A New Kalman Filter-based Power Spectral Density Estimation for Nonstationary Pressure Signals

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Abstract—This paper presents a new Kalman filter-based power spectral density estimation (PSD) algorithm for nonstationary pressure signals. The pressure signal is assumed to be an autoregressive (AR) process, and a stochastically perturbed difference equation constraint model is used to describe the dynamics of the AR coefficients. The proposed Kalman filter frame uses variable number of measurements to estimate the time-varying AR coefficients and yield the PSD estimation with better time-frequency resolution. Simulation results show that the proposed algorithm achieves a better time-frequency resolution than conventional algorithms for nonstationary pressure signals.

I. INTRODUCTION

Power spectral density (PSD) is a popular frequency analysis technique widely applied in biomedical signal processing. There are two kinds of power spectrum estimation: nonparametric and parametric spectrum estimations. Nonparametric spectrum approaches, such as Fourier transform and Lomb periodogram, do not assume a particular model for the signal, but estimate the spectrum entirely by the data. On the contrary, parametric methods suppose that the signal is generated by a specific parametric model (e.g., an AR model). Generally, nonparametric methods have a lower computational complexity than parametric ones, while parametric methods have a higher frequency resolution.

All the above spectral analysis methods only work well for stationary signals because they only provide the frequency information. However, most interesting signals, such as the pressure signals in our paper, contain numerous nonstationary or transient characteristics: drift, trends, abrupt changes, and beginnings and ends of events. To understand the time-frequency property of these nonstationary signals, several nonparametric time-frequency analysis techniques are developed, such as short time Fourier transform (STFT), wavelet transform and windowed Lomb periodogram. In [1]-[3], a parametric spectrum estimation algorithm was developed by Kitagawa and Gersch, and it can give a much higher time-frequency resolution than nonparametric methods. Their algorithm described the nonstationary signal with a smoothness priors AR process, in which the AR coefficients were obtained by a stochastically perturbed difference equation constraint model. The Kalman filtering method was used to track the time-varying AR coefficients and then yield the PSD estimation.

Recently, the parametric spectrum of pressure signals, including arterial, intracranial, cardiac pressure, etc. has been studied by Aboy *et al.* [4, 5]. They constructed a novel statistical model to describe the frequency variability of pressure signals. In this synthetic pressure signal model, the effects of respiration were incorporated on arterial blood or intracranial pressure. Subsequently, a Kalman filter-based PSD estimate algorithm, which is similar to [1]-[3], was developed by Aboy *et al.* for the synthetic pressure signals [5]. Their spectral density estimate had an unsatisfactory resolution in frequency domain, because the conventional Kalman filter only employed one measurement to update the AR coefficients so that the AR coefficient estimates had a large variance. To solve this problem, a simple average operation over short time windows to the PSD estimation was employed in [5]. However, this average operation in the frequency domain blurred the time resolution of PSD.

In this paper, we propose a new Kalman filter-based PSD estimation with variable number of measurements. Following the system model in [1]-[3], the synthetic pressure signal is regarded as a smoothness priors AR process, and the AR coefficients are described with a stochastically perturbed k-th order difference equation constraint model. The system dynamics are given by a linear state-space model, and the AR coefficients are just the system state which is estimated using Kalman filter. The proposed Kalman filter-based algorithm employs a number of measurements to improve the time-frequency resolution through a better tradeoff between bias and variance of state estimate. Basically, a measurement window of appropriate length can help to reduce the variance of estimation due to additive noise, while avoiding excessive bias for nonstationary signals. The intersection of confidence intervals (ICI) rule [6, 7] is employed to determine the optimal number of measurements adaptively in time-frequency domain. The better performance of the proposed PSD estimation for pressure signals is demonstrated with simulation results, a compared with conventional algorithms.

This paper is organized as follows: Section II briefly describes the statistical model for pressure signals. The new Kalman filter with variable number of measurements algorithm is presented in Section III. Section IV is devoted to the adaptive parametric PSD estimation. Simulation results and comparisons are given in Section V. Finally, conclusions are drawn in Section VI.

