

On Einstein clusters as galactic dark matter haloes

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ABSTRACT

We consider global and gravitational lensing properties of the recently suggested Einstein clusters of weakly interacting massive particles (WIMPs) as galactic dark matter haloes. Being tangential pressure dominated, Einstein clusters are strongly anisotropic systems which can describe any galactic rotation curve by specifying the anisotropy. Due to this property, Einstein clusters may be considered as dark matter candidates. We analyse the stability of the Einstein clusters against both radial and non-radial pulsations, and we show that the Einstein clusters are dynamically stable. With the use of the Buchdahl type inequalities for anisotropic bodies, we derive upper limits on the velocity of the particles defining the cluster. These limits are consistent with those obtained from stability considerations. The study of light deflection shows that the gravitational lensing effect is slightly smaller for the Einstein clusters as compared to the singular isothermal density sphere model for dark matter. Therefore, lensing observations may discriminate, at least, in principle, between Einstein cluster and the other dark matter models.

Key words: dark matter.

1 INTRODUCTION

In 1939, Einstein (Einstein 1939) presented a model of a thick spherical shell composed of test particles of equal mass, each moving in a circular geodesic orbit in the field of all the other. Such a system has a spherical symmetry and may be used to model globular clusters of stars, since it gives a good approximation to the average density of matter and the average value of the gravitational field.

Einstein clusters have been studied extensively by Gilbert (1954), Hogan (1973), Banerjee & Som (1981), Comer & Katz (1993) and Comer, Langlois & Peter (1993), and the main physical properties of the cluster have been derived. In particular, the energy momentum tensor components T_i^j for the cluster have been obtained in several representations. Due to the spherical symmetry, the only non-vanishing components of T_i^j are $T_t^t = \rho^{(\text{eff})}$, $T_r^r = -p_r^{(\text{eff})}$ and $T_\theta^\theta = T_\varphi^\varphi = -p_\perp^{(\text{eff})}$, where $\rho^{(\text{eff})}$ is the effective energy density of the cluster and $p_r^{(\text{eff})}$ and $p_\perp^{(\text{eff})}$ are the radial and tangential pressures, respectively. Since the junction conditions require $p_r^{(\text{eff})}$ to be continuous across the boundary of each layer of the shell, it follows that for the Einstein cluster $p_r^{(\text{eff})} = 0$. The non-vanishing energy momentum components can be expressed in terms of the velocity of the particles in the cluster and the density only. It can also be interpreted as giving rise to rotation without introducing global angular momentum.

Therefore, the Einstein clusters are the examples of spherically symmetric systems with an anisotropic energy momentum, for which the radial pressure is different from the tangential one, $p_r^{(\text{eff})} \neq$

$p_\perp^{(\text{eff})}$. Anisotropic spherically symmetric matter distributions have been extensively studied in the past and several important physical characteristics of these systems have been obtained. In particular, several generalizations of the Buchdahl bound on the mass–radius ratio, the redshift and the value of the anisotropy parameter have been obtained, also in the presence of the cosmological constant by Böhmer & Harko (2006). Similar to the case of isotropic systems (Böhmer & Harko 2005b), the presence of the cosmological constant also determines the existence of a minimal mass for anisotropic matter distributions.

An interesting physical application of the Einstein clusters has recently been suggested by Lake (2006). The mass of the Einstein cluster is proportional to the square of the tangential velocity and to the distance to the centre. This behaviour is very similar to that observed in the case of test particles in stable circular orbits around the Galactic Centre. In most galaxies, neutral hydrogen clouds are observed at large distances from the centre, much beyond the extent of the luminous matter (see Binney & Tremaine 1987; Persic, Salucci & Stel 1996). Since these clouds move in circular orbits with velocity $V(r)$, the orbits are maintained by the balance between the centrifugal acceleration $V(r)^2/r$ and the gravitational attraction force $GM(r)/r^2$ of the total mass $M(r)$ contained within the orbit. This allows us to express the mass profile of the galaxy in the form $M(r) = rV^2/G$. This mass profile can be obtained rigorously for the Einstein clusters. Observations show that the rotational velocities increase near the centre of the galaxy and then remain nearly constant at a value of $V_0 \sim 200 \text{ km s}^{-1}$ (Binney & Tremaine 1987; Persic et al. 1996). Consequently, the mass within a distance r from the centre of the galaxy increases linearly with r , even at large distances where very little luminous matter can be detected.

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It is the purpose of the present analysis to show that the Einstein cluster may indeed serve as a physically acceptable dark matter model (Lake 2006), which satisfies all energy conditions, and is also in agreement with the generic results on anisotropic matter distributions, obtained in Böhmer & Harko (2006). As a first step, we obtain the energy momentum tensor for the Einstein cluster and derive the mass–velocity relation in the presence of a cosmological constant. Next, we show that the Einstein cluster is dynamically stable against radial and non-radial perturbations. Several Buchdahl type physical bounds for the Einstein cluster are also considered, leading to some upper bounds for the velocity of the particles in the cluster and to the estimation of its minimum mass.

