Design of FIR Digital Filters With Prescribed Flatness and Peak Error Constraints Using Second-Order Cone Programming

K. M. Tsui, S. C. Chan, *Member, IEEE*, and K. S. Yeung

***Abstract*—This paper studies the design of digital finite impulse response (FIR) filters with prescribed flatness and peak design error constraints using second-order cone programming (SOCP). SOCP is a powerful convex optimization method, where linear and convex quadratic inequality constraints can readily be incorporated. It is utilized in this study for the optimal minimax and least squares design of linear-phase and low-delay (LD) FIR filters with prescribed magnitude flatness and peak design error. The proposed approach offers more flexibility than traditional maximally-flat approach for the tradeoff between the approximation error and the degree of design freedom. Using these results, new LD specialized filters such as digital differentiators, Hilbert Transformers, Mth band filters and variable digital filters with prescribed magnitude flatness constraints can also be derived.**

***Index Terms*—Constrained finite impulse response (FIR) filter design, digital differentiators, low-delay (LD), magnitude and group delay flatness, peak error constraints, second-order cone programming (SOCP).**

I. Introduction

RECENTLY, convex optimization methods such as semidefinite programming (SDP) [1]–[3] and second-order cone programming (SOCP) [4]–[6] have been widely employed in designing digital finite impulse response (FIR) and infinite impulse response (IIR) filters. An important advantage of such methods is its ability to satisfy multiple objectives expressed in terms of a set of linear and convex quadratic constraints. Since SDP and SOCP are convex problems, the optimality of the solution, if it exists, is guaranteed. This motivates us to study in this paper the design of digital FIR filters with more general constraints such as magnitude flatness (such as multiple zeros in magnitude response) and peak design error constraints. More specifically, we shall formulate these design problems as a SOCP [4]–[6]. Alternatively, SDP, which is a generalization of SOCP, can also be used at the expense of higher arithmetic complexity. Conventionally, linear programming has been proposed [7] as a general framework for handling the additional linear equality and inequality constraints for designing linear-phase (LP) FIR filters. Since SOCP is an extension of linear programming, the SOCP-constrained FIR filter design method proposed in this paper can be viewed as its generalization to handle convex quadratic constraints, which allows optimal minimax and least-squares (LS) passband LP FIR filters subject to linear equalities and convex quadratic inequalities to be designed. There were also previous attempts in incorporating linear equality constraints in LS design of FIR filters [8]–[10]. The design problem is usually formulated as a quadratic programming problem with linearly equality constraints (QPLC), generally known as the eigenfilter design method. Advantages of these approaches are their good performance and low design complexities. On the other hand, the SOCP approach is capable of handling more general types of quadratic constraints and design criterion, including the QPLC, as we shall demonstrate in the design example section.

In this paper, we mainly focus on prescribed flatness and peak error constraints [11]. Magnitude flatness and multiple zeros are desirable in designing sample rate converters in order to suppress the alias components and the design of wavelet basis. On the other hand, peak error constraints are useful to limit the sidelobe and undesirable peaks in filters with wide unconstrained transition band. Both the LS and minmax design criteria will be considered. The magnitude flatness constraints are derived through a simple relation between the derivatives of the filter response and its ideal counterpart. This yields a set of linear equality constraints, which can readily be solved using SOCP. A similar set of linear equality constraints can also be derived for a prescribed flatness in the group delay response. In addition, since the FIR filters are not limited to LP, the system delay can further be reduced. Using these results, new low-delay (LD) specialized filters such as digital differentiators (DDs) with the magnitude and/or group delay flatness constraints are derived. The proposed method is also applicable to the design of other specialized filters such as Hilbert transformers, Mth-band filters, complex coefficient FIR filters and variable digital filters with magnitude and group delay flatness, and peak error constraints. Interested readers are referred to [3] and [12] for more details. Within the SOCP framework, these linear equalities and convex quadratic inequalities such as peak design error can be integrated together to yield FIR filters in the minimax and LS design errors. This gives a better tradeoff between magnitude and group delay flatness and passband and stopband ripples over conventional LP maximally flat DDs [13], [14]. Design results show that the SOCP method offers an attractive alternative to traditional design methods because of its optimality, generality, and flexibility. The paper is organized as follows. Section II is devoted to the SOCP formulation of the design problem. Methods for deriving the magnitude flatness, group

Manuscript received July 13, 2003; revised November 1, 2004. This paper was recommended by Associate Editor Y. Lian.

