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Optimal Model Reduction of Stable Delay Systems *

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Abstract

A model reduction method for stable delay systems under L_2 optimality is introduced in this paper. The reduced models may take the form of either a stable finite dimensional system or a delay system with reduced order finite dimensional part. Based on the Routh parametrization of stable systems, the two cases are studied under a unified framework of unconstrained optimization. Numerical examples are used to illustrate the effectiveness of the proposed method.

1. Introduction

Model reduction has been a popular research area and has attracted a lot of attention in the last few decades. Various model reduction methods have been proposed and algorithms of diverse complexity have been presented. The choice of error measures for model reduction is often a compromise between their physical significance and the associated mathematical or computational tractability. The most commonly norm used for measuring model reduction error is the L_2 norm [2,5,6,9,17] . The importance of this error criterion is that the L_2 norm of a system is the expected root-mean-square value of the output when the input is a unit variance white noise process. A well-established approach for treating L_2 optimal model reduction is to establish and utilize the necessary conditions for optimality [5,10,16,17] . However, many of the algorithms derived lack the proof of convergence except in some special cases. The solution technique often applied in optimal L_2 model reduction recently is based on parameter optimization [4,9,18–20] . The main difficulties in formulating an effective solution procedure for any optimal L_2 model reduction is the preservation of stability in the reduced order models when the original models are stable. This complicates the optimization process by imposing certain constraints to the optimization problems [4,9,20] .

In many engineering applications, control systems cannot be described accurately without the introduction of delay element(s). A class of delay systems with delay in an input-output sense takes the form $\exp(-sT_d)G(s)$, where $G(s)$ is a stable strictly proper real rational transfer function matrix, and T_d is the delay time. Many methods have been proposed to approximate $\exp(-sT_d)G(s)$ by using the Padé approximants of $\exp(-sT_d)$, for example, Johnson *et al.* [11] , Marshak *et al.* [14] , and recently Lam [12] . However, when time delay systems are approximated by finite dimensional systems, the order of the reduced models, in many situations, have to be high for good approximations. If a time delay, of different delay time value, is also permitted in the reduced model, the approximation might be substantially improved and the original system can be approximated with fewer parameters. Xue *et al.* [18] and Yang *et al.* [20] proposed methods to obtain a reduced order time delay system of the form $\exp(-s\tau_d)\widetilde{G}(s)$, where $\widetilde{G}(s)$ is a finite dimensional system with order lower than $G(s)$ and $\tau_d > 0$, based on parameter optimization under the L_2 criterion using a gradient-based method and the genetic algorithm respectively. It is worth noting that in [18] , the L_2 error measure is only approximately minimized and the method also fails to ensure the stability of the reduced models. In [20] , though the stability is ensured, the formulation imposed constraints on the admissible class of reduced models.

In this paper, a novel method is proposed to obtain reduced order models for SISO delay systems of the form $\exp(-sT_d)G(s)$. Two cases will be treated in a unified formulation. Namely, the approximation may take the form exp $(-s\tau_d)\widetilde{G}(s)$ with $\widetilde{G}(s)$ finite dimensional and $\tau_d = 0$ or $\tau_d > 0$. The proposed method is based on the Routh parametrization of stable systems which leads to an unconstrained optimization procedure. The optimal parameters are obtained by minimizing the L_2 approximation error through a gradient-based method. The formulas of the L_2 error measure and its gradients are explicitly expressed.

2. Problem Formulation

Consider a linear time-invariant system with delay time $T_d > 0$ described by

$$\mathcal{G}(s) := \exp(-sT_d)G(s) \qquad (1)$$

where $G(s)$ is a stable linear finite-dimensional SISO

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system with minimal realization $(A, \mathbf{b}, \mathbf{c})$, that is,

$$G(s) = \mathbf{c}(sI - A)^{-1}\mathbf{b}$$

with $A \in \mathbb{R}^{n\times n}$, $\mathbf{c} \in \mathbb{R}^{1\times n}$ and $\mathbf{b} \in \mathbb{R}^{n\times 1}$. The *optimal L_2 model reduction problem* is to find an mth order stable reduced order system $\widetilde{G}(s)$ with delay $\tau_d \geq 0$,

