# An inequality between the diameter and the inverse dual degree of a tree

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**Abstract**

Let $R(T), D(T)$ be respectively the radius and diameter of a nontrivial tree $T$ and $I(T) = \sum_{u \in V(T)} 1/\overline{d}(u)$ be the inverse dual degree, where $\overline{d}(u) = (\sum_{v \in N(u)} d(v))/d(u)$ for each $u \in V(T)$. In this note we prove that

$$I(T) \geq \begin{cases} R(T) + 1/3, & \text{if } D(T) \text{ is odd} \\ R(T) + 5/6, & \text{if } D(T) \text{ is even,} \end{cases}$$

with equality if and only if $T$ is a path of at least 4 vertices. This inequality strengthens a conjecture of Graffiti.

Let $G = (V(G), E(G))$ be a simple, connected graph. The *distance* $d(u, v)$ between two vertices $u, v$ of $G$ is the minimal length of a path from $u$ to $v$ in $G$. The *diameter* $D(G)$ of $G$ is the largest distance between any two vertices of $G$. The *radius* $R(G)$ of $G$ is $\min_{u \in V(G)} \max_{v \in V(G)} d(u, v)$. If $S \subseteq V(G)$ and $u \in V(G) \setminus S$, then we denote $d(u, S) = \min_{v \in S} d(u, v)$. The *neighbours* of $u \in V(G)$ are vertices adjacent to $u$ in $G$ and the *neighbourhood* $N(u)$ of $u$ in $G$ is the set of neighbours of $u$. Since $G$ is simple, the degree of $u$ is $d(u) = |N(u)|$. The *dual degree* of $u$ and the *inverse dual degree* of $G$ are respectively $\overline{d}(u) = (\sum_{v \in N(u)} d(v))/d(u)$ and $I(G) = \sum_{u \in V(G)} 1/\overline{d}(u)$ [2]. When ambiguity arises we use $d_G(u), \overline{d}_G(u)$, etc., to emphasize that the underlying graph is $G$.

The main purpose of this note is to prove an inequality between $D(T)$ and $I(T)$. As a consequence we get an inequality involving $R(T)$ and $I(T)$ which strengthens the following conjecture (see [1, 3, 4, 5] for results relating to Graffiti conjectures).

**Graffiti Conjecture 577** For any (nontrivial) tree $T$, $I(T) \geq R(T)$.
(There are examples of $G$ which are not trees such that $I(G) < R(G)$.)

In the following we suppose $T$ is a (nontrivial) tree and $P = v_0 v_1 \ldots v_D$ is a path of maximal length in $T$, where $D = D(T)$. Then $d(v_0) = d(v_D) = 1$. Let

$$a = a(P) = |\{v \in V(T) \setminus V(P) : d(v) \geq 2\}|,$$

$$b = b(P) = \begin{cases} |\{i : d(v_i) \geq 3, 2 \leq i \leq D-2\}|, & \text{if } D \geq 4, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$c = c(P) = \begin{cases} |\{i : d(v_i) \geq 3, i = 1, D-1\}|, & \text{if } D \geq 2, \\ 0, & \text{if } D = 1. \end{cases}$$

**Theorem** $I(T) \geq D(T)/2 + a/3 + +b/10 + c/12 + 5/6$.

A *caterpillar* is a tree with the property that the removal of all degree-one vertices yields a path, called the spine. Note that if $T$ is a caterpillar, then $v_1 \ldots v_{D-1}$ is the spine. To prove the theorem we need the following lemmas.

**Lemma 1** *Suppose $T$ is not a caterpillar (so in particular $D(T) \geq 4$) and $u$ is a vertex not in $P$ such that $d(u) \geq 2$ and $d(u, V(P))$ is as large as possible. Let $T'$ be the subtree obtained from $T$ by deleting all degree-one vertices adjacent to $u$. Then $D(T) = D(T')$ and $I(T) \geq I(T') + 1/3$.*

**Proof** We first note that all but one neighbours of $u$ have degree one, for otherwise there would be a neighbour $w$ of $u$ not in $P$ with $d(w) \geq 2$ and $d(w, V(P)) > d(u, V(P))$, violating the choice of $u$. Suppose $N(u) = \{u_1, \ldots, u_m, v\}$ where $d(u_i) = 1$, for $i \in \{1, \ldots, m\}$ and $d(v) = r$. Then

$D(T) = D(T')$. Let $\sigma = \sum_{x \in N(v)\setminus\{u\}} d(x)$. Since $\sigma + 1 \geq r$, we have

$$\begin{array}{l} I(T) - I(T') \\ = \sum_{i=1}^{m} \frac{1}{d_T(u_i)} + (\frac{1}{d_T(u)} - \frac{1}{d_{T'}(u)}) + (\frac{1}{d_T(v)} - \frac{1}{d_{T'}(v)}) \\ = \frac{m}{m+1} + (\frac{m+1}{m+r} - \frac{1}{r}) + (\frac{r}{m+\sigma+1} - \frac{r}{\sigma+1}) \\ = 1 + \frac{1}{m+r} - \frac{1}{r} - \frac{1}{m+1} + (\frac{m}{m+r} + \frac{r}{m+\sigma+1} - \frac{r}{\sigma+1}) \\ \geq 1 + \frac{1}{m+r} - \frac{1}{r} - \frac{1}{m+1}. \end{array}$$