The authors are with the Department of Electrical and Electronic Engineering, The University of Hong Kong, Hong Kong (e-mail: kmtsui@eee.hku.hk; scchan@eee.hku.hk; ksyeung@eee.hku.hk).

Digital Object Identifier 10.1109/TCSII.2005.850515

delay flatness, and peak design error constraints are also introduced. Design examples are presented in Section III. Finally, conclusions are drawn in Section IV.

II. Constrained FIR Filter Design Using SOCP

A. Problem Formulation

Let $H(z) = \sum_{n=0}^{N-1} h(n)z^{-n}$ be the transfer function of a FIR filter of length N, where $h(n)$'s are the filter coefficients to be determined. The frequency response of $H(z)$ is given by

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-jn\omega} = \boldsymbol{h}^T\{\boldsymbol{c}(\omega) - j\boldsymbol{s}(\omega)\} \qquad (1)$$

where $\boldsymbol{h} = [h(0),\ldots,h(N-1)]^T, \boldsymbol{c}(\omega) = [1,\cos(\omega),\ldots,\cos((N-1)\omega)]^T$, and $\boldsymbol{s}(\omega) = [0,\sin(\omega),\ldots,\sin((N-1)\omega)]^T$. To approximate the desired response $H_d(\omega)$ by $H(z)$ in the minimax sense, we solve the following min-max problem:

$$\min_{h(n)} \max W(\omega)|H(e^{j\omega}) - H_d(\omega)|, \qquad \omega \in \Omega_I \qquad (2)$$

where $W(\omega)$ is a positive weighting function, and $\Omega_I \subset [-\pi,\pi]$ is the frequency interval of interest. Letting $H_{R\_d}(\omega)$ and $H_{I\_d}(\omega)$ be the real and imaginary parts of $H_d(\omega)$, (2) can be reformulated as

$$\min_{\boldsymbol{h}} \delta \quad \text{subject to } \delta - \left[\alpha_R^2(\omega) + \alpha_I^2(\omega)\right]^{1/2} \geq 0 \qquad (3)$$

for $\omega \in \Omega_I$, where $\alpha_R(\omega) = W(\omega)\cdot[\boldsymbol{h}^T\boldsymbol{c}(\omega) - H_{R\_d}(\omega)]$ and $\alpha_I(\omega) = W(\omega)\cdot[\boldsymbol{h}^T\boldsymbol{s}(\omega) + H_{R\_I}(\omega)]$. By digitizing the frequency variable ω over a dense set of frequencies $\{\omega_i, 1 \leq i \leq K_1\}$ on the frequency of interest, (3) can be cast to the following standard SOCP problem:

$$\min_{\boldsymbol{x}} \boldsymbol{c}^T\boldsymbol{x} \quad \text{subject to } \boldsymbol{c}^T\boldsymbol{x} \geq \|\boldsymbol{F}_i\boldsymbol{x} - \boldsymbol{g}_i\|_2 \qquad (4)$$

where $\boldsymbol{c} = [1\ \boldsymbol{O}_N^T]^T, \boldsymbol{x} = [\delta\ \boldsymbol{h}^T]^T, \boldsymbol{F}_i = W(\omega_i)[\begin{smallmatrix}0 & \boldsymbol{c}(\omega_i)^T \\ 0 & \boldsymbol{s}(\omega_i)^T\end{smallmatrix}]$ and $\boldsymbol{g}_i = W(\omega_i)[-H_{R\_d}(\omega_i)\ \ H_{I\_d}(\omega_i)]^T; \boldsymbol{O}_N$ is an N row zero vector and $\|\cdot\|_2$ denotes the Euclidean norm. Alternatively, (2) can be formulated as an SDP problem [1]–[3], [12], which might provide more flexibility but requires a longer design time. For simplicity, only the SOCP formulation is considered below. Instead of using the minimax criterion, the following LS error criterion can be minimized:

$$\begin{aligned} E_{\text{LS}}(\boldsymbol{h}) &= \int_{\Omega_I} W(\omega)|H(e^{j\omega}) - H_d(\omega)|^2 d\omega \\ &= \boldsymbol{h}^T\boldsymbol{Q}h - 2\boldsymbol{h}^T\boldsymbol{p} + k \end{aligned} \qquad (5)$$

