

Miniscule representations, Gauss sum and modular invariance ¹

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Abstract. After explaining the concepts of Langlands dual and miniscule representations, we define an analog of the Gauss sum for any compact, simple Lie group with a simply laced Lie algebra. We then show a reciprocity property when a Lie group is exchanged with its Langlands dual. We also explore the relation with theta functions and modular transformations. In the non-simply laced case, we construct a unitary representation of the Hecke group which involves interesting new phase factors.

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1. Introduction

Duality has played an important role in physics and mathematics. One of the earliest examples is the electric-magnetic duality or S -duality [9, 16], which is exact in the $N = 4$ supersymmetric gauge theory. This leads to the celebrated conjecture of Vafa and Witten [20] on the Euler number of the moduli space of anti-self-dual connections. In this paper, we study some finite dimensional unitary representations of the modular group motivated from the S -duality transformations in [20] and their consequences. We also construct similar representations of the Hecke groups. We leave the interplay with four manifolds to subsequent work.

The paper is organized as follows. In Section 2, we review the concepts of Langlands dual and miniscule representations. In Section 3, we study the simply laced case. We define an analog of the Gauss sum for a compact simple Lie group using miniscule representations. We then show a reciprocity property when the Lie group is exchanged with its Langlands dual. We also explore the relation with theta functions and modular transformations. Section 4 is on the non-simply laced case. We start with the theta functions and derive an analog of the modular transformations for the Hecke group. This enables us to construct a finite dimensional unitary representation of the Hecke group which involves new phase factors that will have interesting implications on S -duality.

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2. Langlands dual and miniscule representations

To explain the concept of dual groups, we start with the Abelian case. Suppose T is a compact real torus with Lie algebra \mathfrak{t} . Then $T = \mathfrak{t}/2\pi\sqrt{-1}\ell$, where $\ell \subset \sqrt{-1}\mathfrak{t}$ is a lattice of full rank. We note that $\ell \cong \text{Hom}(U(1), T)$. On the other hand, the (one-dimensional) irreducible representations of T are classified by the dual lattice $\ell^* \subset \sqrt{-1}\mathfrak{t}^*$. The dual torus is $T^* = \mathfrak{t}^*/2\pi\sqrt{-1}\ell^*$, for which the roles of ℓ and ℓ^* are reversed. That is, $\ell^* \cong \text{Hom}(U(1), T^*)$ whereas ℓ classifies the irreducible representations of T^* .

Now let G be a simple, compact and connected Lie group. The Langlands dual ${}^L G$ of G is characterised by the property that the irreducible representations of ${}^L G$ are in one-to-one correspondence with the homomorphisms from $U(1)$ to G modulo the conjugation in G . We will describe a construction of ${}^L G$ below. Some examples of G and its dual ${}^L G$ are listed as follows.

G	$SU(n)$	$Spin(2n)$	$Sp(n)$	$Spin(2n+1)$	E_8
${}^L G$	$SU(n)/\mathbb{Z}_n$	$SO(2n)/\mathbb{Z}_2$	$SO(2n+1)$	$Sp(n)/\mathbb{Z}_2$	E_8

Let T be a maximal torus of G and $\mathfrak{g}, \mathfrak{t}$, the Lie algebras of G, T , respectively. Recall that the root system $\Delta \subset \sqrt{-1}\mathfrak{t}^*$ is the set of roots and that the root lattice $\Lambda = \text{span}_{\mathbb{Z}}\Delta \subset \sqrt{-1}\mathfrak{t}^*$ is the lattice generated by Δ . For any $\alpha \in \Delta$, the corresponding coroot $\check{\alpha} \in \mathfrak{t}$ is the vector such that $\langle \beta, \check{\alpha} \rangle = 2(\beta, \alpha)/(\alpha, \alpha)$ for any $\beta \in \Delta$. Here $\langle \cdot, \cdot \rangle$ is the pairing between $\sqrt{-1}\mathfrak{t}$ and its dual space $\sqrt{-1}\mathfrak{t}^*$ while (\cdot, \cdot) is an inner product on $\sqrt{-1}\mathfrak{t}^*$ invariant under the Weyl group; such an inner product is unique up to a multiple of positive scalars (since \mathfrak{g} is simple) and the coroot $\check{\alpha}$ does not depend on its choice. The set $\check{\Delta} = \{\check{\alpha} | \alpha \in \Delta\}$ is also a root system. Let $\check{\Lambda} = \text{span}_{\mathbb{Z}}\check{\Delta} \subset \sqrt{-1}\mathfrak{t}$ be the coroot lattice. Then the weight lattice is $\check{\Lambda}^* \subset \sqrt{-1}\mathfrak{t}^*$ and the coweight lattice is $\Lambda^* \subset \sqrt{-1}\mathfrak{t}$. Let $\ell \subset \sqrt{-1}\mathfrak{t}$ be the lattice such that $T = \mathfrak{t}/2\pi\sqrt{-1}\ell$. Then we have the inclusions [5, §4.9]

$$\check{\Lambda} \subset \ell \subset \Lambda^* \subset \sqrt{-1}\mathfrak{t}, \quad \Lambda \subset \ell^* \subset \check{\Lambda}^* \subset \sqrt{-1}\mathfrak{t}^*. \quad (2.1)$$

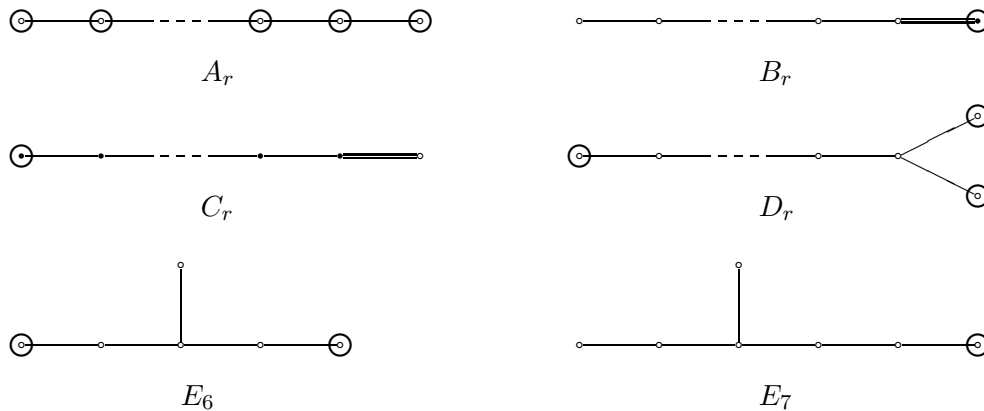
The Lie algebra \mathfrak{g} uniquely determines the universal covering group \tilde{G} of G , which is simply connected and whose centre is $Z(\tilde{G}) \cong \Lambda^*/\check{\Lambda}$. Here is a table of $Z(\tilde{G})$.

