respectively. Then $\Gamma = \gamma \bigcup \sigma_1 \bigcup \gamma_1 \bigcup \sigma_2$ forms a simple closed curve in $\mathbb{C}$ and $\mathbb{C}_\infty \backslash \Gamma$ has two components. Note that each of these two components contains some points in $J(f)$, for otherwise, as D_1 and E_1 are connected, they will be the same component which is a contradiction. Then it follows from the assumption $\partial D = J(f)$ that each component of $\mathbb{C}_\infty \backslash \Gamma$ contains some points in D. Let d_1 and d_2 be two points in D so that they are lying in the two different components of $\mathbb{C}_\infty \backslash \Gamma$. Connect d_1 and d_2 by a path in D. This path must cut Γ at certain point and it is a contradiction as $\Gamma \bigcap D = \varnothing$.

4. *Proof of Theorem* 2·1

The 'if' part of Theorem 2·1 is rather easy to prove. In fact, if there exists a completely invariant component D of $F(f)$, then $\overline{D}$ is also completely invariant. Hence, by the minimality of $J(f)$, we have $J(f) \subset \overline{D}$ and therefore $\partial D = J(f)$. This implies $J_r(f) = \phi$. Now suppose $F(f)$ has two components only, say D_1 and D_2.

Then each of these two components must be a completely invariant component of $F(f^2) = F(f)$. Hence, we have $\partial D_i = J(f^2) = J(f)$ and therefore $J_r(f) = \varnothing$.

It remains to consider the 'only if' part. We shall prove by contradiction. Assume to the contrary that $F(f)$ has no completely invariant components and $F(f)$ consists of more than two components. This implies that $F(f)$ has infinitely many components. Given that $J_r(f) = \varnothing$, then by definition, $J(f) = \bigcup_i \partial D_i$, where D_i are all the Fatou components. In [**29**], Morosawa shows that for rational function f, we actually have $J(f) = \bigcup_i^M \partial D_i$, where D_i are all the periodic components of $F(f)$. This result is then improved by Baker and Domínguez to the following lemma.

Lemma F ([**4**]). If $f \in \mathbf{K}$ has no wandering domains and $J_r(f) = \varnothing$, then there is a periodic Fatou component D such that $\partial D = J(f)$.

Therefore we can assume that $J(f) = \partial D$ for some periodic Fatou component D. Let's consider the periodic cycle $D_1 = D, D_2, \ldots, D_p$. Then each D_i is forward invariant under f^p for $i = 1, 2, \ldots, p$. Note that as $f \in S$, $f^p \in \mathbf{MS} \cap \mathbf{K}$ and $E(f^p)$ contains an isolated point (see the proof of Lemma 3·1). Apply Lemma 3·4 and we conclude that D_i is completely invariant under f^p for $i = 1, \ldots, p$. However by Lemma 3·1, f^p can have at most two completely invariant Fatou components and therefore $p \leqslant 2$. If $p = 1$, then f has a completely invariant Fatou component which is a contradiction. If $p = 2$, then f^2 has two completely invariant Fatou components. By Lemma 3·3, $F(f^2)$ and hence $F(f)$ has only two Fatou components. This is again a contradiction and we are done.

5. *Proof of Theorem* 2·2

We shall repeatedly apply the following result to prove Theorem 2·2.

Theorem G ([**17**, p. 555]). Let $f\colon V \to U$ be a nonconstant meromorphic function between V and U which are subdomains of $\mathbb{C}_\infty$. Let $c(V)$ and $c(U)$ be the connectivity of V and U respectively. If there exists a countable subset E of ∂V such that for any $a \in \partial V \backslash E$, the cluster set of f at a, $C(f, V, a)$ is a subset of ∂U. Then we have

$$\max\{c(V) - 2, 0\} \geqslant \frac{\int_V [f^{\#}(z)]^2 d\sigma}{\mathrm{Area}(U)}(c(U) - 2).$$

We first consider the case that f has no infinitely connected Fatou components. Note that for a transcendental meromorphic function f which is not of the form $\alpha + (z - \alpha)^{-k} e^{g(z)}$, if its Julia set $J(f)$ is disconnected, then $J(f)$ consists of uncountably many components (this can be proved by using the similar arguments in the proof of Theorem 3 in [**37**]). Now as the connectivity of each Fatou component of f is finite, the total number of the boundary components of all Fatou components is countable because there are only countably many Fatou components. This implies that $J(f)$ has uncountably many buried components and we are done.