An observational possibility of testing the idea of the dark matter as an Einstein cluster is the study of the light deflection and lensing. We derive the deflection angle of light in an Einstein cluster and compare its value with the deflection angle in the standard isothermal sphere dark matter model. It is pointed out that both models can, in principle, be distinguished by observation.

The present paper is organized as follows. The energy momentum tensor for the Einstein cluster is derived in Section 2. The stability of the Einstein cluster is considered in Section 3. General bounds on the velocity and the mass of the cluster are obtained in Section 4 and the lensing effect is considered in Section 5. We discuss and conclude our results in the final Section 6. Throughout this paper, we use a system of units so that $c = 1$.

2 THE EINSTEIN CLUSTER MODEL

We consider a system consisting of particles of rest mass m , each describing a circular orbit about the centre O of the cluster, in the presence of a cosmological constant Λ . The effect of the collision between particles is neglected. Since the system is static and has spherical symmetry, the line element is given by

$$ds^2 = e^v dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (1)$$

where we have denoted the coordinates as $x^0 = t$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \varphi$, respectively, v and λ are functions of r only, and $r \leq R$, with $r = R$ being the boundary of the cluster. This metric is joined continuously at $r = R$ with the standard Schwarzschild–de Sitter metric, having the mass parameter M , which contains the mass of the galactic core and the mass of the dark matter halo.

The energy momentum of the particles in the cluster is given by

$$T_i^j = \rho \left\langle g_{ik} \frac{dx^j}{ds} \frac{dx^k}{ds} \right\rangle, \quad (2)$$

where ρ is the energy density and $\langle \rangle$ denotes the mean value of the energy and the momentum of a particle for all the particles in the volume element around an arbitrary point P . In the following, we consider the particles in the cluster moving in the plane $\varphi = \text{constant}$. We define the components v^α of the three velocity of the particles, measured in terms of the proper time, that is, by an observer located at the given point, by (Landau & Lifshitz 1998)

$$v^\alpha = \frac{dx^\alpha}{\sqrt{g_{tt}} dx^t}. \quad (3)$$

The square of the velocity is given by $V^2 = v_\alpha v^\alpha = -g_{\alpha\beta} v^\alpha v^\beta$. Since the particles are in circular motion around the centre O of the cluster, there is only one non-vanishing component of the three velocity

$$v^\theta = v^\theta = e^{-v/2} \frac{d\theta}{dt}, \quad (4)$$

with the magnitude of the velocity having the value

$$V^2 = r^2 e^{-v} \left(\frac{d\theta}{dt} \right)^2. \quad (5)$$

From equation (1) we immediately obtain

$$e^v \left(\frac{dt}{ds} \right)^2 = (1 - V^2)^{-1}. \quad (6)$$

By taking the mean value in equation (2) we then obtain the components of the energy momentum tensor of the Einstein cluster as

$$T_t^t = \rho^{(\text{eff})} = \rho(1 - V^2)^{-1}, \quad T_r^r = -p_r^{(\text{eff})} = 0, \quad (7)$$

$$T_\theta^\theta = T_\varphi^\varphi = -p_\perp^{(\text{eff})} = -\frac{1}{2}\rho V^2(1 - V^2)^{-1}. \quad (8)$$

Anisotropic matter distributions may be described in terms of the anisotropy parameter $\Delta = p_\perp^{(\text{eff})} - p_r^{(\text{eff})}$, which for the Einstein cluster takes the form

$$\Delta = p_\perp^{(\text{eff})} = \frac{1}{2}\rho V^2(1 - V^2)^{-1}. \quad (9)$$

If $\Delta > 0$, $\forall r \neq 0$, as is the case for the Einstein cluster, the body is tangential pressure dominated.

In the case of an anisotropic system with vanishing radial pressure, the condition of the conservation of the energy momentum tensor, $\nabla_j T^{ij} = 0$, gives

$$v' = \frac{4\Delta}{\rho^{(\text{eff})} r}, \quad (10)$$

where the prime denotes differentiation with respect to r , so that for the Einstein clusters we obtain

$$v' = \frac{2}{r} V^2. \quad (11)$$

In static and spherically symmetric space–times the field equations reduce to three independent equations, which imply conservation of energy momentum. Since in the present discussion it is convenient to use the conservation equation directly, it suffices to consider the first two Einstein field equations, which are

$$-e^{-\lambda} \left(\frac{1}{r^2} - \frac{\lambda'}{r} \right) + \frac{1}{r^2} = 8\pi G \rho (1 - V^2)^{-1} + \Lambda \quad (12)$$

and

$$e^{-\lambda} \left(\frac{v'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = -\Lambda, \quad (13)$$

respectively.