II. Pressure Signal Model

In [4], Aboy *et al.* developed a synthetic physiologic simulator for arterial blood pressure (ABP) and intracranial pressure (ICP) signals. In this harmonic model, the effects of pulse pressure variation are regarded as a conventional amplitude modulation (AM) of a multi-frequency pulse pressure carrier with respiration as the modulating signal. Synthetic pressure signals are generated using the following model:

$$y(t) = u_p + [1 + ar_n(t)] \cdot \sum_{n=1}^{N} C_n e^{j2\pi f_c nt} + kr(t). \tag{1}$$

In this model, u_p is the DC component of the pressure signal, $r_n(t)$ is the normalized respiratory signal ($|r_n(t)| \le 1$), and a is the modulation index. The carrier signal is a quasi-periodic signal with an arbitrary pulse morphology that can be approximated as a multi-harmonic periodic signal with a fundamental cardiac frequency f_c. The respiratory signal $r_n(t)$ can also be modeled as a multi-harmonic signal with a fundamental respiratory frequency f_r. To tell the frequency variability of f_c and f_r, they can be both considered as the sum of a constant carrier frequency $\bar{f}$ and a stochastic frequency variation $\lambda(t)$:

$$f_r(t) = \bar{f}_r + \lambda_r(t), \tag{2}$$

$$\text{with } \lambda_r(t) = -\sum_{k=1}^{P1} a_k \lambda_r(t-k) + w(t), \tag{3}$$

$$f_c(t) = \bar{f}_c + \lambda_c(t), \tag{4}$$

$$\text{with } \lambda_c(t) = -\sum_{k=1}^{P2} b_k \lambda_r(t-k) + w(t) + \sum_{k=0}^{Q} h_c(k) \lambda_r(t-k). \tag{5}$$

Here, the cardiac stochastic frequency variation $\lambda_c(t)$ and the respiratory variation $\lambda_r(t)$ are modeled as two correlated AR processes. The pressure signal is passed through a fading multipath channel to incorporate the nonstationary pulse pressure variability of real ICP and ABP data. Finally, the general model equation is given by:

$$y(t) = \sum_{k=0}^{K-1} h(k,t) \left\{ u_p + [1 + a \sum_{l=1}^{K} \gamma_l e^{j2\pi(\sum \bar{f}_r + \lambda_r(t))l(t-k)}] \right.$$
$$\left. \cdot \sum_{n=1}^{N} C_n e^{j2\pi(\sum \bar{f}_c + \lambda_c(t))n(t-k)} + kr(t-k) \right\}. \tag{6}$$

In [4], this pressure signal model was employed to generate a nonstationary synthetic pressure signal for spectral analysis and achieved satisfactory results. Our simulations will also use this model to give the experimental data for our Kalman filter-based spectrum estimation. More details about the pressure signal model can be found in [4, 5].

III. Kalman Filter with Variable Measurements

References [1] – [3] and [5] proposed two similar parametric PSD estimation methods for nonstationary signals. Their Kalman filter-based algorithms only use one current measurement to update the state estimate. In this section, we will propose a new Kalman filter with variable number of measurements to improve the state tracking performance.

The conventional linear state-space model is given by:

$$x(t) = \mathbf{F}(t)x(t-1) + w(t), \tag{7}$$

$$y(t) = \mathbf{H}(t)x(t) + \varepsilon(t), \tag{8}$$

where $x(t)$ and $y(t)$ are respectively the state vector and the observation vector at time instant t. $\mathbf{F}(t)$ and $\mathbf{H}(t)$ are respectively the state transition matrix and the observation matrix. The state noise vector $w(t)$ and the observation noise vector $\varepsilon(t)$ are zero mean Gaussian noise with covariance matrices $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ respectively. Let $\hat{x}(t/k)$ ($k = t-1, t$) present the estimator of $x(t)$ given the measurements up to time instant k, and $\mathbf{P}(t/k)$ be the corresponding error covariance matrix of $\hat{x}(t/k)$. The standard Kalman filter recursions are given by:

$$\hat{x}(t+1/t) = \mathbf{F}(t)\hat{x}(t/t), \tag{9}$$

$$\mathbf{P}(t+1/t) = \mathbf{F}(t)\mathbf{P}(t/t)\mathbf{F}(t)^T + \mathbf{Q}(t), \tag{10}$$

$$e(t) = y(t) - \mathbf{H}(t)\hat{x}(t/t-1), \tag{11}$$

$$\mathbf{K}(t) = \mathbf{P}(t/t-1)\mathbf{H}(t)^T$$
$$\cdot [\mathbf{H}(t)\mathbf{P}(t/t-1)\mathbf{H}(t)^T + \mathbf{R}(t)]^{-1}, \tag{12}$$

$$\hat{x}(t/t) = \hat{x}(t/t-1) + \mathbf{K}(t)e(t), \tag{13}$$

$$\mathbf{P}(t/t) = [\mathbf{I} - \mathbf{K}(t)\mathbf{H}(t)]\mathbf{P}(t/t-1). \tag{14}$$