\mathfrak{g}	A_r	B_r	C_r	D_r	$E_{r=6,7,8}$	F_4	G_2
$Z(\tilde{G})$	\mathbb{Z}_{r+1}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_4	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	\mathbb{Z}_{9-r}	1
				(r odd)	(r even)		

At the opposite extreme of \tilde{G} , the adjoint group $G_{\text{ad}} = \tilde{G}/Z(\tilde{G})$ has the same Lie algebra \mathfrak{g} , but has $Z(G_{\text{ad}}) = 1$ and $\pi_1(G_{\text{ad}}) = Z(\tilde{G})$. Notice that $\ell = \check{\Lambda}$ for $G = \tilde{G}$ and $\ell = \Lambda^*$ for $G = G_{\text{ad}}$. For a general compact group G with Lie algebra \mathfrak{g} , we have $Z(G) = \Lambda^*/\ell$ and $\pi_1(G) = \ell/\check{\Lambda}$.

The construction of the Langlands dual ${}^L G$ is achieved in two steps. First, the Lie algebra ${}^L \mathfrak{g}$ is defined so that its root system is the coroot system $\check{\Delta}$ of \mathfrak{g} . In Cartan's classification, ${}^L \mathfrak{g}$ is of the same type as \mathfrak{g} unless \mathfrak{g} is of type B_r or C_r , in which case ${}^L \mathfrak{g}$ is of type C_r or B_r , respectively. The Lie algebra ${}^L \mathfrak{g}$ then determines the universal covering group $\widetilde{{}^L G}$, whose centre is $Z(\widetilde{{}^L G}) \cong \check{\Lambda}^*/\Lambda$. The latter is also the character group $Z(\tilde{G})^* = \text{Hom}(Z(\tilde{G}), U(1))$ of $Z(\tilde{G})$. Second, the fundamental group of ${}^L G$, which should be a subgroup of $Z(\widetilde{{}^L G}) \cong Z(\tilde{G})^*$, is defined to be $\pi_1({}^L G) = \{\chi \in Z(\tilde{G})^* | \chi(\pi_1(G)) = 1\}$. $\widetilde{{}^L G}$ and $\pi_1({}^L G)$ uniquely determine ${}^L G = \widetilde{{}^L G}/\pi_1({}^L G)$. We have $\pi_1({}^L G) \cong Z(\tilde{G})^* \cong \ell^*/\Lambda$ and $Z({}^L G) \cong \pi_1(G)^* \cong \check{\Lambda}^*/\ell^*$. In particular, the Langlands duals of \tilde{G}, G_{ad} are $({}^L G)_{\text{ad}}, {}^L G$, respectively.

A non-trivial representation of \mathfrak{g} is miniscule if all the weights form a single Weyl group orbit. Their highest weights are called miniscule weights. A miniscule weight must be fundamental, but the converse is not true. For some simple Lie algebras (of types B_r ($r \geq 3$), D_r ($r \geq 4$) and all exceptional types), the adjoint representation is fundamental, but it can not be miniscule because 0 is a weight not in the orbit of the highest weight. A fundamental representation may also contain non-zero weights of different lengths. For example, the adjoint representation of B_r ($r \geq 3$) is fundamental; it is the second exterior product of the defining representation. But all non-zero weights of the latter are the weights of the former. If a representation is miniscule, then so are its conjugate representation and those obtained by (outer) automorphisms of \mathfrak{g} . A complete list of miniscule weights is shown below by highlighting the corresponding vertices in the Dynkin diagrams.



For any simple Lie algebra \mathfrak{g} , the miniscule weights are in one-to-one correspondence with the non-zero elements in $\check{\Lambda}^*/\Lambda$ [4, §VIII.7.3]. To paraphrase this result, notice that the set $\check{\Lambda}^*/\Lambda$ is precisely $Z(\check{L}\tilde{G})$, the centre of (the universal covering of) the Langlands dual group. So there is a one-to-one correspondence between the miniscule representations and the non-trivial elements of $Z(\check{L}\tilde{G})$. The above list of miniscule representations can be compared with the table of the centre.

Let $\mathcal{M}(\mathfrak{g})$ be the set of zero and miniscule weights of \mathfrak{g} . The bijection with $Z(\check{L}\tilde{G}) = Z(\check{G})^*$ means that there is a group structure on the set $\mathcal{M}(\mathfrak{g})$, which we now describe. Two irreducible representations of \mathfrak{g} are congruent [7, §3] if the differences of their weights are in the root lattice Λ . This classifies representations of \mathfrak{g} according to how $Z(\check{G})$ acts. That is, two representations of \mathfrak{g} are in the same congruence class if and only if they, upon exponentiating to \check{G} and restricting to the center, yield the same character. (Compare [7, Theorem 3.1].) For example, the representations of B_r fall into two congruence classes: those that are representations of $SO(2r + 1)$ (such as the defining representation and its tensor powers) and those that are not (such as the spinor representation). The group structure on the congruence classes comes from the tensor product (which is well-defined on such classes) and coincides with the multiplication of characters on $Z(\check{G})$. In each congruent class of representations, the (unique) one of the smallest dimension is the trivial or miniscule representation. So $\mathcal{M}(\mathfrak{g})$ is the set of representatives of the congruence classes of representations. Moreover, it can be shown that given two miniscule representations, their tensor product contains a miniscule or trivial representation as a summand.