where $\boldsymbol{Q} = \int_{\Omega_I} W(\omega)\cdot[\boldsymbol{c}(\omega)\boldsymbol{c}(\omega)^T + \boldsymbol{s}(\omega)\boldsymbol{s}(\omega)^T]d\omega, \boldsymbol{p} = \int_{\Omega_I} W(\omega)[\boldsymbol{c}(\omega)H_{R\_d}(\omega) + \boldsymbol{s}(\omega)H_{I\_d}(\omega)]d\omega$, and $k = \int_{\Omega_I} W(\omega)|H_d(\omega)|^2 d\omega$. The optimal LS solution is given by $\boldsymbol{h}_{\text{LS}} = \boldsymbol{Q}^{-1}\boldsymbol{p}$, which can also be solved by a SOCP as follows:

$$\min_{\boldsymbol{x}} \boldsymbol{c}^T\boldsymbol{x} \quad \text{subject to } \boldsymbol{c}^T\boldsymbol{x} \geq \|\bar{\boldsymbol{Q}}\boldsymbol{x} - \bar{\boldsymbol{p}}\|_2 \qquad (6)$$

where $\bar{\boldsymbol{Q}} = [\boldsymbol{O}_N \quad \boldsymbol{Q}^{1/2}]$ and $\bar{\boldsymbol{p}} = \boldsymbol{Q}^{-1/2}\boldsymbol{p}$. The advantage of employing a convex programming such as SOCP or SDP in the formulation is that the resulting problem is a convex optimization and the optimal solution, if it exists, can be found. In addition, other linear equalities or convex quadratic constraints can be incorporated.

B. Imposing Linear Equality Constraints

When designing digital filters, it is often required to impose certain constraints on the frequency characteristics. One commonly encountered constraint is the linear equality constraints, which includes magnitude and group delay flatness at certain frequency points in the passband. Constraints such as a prescribed number of zeros at the stopband also belong to this category. To incorporate these magnitude flatness constraints into SOCP, the following relation between the derivatives of the design frequency response and its ideal counterparts is employed:

$$\left.\frac{d^i}{dz^i}H(e^{j\omega})\right|_{\omega=\hat{\omega}} = \left.\frac{d^i}{dz^i}H_d(\omega)\right|_{\omega=\hat{\omega}}, \qquad i = 0,1,\ldots,K-1. \qquad (7)$$

Equation (7) tells us that the filter to be designed, $H(e^{j\omega})$, should approximate the desired response at $\omega = \hat{\omega}$ up to the $(K-1)$th derivatives. Two simple examples are given below.

1) Magnitude Flatness Constraint at the Passband: Suppose that the desired passband response is of the form $e^{-j\omega\tau+\phi}$ for any $\phi \in [-\pi,\pi]$, where $\tau = (N-1)/2 - D$ is the group delay, and D is the delay reduction parameter. To impose a magnitude flatness of order $U_{\hat{\omega}_p} - 1$ on $H(z)$ at $\hat{\omega}_p$ in the passband, Ω_p, we have

$$\sum_{n=0}^{N-1} (n-\tau)^u \cdot h(n) \cdot e^{-j\hat{\omega}_p(n-\tau)} = \begin{cases} e^{\phi}, & u = 0 \\ 0, & u = 1,\ldots,U_{\hat{\omega}_p} - 1 \end{cases} \qquad (8)$$

or in matrix form

$$\boldsymbol{U}_{\hat{\omega}_p} \cdot \boldsymbol{h} = \boldsymbol{d}_{\hat{\omega}_p}, \qquad u = 0,1,\ldots,U_{\hat{\omega}_p} - 1 \qquad (9)$$

where $[\boldsymbol{U}_{\hat{\omega}_p}]_{u,n} = (n-\tau)^u e^{-j\hat{\omega}_p(n-\tau)}$ and $\boldsymbol{d}_{\hat{\omega}_p} = [e^{\phi} \quad \boldsymbol{O}_{U_{\hat{\omega}_p}-1}^T]^T$. Here, $[\boldsymbol{A}]_{m,n}$ denotes the (m,n)th entry of matrix $\boldsymbol{A}$. The constraints in (7) can also be applied to other forms of desired passband response, say DD, as we shall illustrate in Section III-B.