$$\widetilde{\mathcal{G}}(s) = \exp(-s\tau_d)\widetilde{G}(s) \text{ where } \widetilde{G}(s) = \widetilde{\mathbf{c}}(sI - \widetilde{A})^{-1}\widetilde{\mathbf{b}}$$

with $\widetilde{A} \in \mathbb{R}^{m\times m}$, $\widetilde{\mathbf{c}} \in \mathbb{R}^{1\times m}$ and $\widetilde{\mathbf{b}} \in \mathbb{R}^{m\times 1}$, such that the L_2 error

$$E = \left\| \exp(-sT_d)G(s) - \exp(-s\tau_d)\widetilde{G}(s) \right\|_2$$

is minimized. To solve the optimal L_2 model reduction problem for $\mathcal{G}(s)$, a parametrization of the stable reduced order systems is employed and then a gradient-based unconstrained optimization method to obtain the optimal parameters is applied.

Suppose that the stable linear $\widetilde{G}(s)$ is expressed as

$$\widetilde{G}(s) = \frac{b_1 s^{m-1} + \cdots + b_m}{a_0 s^m + a_1 s^{m-1} + \cdots + a_m}$$

Hutton and Friedland [8] studied the Routh approximation by expanding $\widetilde{G}(s)$ in continued-fraction form given by

$$\widetilde{G}(s) = \cfrac{1}{1 + \alpha_1 s + \cfrac{1}{\alpha_2 s + \cfrac{\ddots}{+\frac{1}{\alpha_m s}}}} \times$$

$$\left\{ \beta_1 + \cfrac{1}{\alpha_2 s + \cfrac{1}{\alpha_3 s + \cfrac{\ddots}{+\frac{1}{\alpha_m s}}}} \left[\beta_2 + \cdots + \left(\frac{\beta_m}{\alpha_m s} \right) \cdots \right] \right\}$$

where α_i and β_i ($i = 1, 2, \ldots, m$) are scalars obtained from the alpha and beta tables. A state-space realization of $\widetilde{G}(s)$ (see [7]) is $(\widetilde{A}, \widetilde{\mathbf{b}}, \widetilde{\mathbf{c}})$ with

$$\widetilde{A} = \begin{bmatrix} -\frac{1}{\gamma_1^2} & -\frac{1}{\gamma_1^2} & & & \\ \frac{1}{\gamma_2^2} & 0 & -\frac{1}{\gamma_2^2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{\gamma_{m-1}^2} & 0 & -\frac{1}{\gamma_{m-1}^2} \\ & & & \frac{1}{\gamma_m^2} & 0 \end{bmatrix} \quad \widetilde{\mathbf{b}} = \begin{bmatrix} \frac{1}{\gamma_1^2} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$\widetilde{\mathbf{c}} = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \cdots & \beta_m \end{bmatrix} \tag{2}$$

where

$$\gamma_i^2 = \alpha_i \quad , \quad i = 1, 2, \ldots, m$$

In the case where $\widetilde{G}(s)$ is restricted to have numerator polynomial of degree r, then one can simply take $\beta_i = 0$ for $i = 1, 2, \ldots, (m - r)$.The following result relates the stability of $\widetilde{G}(s)$ to the signs of the parameters.

Proposition 1 [8] *$\widetilde{G}(s)$ is asymptotically stable if and only if $\alpha_i > 0$, $i = 1, 2, \ldots, m$.*

With these properties, we have a parametrization method, using two parameter vectors $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$, where

$$\begin{aligned} \boldsymbol{\gamma} &= \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \cdots & \gamma_m \end{bmatrix}^T \quad , \quad \gamma_i \neq 0 \\ \boldsymbol{\beta} &= \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \cdots & \beta_m \end{bmatrix}^T \end{aligned}$$

to describe all stable strictly proper $\widetilde{G}(s)$. Since τ_d is also required to be nonnegative in the approximation, it is expressed in terms of τ such that $\tau_d = \tau^2$. With this and the Routh parametrization of stable systems, the following unconstrained optimization problem

$$\min_{\gamma,\beta,\tau} E^2 \tag{3}$$

is formulated. It can be easily seen that the set of optimization parameters is open and dense in $\mathbb{R}^{2m+1}$.