The latter equation (13) leads to

$$e^{-\lambda} = (1 - \Lambda r^2)(2V^2 + 1)^{-1}, \quad (14)$$

whereas from equation (12) we obtain

$$e^{-\lambda} = 1 - \frac{2Gm(r)}{r} - \frac{\Lambda}{3} r^2, \quad (15)$$

where we have defined the mass inside the radius r by

$$m(r) = 4\pi \int_0^r \rho (1 - V^2)^{-1} r'^2 dr'. \quad (16)$$

The metric function λ can be eliminated from equations (14) and (15) to obtain the explicit dependence of $m(r)$ on V^2 and r as

$$\frac{2Gm(r)}{r} = \frac{2V^2 + \Lambda r^2}{1 + 2V^2} - \frac{\Lambda}{3} r^2. \quad (17)$$

Hence, the explicit dependence of the velocity on the mass of the cluster is

$$V^2 = \left[\frac{Gm(r)}{r} - \frac{\Lambda}{3}r^2 \right] \left[1 - \frac{2Gm(r)}{r} - \frac{\Lambda}{3}r^2 \right]^{-1}. \quad (18)$$

For the case of low rotational velocities $V^2 \ll 1$ (the Newtonian limit) the expression for the tangential velocity becomes

$$V^2 = \frac{Gm(r)}{r} - \frac{\Lambda}{3}r^2. \quad (19)$$

In the same approximation, the metric tensor coefficient e^v is given by

$$e^{v(r)} = \exp \left[2 \int \left(\frac{Gm(r')}{r'^2} - \frac{\Lambda}{3}r' \right) dr' \right]. \quad (20)$$

In the Einstein cluster interpretation of the dark matter, the tangential velocity tends to a constant value for large radii $\lim_{r \rightarrow \infty} V(r) = V_0 = \text{constant}$. Therefore, in this region the density of the Einstein cluster is given by

$$\rho(r) = \frac{V_0^2}{4\pi G r^2} + \frac{\Lambda}{4\pi G}. \quad (21)$$

3 STABILITY OF THE EINSTEIN CLUSTERS AGAINST RADIAL AND NON-RADIAL PERTURBATIONS

The issue of stability is of great importance in the study of Newtonian and general relativistic models of self-gravitating objects. Its relevance becomes evident if we recall that any astrophysical model is physically uninteresting if it is unstable against small perturbations, and different degrees of stability/instability will lead to different patterns of evolution in the collapse of self-gravitating objects. Therefore, as realistic dark matter models the Einstein clusters must be dynamically stable.

The stability of astrophysical objects has been the subject of many investigations. Most studies, however, assume that the matter is described by an isotropic perfect fluid. On the other hand, one expects that the anisotropy will change the evolution of self-gravitating systems (Hillebrandt & Steinmetz 1976; Herrera & Santos 1997).

In the present section, we analyse the stability of the Einstein clusters under small radial and non-radial perturbations. In both the cases, we neglect the effect of the cosmological constant since its effects on the dynamical stability is small for the present model (see e.g. Böhmer & Harko 2005a; Hledik, Stuchlik & Mrazova 2007).

3.1 Stability against radial perturbations

We assume that initially the equilibrium configuration of the Einstein cluster is spherically symmetric and that the perturbations preserve this symmetry. Under these perturbations only radial motions will ensue. Hence, the metric of the space–time is still given by equation (1), but with v and λ functions of both t and r , $v = v(t, r)$, $\lambda = \lambda(t, r)$. If we consider the time derivatives as small quantities and neglect the terms of the order V^2 , the field equations are given by Hillebrandt & Steinmetz (1976) and Herrera & Santos (1997) as

$$e^{-\lambda} \left(\frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = 8\pi G \rho^{(\text{eff})}, \quad (22)$$

$$e^{-\lambda} \left(\frac{v'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = 8\pi G p_r^{(\text{eff})}, \quad (23)$$

$$\frac{1}{2}e^{-\lambda} \left(v'' + \frac{v'^2}{2} + \frac{v' - \lambda'}{r} - \frac{v'}{\lambda} 2 \right) - \frac{1}{2}e^{-\lambda} \ddot{\lambda} = 8\pi G p_{\perp}^{(\text{eff})}, \quad (24)$$

$$e^{-\lambda/2} \frac{\dot{\lambda}}{r} = 8\pi G [\rho^{(\text{eff})} + p_r^{(\text{eff})}] \dot{r}, \quad (25)$$

where a dot means the derivative with respect to t .