2) Magnitude Zero Constraint at the stopband: Similarly, to impose $V_{\hat{\omega}_s} - 1$ zeros on $H(z)$ at $\hat{\omega}_s \in \Omega_s$ (say in the stopband), we have

$$\sum_{n=0}^{N-1} n^v \cdot h(n) \cdot e^{-j\hat{\omega}_s n} = 0, \qquad v = 0,1,\ldots,V_{\hat{\omega}_s} - 1 \qquad (10)$$

or in matrix form

$$\boldsymbol{V}_{\hat{\omega}_s} \cdot \boldsymbol{h} = \boldsymbol{O}_{V_{\hat{\omega}_s}}, \qquad v = 0,1,\ldots,V_{\hat{\omega}_s} - 1 \qquad (11)$$

where $[\boldsymbol{V}_{\hat{\omega}_s}]_{v,n} = n^v e^{-j\hat{\omega}_s n}$. These constraints can be combined and written in the following matrix representation:

$$\boldsymbol{A}\boldsymbol{h} = \boldsymbol{b} \tag{12}$$

where $\boldsymbol{A}$ is an $(T \times N)$ matrix and $\boldsymbol{b}$ is an $(T \times 1)$ vector. The problems in (4) and (6) can then be solved subject to these linear equality constraints using SOCP.

C. Group Delay Flatness Constraints

Other than the magnitude flatness constraints, a prescribed group delay flatness at the passband can also be imposed. Let us consider the phase response $\theta(\omega)$ of $H(z)$

$$\tan(\theta(\omega)) = \sum_{n=0}^{N-1} h(n)\sin(n\omega) \Big/ \sum_{n=0}^{N-1} h(n)\cos(n\omega). \tag{13}$$

The error in approximating the ideal phase response $\theta_d(\omega)$ can be written as follows:

$$\begin{aligned} e(\omega) &= \tan(\theta(\omega)) - \tan(\theta_d(\omega)) \\ &= \sum_{n=0}^{N-1} h(n)\sin[(n-\tau)\omega] \Big/ \left[\cos(\tau\omega)\cdot\sum_{n=0}^{N-1} h(n)\cos(n\omega)\right]. \end{aligned} \tag{14}$$

To impose a group delay flatness of order $Y_{\hat{\omega}_p}$ on $H(z)$ at $\hat{\omega}_p$ in the passband is equivalent to

$$\left.\frac{d^y}{d\omega^y} e(\omega)\right|_{\omega=\hat{\omega}_p} = 0, \qquad y = 1, 2, \ldots, Y_{\hat{\omega}_p}. \tag{15}$$

Substituting (14) into (15), one obtains

$$\boldsymbol{Y}_{\hat{\omega}_p} \cdot \boldsymbol{h} = \boldsymbol{O}_{Y_{\hat{\omega}_p}-1} \tag{16}$$

where

$$[\boldsymbol{Y}_{\hat{\omega}_p}]_{y,n} = \begin{cases} (n-\tau)^y \cos[\hat{\omega}_p(n-\tau)], & y \text{ odd} \\ (n-\tau)^y \sin[\hat{\omega}_p(n-\tau)], & y \text{ even} \end{cases}, \quad y = 1, \ldots, Y_{\hat{\omega}_p}$$

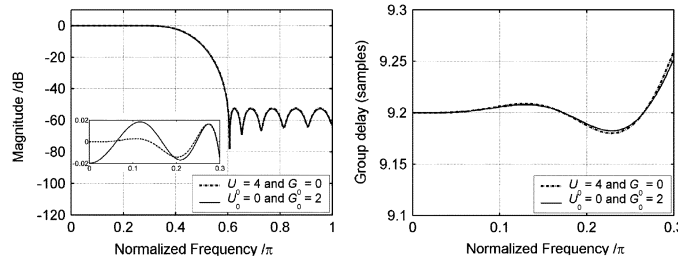
and $n = 0, 1, \ldots, N-1$. Again, (16) can be readily incorporated into the SOCP formulation. Moreover, it is interesting to note that the group delay flatness constraints are related to magnitude flatness constraints. Comparing (16) with the first or higher order magnitude flatness constraints in (9), it can be observed that either the real or imaginary part of (9) can satisfy the group delay flatness in (16) of the same order. Therefore, if a magnitude flatness constraint up to at least the first order is imposed, the same order of group delay flatness is automatically satisfied.