Remark 1 *It is known that the Schwarz canonical realization* [13] *of $\widetilde{G}(s)$ is similarity equivalent to the Routh canonical realization. Thus, the idea of the present development is also applicable to a Schwarz realization parametrization.*

3. Error and Gradient Formulas

3.1 Error Expression

Let $g(t)$ and $\widetilde{g}(t)$ be the impulse response of $\exp(-sT_d)G(s)$ and $\exp(-s\tau^2)\widetilde{G}(s)$ respectively, then

$$E = \sqrt{\int_0^\infty (\widetilde{g}(t) - g(t))(\widetilde{g}(t) - g(t))^T dt} \tag{4}$$

Since

$$\widetilde{g}(t) = \begin{cases} 0, & 0 \leq t < \tau^2 \\ \widetilde{\mathbf{c}} \exp(\widetilde{A}(t - \tau^2))\widetilde{\mathbf{b}}, & \tau^2 \leq t \end{cases}$$

and

$$g(t) = \begin{cases} 0, & 0 \leq t < T_d \\ \mathbf{c} \exp(A(t - T_d))\mathbf{b}, & T_d \leq t \end{cases}$$

Hence, E can be obtained by the sum as follows,

$$E^2 = E_1^2 + E_2^2 \tag{5}$$

where $E_1 \geq 0$ and $E_2 \geq 0$ are given by

$$E_1 := \sqrt{\int_{\tau^2}^{T_d} \widetilde{g}(t)\widetilde{g}(t)^T dt}$$

$$E_2 := \sqrt{\int_{T_d}^{\infty} (\widetilde{g}(t) - g(t))(\widetilde{g}(t) - g(t))^T dt}$$

for $\tau^2 \leq T_d$, and

$$E_1 := \sqrt{\int_{T_d}^{\tau^2} g(t)g(t)^T dt}$$

$$E_2 := \sqrt{\int_{\tau^2}^{\infty} (\widetilde{g}(t) - g(t))(\widetilde{g}(t) - g(t))^T dt}$$

for $\tau^2 > T_d$.

Theorem 1 *With the notation above, the L_2 model reduction error between $\mathcal{G}(s)$ and $\widetilde{\mathcal{G}}(s)$ is given by*

$$E = \begin{cases} \sqrt{\mathbf{c}P_1\mathbf{c}^T - 2\mathbf{c}\widetilde{P}_0 \exp(\widetilde{A}^T(T_d-\tau^2))\widetilde{\mathbf{c}}^T + \widetilde{\mathbf{c}}\widetilde{P}_2\widetilde{\mathbf{c}}^T}, & \text{if } \tau^2 \le T_d \\ \sqrt{\mathbf{c}P_1\mathbf{c}^T - 2\mathbf{c}\exp(A(\tau^2-T_d))\widetilde{P}_0\widetilde{\mathbf{c}}^T + \widetilde{\mathbf{c}}\widetilde{P}_2\widetilde{\mathbf{c}}^T} & \text{if } \tau^2 > T_d \end{cases} \quad (6)$$

where P_1, $\widetilde{P}_0$ and $\widetilde{P}_2$ are respectively the solutions of

$$AP_1 + P_1A^T + \mathbf{b}\mathbf{b}^T = 0 \quad (7)$$

$$A\widetilde{P}_0 + \widetilde{P}_0\widetilde{A}^T + \mathbf{b}\widetilde{\mathbf{b}}^T = 0 \quad (8)$$

$$\widetilde{A}\widetilde{P}_2 + \widetilde{P}_2\widetilde{A}^T + \widetilde{\mathbf{b}}\widetilde{\mathbf{b}}^T = 0 \quad (9)$$