Note that $1/(m + x) - 1/x$ is an increasing function of $x$ and $r \geq 2, m \geq 1$. We have from the inequality above that

$$\begin{array}{l} I(T) - I(T') \\ \geq 1 + \frac{1}{m+2} - \frac{1}{2} - \frac{1}{m+1} \\ = \frac{1}{2} - \frac{1}{(m+1)(m+2)} \\ \geq \frac{1}{3}. \end{array}$$

Q.E.D.

**Lemma 2** *Suppose $T$ is a caterpillar but not a path and $D = D(T) \geq 4$. If $d(v_1) \geq 3$ (respectively $d(v_{D-1}) \geq 3$) and let $T'$ be the subtree obtained from $T$ by deleting all degree-one vertices adjacent to $v_1$ (respectively $v_{D-1}$) excepting $v_0$ (respectively $v_D$). Then $D(T) = D(T')$ and $I(T) \geq I(T') + 1/12$.*

**Proof** Suppose $d(v_1) = m + 2 \geq 3, d(v_2) = r, d(v_3) = s$. Then $r, s \geq 2$ and $D(T) = D(T')$. We have

$$\begin{array}{l} I(T) - I(T') \\ = \sum_{i=1}^{m} \frac{1}{d_T(u_i)} + \sum_{i=0}^{2} (\frac{1}{d_T(v_i)} - \frac{1}{d_{T'}(v_i)}) \\ = \frac{m}{m+2} + (\frac{1}{m+2} - \frac{1}{2}) + (\frac{m+2}{m+r+1} - \frac{2}{r+1}) + (\frac{r}{m+r+s} - \frac{r}{r+s}) \\ = m[(\frac{1}{2(m+2)} - \frac{1}{(r+1)(m+r+1)}) + r(\frac{1}{(r+1)(m+r+1)} - \frac{1}{(r+s)(m+r+s)})] \\ \geq m(\frac{1}{2(m+2)} - \frac{1}{(r+1)(m+r+1)}) \\ \geq m(\frac{1}{2(m+2)} - \frac{1}{3(m+3)}) \\ = \frac{m(m+5)}{6(m+2)(m+3)} \\ \geq \frac{1}{12}. \end{array}$$

Q.E.D.

**Lemma 3** *Suppose $T$ is a caterpillar but not a path and $D = D(T) \geq 4$. If $d(v_1) = d(v_{D-1}) = 2$ and let $T'$ be the subtree obtained by deleting all degree-one neighbours of $v_\alpha$, where $v_\alpha$ is the vertex nearest to one terminal vertex of $P$ such that $d(v_i) \geq 3$. Then $D(T) = D(T')$ and $I(T) \geq I(T') + 1/10$.*

**Proof** Without loss of generality we may assume $\alpha \le \lfloor \frac{D}{2} \rfloor$. Let $u_1, \dots, u_m$ be all the degree-one neighbours of $v_\alpha$. We have $d(v_{\alpha-1}) = 2$. Let $d(v_{\alpha-2}) = r$ ($r = 1$ if $\alpha = 2$ and $r = 2$ otherwise), $d(v_{\alpha+1}) = s, d(v_{\alpha+2}) = t$. Clearly we have $D(T) = D(T')$. If $D \ge 5$, then $t \ge 2$, hence we have

$$\begin{aligned}
&I(T) - I(T') \\
&= \sum_{i=1}^{m} \frac{1}{d_T(u_i)} + \sum_{i=\alpha-1}^{\alpha+1} \left(\frac{1}{d_T(v_i)} - \frac{1}{d_{T'}(v_i)}\right) \\
&= \frac{m}{m+2} + \left(\frac{2}{m+r+2} - \frac{2}{r+2}\right) + \left(\frac{m+2}{m+s+2} - \frac{2}{s+2}\right) \\
&\quad + \left(\frac{s}{m+s+t} - \frac{s}{s+t}\right) \\
&= \frac{m}{m+2} + \left(\frac{2}{m+3} - \frac{2}{3}\right) + \left(\frac{m+2}{m+s+2} - \frac{2}{s+2}\right) \\
&\quad + \left(\frac{s}{m+s+2} - \frac{s}{s+2}\right) \\
&= \frac{m(m+5)}{3(m+2)(m+3)} \\
&\ge \frac{1}{6} \ge \frac{1}{10}.
\end{aligned}$$

If $D = 4$, then a straightforward calculation shows that

$$\begin{aligned}
&I(T) - I(T') \\
&= \frac{1}{6} + \frac{4}{m+3} - \frac{2}{m+2} - \frac{2}{m+4} \\
&= \frac{1}{6} - \frac{4}{(m+2)(m+3)(m+4)} \\
&\ge \frac{1}{10}.
\end{aligned}$$

Q.E.D.