D. Peak Error and Convex Quadratic Constraints

Apart from linear equality constraints, linear and convex quadratic inequality constraints can easily be incorporated in the SOCP formulation. As an illustration, we shall consider the optimal design of LD FIR filters with LS stopband attenuation and a prescribed peak ripple constraints. Letting ε be the peak stopband ripple to be imposed in a frequency band $\omega \in [\omega_{s1}, \omega_{s2}]$ (a collection of frequency bands is also feasible), then the peak error constraint can be written as $|H(e^{j\omega})| \le \varepsilon, \omega \in [\omega_{s1}, \omega_{s2}]$. Similar to the minimax case, one obtains

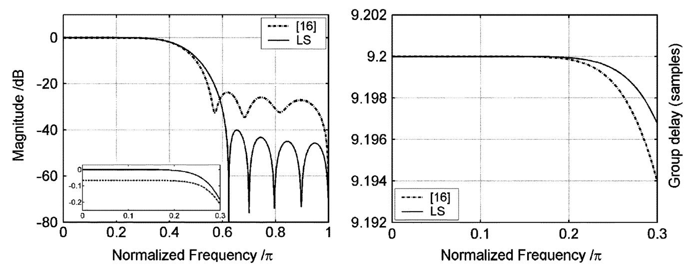
$$\varepsilon \ge \|\boldsymbol{R}\boldsymbol{x}\|_2, \qquad \text{where } \boldsymbol{R} = \begin{bmatrix} 0 & \boldsymbol{c}(\omega)^T \\ 0 & \boldsymbol{s}(\omega)^T \end{bmatrix}. \tag{17}$$

Discretizing (17), the resulting constraints on the peak ripples can be augmented to the existing constraints in (4) and (6) for the minimax and LS criteria, respectively.



(a) (b)

Fig. 1. Design results of fractional delay LPFs in example 1 (design criterion: minimax): verification of the relation between magnitude and group delay flatness for LPFs. (a) Frequency response (passband details in smaller figure). (b) Group delay response.



(a) (b)

Fig. 2. Design results of fractional delay LPFs in example 1 (linear equality constraint: $U_0 = 10, V_p = 1$ and $G_0 = 5$). (a) Frequency response (passband details are in the smaller figure). (b) Group delay response.

III. Design Examples and Results

In this section, LD FIR filters including low-pass filter (LPFs) and DDs are considered. All of the design problems were solved by the *SeDuMi* Matlab Toolbox [15] on a Pentium 4 PC.

A. Low-Delay FIR LPFs

Example 1: In this example, fractional passband delay FIR filters with prescribed magnitude flatness and group delay flatness at $\omega = 0$ and a prescribed number of zeros at $\omega = \pi$ are designed. The number of sample points at the passband and stopband are 100. The specifications are as follows: $\omega_p = 0.3\pi, \omega_s = 0.6\pi, D = 0.3$, and $N = 20$. According to the discussion in Section II-C, if a prescribed group delay flatness at $\omega = 0$ of order Y_0 is imposed, then (15) reduces to

$$\left.\frac{d^{2g+1}}{d\omega^{2g+1}} e(\omega)\right|_{\omega=0} = \sum_{n=0}^{N-1} h(n)(n-\tau)^{2g+1} = 0 \tag{18}$$

for $g = 0, 1, \ldots, G_0 - 1$, where $G_0 = \lfloor (Y_0 - 1)/2 \rfloor + 1$ and $\lfloor \alpha \rfloor$ denotes the integer just less than or equal to α. Equation (18) can be written more compactly in matrix form as $\boldsymbol{G}_0 \cdot \boldsymbol{h} =$

TABLE I

SPECIFICATIONS AND DESIGN RESULTS OF THE LD LPFS IN EXAMPLES 1 AND 2. δ_p: PASSBAND ERROR (dB); δ_s: STOPBAND ERROR (dB); δ_g: GROUP DELAY ERROR (SAMPLES)

Fig.	N	D	U_0	V_π	G_0	ε	Design criteria	δ_p	δ_s	δ_g	Design time (s)
1	20	0.3	4	0	0	N/A	Minimax	0.016	52.345	0.0593	1.672
			0	0	2	N/A	Minimax	0.019	52.793	0.0518	1.532
2	20	0.3	10	1	5	N/A	[16]	0.207	23.862	0.0060	N/A
						N/A	LS	0.206	33.337	0.0030	0.903
Not shown	20	2	3	3	0	N/A	Minimax	0.023	50.951	0.1285	1.406
						N/A	LS	0.023	42.183	0.1105	0.656
						-50dB	LS	0.029	50.000	0.1349	0.864