Proof: Notice that P_1 and $\widetilde{P}_2$ are the controllability grammians of $G(s)$ and $\widetilde{G}(s)$ respectively. For $\tau^2 \le T_d$, we have E_1^2 given by

$$\int_{\tau^2}^{T_d} \widetilde{g}(t)\widetilde{g}(t)^T dt$$

$$= \widetilde{\mathbf{c}}\widetilde{P}_2\widetilde{\mathbf{c}}^T - \widetilde{\mathbf{c}}\exp(\widetilde{A}(T_d-\tau^2))\widetilde{P}_2\exp(\widetilde{A}^T(T_d-\tau^2))\widetilde{\mathbf{c}}^T$$

where

$$\widetilde{P}_2 = \int_0^\infty \exp(\widetilde{A}t)\widetilde{\mathbf{b}}\widetilde{\mathbf{b}}^T\exp(\widetilde{A}^Tt)dt$$

satisfies the Lyapunov equation (9). And for E_2^2 in (5), we have E_2^2 given by

$$\int_{T_d}^{\infty} \left(\mathbf{c}\exp(A(t-T_d))\mathbf{b} - \widetilde{\mathbf{c}}\exp(\widetilde{A}(t-\tau^2))\widetilde{\mathbf{b}}\right) \times \left(\mathbf{c}\exp(A(t-T_d))\mathbf{b} - \widetilde{\mathbf{c}}\exp(\widetilde{A}(t-\tau^2))\widetilde{\mathbf{b}}\right)^T dt$$

$$= \begin{bmatrix} \mathbf{c} & -\widetilde{\mathbf{c}} \end{bmatrix} P \begin{bmatrix} \mathbf{c}^T \\ -\widetilde{\mathbf{c}}^T \end{bmatrix}$$

where P satisfies

$$\begin{bmatrix} A & 0 \\ 0 & \widetilde{A} \end{bmatrix} P + P \begin{bmatrix} A^T & 0 \\ 0 & \widetilde{A}^T \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \exp(\widetilde{A}(T_d-\tau^2))\widetilde{\mathbf{b}} \end{bmatrix} \begin{bmatrix} \mathbf{b}^T & \widetilde{\mathbf{b}}^T\exp(\widetilde{A}^T(T_d-\tau^2)) \end{bmatrix} = 0 \quad (10)$$

Suppose

$$P = \begin{bmatrix} P_1 & P_0 \\ P_0^T & P_2 \end{bmatrix}$$

therefore (10) becomes

$$AP_1 + P_1A^T + \mathbf{b}\mathbf{b}^T = 0$$

$$AP_0 + P_0\widetilde{A}^T + \mathbf{b}\widetilde{\mathbf{b}}^T\exp(\widetilde{A}^T(T_d-\tau^2)) = 0 \quad (11)$$

$$\widetilde{A}P_2 + P_2\widetilde{A}^T + \exp(\widetilde{A}(T_d-\tau^2))\widetilde{\mathbf{b}}\,\widetilde{\mathbf{b}}^T\exp(\widetilde{A}^T(T_d-\tau^2)) = 0 \quad (12)$$

From (9) and (12), it is noted that

$$P_2 = \exp(\widetilde{A}(T_d-\tau^2))\widetilde{P}_2\exp(\widetilde{A}^T(T_d-\tau^2))$$

Substitute E_1^2 and E_2^2 into (5), and notice the relationship between P_2 and $\widetilde{P}_2$, we have E^2 given by

$$\widetilde{\mathbf{c}}\widetilde{P}_2\widetilde{\mathbf{c}}^T - \widetilde{\mathbf{c}}\exp(\widetilde{A}(T_d-\tau^2))\widetilde{P}_2\exp(\widetilde{A}^T(T_d-\tau^2))\widetilde{\mathbf{c}}^T + \begin{bmatrix} \mathbf{c} & -\widetilde{\mathbf{c}} \end{bmatrix} \begin{bmatrix} P_1 & P_0 \\ P_0^T & P_2 \end{bmatrix} \begin{bmatrix} \mathbf{c}^T \\ -\widetilde{\mathbf{c}}^T \end{bmatrix}$$

$$= \widetilde{\mathbf{c}}\widetilde{P}_2\widetilde{\mathbf{c}}^T + \mathbf{c}P_1\mathbf{c}^T - 2\mathbf{c}P_0\widetilde{\mathbf{c}}^T$$

Now if we let

$$P_0 = \widetilde{P}_0\exp(\widetilde{A}^T(T_d-\tau^2))$$

then from (11), $\widetilde{P}_0$ satisfies (8). The result for $\tau^2 \le T_d$ follows. The case with $\tau^2 > T_d$ also follows similary and hence omitted. ■