We perturb the equilibrium solution

$$\rho^{(\text{eff})} = \rho_0^{(\text{eff})} + \delta\rho^{(\text{eff})}, \quad (26)$$

$$p_r^{(\text{eff})} = p_{0r}^{(\text{eff})} + \delta p_r^{(\text{eff})}, \quad (27)$$

$$p_{\perp}^{(\text{eff})} = p_{0\perp}^{(\text{eff})} + \delta p_{\perp}^{(\text{eff})}, \quad (28)$$

$$v = v_0 + \delta v, \quad \lambda = \lambda_0 + \delta\lambda, \quad (29)$$

where the index 0 refers to the equilibrium quantities. By expanding equations (22)–(25) to first order in the perturbations we obtain (Hillebrandt & Steinmetz 1976)

$$8\pi G \delta p_r^{(\text{eff})} = \frac{e^{-\lambda_0}}{r^2} [r(\delta v' - v_0 \delta\lambda') - \delta\lambda], \quad (30)$$

$$8\pi G \delta p_{\perp}^{(\text{eff})} = \frac{1}{2} r \frac{\partial \delta p_r^{(\text{eff})}}{\partial r} + \delta p_r^{(\text{eff})} + \frac{1}{4} v_0' r [\delta\rho^{(\text{eff})} + \delta p_r^{(\text{eff})}] + [\rho_0^{(\text{eff})} + p_{0r}^{(\text{eff})}] \left(\frac{1}{2} e^{\lambda_0} r \ddot{r} + \frac{1}{4} r \delta v' \right), \quad (31)$$

$$8\pi G \delta\rho^{(\text{eff})} = \frac{1}{r^2} \frac{\partial}{\partial r} (r e^{-\lambda_0} \delta\lambda), \quad (32)$$

$$8\pi G [\rho_0^{(\text{eff})} + p_{0r}^{(\text{eff})}] r \dot{r} = -e^{-\lambda_0} \delta\dot{\lambda}. \quad (33)$$

In the case of the Einstein cluster, we have $p_{0r}^{(\text{eff})} = 0$, and since the spherical symmetry of the perturbed system is preserved, we may also take $\delta p_r^{(\text{eff})} = 0$. By assuming that all the perturbations are due to a small perturbation in $r = r(r, t) = r_0 + \delta r(r, t)$ and after introducing the ansatz $\delta r(r, t) = e^{i\omega t} \xi(r)$ and denoting $v(r) = e^{-i\omega t} r^2 e^{-v/2} \delta r(r, t)$, the equation governing the radial pulsations of the Einstein cluster reduces to the simple algebraic condition

$$8\pi G \rho_0^{(\text{eff})} e^{\lambda_0 - v_0} \left[\frac{e^{5v_0/2 + \lambda_0/2}}{r^2} - \omega^2 \right] v(r) = 0. \quad (34)$$

Since $v(r) \neq 0$, we obtain the condition

$$\omega^2 = \frac{e^{5v_0/2 + \lambda_0/2}}{r^2} > 0, \quad \forall r \in [0, R]. \quad (35)$$

Stability means that the eigenfrequency ω of the lowest mode is real and positive, $\omega^2 > 0$ (Hillebrandt & Steinmetz 1976), which taking into account the previous condition, is obviously satisfied for the Einstein cluster. We therefore can conclude that the Einstein cluster is stable against small radial perturbations [a similar result was obtained by Gilbert (1954) by using a different method]. The frequency of the radial oscillations of the cluster is related to its radius R and mass M by the relation $\omega^2 = (1 - 2GM/R)^2/R^2$.

3.2 Stability against non-radial perturbations

For isotropic spherically symmetric systems the stability against radial pulsations implies the stability against all small adiabatic oscillations (Binney & Tremaine 1987). For anisotropic models this need not to be true since the anisotropy may serve as a source of

instability once the exact spherical symmetry is broken (Hillebrandt & Steinmetz 1976). In the case of Einstein clusters, the condition of the exact circular orbits implies the vanishing of the radial pressure. However, as far as this condition is broken, and the trajectories of the particles are not perfectly circular, a radial pressure may be generated, and this pressure, together with the tangential pressure, may destabilize the system. Therefore, it is important to also analyse the stability of the Einstein cluster under non-radial perturbations. Since the general relativistic analysis of this problem is extremely complicated and analytic solutions are difficult to be obtained, we restrict ourselves to the Newtonian case and follow the analysis given by Hillebrandt & Steinmetz (1976).