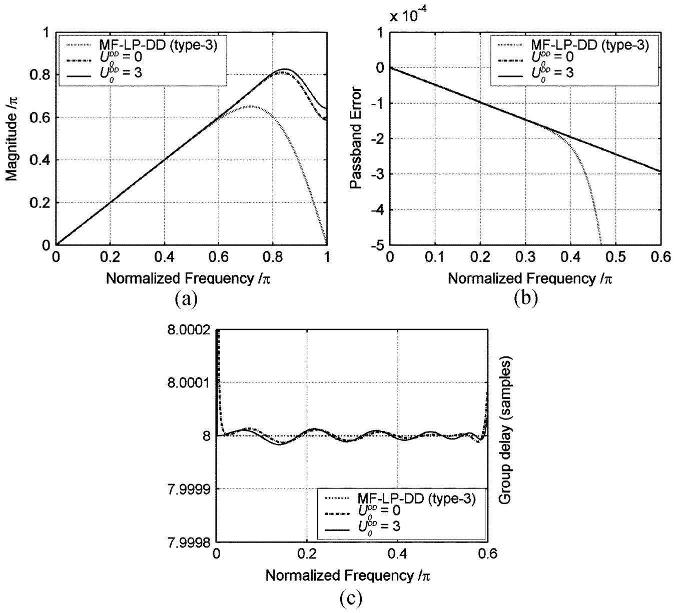


Fig. 3. Design results of LD odd-length DDs in example 3 (design criterion: minimax). Comparison with different orders of magnitude linearity for odd-length DDs. (a) Magnitude response. (b) Passband errors. (c) Group delay response. (MF-LP-DD: maximally flat linear-phase digital differentiator.)

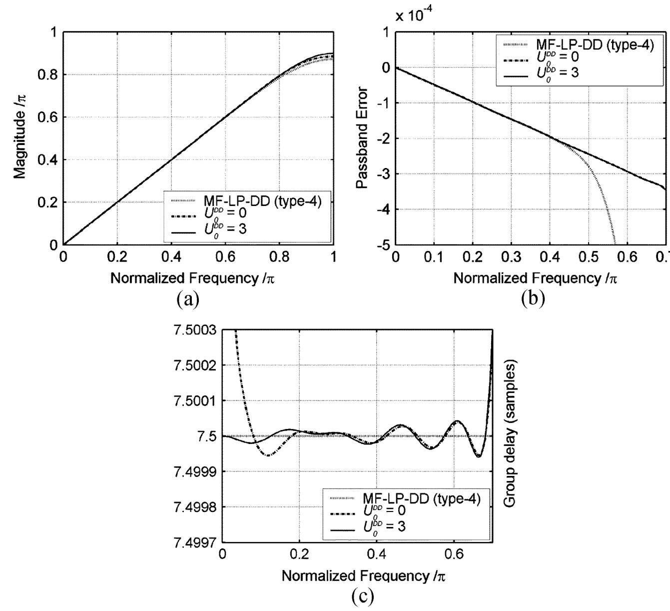


Fig. 4. Design results of LD even-length DDs in example 3 (design criterion: minmax). Comparison with different orders of magnitude linearity for even-length DDs. (a) Magnitude response. (b) Passband errors. (c) Group delay response. (MF-LP-DD: maximally flat linear-phase digital differentiator.)

$\boldsymbol{O}_{G_0}$, where $[\boldsymbol{G}_0]_{g,n} = (n-\tau)^{2g+1}$. The dotted line in Fig. 1 shows the frequency and group delay responses of the LPFs with $U_0 = 4, V_\pi = 4$, and $G_0 = 0$. As mentioned earlier, if $U_0 \geq 2$ magnitude flatness constraints at $\omega = 0$ (i.e., the first or higher order of the magnitude flatness) are imposed, then (18) will also be satisfied for $g = 0, 1, \ldots, G' - 1$ where $G' = \lfloor (U_0 - 2)/2 \rfloor + 1$. Therefore, if $U_0 = 4$, then a group delay flatness of order $3 (G_0 = 2)$ will also be satisfied. To verify this, an LPF with the same specification, but different flatness parameters of $U_0 = 0, V_\pi = 0$, and $G_0 = 2$, is designed and plotted as the solid line in Fig. 1. It can be seen from the group delay responses in Fig. 1(b) that both filters achieve the same degree of group delay flatness at $\omega = 0$. As a comparison, fractional delay LPFs with $U_0 = 10, V_\pi = 1$ and $G_0 = 5$ are also designed using the LS criterion and the technique recently proposed in [16]. It can be seen from Fig. 2 that the deviation at $\omega = 0$, the stopband, and the group delay errors in this approach are relatively higher. One advantage of our approach over the method in [16] is that the relation between magnitude and group delay flatness is explored so that the remaining freedom can be used to improve the stopband and passband responses. Moreover, the worst-case passband deviation and stopband attenuation can further be improved by applying the peak error constraints to the LS solution or using the minmax criterion, as we shall demonstrate in next example.