At the group level, we say that a representation of G is miniscule if the induced representation of the Lie algebra \mathfrak{g} is so. Let $\mathcal{M}(G)$ be the set of highest weights of the trivial or miniscule representations of G . Clearly, $\mathcal{M}(G)$ is a subgroup of $\mathcal{M}(\mathfrak{g})$ under the tensor product of congruence classes, and $\mathcal{M}(\tilde{G}) = \mathcal{M}(\mathfrak{g})$, $\mathcal{M}(G_{\text{ad}}) = \{0\}$. For example, if $G = SU(pq)/\mathbb{Z}_p$, then $\mathcal{M}(G) \cong \mathbb{Z}_q$. In general, $\mathcal{M}(G)$ is in one-to-one correspondence with the subset ℓ^*/Λ of $\check{\Lambda}^*/\Lambda \cong Z(\check{L}G)$. Therefore we have the following

Lemma 2.1 *There is an isomorphism between the groups $\mathcal{M}(G)$ and $\pi_1(LG) \cong Z(G)^*$.*

3. Simply laced case: Gauss sum and modular invariance

As before, let G be a compact, simple Lie group. Suppose the Lie algebra \mathfrak{g} is simply laced, that is, all the roots are of the same length. We consider the Weyl group invariant inner product $(\cdot|\cdot)$ on $\sqrt{-1}\mathfrak{t}^*$ so that $(\alpha|\alpha) = 2$ for all $\alpha \in \Delta$. We define an analog of the Gauss sum

$$\mathcal{G}(G) = \frac{1}{\sqrt{|Z(G)|}} \sum_{\mu \in \mathcal{M}(G)} e^{\pi\sqrt{-1}(\mu|\mu)} \quad (3.1)$$

and set $\mathcal{G}(\mathfrak{g}) = \mathcal{G}(\tilde{G})$. For example, $\mathcal{G}(E_8) = 1$ since $Z(E_8) = 1$ and

$$\mathcal{G}(SU(n)) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{\pi\sqrt{-1} \frac{k(n-k)}{n}}. \quad (3.2)$$

The latter can be compared with the classical Gauss sum

$$\sum_{k=0}^{n-1} e^{2\pi\sqrt{-1} \frac{k^2}{n}} = \sqrt{n} \frac{1 + (-\sqrt{-1})^n}{1 - \sqrt{-1}}. \quad (3.3)$$

Identifying $\sqrt{-1}\mathfrak{t}$ with $\sqrt{-1}\mathfrak{t}^*$ using the inner product $(\cdot|\cdot)$, we have $\check{\Lambda} = \Lambda \subset \check{\Lambda}^* = \Lambda^*$. The following is a relation between the Gauss sum of G and that of the Langlands dual ${}^L G$.

Theorem 3.1 *If \mathfrak{g} is simply laced and of rank r , then*

$$\mathcal{G}(G) = \overline{\mathcal{G}({}^L G)} e^{\frac{\pi\sqrt{-1}}{4}r}. \quad (3.4)$$

In particular,

$$\sum_{\mu \in \mathcal{M}(\mathfrak{g})} e^{\pi\sqrt{-1}(\mu|\mu)} = \sqrt{|Z(G)|} e^{\frac{\pi\sqrt{-1}}{4}r}. \quad (3.5)$$

Examples of (3.5) are below.

A_r	$\sum_{k=0}^r e^{\pi\sqrt{-1} \frac{k(r+1-k)}{r+1}} = \sqrt{r+1} e^{\frac{\pi\sqrt{-1}}{4}r}$
D_r	$1 + e^{\pi\sqrt{-1}} + 2e^{\pi\sqrt{-1} \frac{r}{4}} = \sqrt{4} e^{\frac{\pi\sqrt{-1}}{4}r}$
E_6	$1 + 2e^{\pi\sqrt{-1} \frac{4}{3}} = \sqrt{3} (e^{\frac{\pi\sqrt{-1}}{4}})^6$
E_7	$1 + e^{\pi\sqrt{-1} \frac{3}{2}} = \sqrt{2} (e^{\frac{\pi\sqrt{-1}}{4}})^7$
E_8	$1 = (e^{\frac{\pi\sqrt{-1}}{4}})^8$

If $Z(\tilde{G}) = 1$, then (3.5) is consistent only if the rank $r = 0 \pmod{8}$, which is the case for E_8 . A non-trivial example of (3.4) is

$$\mathfrak{G}(p, q) = \sqrt{\frac{q}{p}} e^{\frac{\pi\sqrt{-1}}{4}(pq-1)} \overline{\mathfrak{G}(q, p)} \quad (3.6)$$

when $G = SU(pq)/\mathbb{Z}_p$, ${}^L G = SU(pq)/\mathbb{Z}_q$, $r = pq - 1$. Here

$$\mathfrak{G}(p, q) = \sum_{k=1}^{q-1} e^{\pi\sqrt{-1}p\frac{k(q-k)}{q}} \quad (3.7)$$

is the generalised Gauss sum [6, §V.2]. If p, q are distinct odd primes, since $\mathfrak{G}(p, q) = \left(\frac{p}{q}\right) \mathfrak{G}(1, q)$, where $\left(\frac{p}{q}\right)$ is the Legendre symbol, (3.6) implies (see for example [6, §V.3]) the celebrated quadratic reciprocity of Gauss

$$\left(\frac{p}{q}\right) \left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{4}}. \quad (3.8)$$

The formula (3.5) for $G = SU(n)$ appears explicitly in [20]; the generalisation to any simply laced \mathfrak{g} is straightforward. As observed in [20], (3.5) is a special case of the following. Let V be a real vector space with a (not necessarily positive) non-degenerate quadratic form (\cdot, \cdot) , which identifies V^* with V . If L is an even lattice in V , then [2] (see also [15, Appen. 4])

$$\sum_{\mu \in L^*/L} e^{\pi\sqrt{-1}(\mu, \mu)} = \sqrt{|L^*/L|} e^{\frac{\pi\sqrt{-1}}{4}\sigma}, \quad (3.9)$$

where σ is the signature of (\cdot, \cdot) .