We assume that the perturbed Einstein cluster with non-spherical symmetry has a density ρ , a radial pressure p_r , and a tangential pressure p_\perp , and that generally $p_r \neq p_\perp$. As a result of the perturbation, the Newtonian equations of motion of the perturbed system, moving with velocity \mathbf{v} under the action of the hydrodynamic force \mathbf{F} in a gravitational field with potential Φ , will be the anisotropic Euler equation, the continuity equation and the Poisson equation, given by

$$\rho \frac{d\mathbf{v}}{dt} = \mathbf{F} - \rho \nabla \Phi, \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (36)$$

and

$$\nabla^2 \Phi = 4\pi G \rho, \quad (37)$$

respectively. In spherical coordinates (r, θ, φ) , the components of the hydrodynamical force F are

$$F_r = -\frac{\partial p_r}{\partial r} + \frac{2\Delta}{r},$$

$$F_\theta = -\frac{1}{r} \frac{\partial p_\perp}{\partial \theta}, \quad F_\varphi = -\frac{1}{r \sin \theta} \frac{\partial p_\perp}{\partial \varphi}. \quad (38)$$

To close the system of equations, we need an equation of state of the radial pressure, which we assume to be of the barotropic form, $p_r = p_r(\rho)$, and an equation of state for the anisotropy parameter Δ , which we take as $\Delta = \beta(\rho) p_r$, with $\beta(\rho)$ an arbitrary function. We also denote $\gamma = (\rho/p_r) (\partial p_r / \partial \rho)$. In the following, we will assume only axially symmetric perturbations, so that $\delta\varphi = 0$.

By assuming that $\delta \mathbf{r}(\mathbf{r}, t) = e^{i\sigma t} \delta \mathbf{r}(\mathbf{r})$, the linearized equations of motion become

$$\sigma^2 \delta r = \frac{\partial \delta \Phi}{\partial r} - \frac{\delta \rho}{\rho^2} \frac{\partial p_r}{\partial r} + \frac{1}{\rho} \frac{\partial \delta p_r}{\partial r} - \frac{2}{r\rho} \left(\delta \Delta - \frac{\delta r}{r} \Delta - \frac{\delta \rho}{\rho} \Delta \right), \quad (39)$$

$$\sigma^2 \delta \theta = \frac{1}{r} \frac{\partial}{\partial \theta} \left(\delta \Phi + \frac{\delta p_\perp}{\rho} \right), \quad (40)$$

$$\frac{\delta \rho}{\rho} + \frac{\delta r}{r} \frac{\partial \rho}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \delta r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r \sin \theta \delta \theta) = 0 \quad (41)$$

and

$$\nabla^2 \delta \Phi = 4\pi G \delta \rho, \quad (42)$$

respectively, where all the quantities without δ are equilibrium quantities. We now express all the perturbations in terms of the Legendre polynomials $P_l(\cos \theta)$, so that

$$\delta \Phi = \delta \Phi^*(r) P_l(\cos \theta), \quad \delta \rho = \delta \rho^* P_l(\cos \theta), \quad \text{etc.} \quad (43)$$

Equation (39) thus becomes

$$\sigma^2 \delta r^* = \frac{\partial \delta \Phi^*}{\partial r} - \frac{\delta \rho^*}{\rho^2} \frac{\partial p_r}{\partial r} + \frac{1}{\rho} \frac{\partial \delta p_r^*}{\partial r} - \frac{2}{r\rho} \left(\delta \Delta^* - \frac{\delta r^*}{r} \Delta - \frac{\delta \rho^*}{\rho} \Delta \right). \quad (44)$$

By introducing the new variables $\xi = \gamma p_r \delta \rho^* / \rho$, $\varepsilon = \delta \Phi^*$ and $\eta = r^2 \delta r^*$, we obtain the non-radial perturbation equations for an anisotropic system as (Hillebrandt & Steinmetz 1976)

$$\eta' = -\frac{p_r'}{\gamma p_r} + \frac{l(l+1)}{\sigma^2} \varepsilon - \left\{ \frac{r^2}{\gamma p_r} - \frac{l(l+1)}{\sigma^2 \rho} \left[(\beta+1) + \frac{\rho}{\gamma} \left(\frac{\partial \beta}{\partial \rho} \right) \right] \right\} \xi, \quad (45)$$

$$\varepsilon'' = 4\pi G \frac{\rho}{\gamma p_r} \xi - \frac{2}{r} \varepsilon' + \frac{l(l+1)}{r^2} \varepsilon, \quad (46)$$

$$\xi' = \left(\sigma^2 \rho + \frac{2\beta}{r^2} p_r \right) \frac{\eta}{r^2} - \rho \varepsilon' - \frac{2}{r} \left[\frac{\beta}{\rho} \left(1 - \frac{1}{\gamma} \right) + \frac{1}{\gamma} \left(\frac{\partial \beta}{\partial \rho} \right) + \frac{1}{\gamma \rho} \frac{p_r'}{p_r} \right] \xi. \quad (47)$$

Equations (45)–(47) must be solved together with the boundary conditions $\xi(0) = \eta(0) = \varepsilon(0) = 0$, ε continuous across the boundary R and $\Delta p_r^*(R) = [\delta p_r^*(r) + p_r' \delta r^*(r)]_{r=R} = 0$. Any solution satisfying these conditions determines one eigenvalue σ^2 (Hillebrandt & Steinmetz 1976).