3.2 Gradient Formulas

From (6), for $\tau^2 < T_d$, the partial derivatives of E^2 are given by

$$\frac{\partial E^2}{\partial \gamma_i} = -2\mathbf{c}\frac{\partial \widetilde{P}_0}{\partial \gamma_i}\exp(\widetilde{A}^T(T_d-\tau^2))\widetilde{\mathbf{c}}^T - 2\mathbf{c}\widetilde{P}_0\frac{\partial \exp(\widetilde{A}^T(T_d-\tau^2))}{\partial \gamma_i}\widetilde{\mathbf{c}}^T + \widetilde{\mathbf{c}}\frac{\partial \widetilde{P}_2}{\partial \gamma_i}\widetilde{\mathbf{c}}^T \quad (13)$$

$$\frac{\partial E^2}{\partial \beta_i} = -2\mathbf{c}\widetilde{P}_0\exp(\widetilde{A}^T(T_d-\tau^2))\frac{\partial \widetilde{\mathbf{c}}^T}{\partial \beta_i} + 2\frac{\partial \widetilde{\mathbf{c}}}{\partial \beta_i}\widetilde{P}_2\widetilde{\mathbf{c}}^T \quad (14)$$

$$\frac{\partial E^2}{\partial \tau} = -4\tau\mathbf{c}\widetilde{P}_0\exp(\widetilde{A}^T(T_d-\tau^2))\widetilde{A}^T\widetilde{\mathbf{c}}^T \quad (15)$$

for $i = 1, 2, \ldots, m$. While for $\tau^2 > T_d$, the partial derivatives of E^2 are

$$\frac{\partial E^2}{\partial \gamma_i} = -2\mathbf{c}\exp(A(\tau^2-T_d))\frac{\partial \widetilde{P}_0}{\partial \gamma_i}\widetilde{\mathbf{c}}^T + \widetilde{\mathbf{c}}\frac{\partial \widetilde{P}_2}{\partial \gamma_i}\widetilde{\mathbf{c}}^T \quad (16)$$

$$\frac{\partial E^2}{\partial \beta_i} = -2\mathbf{c}\exp(A(\tau^2-T_d))\widetilde{P}_0\frac{\partial \widetilde{\mathbf{c}}^T}{\partial \beta_i} + 2\frac{\partial \widetilde{\mathbf{c}}}{\partial \beta_i}\widetilde{P}_2\widetilde{\mathbf{c}}^T \quad (17)$$

$$\frac{\partial E^2}{\partial \tau} = -4\tau\mathbf{c}\exp(A(\tau^2-T_d))A\widetilde{P}_0\widetilde{\mathbf{c}}^T \quad (18)$$

for $i = 1, 2, \ldots, m$ where $\frac{\partial \widetilde{P}_0}{\partial \gamma_i}$ and $\frac{\partial \widetilde{P}_2}{\partial \gamma_i}$ are obtained from

$$A\frac{\partial \widetilde{P}_0}{\partial \gamma_i} + \frac{\partial \widetilde{P}_0}{\partial \gamma_i}\widetilde{A}^T + \left(\widetilde{P}_0\frac{\partial \widetilde{A}}{\partial \gamma_i}^T + \mathbf{b}\frac{\partial \widetilde{\mathbf{b}}}{\partial \gamma_i}^T\right) = 0 \quad (19)$$

$$\widetilde{A}\frac{\partial \widetilde{P}_2}{\partial \gamma_i} + \frac{\partial \widetilde{P}_2}{\partial \gamma_i}\widetilde{A}^T + \left(\frac{\partial \widetilde{A}}{\partial \gamma_i}\widetilde{P}_2 + \widetilde{P}_2\frac{\partial \widetilde{A}}{\partial \gamma_i}^T + 2\widetilde{\mathbf{b}}\frac{\partial \widetilde{\mathbf{b}}}{\partial \gamma_i}^T\right) = 0 \quad (20)$$