Example 2: In this example, LD LPFs with magnitude and group delay flatness are designed. Both minimax and LS stopband criteria are minimized. The specifications are identical to example 1 except that $D = 2$ and the constraint parameters are $U_0 = 3$ and $V_\pi = 3$. Note that the LS design is a quadratic programming problem with linear equality constraints, which can be solved either by the eigenfilter method [8], [9] or the SOCP method. The worst-case stopband attenuation is 42.183 dB for the LS design, as compared to 50.951 dB for the minimax design. To further illustrate the flexibility of the SOCP method, peak stopband constraints, which are convex quadratic inequality constraints, are imposed to limit the sidelode at $\omega \in [0.6\pi, 0.7\pi]$ to 50 dB. From its frequency response and pole-zero plot (not shown here due to page limitation), it is noticed that both equality and inequality constraints are satisfied. The parameters and results of the LPFs in examples 1 and 2 are summarized in Table I.

TABLE II
SPECIFICATIONS AND DESIGN RESULTS OF THE LD DDs IN EXAMPLE 3

Fig.	N	Passband Cutoff frequencies ω_{p1}	Passband Cutoff frequencies ω_{p2}	Design criteria	U_0^{DD}	Passband error (1×10^{-4})	Design Time (s)
3	17	N/A	N/A	Maximally-flat linear-phase [20]	N/A	-83.78 ($\omega = 0.6\pi$)	N/A
	21	0	0.6π	Minimax	0	-2.919	1.206
				Minimax	3	-2.923	1.328
4	16	N/A	N/A	Maximally-flat linear-phase [20]	N/A	-36.67 ($\omega = 0.7\pi$)	N/A
	20	0	0.7π	Minimax	0	-3.456	1.156
				Minimax	3	-3.463	1.297

B. Low-Delay Digital Differentiators

Example 3: In this example, LD DDs with prescribed magnitude linearity at $\omega = 0$ are designed. Due to space limitation, only the minimax design is considered. The desired frequency response of a DD [7] is given by $H_d^{\mathrm{DD}}(\omega) = j\omega e^{-j\omega\tau}, 0 \leq \omega \leq \omega_p$, where $\tau = (N-1)/2 - D$ is an additional group delay term over the passband $\Omega_p \in [0, \omega_p]$. When $D = 0$, the proposed DDs with odd and even lengths correspond to the traditional type-3 and type-4 LP DDs, respectively. By choosing $D > 0$, DDs with low system delay can be obtained. From (7), a magnitude flatness constraints in its error response (or magnitude linearity constraints) of order $U_0^{\mathrm{DD}} - 1$ at $\omega = 0$ can be written as $\boldsymbol{U}_0^{\mathrm{DD}} \cdot \boldsymbol{h} = \boldsymbol{d}_0^{\mathrm{DD}}$, where $[\boldsymbol{U}_0^{\mathrm{DD}}]_{u,n} = n^u$ and $[\boldsymbol{d}_0^{\mathrm{DD}}]_u = -\tau^{u-1} \cdot u$ for $u = 0, 1, \ldots, U_0^{\mathrm{DD}} - 1$. As an illustration, LD DDs with $D = 2$ are designed. The odd-length low-delay DD has a filter length of $N = 21$ and the passband is from 0 to 0.6π whereas the even-length LD DD has a filter length $N = 20$ and its passband is from 0 to 0.7π. No stopband sample is required and the number of passband sample is 200. Different degrees of magnitude linearity for $U_0^{\mathrm{DD}} = 0$ and 3 are imposed at $\omega = 0$. Figs. 3 and 4 show the corresponding magnitude and group delay responses of the odd- and even-length LD DDs, respectively. Note that the new odd-length LD DDs do not necessarily have a zero at $\omega = \pi$, unlike its LP counterparts, and this gives rise to a larger possible passband. From Table II and Figs. 3 and 4, it can be seen that the passband ripples are smaller for the designs with a lower order of magnitude linearity, which is to be expected. Also, although the group delay of the design with $U_0^{\mathrm{DD}} = 3$ appears to be rather large around $\omega = 0$, its magnitude response is close to zero and it has a better linearity at $\omega = 0$. As a comparison, the conventional type-3 and type-4 maximally flat LP DDs (MF-LP-DD) [13] are also designed with the same group delay as the proposed DDs. It can be seen from Figs. 3(b) and 4(b) that the passband errors of the maximally flat solutions increase as ω increases, and they are much higher than that of the proposed DDs at the band edges. This suggests that the prescribed flatness approach offers more flexibility and freedom than the maximally flat solution in satisfying different passband and stopband requirements. The design results in this example are summarized in Table II.