(3.4) can be proved using the Fourier transform on finite Abelian groups (see for example [18, §I.4]). To explain the general setting, let A be a finite Abelian group and let $A^* = \text{Hom}(A, U(1))$, the group of characters. The Fourier transform of a function $f: A \rightarrow \mathbb{C}$ is $\hat{f}: A^* \rightarrow \mathbb{C}$ given by $\hat{f}(\chi) = \sum_{a \in A} \chi(a) f(a)$, $\chi \in A^*$. The inverse transform is $f = \frac{1}{|A|} \sum_{\chi \in A^*} \hat{f}(\chi) \chi$. In particular, $f(0) = \frac{1}{|A|} \sum_{\chi \in A^*} \hat{f}(\chi)$. For any subgroup B of A , we have the (discrete) Poisson summation formula [18, Theorem 4.11]

$$\frac{1}{|B|} \sum_{a \in B} f(a) = \frac{1}{|A|} \sum_{\chi \in (A^*)^B} \hat{f}(\chi), \quad (3.10)$$

where $(A^*)^B \cong (A/B)^*$ is the set of characters of A that are trivial on B . To apply to our situation, let $A = \Lambda^*/\Lambda$, $B = \ell/\Lambda \cong \mathcal{M}({}^L G)$. A^* can be identified with A by $u \mapsto \chi_u$, where $\chi_u(v) = e^{2\pi\sqrt{-1}(u|v)}$ ($u, v \in A$). Then $(A^*)^B = \ell^*/\Lambda \cong \mathcal{M}(G)$. If $f(u) = e^{\pi\sqrt{-1}(u|u)}$ ($u \in A$), then by (3.5), for any $v \in A$,

$$\hat{f}(\chi_v) = \sum_{u \in \Lambda^*/\Lambda} e^{\pi\sqrt{-1}(u|u)\tau + 2\pi\sqrt{-1}(u|v)} = |Z(\tilde{G})| e^{-\pi\sqrt{-1}(v|v)} e^{\pi\sqrt{-1}\frac{\tau}{4}}. \quad (3.11)$$

Hence (3.4) follows from (3.10).

We now explain the relation of Theorem 3.1 with modular invariance. Recall that the modular group $\Gamma = SL(2, \mathbb{Z})$ is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix}$ satisfying the relations

$$S^2 = (ST)^3 \in Z(\Gamma), \quad S^4 = I. \quad (3.12)$$

Thus $\bar{\Gamma} = PSL(2, \mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$ is the free product of \mathbb{Z}_2 and \mathbb{Z}_3 . We construct a representation of Γ on the group algebra $\mathbb{C}[\mathcal{M}(\mathfrak{g})] = \text{span}_{\mathbb{C}}\{|u\rangle \mid u \in \Lambda^*/\Lambda\}$, where S and T act as (using the same notations)

$$\begin{aligned} S|u\rangle &= \frac{1}{\sqrt{|Z(\tilde{G})|}} \sum_{v \in \mathcal{M}(\mathfrak{g})} e^{-2\pi\sqrt{-1}(u|v)} |v\rangle, \\ T|u\rangle &= e^{-\frac{\pi\sqrt{-1}}{12}r + \pi\sqrt{-1}(u|u)} |u\rangle. \end{aligned} \quad (3.13)$$

The matrix of S is a discrete Fourier transform, that of T is diagonal, and both are unitary under the basis $\{|u\rangle\}$. The definition (3.13) is motivated by the S-duality transformations in the $N = 4$ supersymmetric gauge theory [20]. It also coincides with the modular transformations on the characters of affine Lie algebras at level 1 [11, 8], where r plays the role of the central charge.

One way to show that (3.13) defines a representation of Γ is to check that the relations in (3.12) are satisfied. It is easy to show that $S^2|u\rangle = |-u\rangle$ for any $u \in \Lambda^*/\Lambda$. Hence S^2 commutes with T and $S^4 = \text{id}$. A straightforward calculation yields that, for any $u \in \Lambda^*/\Lambda$,

$$\begin{aligned} T^{-1}S^{-1}T^{-1}|u\rangle &= \frac{e^{\frac{\pi\sqrt{-1}}{6}r}}{\sqrt{|Z(\tilde{G})|}} \sum_{v \in \Lambda^*/\Lambda} e^{-\pi\sqrt{-1}(u-v|u-v)} |v\rangle, \\ STS|u\rangle &= \frac{e^{-\frac{\pi\sqrt{-1}}{12}r}}{|Z(\tilde{G})|} \sum_{w \in \Lambda^*/\Lambda} e^{\pi\sqrt{-1}(w|w)} \sum_{v \in \Lambda^*/\Lambda} e^{-\pi\sqrt{-1}(u+v|u+v)} |v\rangle. \end{aligned} \quad (3.14)$$

So (3.13) satisfies $(ST)^3 = S^2$ if and only if (3.5) holds [20].

For the Lie group G whose Lie algebra is \mathfrak{g} , we can define a vector

$$|G\rangle = \frac{1}{\sqrt{|\pi_1(G)|}} \sum_{u \in \pi_1(G)} |u\rangle \in \mathbb{C}[\Lambda^*/\Lambda]. \quad (3.15)$$

Then we have the following

Proposition 3.2 *Under the above notations, we have*

$$S|G\rangle = |{}^L G\rangle, \quad S|{}^L G\rangle = |G\rangle. \quad (3.16)$$

For example, $|\tilde{G}\rangle = |0\rangle$ and

$$S|0\rangle = \frac{1}{\sqrt{|Z(\tilde{G})|}} \sum_{u \in \mathcal{M}(\mathfrak{g})} |u\rangle = |G_{\text{ad}}\rangle. \quad (3.17)$$

The proof of (3.16) is a similar but more general calculation.

$$\begin{aligned} S|G\rangle &= \frac{1}{\sqrt{|\pi_1(G)|}} \sum_{u \in \pi_1(G)} \frac{1}{\sqrt{|Z(\tilde{G})|}} \sum_{v \in \mathcal{M}(\mathfrak{g})} e^{-2\pi\sqrt{-1}(u|v)} |v\rangle \\ &= \sqrt{\frac{|\pi_1(G)|}{|Z(\tilde{G})|}} \sum_{v \in \pi_1({}^L G)} |v\rangle = |{}^L G\rangle. \end{aligned} \quad (3.18)$$