Now let us prove the main theorem. If $T = P_n$, the path with $n$ vertices, then

$$I(P_n) - D(P_n)/2 = \begin{cases} 3/2, & n = 2 \\ 1, & n = 3 \\ 5/6, & n \ge 4. \end{cases}$$

If $D(T) = 2$, then $T$ is a star with $a = b = 0$, $c = 1$ and $I(T) - D(T)/2 = 1 \ge c/12 + 5/6$. If $D(T) = 3, T \ne P_4$, then $a = b = 0$ and $T$ has exactly two vertices with degree $\ge 2$. Suppose the degrees of them are $l + 1, m + 1$. Then $\max\{l, m\} \ge 2$ and

$$\begin{aligned}
&I(T) - D(T)/2 \\
&= \frac{\ell+m+2}{l+m+1} + \frac{l}{l+1} + \frac{m}{m+1} - \frac{3}{2} \\
&= \frac{1}{l+m+1} - \frac{1}{l+1} - \frac{1}{m+1} + \frac{3}{2} \\
&\ge \frac{c}{12} + \frac{5}{6}.
\end{aligned}$$

In the following we suppose $T$ is not a path and $D = D(T) \ge 4$. If $T$ is not a caterpillar, let $u$ be the vertex not in $P$ such that $d(u) \ge 2$ and $d(u, V(P))$ is as large as possible. Then all but one neighbours of $u$ have degree one. Removing from $T$ all the degree-one neighbours of

$u$ we get a subtree $T_1$ with $D(T) = D(T_1), I(T) \geq I(T_1) + 1/3$, according to Lemma 1. If $T_1$ is not a caterpillar, then repeat this procedure until a caterpillar is obtained. It is clear that after $a$ steps we get a sequence $T = T_0, T_1, \ldots, T_a$ such that each $T_{i+1}$ is a subtree of $T_i$ and $D(T_i) = D(T_{i+1})$ and $I(T_i) \geq I(T_{i+1}) + 1/3$. So we have $D(T) = D(T_a)$ and $I(T) \geq I(T_a) + a/3$.

If $d(v_1) \geq 3$ in $T_a$, then delete all the degree-one neighbours of $v_1$ except $v_0$. We get $T_{a+1}$ with the same diameter as $T$ such that $I(T_a) \geq I(T_{a+1}) + 1/12$, according to Lemma 2. If $d(v_{D-1}) \geq 3$, we do the same thing. In this way $c$ subtrees are added to the sequence above and we get $T = T_0, T_1, \ldots, T_a, \ldots, T_{a+c}$ with $D(T) = D(T_{a+c})$ and $I(T) = I(T_{a+c}) + a/3 + c/12$.

Now we have $d_{T_{a+c}}(v_1) = d_{T_{a+c}}(v_{D-1}) = 2$ and $d_{T_{a+c}}(v_i) = d(v_i)$, $i \notin \{1, D-1\}$. If $T_{a+c}$ is not a path, then according to Lemma 3 we can delete all degree-one neighbours of some $v_\alpha$ and obtain a subtree $T_{a+c+1}$ with $I(T_{a+c}) \geq I(T_{a+c+1}) + 1/10$. Repeat the procedure until we obtain a path $P$. When the process stops we get a sequence $T = T_0, T_1, \ldots, T_a, \ldots, T_{a+c}, \ldots, T_{a+c+b} = P$ with $I(T) \geq I(T_{a+c}) + a/3 + c/12 \geq I(P) + a/3 + b/10 + c/12$. Since $I(P) = D(P)/2 + 5/6$, as we have just proved it for paths, and since $D(P) = D(T)$, we get $I(T) \geq D(T)/2 + a/3 + b/10 + c/12 + 5/6$. This completes the proof.

Note that $R(T) = \lceil D(T)/2 \rceil$ for any tree $T$ and $a, b, c$ are non-negative integers. Hence we have the following corollary.

**Corollary** For any (nontrivial) tree $T$

$$I(T) \geq \begin{cases} R(T) + 1/3, & \text{if } D(T) \text{ is odd} \\ R(T) + 5/6, & \text{if } D(T) \text{ is even,} \end{cases}$$

with equality if and only if $T$ is a path of at least four vertices.

This corollary tells us that $I(T) - R(T)$ is bounded below. We point out that it is unbounded above. In fact, for the full binary tree $T$ of height $h \geq 3$ we have $I(T) - R(T) = 2^{h+2}/5 - h - 1/4$, which can be arbitrarily large as $h$ tends to infinity.

## References


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