IV. CONCLUSION

A design approach for LD FIR filters with prescribed flatness and peak design error constraints using SOCP has been presented. Design results show that the SOCP method is an attractive alternative to traditional design methods in tackling a wide range of filter design problems, because of its optimality, generality, and flexibility.

REFERENCES

[1] W.-S. Lu, "Design of nonlinear-phase FIR digital filters: A semidefinite programming approach," *Proc. ISCAS*, vol. III, pp. 263–266, May 1999.

[2] S.-P. Wu, S. Boyd, and L. Vandenberghe, "FIR filter design via semidefinite programming and spectral factorization," in *Proc. 5th Conf. Decision and Control*, Kobe, Japan, Dec. 1996, pp. 271–276.

[3] S. C. Chan and K. M. Tsui, "On the design of real and complex FIR filters with flatness and peak error constraints using semidefinite programming," in *Proc. ISCAS*, vol. 3, May 2004, pp. 125–128.

[4] J. O. Coleman and D. P. Scholnik, "Design of nonlinear phase FIR filters with second-order cone programming," in *Proc. MWSCAS*, vol. 1, Aug. 1999, pp. 409–412.

[5] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2004.

[6] W. S. Lu and T. Hinamoto, "Optimal design of FIR frequency-response-masking filters using second-order cone programming," in *Proc. ISCAS*, vol. 3, May 2003, pp. 878–881.

[7] S. K. Mittra and J. F. Kaiser, *Handbook for Digital Signal Processing*. New York: Wiley, 1993.

[8] S. C. Pei, C. C. Tseng, and W. S. Yang, "FIR filter designs with linear constraints using the eigenfilter approach," *IEEE Trans. Circuits Syst. II, Analog. Digit. Signal Process.*, vol. 45, no. 2, pp. 232–237, Feb. 1998.

[9] T. Chen and P. P. Vaidyanathan, "Design of IFIR eigenfilters," in *Proc. ISCAS*, vol. 1, Jun. 1991, pp. 264–267.

[10] Y. Wisutmethangoon and T. Q. Nguyen, "A method for design of Mth-band filters," *IEEE Trans. Signal Process.*, vol. 47, no. 6, pp. 1669–1678, Jun. 1999.

[11] J. W. Adams, "FIR digital filters with least-squares stopbands subject to pain-gain constraints," *IEEE Trans. Circuits Syst.*, vol. 39, no. 4, pp. 376–388, Nov. 1991.

[12] S. C. Chan, K. M. Tsui, and K. S. Yeung, "Design of FIR Digital Filters with Prescribed Flatness and Peak Error Constraints Using Convex Programming," The University of Hong Kong, Internal Report, 2003.

[13] I. W. Selesnick, "Maximally flat low-pass digital differentiators," *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.*, vol. 49, no. 3, pp. 219–223, Mar. 2002.

[14] E. Hermanowicz and M. Rojewski, "Design of FIR first order digital differentiators of variable fractional sample delay using maximally flat error criterion," *Electron. Lett.*, vol. 30, no. 1, pp. 17–18, Jan. 6, 1994.

[15] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optim. Meth. Softw.*, vol. 11–12, pp. 625–653, 1999.

[16] R. Gopinath, "Lowpass delay filters with flat magnitude and group delay constraints," *IEEE Trans. Signal Process.*, vol. 51, no. 1, pp. 182–192, Jan. 2003.