To give a more concrete understanding of the consistency of (3.13), we consider the theta functions [11, 8]

$$\vartheta_u(z, \tau) = \sum_{\xi \in \Lambda + u} e^{\pi\sqrt{-1}(\xi|\xi)\tau + 2\pi\sqrt{-1}(\xi|z)}, \quad (3.19)$$

where $u \in \Lambda^*/\Lambda$, $z \in (\sqrt{-1}\mathfrak{t})^{\mathbb{C}}$ and τ is in the upper half plane. As

$$\begin{aligned}\vartheta_u(z + \lambda, \tau) &= e^{2\pi\sqrt{-1}(\lambda|u)} \vartheta_u(z, \tau), & \forall \lambda \in \Lambda^*, \\ \vartheta_u(z + \tau\eta, \tau) &= e^{-\pi\sqrt{-1}[(\eta|\eta)\tau + 2(z|\eta)]} \vartheta_u(z, \tau), & \forall \eta \in \Lambda,\end{aligned}\tag{3.20}$$

the theta functions ϑ_u can be considered either as holomorphic sections of $|Z(\tilde{G})|$ line bundles \mathcal{L}_u (which differ from each other by flat line bundles) over the Abelian variety $A_{\tilde{G}} = (\sqrt{-1}\mathfrak{t})^{\mathbb{C}}/\Lambda^* \oplus \tau\Lambda$ or as sections of a single line bundle \mathcal{L} , which is the pull-back of each \mathcal{L}_u from $A_{\tilde{G}}$ to $A = (\sqrt{-1}\mathfrak{t})^{\mathbb{C}}/\Lambda \oplus \tau\Lambda$. Moreover, each ϑ_u spans $H^0(A_{\tilde{G}}, \mathcal{O}(\mathcal{L}_u))$ and $\{\vartheta_u\}$ is a basis of $H^0(A, \mathcal{O}(\mathcal{L}))$.

The modular transformation properties of ϑ_u are

$$\begin{aligned}\vartheta_u(z, \tau + 1) &= e^{\pi\sqrt{-1}(u|u)} \vartheta_u(z, \tau), \\ \vartheta_u\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) &= \frac{1}{\sqrt{|Z(\tilde{G})|}} \left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{r}{2}} e^{\pi\sqrt{-1}\frac{(z|z)}{\tau}} \sum_{v \in \Lambda^*/\Lambda} e^{-2\pi\sqrt{-1}(u|v)} \vartheta_v(z, \tau);\end{aligned}\tag{3.21}$$

the latter is a consequence of the Poisson summation formula. $\{\vartheta_u\}$ is an example of vector-valued modular forms [12]. To reproduce the actions of S and T in (3.13), we define

$$\hat{\vartheta}_u(z, \tau, \delta) = e^{-2\pi\sqrt{-1}\delta} \eta(\tau)^{-r} \vartheta_u(z, \tau),\tag{3.22}$$

where $\delta \in \mathbb{C}$ and $\eta(\tau) = e^{\frac{\pi\sqrt{-1}\tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi\sqrt{-1}n\tau})$ is the Dedekind eta-function. Then

$$\begin{aligned}\hat{\vartheta}_u(z, \tau + 1, \delta) &= e^{-\frac{\pi\sqrt{-1}}{12}r + \pi\sqrt{-1}(u|u)} \hat{\vartheta}_u(z, \tau, \delta), \\ \hat{\vartheta}_u\left(\frac{z}{\tau}, -\frac{1}{\tau}, \delta + \frac{(z|z)}{2\tau}\right) &= \frac{1}{\sqrt{|Z(\tilde{G})|}} \sum_{v \in \Lambda^*/\Lambda} e^{-2\pi\sqrt{-1}(u|v)} \hat{\vartheta}_v(z, \tau, \delta),\end{aligned}\tag{3.23}$$

in conformity with (3.13). Notice that the following is an action of Γ on the triple:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (z, \tau, \delta) \mapsto \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}, \delta + \frac{c(z|z)}{2(c\tau + d)} \right).\tag{3.24}$$

We can define, for any compact Lie group G with Lie algebra \mathfrak{g} ,

$$\begin{aligned}\vartheta_G(z, \tau) &= \frac{1}{\sqrt{|\pi_1(G)|}} \sum_{u \in \ell/\Lambda} \vartheta_u(z, \tau) \\ &= \frac{1}{\sqrt{|\pi_1(G)|}} \sum_{\xi \in \ell} e^{\pi\sqrt{-1}(\xi|\xi)\tau + 2\pi\sqrt{-1}(z|\xi)}.\end{aligned}\tag{3.25}$$

Then ϑ_G is a holomorphic section of a line bundle \mathcal{L}_G over $A_G = (\sqrt{-1}\mathfrak{t})^{\mathbb{C}}/(\ell^* \oplus \tau\ell)$. For example, $\vartheta_{\tilde{G}} = \vartheta_0 \in H^0(A_{\tilde{G}}, \mathcal{O}(\mathcal{L}_0))$. In fact, ϑ_G spans $H^0(A_G, \mathcal{O}(\mathcal{L}_G))$. By Poisson summation,

$$\vartheta_G\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = \left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{r}{2}} e^{\pi\sqrt{-1}\frac{(z|z)}{\tau}} \vartheta_{L_G}(z, \tau),\tag{3.26}$$

which reflects (3.16) in Proposition 3.2.

With the concrete representation of Γ on $H^0(A, \mathcal{O}(\mathcal{L}))$, the consistency of (3.13) is automatic and can be used to prove (3.5). We give a direct proof of (3.4) from (3.26) using the technique of M.

Landsberg (see [1, §29]). Take $\tau = 1 + \sqrt{-1}\epsilon$, where $\epsilon > 0$. Then

$$\begin{aligned}\vartheta_{LG}(0, \tau) &= \frac{1}{\sqrt{|\pi_1(LG)|}} \sum_{\lambda \in \ell^*} e^{\pi\sqrt{-1}(\xi|\xi)\tau} \\ &= \frac{1}{\sqrt{|Z(G)|}} \sum_{\mu \in \ell^*/\Lambda} e^{\pi\sqrt{-1}(\mu|\mu)} \sum_{\xi \in \Lambda} e^{-\pi\epsilon(\xi+\mu|\xi+\mu)}\end{aligned}\quad (3.27)$$

because $(\xi|\xi) \in 2\mathbb{Z}$ and $(\xi|\mu) \in \mathbb{Z}$. As $\epsilon \rightarrow 0^+$, the sum over $\xi \in \Lambda$, multiplied by $\epsilon^{\frac{r}{2}}$, turns to a Gaussian integral over $\xi \in \sqrt{-1}\mathfrak{t}^*$, whose value is $\frac{1}{\sqrt{\text{vol}(\Lambda)}}$. Therefore to the leading order in ϵ ,

$$\vartheta_{LG}(0, \tau) \sim \frac{1}{\sqrt{|Z(G)|}} \sum_{\mu \in \mathfrak{M}(G)} e^{\pi\sqrt{-1}(\mu|\mu)} \frac{1}{\epsilon^{\frac{r}{2}} \sqrt{\text{vol}(\Lambda)}} = \frac{1}{\epsilon^{\frac{r}{2}} \sqrt{\text{vol}(\Lambda)}} \mathfrak{G}(G). \quad (3.28)$$

On the other hand, since $-\frac{1}{\tau} \sim -1 + \sqrt{-1}\epsilon$, we have

$$\vartheta\left(0, -\frac{1}{\tau}\right) \sim \frac{1}{\epsilon^{\frac{r}{2}} \sqrt{\text{vol}(\Lambda)}} \overline{\mathfrak{G}(LG)}, \quad (3.29)$$

and (3.4) follows.