On the other hand, $\frac{\partial \widetilde{A}}{\partial \gamma_i}$, $\frac{\partial \widetilde{\mathbf{b}}}{\partial \gamma_i}$ and $\frac{\partial \widetilde{\mathbf{c}}}{\partial \beta_i}$ are given by

$$\frac{\partial \widetilde{A}}{\partial \gamma_1} = \frac{2}{\gamma_1^3}\left(\mathbf{e}_1\mathbf{e}_1^T + \mathbf{e}_1\mathbf{e}_2^T\right)$$

$$\frac{\partial \widetilde{A}}{\partial \gamma_i} = \frac{2}{\gamma_i^3}\left(-\mathbf{e}_i\mathbf{e}_{i-1}^T + \mathbf{e}_i\mathbf{e}_{i+1}^T\right), \quad i = 2, 3, \ldots, m$$

$$\frac{\partial \widetilde{\mathbf{b}}}{\partial \gamma_1} = -\frac{2}{\gamma_1^3}\mathbf{e}_1, \quad \frac{\partial \widetilde{\mathbf{b}}}{\partial \gamma_i} = 0, \quad i = 2, 3, \ldots, m$$

$$\frac{\partial \widetilde{\mathbf{c}}}{\partial \beta_i} = \mathbf{e}_i^T, \quad i = 1, 2, \ldots, m$$

where $\mathbf{e}_i$, $i = 1, 2, \ldots, m$ is the ith standard basis vector of $\mathbb{R}^m$.

As for $\frac{\partial \exp(\widetilde{A}(T_d-\tau^2))}{\partial \gamma_i}$, [15, Proposition 4.10] gives a direct method expressed in the following equation

$$\exp\begin{bmatrix} \widetilde{A}(T_d-\tau^2) & \frac{\partial(\widetilde{A}(T_d-\tau^2))}{\partial \gamma_i} \\ 0 & \widetilde{A}(T_d-\tau^2) \end{bmatrix} =$$

$$\begin{bmatrix} \exp(\tilde{A}(T_d - \tau^2)) & \frac{\partial \exp(\tilde{A}(T_d-\tau^2))}{\partial \gamma_i} \\ 0 & \exp(\tilde{A}(T_d - \tau^2)) \end{bmatrix} \quad (21)$$

In (21), only $\frac{\partial \exp(\tilde{A}(T_d-\tau^2))}{\partial \gamma_i}$ is unknown and it is easy to obtain.

4. Algorithm and Examples

With the expression of E^2 and its gradient formulas, existing gradient-based globally convergent optimization methods [3] can be applied to solve the unconstrained optimization problem (3). Though one cannot guarantee that local optimal solutions are in fact global, numerical tests indicated that the algorithm given below is effective when the initial choice of γ, β correspond to a good initial approximation model such as, for example, a Routh approximation.

Remark 2 *It can be seen that $\tau = 0$ is a solution of (15) or (18), so if $\tau = 0$ is chosen as the initial value, an mth finite dimensional approximation $\tilde{G}(s)$ of time delay system $\mathcal{G}(s)$ is obtained. In this case, it is not necessary to constrain $m < n$.*

4.1 Model Reduction Algorithm

In this subsection, a model reduction algorithm is summarized as follows.

Step 1 Generate the initial values of γ, β and τ .

(a) Set $\tau = 0$ for obtaining a finite dimensional approximation: Form a finite dimensional approximation model of $\mathcal{G}(s)$ based on Padé approximation of $\exp(-sT)$ [1] and then obtain initial values of γ and β from the Routh table. *Or*

(b) Set $\tau^2 = T_d$ for obtaining a reduced order system with delay: Obtain an mth order Routh approximation from $G(s)$ and then obtain initial values of γ and β from the Routh table. (Notice that the left-derivative of E with respect to τ is used for calculating the initial gradient.)

Step 2 Obtain the optimal parameters γ, β and τ by solving problem (3).

(I) Calculate the objective function E^2 of problem (3) by (6), (7), (8) and (9), where $\tilde{A}$, $\tilde{b}$ and $\tilde{c}$ are given in (2).

(II) Obtain the gradients of E^2 with respect to γ, β and τ given by (13), (14), (15), (16), (17) and (18) via (19), (20) and (21).

