Discussion on the paper "Analyzing short time series data from periodically fluctuating rodent populations by threshold models: A nearest block bootstrap approach"

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The authors are to be congratulated for an innovative paper in terms of both modelling methodology and subject matter significance. The analysis of short time series is known to be difficult even for linear models. In case that nonlinearity is present, the TAR or CTAR models are clearly obvious choices for describing the dependence structure in the data because of their structural simplicity and interpretability. This is well demonstrated in the present paper. Nevertheless, there are viable alternatives that may offer a somewhat different view and interpretation of the data. One example is the mixture type of models in [1, 2] . Wong and Li[1] considered a mixture autoregressive model of K components that can be represented, in terms of its conditional cumulative distribution $F(\cdot)$, as follows:

$$F(x_t|\mathcal{F}_{t-1}) = \sum_{k=1}^{K} \alpha_k \Phi\left(\frac{x_t - \phi_{k0} - \phi_{k1}x_{t-1} - \cdots - \phi_{kp_k}x_{t-p_k}}{\sigma_k}\right), \tag{1}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution, $\mathcal{F}_{t-1}$ is the information set up to time $t-1$, $\alpha_1 + \cdots + \alpha_K = 1$, $\alpha_k > 0$, $k = 1, \ldots, K$. Clearly $\Phi(\cdot)$ can be replaced by other cumulative distribution functions. The α_k's in (1) can also be made to depend on previous observations of x_t or other variables (see [2]). For example, when $K = 2$ we may have

$$\log\left(\frac{\alpha_{1,t}}{\alpha_{2,t}}\right) = \beta_0 + \beta_1 v_{1t} + \cdots + \beta_m v_{mt},$$

where $\alpha_{1,t}$ and $\alpha_{2,t}$ are the mixing proportions in

$$F(x_t|\mathcal{F}_{t-1}, \Omega_t) = \sum_{k=1}^{2} \alpha_{k,t}\Phi(e_{k,t}/\sigma_k), \qquad e_{kt} = x_t - \varphi_{k0} - \sum_{i=1}^{p_k} \varphi_{ki}x_{t-i},$$

and Ω_t is the information at time t generated by $v_{1t}, \ldots, v_{mt}$. Note that $\{v_{it}\}$ could be a subset of $\{x_t, x_{t-1}, \ldots\}$. This results in a dynamic mixture autoregressive model. Extensions to mixture autoregressive conditional heteroscedasticity (ARCH) models and generalized ARCH (GARCH) models can be found in [3, 4]. A multivariate extension can be found in [5]. A dynamical mixture GARCH model has been considered by Cheng et al.[6].

One advantage of the mixture type of models is that some components of (1) can be nonstationary on its own but yet the entire series can still be stationary. Another advantage is that the predictive distribution of x_t can be multi-modal but this may not be the case with TAR models. As pointed out in [1] a multimodal predictive density will be very useful in areas like financial risk management.

Given the possibility of other alternatives the problem of model selection arises naturally. However, classical tools like likelihood ratio test may not be applicable because the classes of models may not be nested. For non-nested model selection the classical tool, as pointed out by the authors, has been the Cox statistic[7],

$$T_f = L_f(\hat{\alpha}) - L_g(\hat{\beta}) - E_{\hat{\alpha}}\{L_f(\hat{\alpha}) - L_g(\hat{\beta})\}, \tag{2}$$

where $f(\alpha)$ and $g(\beta)$ are probability density functions belonging to two separate and possibly non-overlapping families and $\hat{\alpha}$, $\hat{\beta}$ are the respective maximum likelihood estimators; $E_{\hat{\alpha}}(\cdot)$ denotes expectation under $f(\hat{\alpha})$. For time series data the null distribution of (2) (under H_0 : $f(\alpha)$ is the correct distributional density) is very difficult to obtain and Li[8,9] advocated the use of the bootstrap method to estimate the unknown distribution. However, the bootstrap approach in [8] is based on residuals under the null and alternative hypotheses as in regression analysis. The Nearest Neighbor Bootstrap seems promising in offering a better approach in finding an approximate distribution of the Cox-statistic (2). In fact, the bootstrap statistic is given by

$$T_f^* = L_f(\hat{\alpha}^*) - L_g(\hat{\beta}^*),$$

where $\hat{\alpha}^*$ and $\hat{\beta}^*$ refers to the respective maximum likelihood estimators for the bootstrapped time series. Since the bootstrap sample is obtained non-parametrically the resulting statistics should be more accurate in representing the truth. We therefore look forward to seeing further development of the theory of NBB methodology by the authors in the future.

Finally, we would like to applaud the authors again for an interesting piece of work.

References

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