4. Non-simply laced case: the Hecke groups

For a non-simply laced simple Lie algebra \mathfrak{g} , let $n_{\mathfrak{g}}$ be the ratio of the square lengths of the long and short roots. Either $n_{\mathfrak{g}} = 2$ (for B_r, C_r, F_4) or $n_{\mathfrak{g}} = 3$ (for G_2). Denote by $(\cdot|\cdot)$ the Weyl group invariant inner product on $\sqrt{-1}\mathfrak{t}^*$ such that the square length of the long roots is 2. We continue to use it to identify $\sqrt{-1}\mathfrak{t}$ with $\sqrt{-1}\mathfrak{t}^*$. The root system of the Langlands dual Lie algebra ${}^L\mathfrak{g}$ is ${}^L\Delta = \frac{1}{\sqrt{n_{\mathfrak{g}}}}\check{\Delta} \subset \sqrt{-1}\mathfrak{t}$. The (co)root and (co)weight lattices of ${}^L\mathfrak{g}$ are

$${}^L\Lambda = \frac{1}{\sqrt{n_{\mathfrak{g}}}}\check{\Lambda}, \quad ({}^L\Lambda)^{\vee} = \sqrt{n_{\mathfrak{g}}}\Lambda, \quad (({}^L\Lambda)^{\vee})^* = \frac{1}{\sqrt{n_{\mathfrak{g}}}}\Lambda^*, \quad ({}^L\Lambda)^* = \sqrt{n_{\mathfrak{g}}}\check{\Lambda}^*. \quad (4.1)$$

If the maximal torus of G is $T = \mathfrak{t}/2\pi\sqrt{-1}\ell$ and that of ${}^L G$ is ${}^L T = \mathfrak{t}^*/2\pi\sqrt{-1}{}^L\ell$, then ${}^L\ell = \sqrt{n_{\mathfrak{g}}}\ell^*$. We have the inclusion relations

$$({}^L\Lambda)^{\vee} \subset {}^L\ell \subset ({}^L\Lambda)^* \subset \sqrt{-1}\mathfrak{t}^*, \quad {}^L\Lambda \subset ({}^L\ell)^* \subset (({}^L\Lambda)^{\vee})^* \subset \sqrt{-1}\mathfrak{t}. \quad (4.2)$$

We make a digression on Coxeter and dual Coxeter numbers. For every simple Lie algebra \mathfrak{g} , the Coxeter number h and the dual Coxeter number \check{h} of \mathfrak{g} are listed as follows.

\mathfrak{g}	A_r	B_r	C_r	D_r	E_6	E_7	E_8	F_4	G_2
h	$r+1$	$2r$	$2r$	$r+1$	12	18	30	12	6
\check{h}	$r+1$	$2r-1$	$r+1$	$2r-2$	12	18	30	9	4

We have $\check{h} = 1 + \langle \rho, \check{\theta} \rangle$, where ρ is half the sum of positive roots and θ , the highest root. If \mathfrak{g} is simply laced, then $h = \check{h}$. If \mathfrak{g} is non-simply laced, let $\rho_{\text{long}}, \rho_{\text{short}}$ be half the sums of long, short positive roots, respectively, and let $h_{\text{long}} = 1 + \langle \rho_{\text{long}}, \check{\theta} \rangle$, $h_{\text{short}} = n_{\mathfrak{g}} \langle \rho_{\text{short}}, \check{\theta} \rangle$. Then $h = h_{\text{long}} + h_{\text{short}}$ and

$\check{h} = h_{\text{long}} + n_{\mathfrak{g}}^{-1} h_{\text{short}}$. When Δ is exchanged with $\check{\Delta}$, or \mathfrak{g} with ${}^L\mathfrak{g}$, h_{long} and h_{short} are also exchanged [19]. Thus the Coxeter number ${}^L h$ and the dual Coxeter number ${}^L \check{h}$ of the Langlands dual ${}^L\mathfrak{g}$ satisfy

$${}^L h = h, \quad \check{h} + {}^L \check{h} = (1 + n_{\mathfrak{g}}^{-1}) h. \quad (4.3)$$

It is well known [3, Théorème V.6.2.1(ii), Exer. VI.1.20] that $|\Delta| = r h$, $|\Delta_{\text{long}}| = r_{\text{long}} h$, $|\Delta_{\text{short}}| = r_{\text{short}} h$, where Δ_{long} , Δ_{short} are the sets of long, short roots and r_{long} , r_{short} are the numbers of long, short simple roots, respectively. In addition, we have $|\Delta_{\text{long}}| = r h_{\text{long}}$, $|\Delta_{\text{short}}| = r h_{\text{short}}$ [19].