Now we may assume that f has at least one infinitely connected Fatou component U. Then it follows from a result of Domínguez ([**21**, theorem A]), that the singleton components of $J(f)$ are dense in $J(f)$.

Let a be any point in $J(f)$ and r be any positive number. Consider the open disc $\mathbb{D}(a, r)$ with center a and radius r. We would like to show that $\mathbb{D}(a, r)$ contains at least one buried component of $J(f)$. Assume to the contrary that $\mathbb{D}(a, r)$ contains no buried components. Since the singleton components of $J(f)$ are dense in $J(f)$, we can find some singleton component $\{b_0\}$ of $J(f)$ such that $b_0 \in \mathbb{D}(a,r)$. The assumption that $\mathbb{D}(a,r)$ contains no buried components implies that $\{b_0\}$ is a boundary component of some Fatou component U_0. Let γ be a closed Jordan curve in U_0 such that b_0 is contained in $int(\gamma)$, the interior of γ. It follows that γ is not homotopic to a point in U_0. Note that as $b_0 \in J(f)$, there exists a sequence $\{\beta_n\} \subset int(\gamma)$ of singleton components of $J(f)$ such that $\beta_n \to b_0$. This implies that U_0 is infinitely connected.

In what follows, we show that $int(\gamma)$, the interior of γ contains another infinitely connected Fatou component. As U_0 cannot be completely invariant, there is a component V of $f^{-1}(U_0)$ such that V and U_0 are disjoint. By Theorem G, V is infinitely connected. As we shall see later, there is no harm to assume that $U_0 \cup V$ does not contain any Picard exceptional values of f. If f is entire, then both infinitely connected domains U_0 and V must be bounded. In this case, by the expanding properties of Julia sets, there exists some $m \in \mathbb{N}$ such that $f^m(int(\gamma)) \supset U_0 \cup V$. Now if f has poles, since f is not of the form $\alpha + (z-\alpha)^{-k}e^{g(z)}$, $J(f)$ is the closure of the backward orbit $O^-(\infty)$. As $int(\gamma) \cap J(f) \neq \varnothing$, there exists some $c \in int(\gamma)$ and $m \in \mathbb{N}$ such that $f^{m-1}(c) = \infty$. Hence f^m has an essential singularity at c and by Picard Theorem, we also have $f^m(int(\gamma)) \supset U_0 \cup V$. It follows that there exist $U_1, U_1^{'} \subset F(f)$ such that both $U_1 \cap int(\gamma)$ and $U_1^{'} \cap int(\gamma)$ are non-empty and $f^m(U_1) = U_0$, $f^m(U_1^{'}) = V$. Obviously, U_1 and $U_1^{'}$ are disjoint and therefore one of them (say U_1) is not equal to U_0. This implies that U_1 must lie inside $int(\gamma)$. By Theorem G, U_1 is infinitely connected. Hence, $int(\gamma)$ contains an infinitely connected Fatou component U_1 which is different from U_0.

Since U_1 is infinitely connected, just as before we can find some singleton component $\{b_1\}$ of $J(f)$ and some non-contractible closed Jordan curve γ_1 such that $b_1 \in int(\gamma_1)$ and the diameter $diam(\gamma_1) < r/2$. Now replace U by U_1 and repeat the previous argument, then $int(\gamma_1)$ will again contain an infinitely connected Fatou component U_2 which is different from U_1 and with diameter less than $r/2$.

In this way, we can obtain a sequence of Fatou components $\{U_n\}$, a sequence of singleton components $\{b_n\}$ of $J(f)$ and a sequence of non-contractible closed curves γ_n such that $b_{n+1} \in U_{n+1} \subset int(\gamma_n)$ and $diam(U_n) \to 0$ as $n \to \infty$. Therefore, $\bigcap_{n=1}^{\infty} U_n = \{b\}$. Note that b is inside $J(f)$ as b is the limit point of $b_n \in J(f)$. By our construction, $\gamma_n \subset F(f)$ and $diam(\gamma_n) \to 0$ as $n \to \infty$. As $b \in int(\gamma_n)$, $\{b\}$ is a singleton component of $J(f)$. Now if $\{b\}$ is a boundary component of some Fatou component W, then $\gamma_n \cap W \neq \varnothing$ for all sufficiently large n which is impossible as each γ_n belongs to different Fatou components U_n. Therefore, $\{b\}$ is a singleton buried component of $J(f)$. As a and r are arbitrary, the singleton buried components are dense in $J(f)$.