We consider now the Einstein cluster as our equilibrium model. Moreover, we will consider only quadrupole oscillations, thus taking $l=2$. Since in this case $p_r = 0$, it immediately follows that $\xi = 0$. Hence, equation (46) becomes

$$\varepsilon'' + \frac{2}{r} \varepsilon' - \frac{6}{r^2} \varepsilon = 0, \quad (48)$$

with the solution $\varepsilon(r) = C_1 r^2$, with C_1 an arbitrary constant of integration. Equation (47) gives $\varepsilon' = \sigma^2 \eta / r^2$ and after taking the derivative of equation (45) it follows that η satisfies the second-order differential equation

$$\eta'' - \frac{6}{r^2} \eta = 0, \quad (49)$$

with the solution $\eta(r) = C_2 r^3$. From the definition of η , we thus obtain $\delta r^* = C_2 r$. The continuity of the gravitational potential at the boundary $r = R$ gives the ratio of the integration constants as $C_1/C_2 = GM/R^3$. In the case of the Einstein cluster, the condition $\Delta p_r^*(R) = 0$ gives $\Delta p_r^*(R) = 0$.

For the Einstein cluster $\Delta \approx \rho V_0^2/2$ and $V_0^2 \approx Gm(r)/r$, where $V_0 = \text{constant}$ is the speed in the constant velocity region. By estimating equation (44) near $r = R$ gives

$$\sigma^2(r)|_{r=R} = \frac{2GM}{R^3} + Gm(r)r^3 \Big|_{r=R} > 0. \quad (50)$$

Since σ^2 is always positive, it follows that the Einstein clusters are stable with respect to the quadrupole oscillations.

4 PHYSICAL BOUNDS FOR THE EINSTEIN CLUSTERS

In realistic physical models for general relativistic matter distributions, the anisotropy Δ should be finite, positive and should satisfy the dominant energy condition (DEC) $\Delta \leq \rho$ and the strong energy condition (SEC) $2\Delta \leq \rho$ (see e.g. Böhmer & Harko 2006). These

conditions may be written together as $\Delta \leq n\rho$, where $n = 1$ for DEC and $n = 1/2$ for SEC, and they are automatically satisfied by the components of the energy momentum tensor of the Einstein cluster.

If the function $\Delta e^{v/2}/r$ is monotonically decreasing for all r , then, as was proved rigorously in Böhmer & Harko (2005b), inside the anisotropic matter distribution the following inequality always holds:

$$\sqrt{1 - \frac{2Gm(r)}{r} - \frac{\Lambda}{3}r^2} \geq \frac{1}{3} \left(1 - \frac{\Lambda}{\langle \rho \rangle}\right) \frac{1}{1+f}, \quad (51)$$

where $f = f(m, r, \Lambda, \Delta)$, and we have already inserted that $p_r = 0$. The function f explicitly reads as

$$f = 2 \frac{\Delta(r)}{\langle \rho \rangle} \left\{ \frac{\arcsin \sqrt{2G\alpha(r)m(r)/r}}{\sqrt{2G\alpha(r)m(r)/r}} - 1 \right\}, \quad (52)$$

where we have denoted

$$\alpha(r) = 1 + \frac{\Lambda r^3}{6Gm(r)}, \quad (53)$$

and where $\langle \rho \rangle$ is the mean density of the matter $\langle \rho \rangle = 3m(r)/4\pi r^3$ inside the radius r .

For small values of the argument, we can use the series expansion $\arcsin x/x \approx 1 + x^2/6 + \dots$, and obtain

$$f \approx \frac{2}{3} \frac{\Delta(r)}{\langle \rho \rangle} \frac{G\alpha(r)m(r)}{r}. \quad (54)$$

Since for the Einstein cluster the function $\Delta e^{v/2}/r = 1/r^{3-V_0^2}$ is monotonically decreasing, the above bound can be applied to the constant velocity region. Let us estimate equation (51) at the boundary $r = R$ of the cluster and denote by $M = m(R)$ its total mass. By neglecting small terms ($\Lambda r^2 \ll 1$, $\Lambda V_0^2 \ll 1$) the above bound reduces to

$$\sqrt{1 - \frac{2GM}{R}} \geq \frac{1}{3} \left[1 - \frac{\rho(R)}{3\langle \rho \rangle} V_0^2 \frac{GM}{R}\right]. \quad (55)$$