To find the counterpart of the modular group and the correct analog of (3.13) in the non-simply laced case, we proceed with the theta functions. In fact, we need two sets of such. For each $u \in \Lambda^*/\check{\Lambda}$ and $\mu \in ({}^L\Lambda)^*/({}^L\Lambda)^\vee$, let

$$\begin{aligned} \vartheta_u(z, \tau) &= \sum_{x \in \Lambda + u} e^{\pi\sqrt{-1}(x|x)\tau + 2\pi\sqrt{-1}(x|z)}, \\ \vartheta_\mu(z, \tau) &= \sum_{\lambda \in ({}^L\Lambda)^\vee + \mu} e^{\pi\sqrt{-1}(\lambda|\lambda)\tau + 2\pi\sqrt{-1}(\lambda|z)}. \end{aligned} \quad (4.4)$$

Here $z \in (\sqrt{-1}\mathfrak{t})^{\mathbb{C}} \cong (\sqrt{-1}\mathfrak{t}^*)^{\mathbb{C}}$, where the identification is via $(\cdot|\cdot)$. As in the simply laced case, the periodic properties

$$\begin{aligned} \vartheta_u(z + \lambda, \tau) &= e^{2\pi\sqrt{-1}(\lambda|u)} \vartheta_u(z, \tau), & \forall \lambda \in \check{\Lambda}^*, \\ \vartheta_u(z + \tau y, \tau) &= e^{-\pi\sqrt{-1}[(y|y)\tau + 2(z|y)]} \vartheta_u(z, \tau), & \forall y \in \check{\Lambda} \end{aligned} \quad (4.5)$$

mean that for each $u \in \Lambda^*/\check{\Lambda}$, ϑ_u spans $H^0(A_0, \mathcal{O}(\mathcal{L}_u))$, where \mathcal{L}_u is a line bundle over $A_{\check{G}} = (\sqrt{-1}\mathfrak{t})^{\mathbb{C}}/\check{\Lambda}^* \oplus \check{\Lambda}\tau$ and that $\{\vartheta_u\}$ is a basis of $H^0(A, \mathcal{O}(\mathcal{L}))$, where \mathcal{L} is the pull-back of (any) \mathcal{L}_u to $A = (\sqrt{-1}\mathfrak{t})^{\mathbb{C}}/\Lambda \oplus \check{\Lambda}\tau$. Similarly, for each $\mu \in ({}^L\Lambda)^*/({}^L\Lambda)^\vee$, ϑ_μ spans $H^0({}^L A_{\check{G}}, \mathcal{O}({}^L\mathcal{L}_\mu))$ and $\{\vartheta_\mu\}$ is a basis of $H^0({}^L A, \mathcal{O}({}^L\mathcal{L}))$. Here ${}^L\mathcal{L} \rightarrow {}^L A = (\sqrt{-1}\mathfrak{t})^{\mathbb{C}}/{}^L\Lambda \oplus \tau({}^L\Lambda)^\vee$ is the pull-back of the line bundle ${}^L\mathcal{L}_\mu \rightarrow {}^L A_{\check{G}} = (\sqrt{-1}\mathfrak{t})^{\mathbb{C}}/({}^L\Lambda)^\vee \oplus \tau({}^L\Lambda)^\vee$.

It is easy to see that, using $(\check{\Lambda}|\check{\Lambda}) \subset 2\mathbb{Z}$, $(\Lambda^*|\check{\Lambda}) \subset \mathbb{Z}$ and similar facts for ${}^L\Lambda$, for any $u \in \Lambda^*/\check{\Lambda}$ and $\mu \in ({}^L\Lambda)^*/({}^L\Lambda)^\vee$,

$$\begin{aligned} \vartheta_u(z, \tau + 1) &= e^{\pi\sqrt{-1}(u|u)} \vartheta_u(z, \tau), \\ \vartheta_\mu(z, \tau + 1) &= e^{\pi\sqrt{-1}(\mu|\mu)} \vartheta_\mu(z, \tau). \end{aligned} \quad (4.6)$$

Poisson summation yields

$$\begin{aligned} \vartheta_u\left(\frac{z}{\sqrt{n_{\mathfrak{g}}\tau}}, -\frac{1}{n_{\mathfrak{g}}\tau}\right) &= \sqrt{\frac{n_{\mathfrak{g}}^{r_{\text{long}}}}{|Z(\check{G})|}} \left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{r}{2}} e^{\pi\sqrt{-1}\frac{(z|z)}{\tau}} \sum_{\mu \in ({}^L\Lambda)^*/({}^L\Lambda)^\vee} e^{-2\pi\sqrt{-1}\frac{\langle \mu, u \rangle}{\sqrt{n_{\mathfrak{g}}}}} \vartheta_\mu(z, \tau), \\ \vartheta_\mu\left(\frac{z}{\sqrt{n_{\mathfrak{g}}\tau}}, -\frac{1}{n_{\mathfrak{g}}\tau}\right) &= \sqrt{\frac{n_{\mathfrak{g}}^{r_{\text{short}}}}{|Z({}^L G)|}} \left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{r}{2}} e^{\pi\sqrt{-1}\frac{(z|z)}{\tau}} \sum_{u \in \Lambda^*/\check{\Lambda}} e^{-2\pi\sqrt{-1}\frac{\langle \mu, u \rangle}{\sqrt{n_{\mathfrak{g}}}}} \vartheta_u(z, \tau). \end{aligned} \quad (4.7)$$

Here the factor $n_{\mathfrak{g}}$ appears in order to convert the sum over $\check{\Lambda}^*$ to $({}^L\Lambda)^* = \sqrt{n_{\mathfrak{g}}}\check{\Lambda}^*$. We define

$$\begin{aligned} \hat{\vartheta}_u(z, \tau, \delta) &= e^{-2\pi\sqrt{-1}\delta} \eta(\tau)^{-r_{\text{long}}} \eta(n_{\mathfrak{g}}\tau)^{-r_{\text{short}}} \vartheta_u(z, \tau), \\ \hat{\vartheta}_\mu(z, \tau, \delta) &= e^{-2\pi\sqrt{-1}\delta} \eta(\tau)^{-r_{\text{short}}} \eta(n_{\mathfrak{g}}\tau)^{-r_{\text{long}}} \vartheta_\mu(z, \tau). \end{aligned} \quad (4.8)$$