Our Kalman filter with variable measurements algorithm originates from the work of Durović and Kovačević [8], where they proposed a new Kalman filtering recursion using the equivalence between the Kalman filter and a particular least-squares (LS) regression problem. Combining (7) and (8) together, we get the following equivalent linear model:

$$\begin{bmatrix} \mathbf{I} \\ \mathbf{H}(t) \end{bmatrix} x(t) = \begin{bmatrix} \mathbf{F}\hat{x}(t-1/t-1) \\ y(t) \end{bmatrix} + \mathbf{E}(t), \tag{15}$$

where $\mathbf{E}(t) = \begin{bmatrix} \mathbf{F}[x(t-1) - \hat{x}(t-1/t-1)] + w(t-1) \\ -\varepsilon(t) \end{bmatrix}$ and

$E[\mathbf{E}(t)\mathbf{E}^T(t)] = \begin{bmatrix} \mathbf{P}(t/t-1) & 0 \\ 0 & \mathbf{R}(t) \end{bmatrix} = \mathbf{S}(t)\mathbf{S}^T(t)$. $\mathbf{S}(t)$ can be computed from the Cholesky decomposition of $E[\mathbf{E}(t)\mathbf{E}^T(t)]$. By multiplying both sides of (15) by $\mathbf{S}^{-1}(t)$, we get:

$$\mathbf{Y}(t) = \mathbf{X}(t)\beta(t) + \xi(t), \tag{16}$$

where $\mathbf{X}(t) = \mathbf{S}^{-1}(t)\begin{bmatrix} \mathbf{I} \\ \mathbf{H}(t) \end{bmatrix}$, $\mathbf{Y}(t) = \mathbf{S}^{-1}(t)\begin{bmatrix} \mathbf{F}\hat{x}(t-1) \\ y(t) \end{bmatrix}$, $\beta(t) = x(t)$, and $\xi(t) = -\mathbf{S}^{-1}(t)\mathbf{E}(t)$. Note that $\mathbf{E}(t)$ is whitened by $\mathbf{S}^{-1}(t)$ and the residual $\xi(t)$ satisfies $E[\xi(t)\xi^T(t)] = \mathbf{I}$. Equation (16) is a standard linear regression problem with LS solution:

$$\hat{\beta}(t) = \hat{x}(t/t) = (\mathbf{X}^T(t)\mathbf{X}(t))^{-1}\mathbf{X}^T(t)\mathbf{Y}(t), \tag{17}$$

and the covariance matrix of estimating $\hat{\beta}(t)$ is

$$E[(\beta(t)-\hat{\beta}(t))(\beta(t)-\hat{\beta}(t))^T]$$
$$= \mathbf{P}(t/t) = (\mathbf{X}^T(t)\mathbf{X}(t))^{-1}. \tag{18}$$

That is to say, the Kalman filter can be thought of as the solution to a weighted LS problem with $\hat{\beta}(t) = \hat{x}(t/t)$ and $\mathbf{P}(t/t) = \text{cov}(\hat{\beta}(t))$. Equations (16)–(18) form an equivalent Kalman filtering algorithm based on LS criterion.

To derive the proposed Kalman filter with variable measurement equations, let's rewrite (16) as

$$\left\{ \mathbf{S}^{-1}(t)\begin{bmatrix} \mathbf{F}\hat{x}(t-1) \\ y(t) \end{bmatrix} \right\} = \left\{ \mathbf{S}^{-1}(t)\begin{bmatrix} \mathbf{I} \\ \mathbf{H}(t) \end{bmatrix} \right\}x(t)+\xi(t). \tag{19}$$

The lower part of the equation is a conventional LS estimation of $x(t)$ from the current measurement $y(t)$. The upper part is a regularization term that imposes a smoothness constraint from the state dynamic into the LS problem. If $\mathbf{F}$ is an identity matrix, (19) is equivalent to the LMS algorithm with diagonal loading.