(III) Find the optimal parameters γ, β and τ.

Step 3 Form the optimal finite dimensional approximation or reduced order delay system by substituting the optimal γ and β into (2).

4.2 Numerical Examples

Example 1: Consider a delay system [12] with transfer function given by

$$\mathcal{G}(s) = \frac{1}{(s+1)^2}\exp(-s)$$

The objective of this example is to obtain finite dimensional models $\tilde{G}(s)$ to approximate $\mathcal{G}(s)$ and $\tau = 0$ is fixed.

For different orders of reduction, the corresponding optimal L_2 errors E obtained by the proposed method are summarized and compared in Table 1. $E_{[r-1/r]}$ and $E_{[r/r]}$ are the L_2 errors given in [12] corresponding to the cases where $\exp(-sT_d)$ are approximated by the $[r-1/r] =: R_{[r-1/r]}$ and $[r/r] =: R_{[r/r]}$ Padé approximants of $\exp(-sT_d)$ respectively with $r = n-2$.

n	$E_{[r-1/r]}$	$E_{[r/r]}$	E
3	0.1537	0.1087	0.0627
4	0.0558	0.0499	0.0308
5	0.0293	0.0295	0.0177
6	0.0186	0.0200	0.0114
7	0.0132	0.0146	0.0080
8	0.0099	0.0112	0.0059
9	0.0079	0.0090	0.0046
10	0.0064	0.0074	0.0037
11	0.0053	0.0062	0.0030

Table 1. Comparison of approximation errors

The frequency response errors $|\mathcal{G}(j\omega) - R_{[r-1/r]}(j\omega)G(j\omega)|$ $|\mathcal{G}(j\omega) - R_{[r/r]}(j\omega)G(j\omega)|$ and $\left|\mathcal{G}(j\omega) - \tilde{G}(j\omega)\right|$, over $\omega \in [4, 100]$ for $m = 10$ is shown in Figure 1. It is observed that the E is significantly smaller when compared with the Padé approach.

Example 2: Consider a time delay system [18] given by

$$\mathcal{G}(s) = \frac{(s+1)(s-1)(s+10)}{(s+2)^3(s+3)(s+4)}\exp(-0.5s)$$

The delay system is to be approximated by $\exp(-s\tau_d)\tilde{G}(s)$ where $\tilde{G}(s)$ is second order. The reduced time delay system is

$$\tilde{\mathcal{G}}(s) = \frac{0.2032s - 0.2365}{s^2 + 1.6704s + 2.4444}\exp(-0.6371s)$$

Its associated L_2 error is 0.0414. In [18] , six approximation models with delay for $m = 2$ are given. The one with the smallest L_2 error calculated by (6) is

$$\mathcal{G}_2(s) = \frac{0.3016s - 0.3075}{s^2 + 2.4228s + 2.9518}\exp(-0.6823s)$$

which corresponds to an L_2 error equals 0.0571. The present approach gives a smaller L_2 error.

5. Conclusion

In this paper, a new model reduction method for time delay systems has been presented. A stable finite dimensional system or a delay system with reduced order finite dimensional part can be obtained to approximate a stable time delay systems with L_2 optimality via an unconstrained gradient-based optimization procedure. The effectiveness of the approach is demonstrated via numerical examples.

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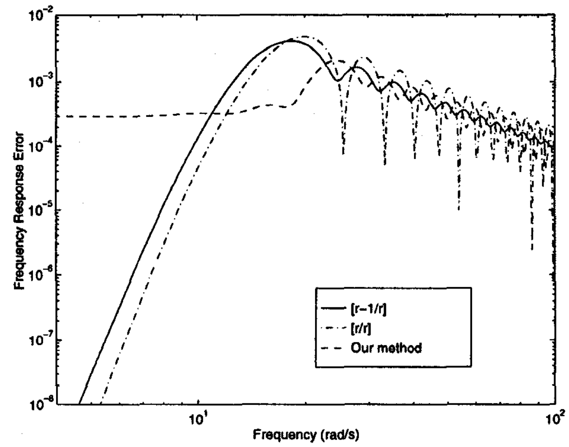

Figure 1: Frequency response error comparison for $m = 10$

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