6. *Buried points of Newton's maps*

In this and the next section we give more examples of the Julia set with buried points but without any buried components. The example considered in this section comes from the Newton's map of a polynomial. The reason that these rational functions do not have any buried components is that their Julia sets are connected. The following result of Shishikura gives a sufficient condition for a Julia set to be connected.

Theorem H ([**36**]). Let f be a rational function of degree at least two. $J(f)$ is connected if f has only one repelling fixed point.

Newton's method involves iterations and is used for finding roots of polynomials. Given a polynomial p, its Newton's map N_p is defined by

$$N_p(z) = z - \frac{p(z)}{p'(z)}.$$

Note that the zeros of a polynomial p are the fixed points of N_p.

Concerning the structure of the Julia set of N_p, we have the following:

Theorem I. $J(N_p)$ is connected for any polynomial p of degree at least two.

This result follows from the above result of Shishikura as it can be shown that N_p has only one repelling fixed point which is ∞. As $J(N_p)$ is connected, there is no buried components in $J(N_p)$. However, $J(N_p)$ may have buried points. For example, we have the following result.

Theorem 6·1. *If $p(z) = z^n - 1$ and $n \geqslant 3$, then $J_r(N_p) \neq \phi$ and $J(N_p)$ has no buried components.*

Proof. For $k = 1, \ldots, n$, let $\omega_k = e^{\frac{2\pi(k-1)}{n}i}$. Then $\omega_1, \omega_2, \ldots, \omega_n$ are the n distinct zeros of p. Note that $N_p(z) = ((n-1)z^n + 1)/nz^{n-1}$ and $N_p'(z) = ((n-1)/n)/[(z^n - 1)/z^n]$. It follows that each ω_k is a critical fixed point of N_p and therefore belongs to one of the immediate superattracting basins of N_p. Hence N_p has at least n immediate superattracting basins. It can be easily checked that $\{\omega_1, \ldots, \omega_n, 0\}$ is the set of all critical points of N_p and ∞ is the repelling fixed point of N_p as $N_p'(\infty) = n/(n-1)$. Therefore $\infty \in J(N_p)$. As $N_p(0) = \infty$, we also have $0 \in J(N_p)$. Hence N_p has only n critical points in $F(N_p)$ and each of them lies in the corresponding immediate superattracting basin of N_p. As the period of the n immediate superattracting basins is one, there exist at least n different periodic cycles. As 0 is the unique critical point of N_p lying in $J(N_p)$ and its forward orbit is finite, N_p must have no Herman rings or Siegel disks. If there is another periodic cycle, then there is a critical point in the cycle. However, all critical points are already in the immediate superattracting basins. Therefore, N_p has exactly n periodic Fatou cycles. As each immediate superattracting basin correspond to a periodic cycle, N_p has exactly n periodic Fatou components which are the n immediate superattracting basins.

Now we show that $F(N_p)$ does not have any completely invariant components. As $J(N_p)$ is connected, all Fatou components of N_p are simply connected. By Riemann-Hurwitz formula, it is easy to show that N_p is a two fold map on each immediate superattracting basin. As $\deg(N_p) = n \geqslant 3$, for each $\alpha \in \mathbb{C}_\infty$, it has n preimages under N_p counting with multiplicities. However, as N_p is a two fold map on each immediate superattracting basin and $\deg(N_p) \geqslant 3$, for each point α in a immediate superattracting basin, there is at least one preimage of α under N_p which is not lying in the basin. Therefore, $F(N_p)$ does not have any completely invariant components.

Since the forward orbit of the critical points in $F(N_p)$ accumulate at the superattracting basins and the forward orbit of the critical point in $J(N_p)$ is eventually periodic, N_p is subhyperbolic. As mentioned in Section 2, the Julia set of a subhyperbolic rational function is locally connected. Thus, $J(N_p)$ is locally connected. Now, assume $J_r(N_p) = \varnothing$. Then by Lemma F, there is a periodic Fatou component D such that $\partial D = J(N_p)$. By Lemma 3·4, D is completely invariant. However, by the above analysis, N_p has no completely invariant Fatou components, and therefore we must have $J_r(N_p) \neq \varnothing$. As $J(N_p)$ is connected, $J(N_p)$ has no buried components.

In view of Theorem 6·1, we may consider the following:

Problem. Let p be a polynomial with $\deg(p) \geqslant 3$. Find some sufficient conditions on the polynomial p so that $J_r(N_p)$ is nonempty.