Both terms containing M/R can be replaced by the rotational velocity in view of equation (19), where the cosmological constant is neglected again. Hence, in terms of V_0 , the condition given by equation (55) can be reformulated as

$$V_0^2 \leq \frac{4}{9} \left[1 + \frac{4}{9} \frac{\rho(R)}{27\langle \rho_R \rangle}\right], \quad (56)$$

where $\langle \rho_R \rangle$ is the average density of the whole cluster. On the other hand, by using the definition of the mean density and equations (19) and (21) we have

$$\frac{\rho(R)}{\langle \rho_R \rangle} = \frac{1}{3}. \quad (57)$$

Hence, one obtains the following absolute bound for the rotational velocity of the massive particles in the Einstein cluster

$$V_0^2 \leq \frac{2}{3} \sqrt{1 + \frac{2^2}{3^6}}. \quad (58)$$

Therefore, we find the condition $V_0 \leq 2/3 \approx 2 \times 10^8 \text{ m s}^{-1}$. This result is consistent with the bound obtained in Gilbert (1954) from stability considerations, requiring that the velocity of particles in stable circular orbits in Einstein clusters must be smaller than half of the speed of light. On the other hand, the Buchdahl limit for the mass–radius ratio for the Einstein cluster (insert GM/R for V_0^2) gives no physically relevant restriction since for typical galaxies $M/R \ll 1$. The same analysis can, in principle, be repeated without neglecting the contributions due to the cosmological constant and the anisotropy in equation (51). However, the resulting bound on

the tangential velocity is only mildly affected and the main result is unchanged.

Following the analysis of Böhmer & Harko (2006), it is natural to consider next the behaviour of the invariant curvature scalars $r_0 = R$, $r_1 = R_{ab}R^{ab}$ and $r_2 = R_{abcd}R^{abcd}$, which are, in general, decreasing functions with respect to the radius for regular relativistic matter distributions. For the Einstein cluster, these invariants only yield the speed of light as an upper bound on the velocity and no further condition emerges.

In the presence of the cosmological constant, the Buchdahl inequality (51) not only leads to upper bounds on the mass, but also implies the existence of a minimal mass (Böhmer & Harko 2005b). For Einstein clusters with radii R_g in the range $R_g \approx 10\text{--}100 \text{ kpc}$ this yields

$$M_{\min} = \frac{4\pi}{3} \frac{\Lambda}{16\pi G} R_g^3 \approx 4.5 \times 10^5 - 4.5 \times 10^8 M_{\odot}, \quad (59)$$

which is perfectly consistent with present mass estimates of galaxies. In fact, this minimal mass is roughly of the order of the total (virial) mass of the galaxies, and further supports that the minimal mass due to Λ is an important physical quantity in an astrophysical context (see e.g. Balaguera-Antolinez & Nowakowski 2005; Balaguera-Antolinez, Böhmer & Nowakowski 2005, 2006).

5 LIGHT DEFLECTION AND LENSING BY EINSTEIN CLUSTERS

One of the ways we could, in principle, test the possible existence of Einstein clusters as astrophysical systems would be by studying the light deflection in the cluster and, in particular, by studying the deflection of photons passing through the region where the rotation curves are flat, with $V = V_0 = \text{constant}$.

The metric coefficient $\exp(\nu)$ can be found from equation (11), and is given by

$$e^{\nu} = \left(\frac{r}{R_c}\right)^{2V_0^2}, \quad (60)$$

where R_c is a constant of integration. Its value can be found by matching the metric tensor at the boundary $r = R$ with the standard Schwarzschild–de Sitter metric. Thus, we obtain

$$e^{\nu} = \left(1 - 2V_0^2 - \frac{\Lambda}{3}R^2\right) \left(\frac{r}{R}\right)^{2V_0^2}. \quad (61)$$

Let us consider a photon approaching a galaxy from far distance. The bending of light by the gravitational field results in a deflection angle $\Delta\phi$ given by

$$\Delta\phi = 2|\phi(r_0) - \phi_{\infty}| - \pi, \quad (62)$$

where ϕ_{∞} is the incident direction and r_0 is the coordinate radius of the closest approach to the centre of the galaxy. Generally, one finds (Weinberg 1972)

$$\phi(r_0) - \phi_{\infty} = \int_{r_0}^{\infty} \frac{e^{\lambda(r)/2}}{\sqrt{e^{\nu(r_0)-\nu(r)}(r/r_0)^2 - 1}} \frac{dr}{r}. \quad (63)$$

For the Einstein cluster in the constant velocity region, we obtain

$$\phi(r_0) - \phi_{\infty} = \frac{\int_{r_0}^{\infty} \left[(r/r_0)^{2(1-V_0^2)} - 1\right]^{-1/2} dr/r}{\sqrt{1 - 2V_0^2}}. \quad (64)$$