Because only one measurement is used to update the state vector in (19), the bias error is low especially when the system is fast time-varying. On the other hand, if the system is time-invariant or slow time-varying, more measurements used for tracking state vector can reduce the estimation variance. Given the block of measurements as $[y(t-L),\cdots,y(t),\cdots,y(t+L)]$, where the total number of measurements is $h=2L+1$. Including all these measurements in (16) or (19) gives:

$$\mathbf{Y}(t) = \mathbf{S}^{-1}(t)[\{\mathbf{F}\hat{x}(t-1)\}^T, y(t-L),\cdots,y(t),\cdots,y(t+L)]^T, \tag{20}$$

$$\mathbf{X}(t) = \mathbf{S}^{-1}(t)[\mathbf{I}, \mathbf{H}^T(t-L),\cdots,\mathbf{H}^T(t),\cdots,\mathbf{H}^T(t+L)]^T, \tag{21}$$

and. $\mathbf{S}^{-1}(t)$ is now obtained from

$$\begin{bmatrix} \mathbf{P}(t/t-1) & \mathbf{0} \\ \mathbf{0} & diag\{\mathbf{R}(t-L),\cdots,\mathbf{R}(t),\cdots,\mathbf{R}(t+L)\} \end{bmatrix}$$ in the

new algorithm. The Kalman filter with variable measurements problem can be solved using (17), with $\mathbf{Y}(t)$ in (20) and $\mathbf{X}(t)$ in (21).

IV. ADAPTIVE PARAMETRIC SPECTRUM ESTIMATION

The synthetic nonstationary pressure signal $y(t)$ in (1) can be characterized with an M-order AR model:

$$y(t) = \sum_{i=1}^{M} a(i,t)y(t-i) + \eta(t), \tag{22}$$

where $a(i,t)$ is the time-varying AR coefficient and $\eta(t)$ is assumed to be a zero mean Gaussian white noise sequence with variance σ^2. A stochastically perturbed k-th order difference equation constraint model is used to describe the change of the AR coefficients in [1] − [3]:

$$\nabla^k a(i,t) = \delta(i,t), i=1,\cdots,M, \tag{23}$$

where $\delta(i,t)$ is assumed to be a zero mean Gaussian white noise sequence with variance $\tau_i^2 = \tau^2$, $i=1,\cdots,M$. For

convenience, k is assumed to be 1 in this paper. That is to say, the AR coefficients $a(i,t)$ can be seen as a one-order AR process. When $k=1$, the difference equation constraint in (23) becomes:

$$a(i,t) = a(i,t-1) + \delta(i,t). \tag{24}$$

Define the state vector $x(t)$ by $x(t) = [a(1,t),\cdots,a(M,t)]^T$, (24) can be rewritten in the form of the state equation in the state-space model:

$$x(t) = \mathbf{F}(t)x(t-1) + \delta(t), \tag{25}$$

where the state transition matrix $\mathbf{F}(t)$ is a $M \times M$ identity matrix, $\delta(t) = [\delta(1,t),\cdots,\delta(M,t)]^T$ is the state noise vector. Similarly, the space equation is obtained from (22):

$$y(t) = \mathbf{H}(t)x(t-1) + \eta(t), \tag{26}$$

where $\mathbf{H}(t) = [y(t-1),\cdots,y(t-M)]$ is the observation matrix. We can see that the state noise variance is $\mathbf{Q}(t) = diag(\tau^2,\cdots,\tau^2)$ and the observation noise variance is $\mathbf{R}(t) = \sigma^2$. So, given the linear state-space model composed of (25) and (26), the state vector $x(t)$ or the AR coefficients $a(i,t)$ can be estimated using the Kalman filter recursion. Using the estimated $a(i,t)$, the instantaneous PSD is calculated by:

$$P(t,f) = \hat{\sigma}^2(t) \bigg/ \left| 1 - \sum_{i=1}^{M} a(i,t)e^{-2\pi i j f} \right|. \tag{27}$$

where $j = \sqrt{-1}$, and $\hat{\sigma}^2(t)$ is the measurement noise variance estimate.

As mentioned earlier, if we use the proposed Kalman filter to estimate the AR coefficients with a block of measurements, the corresponding PSD will be affected by the number of measurements h. If h is given a small value, the PSD estimation gives a good time resolution. In other words, for fast varying time series, a small block size is preferred. On the contrary, when a large block size is chosen, the time resolution of PSD will be reduced, but the frequency resolution will be improved. So, if the instantaneous frequency of the time series changes slowly, a larger block size h is preferred.

Like the adaptive Lomb periodogram [8], ICI rule can be employed to choose the number of measurements adaptively in time-frequency plane. First, we need to calculate a series of PSDs with a series of h's. The ICI rule examines a sequence of confidence intervals of the PSDs to determine the optimal h^+ and the corresponding adaptive PSD. The details of ICI rule and adaptive spectrum estimation are omitted to save space, and more information can be found in [6, 7].

V. SIMULATION RESULTS

Using the pressure signal model introduced in Section II, we first generate a nonstationary pressure signal with the instantaneous frequency components in Fig. 1. The sample rate is 12.5Hz, and the time duration is 20s.