7. *Buried points of meromorphic functions with Sierpinki curve Julia sets*

In this section, we consider some examples of meromorphic functions which have Sierpinki curve Julia sets and show that these functions have buried points. The usual definition of Sierpinski curve requires that the set is planar. However, since the Julia set of a meromorphic function is a subset in $\mathbb{C}_\infty$, we have to modify the definition of Sierpinski curve. Following [**30**] and [**25**], we say that a closed subset in $\mathbb{C}_\infty$ is a Sierpinski curve if it is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by pairwise disjoint simple closed curves. One example of Sierpinski curve is the famous *Sierpinski carpet* and any Sierpinski curve is actually homeomorphic to it.

Examples of rational, entire and meromorphic functions with Sierpinki curve Julia sets can be found in [**16**], [**30**] and [**25**] respectively. In [**16**], it is shown that there exists some λ such that the Julia set of the function $f_\lambda(z) = z^2 + \lambda/z^2$ is a Sierpinski curve. Similar result is also obtained by Morosawa [**30**] for functions of the form $g_a(z) = ae^a\{z - (1-a)\}e^z$, where $a > 1$. Very recently, Hawkins and Look [**25**] show that there exist Weiertrass elliptic $\wp$ functions which have Sierpinki curve Julia sets. Note that all these functions belong to the class **S**. It follows from the following result as well as the fact that class **S** functions have no wandering domains that all these functions have buried points.

PROPOSITION 7·1. *Let f be a function in the class* **K**. *If f has no wandering domains and its Julia set is a Sierpinski curve, then $J_r(f) \neq \phi$.*

Proof. Assume to the contrary that $J_r(f) = \phi$, then by Lemma F, there is a periodic Fatou component D such that $\partial D = J(f)$. Since $J(f)$ is a Sierpinski curve (which is homeomorphic to the standard Sierpinski carpet), $F(f) = \mathbb{C}_\infty \backslash J(f)$ must contain infinitely many components. Let $E \neq D$ be one of these components. By the definition of Sierpinski curve, D and E are bounded by pairwise disjoint simple closed curves which implies that $\partial D \bigcap \partial E = \phi$. This is a contradiction as $\partial E \subset J(f) = \partial D$.

REFERENCES

[**1**] W. ABIKOFF. Some remarks on Kleinian groups. Advances in the theory of Riemann surfaces. *Ann of Math. Studies* **66** (1971), 1–5.
[**2**] W. ABIKOFF. The residual limit sets of Kleinian groups. *Acta Math.* **130** (1973), 127–144.
[**3**] I. N. BAKER. Wandering domains for maps of punctured plane. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **12** (1987), 191–198.