By introducing the new variable $\eta = r/r_0$ this leads to

$$\phi(r_0) - \phi_{\infty} = \frac{\int_1^{\infty} \left[\eta^{2(1-V_0^2)} - 1\right]^{-1/2} d\eta/\eta}{\sqrt{1 - 2V_0^2}}, \quad (65)$$

and this integral can be evaluated exactly. Thus, we find the deflection angle for the Einstein cluster as

$$\Delta\phi_{\text{EC}} = \frac{2\pi V_0^2}{\sqrt{1-2V_0^2}(1-V_0^2)} \approx 2\pi V_0^2. \quad (66)$$

In the standard approach to dark matter lensing it is assumed that the deflection angle $\Delta\phi_{\text{DM}}$ is given by $\Delta\phi_{\text{DM}} = 4GM(r)/r$, where $M(r)$ is the mass inside a radius r (Wambsganss 1998). In the constant velocity region, the usual deflection angle therefore becomes

$$\Delta\phi_{\text{DM}} = 4V_0^2, \quad (67)$$

which differs from the Einstein cluster deflection angle by a factor of $\pi/2$.

In the generic case of a three-dimensional mass distribution, the density $\rho(\mathbf{r})$ can be projected along the line of sight into the lens plane to obtain the two-dimensional surface mass density distribution $\Sigma(\xi) = \int_0^{D_S} \rho(\mathbf{r}) dz$, where D_S is the distance from the source to the observer and ξ is a two-dimensional vector in the lens plane (Wambsganss 1998).

If one assumes for the galaxy lenses the singular isothermal density sphere model, with density varying as $\rho(r) = V_0^2/4\pi Gr^2$, the circularly symmetric mass distribution is given by $\Sigma(\xi) = V_0^2/4G\xi$. Since $M(\xi) = \int_0^\xi \Sigma(\xi') 2\pi\xi' d\xi' = \pi V_0^2/2G\xi$, we find $\alpha_{\text{DM}}(\xi) = 2\pi V_0^2$, which is constant, that is, independent of the impact parameter ξ (Wambsganss 1998). In the case of the Einstein cluster we obtain

$$\alpha_{\text{EC}}(\xi) = \frac{\pi(\pi/2 + 1)V_0^2}{\sqrt{1-\pi V_0^2}(1-V_0^2)} \approx \pi \left(\frac{\pi}{2} + 1 \right) V_0^2. \quad (68)$$

Therefore, lensing effects can, in principle, discriminate between two different dark models. For the Einstein cluster model in comparison with the isothermal dark matter halo, we find

$$\frac{\alpha_{\text{DM}}}{\alpha_{\text{EC}}} = \frac{4}{\pi + 2} \approx 0.77. \quad (69)$$

This suggests that a galactic dark matter halo consisting of weakly interacting massive particles in the form of an Einstein cluster predicts slightly smaller gravitational lensing effects. Thus, lensing seems to be a prime experimental tool to test different dark matter models since they, in general, predict different deflection angles.

6 CONCLUSIONS

The flattened galactic rotation curves and the absence of sufficient luminous matter to explain them continue to pose a challenge to present-day astrophysics. One would like to have a better understanding of some of the phenomena associated with them, such as their universality and the very good correlation between the amount of dark matter and the luminous matter in galaxies.

In this paper, we have further developed an alternative view of the dark matter problem proposed by Lake (2006), namely, the possibility that the dark matter is in the form of an Einstein cluster of WIMPs. This assumption can explain the observed linearly increasing mass profile outside galaxies, without introducing any supplementary conditions. In fact, this model can reproduce any velocity profile (Lake 2006). We analysed the stability of the Ein-

stein clusters against both radial and non-radial perturbations, and we showed that the system is dynamically stable.

The Buchdahl bounds for the clusters have also been obtained, and thus we showed that there is a maximal bound for the velocity of the particles and a minimum mass for the Einstein cluster. This minimal mass is of the same order as the total (virial) masses of the galaxies. The existence of the minimum mass is a direct consequence of the presence of a non-zero cosmological constant (Böhmer & Harko 2005b).

All the relevant physical quantities have been expressed in terms of observable parameters (mass, radius and velocity dispersion). This allows an in-depth comparison of the predictions of the Einstein cluster model with observational results.

A possibility of observationally testing the viability of the Einstein cluster as a dark matter model is via gravitational lensing. We studied the lensing effect for Einstein clusters and compared it with the standard dark matter model, the singular isothermal sphere. Even though the differences in the deflection angles are small, a significant improvement in the observational techniques may allow to discriminate between the Einstein cluster and the other dark matter models.

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