The number of measurements h's used to compute a series of PSDs are: 1, 9, 17, and 25. The parametric PSDs with $h = 1$ and $h = 25$ are shown in Fig. 2 and Fig. 3, respectively. Comparing Fig. 2 and 3, we can see that the PSD using the conventional Kalman filter ($h = 1$), has a better time resolution, but the frequency resolution is unsatisfactory because of the large variance of AR coefficient estimates. Fig. 3 with $h = 25$ exhibits a much better frequency resolution, which means a large number of measurements are used reduce the estimation variance. However, in the time segment when the frequency contents change rapidly ($10s < t < 11s$), a large h may lead to a blurred time resolution, as seen in Fig. 3.

Fig. 4 shows the smoothed PSD after an average operation over 10 samples. This smoothing method was proposed in [5], and we can see that the time resolution becomes poor, although the frequency resolution is improved. To achieve a better time and frequency resolutions at the same time, we need ICI rule to select the adaptive number of measurements. Fig. 5 illustrates that the proposed adaptive PSD with ICI rule is able to achieve good time as well as frequency resolutions.

VI. CONCLUSION

A new Kalman filter with variable measurements algorithm is applied in this paper to estimate the power spectral density of pressure signals. Including more measurements in Kalman filter, this proposed algorithm can achieve a better tracking performance than the conventional Kalman filter. ICI rule can be used to choose the optimal number of measurements adaptively, and yield the spectrum estimation with better time-frequency resolution than the conventional algorithms.

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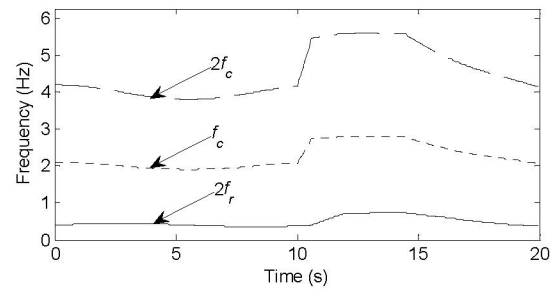
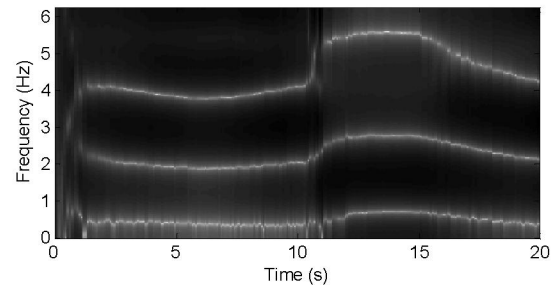Fig. 1. Theoretical time-frequency content.
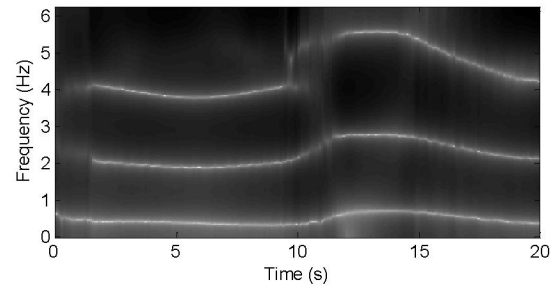

Fig. 2. PSD using Kalman filter with h=1.
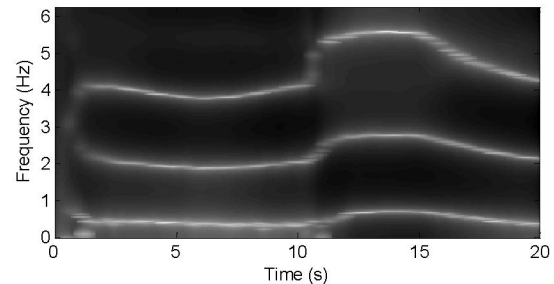

Fig. 3. PSD using Kalman filter with h=25.

Fig. 4. PSD using Kalman filter with h=1, after smoothing in frequency domain.
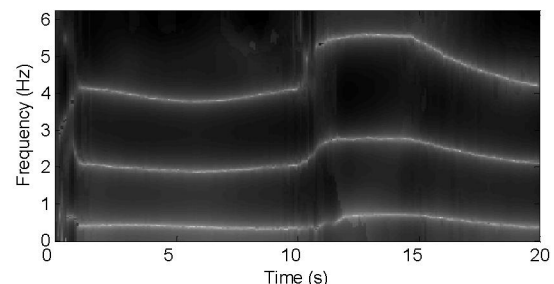

Fig. 5. PSD using Kalman filter with adaptive h.