[4] I. N. Baker and P. Domínguez. Residual Julia sets. *J. Analysis* **8** (2000), 121–137.
[5] I. N. Baker, P. Domínguez and M. E. Herring. Dynamics of functions meromorphic outside a small set. *Ergodic Theory Dynam. Systems* **21** (2001), 647–672.
[6] I. N. Baker, P. Domínguez and M. E. Herring. Functions meromorphic outside a small set: completely invariant domains. *Complex Variables Theory Appl.* **49** (2004), 95–100.
[7] I. N. Baker, J. Kotus and Y. Lü. Iterates of meromorphic functions. I. *Ergodic Theory Dynam. Systems* **11** (1991), 241–248.
[8] I. N. Baker, J. Kotus and Y. Lü. Iterates of meromorphic functions. II. Examples of wandering domains. *J. London Math. Soc.* **42** (1990), 267–278.
[9] I. N. Baker, J. Kotus and Y. Lü. Iterates of meromorphic functions. III. Preperiodic domains. *Ergodic Theory Dynam. Systems* **11** (1991), 603–618.
[10] I. N. Baker, J. Kotus and Y. Lü. Iterates of meromorphic functions. IV. Critically finite functions. *Results Math.* **22** (1992), 651–656.
[11] A. F. Beardon. The components of a Julia set. *Ann. Acad. Sci. Fenn. Series A. I. Mathematica* **16** (1991), 173–177.
[12] A. F. Beardon. *Iteration of Rational Functions* (Springer-Verlag, 1991).
[13] W. Bergweiler. Iteration of meromorphic functions. *Bull. Amer. Math. Soc. (N.S.)* **29** (1993), 151–188.
[14] W. Bergweiler and A. Eremenko. Meromorphic functions with two completely invariant domains. *Transcendental Dynamics and Complex Analysis*. LMS Lecture Note Series. (Cambridge University Press), to appear.
[15] W. Bergweiler and S. Morosawa. Semihyperbolic entire functions. *Nonlinearity.* **15** (2002), 1673–1684.
[16] P. Blanchard, R. L. Devaney, D. M. Look, P. Seal and Y. Shapiro. Sierpinski curves Julia sets and singular perturbations of complex polynomials. *Ergodic Theory Dynam. Systems* **25** (2005), 1047–1055.
[17] A. Bolsch. Periodic Fatou components of meromorphic functions. *Bull. London Math. Soc.* **31** (1999), 543–555.
[18] A. Bolsch. Repulsive periodic points of meromorphic functions. *Complex Variables Theory Appl.* **31** (1996), 75–79.
[19] C. L. Cao and Y. F. Wang. On completely invariant Fatou components. *Ark. Mat.* **41** (2003), no. 2, 253–265.
[20] P. Domínguez. Connectedness properties of Julia sets of transcendental entire functions. *Complex Variables Theory Appl.* **32** (1997), 199–215.
[21] P. Domínguez. Dynamics of transcendent meromorphic functions. *Ann. Acad. Sci. Fenn. Ser. A. I. Math.* **23** (1998), 225–250.
[22] P. Domínguez and S. Guillermo. A study of the dynamics of $\lambda \sin z$. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **12** (2002), 2869–2883.
[23] P. Domínguez and N. Fagella. Residual Julia sets of rational and transcendental functions. *Transcendental Dynamics and Complex Analysis*. LMS Lecture Note Series. (Cambridge University Press), to appear.
[24] A. E. Eremenko and M. Yu. Lyubich. The dynamics of analytic transformations. *Leningrad Math. J.* **1** (1990), 563–634.
[25] J. M. Hawkins and D. M. Look. Locally Sierpinski Julia sets of Weierstrass elliptic $\wp$ functions. Preprint (2005).
[26] X. H. Hua and C. C. Yang. *Dynamics of transcendental functions.* Asian Mathematics Series. **1** (Gordon and Breach Science Publishers, 1998).
[27] J. Milnor. *Dynamics in one Complex Variable* (Vieweg, 1999).
[28] C. T. McMullen. Automorphisms of rational maps. *Holomorphic Functions and Moduli I (Berkeley, CA,* 1986*)*. MSRI Publications **10** (Springer-Verlag, 1998), pp. 31–60.
[29] S. Morosawa. On the Residual Julia sets of rational functions. *Ergodic Theory Dynam. Systems* **17** (1997), 205–210.
[30] S. Morosawa. Local connectedness of Julia sets for transcendental entire functions. *Nonlinear Analysis and Convex Analysis* (*Niigata,* 1998). (World Sci. Publishing, 1999), 266–273.
[31] S. Morosawa. Julia sets of subhyperbolic rational functions. *Complex Variables Theory Appl.* **41** (2000), 151–162.
[32] S. Morosawa, Y. Nishimura, M. Taniguchi and T. Ueda. *Holomorphic Dynamics* (Cambridge University Press, 2000).

[**33**] J. Y. Qiao. The Julia set of the mapping $z \to zexp(z + \mu)$. *Chinese Science Bulletin* **39** (1994), 529–533.

[**34**] J. Y. Qiao. The buried points on the Julia sets of rational and entire functions. *Scientia Sinica Ser. A*. **38** (1995), 1409–1419.

[**35**] J. Y. Qiao. Topological complexity of Julia sets. *Scientia Sinica Ser. A*. **40** (1997), 1158–1165.

[**36**] M. Shishikura. The connectivity of the Julia set of rational maps and fixed points. Preprint, *Inst. Hautes Études Sci*. (1990).

[**37**] N. Steinmetz. *Rational Iteration*. de Gruyter studies in Mathematics (Walter de Gruyter, 1993).

[**38**] D. Sullivan. Quasiconformal homeomorphisms and dynamics I: Solutions of Fatou-Julia problem on wandering domains. *Ann of Math.* **122** (1985), 401–418.

[**39**] Y. Sun and C. C. Yang. Buried components of a Julia set. *Appl. Math. J. Chinese Univ. Ser. B*. **17** (2002), 431–434.

[**40**] Y. Sun and C. C. Yang. Buried points and Lakes of Wada continua. *Discrete Contin. Dyn. Syst.* **9** (2003), 379–382.