Then

$$\begin{aligned}
\hat{\vartheta}_u(z, \tau + 1, \delta) &= e^{-\frac{\pi\sqrt{-1}}{12}n_{\mathfrak{g}}\frac{L\hbar}{\hbar}r+\pi\sqrt{-1}(u|u)}\hat{\vartheta}_u(z, \tau, \delta), \\
\hat{\vartheta}_\mu(z, \tau + 1, \delta) &= e^{-\frac{\pi\sqrt{-1}}{12}n_{\mathfrak{g}}\frac{\hbar}{\hbar}r+\pi\sqrt{-1}(\mu|\mu)}\hat{\vartheta}_\mu(z, \tau, \delta), \\
\hat{\vartheta}_u\left(\frac{z}{\sqrt{n_{\mathfrak{g}}\tau}, -\frac{1}{n_{\mathfrak{g}}\tau}, \delta + \frac{(z|z)}{2\tau}\right) &= \frac{1}{\sqrt{|Z(\tilde{G})|}} \sum_{\mu \in (L\Lambda)^*/(L\Lambda)^-} e^{-2\pi\sqrt{-1}\frac{\langle \mu, u \rangle}{\sqrt{n_{\mathfrak{g}}}}} \hat{\vartheta}_\mu(z, \tau, \delta), \\
\hat{\vartheta}_\mu\left(\frac{z}{\sqrt{n_{\mathfrak{g}}\tau}, -\frac{1}{n_{\mathfrak{g}}\tau}, \delta + \frac{(z|z)}{2\tau}\right) &= \frac{1}{\sqrt{|Z(\tilde{G})|}} \sum_{u \in \Lambda^*/\tilde{\Lambda}} e^{-2\pi\sqrt{-1}\frac{\langle \mu, u \rangle}{\sqrt{n_{\mathfrak{g}}}}} \hat{\vartheta}_u(z, \tau, \delta).
\end{aligned} \tag{4.9}$$

We can also define the theta functions

$$\begin{aligned}
\vartheta_G(z, \tau) &= \frac{1}{\sqrt{|\pi_1(G)|}} \sum_{u \in \ell/\tilde{\Lambda}} \vartheta_u(z, \tau) \\
&= \frac{1}{\sqrt{|\pi_1(G)|}} \sum_{x \in \ell} e^{\pi\sqrt{-1}(x|x)\tau+2\pi\sqrt{-1}(z|x)}, \\
\vartheta_{L_G}(z, \tau) &= \frac{1}{\sqrt{|\pi_1(L_G)|}} \sum_{\mu \in L\ell/(L\Lambda)^-} \vartheta_\mu(z, \tau) \\
&= \frac{1}{\sqrt{|\pi_1(L_G)|}} \sum_{\lambda \in L\ell} e^{\pi\sqrt{-1}(\lambda|\lambda)\tau+2\pi\sqrt{-1}(z|\lambda)}.
\end{aligned} \tag{4.10}$$

Poisson summation yields

$$\vartheta_G\left(\frac{z}{\sqrt{n_{\mathfrak{g}}\tau}, -\frac{1}{n_{\mathfrak{g}}\tau}\right) = \sqrt{n_{\mathfrak{g}}^{\tau_{\text{long}}}} \left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{\tau}{2}} e^{\pi\sqrt{-1}\frac{(z|z)}{\tau}} \vartheta_{L_G}(z, \tau). \tag{4.11}$$

ϑ_G is a holomorphic section of a line bundle $\mathcal{L}_G \rightarrow A_G = (\sqrt{-1}\mathfrak{t})^{\mathbb{C}}/\ell^* \oplus \tau\ell$ while ϑ_{L_G} is that of a line bundle $\mathcal{L}_{L_G} \rightarrow A_{L_G} = (\sqrt{-1}\mathfrak{t})^{\mathbb{C}}/(L\ell)^* \oplus \tau^L\ell$. It is curious to observe that, for both simply laced and non-simply laced cases, the Abelian variety A_{L_G} for τ is dual to A_G for $-\frac{1}{n_{\mathfrak{g}}\tau}$ and that \mathcal{L}_{L_G} is the Fourier-Mukai transform [17] of \mathcal{L}_G .

The transformations $\tau \mapsto \tau + 1$, $\tau \mapsto -\frac{1}{n_{\mathfrak{g}}\tau}$ generate what is called the Hecke group [10]. Recall that the Hecke group $G(\lambda_m)$ ($3 \leq m \leq +\infty$) is the subgroup of $SL(2, \mathbb{R})$ generated by S and $\tilde{T} = \begin{pmatrix} 1 & \lambda_m \\ 0 & 1 \end{pmatrix}$, where $\lambda_n = 2 \cos \frac{\pi}{m}$, satisfying the relations

$$S^2 = (S\tilde{T})^m \in Z(G(\lambda_m)), \quad S^4 = 1. \tag{4.12}$$

The image $\tilde{G}(\lambda_m)$ in $PSL(2, \mathbb{R})$ is the free product of \mathbb{Z}_2 and \mathbb{Z}_m . When $m = 3$, we have the classical modular group $\Gamma = SL(2, \mathbb{Z})$. When $m = 4, 6$, the groups are $G(\sqrt{2})$, $G(\sqrt{3})$, respectively. Consider also the subgroup $\Gamma^*(n)$ of $SL(2, \mathbb{R})$ generated by $\tilde{S} = \begin{pmatrix} 0 & -1/\sqrt{n} \\ \sqrt{n} & 0 \end{pmatrix}$ and T , which define the desired fractional linear transformations $\tau \mapsto \tau + 1$, $\tau \mapsto -\frac{1}{n\tau}$ on τ . The group $\Gamma^*(n)$ contains $\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \pmod{n} \right\}$ as a subgroup of index 2. When n is prime, $\Gamma^*(n)$ is the normaliser of $\Gamma_0(n)$ in $SL(2, \mathbb{R})$ [13]. When $n = n_{\mathfrak{g}} = 2, 3$, $\Gamma^*(n_{\mathfrak{g}})$ is isomorphic to $G(\sqrt{n_{\mathfrak{g}}})$ by a conjugation with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{n_{\mathfrak{g}}} \end{pmatrix}$, under which the generators S, \tilde{T} of $G(\sqrt{n_{\mathfrak{g}}})$ are mapped to \tilde{S} and T , respectively. Hence \tilde{S} and T satisfy the same relations

$$\tilde{S}^2 = (\tilde{S}T)^{2n_{\mathfrak{g}}} \in Z(\Gamma^*(n_{\mathfrak{g}})), \quad \tilde{S}^4 = 1. \tag{4.13}$$

These two groups, together with $G(1) = \Gamma$ and $G(2) = \Gamma(2)$, are the only Hecke groups $G(\lambda_m)$ that are commensurable with the modular group [14].

By the transformations (4.9), the Hecke group $\Gamma^*(n_{\mathfrak{g}})$ acts on the direct sum $H^0(A, \mathcal{O}(\mathcal{L})) \oplus H^0(LA, \mathcal{O}(L\mathcal{L}))$. The subgroup $\Gamma_0(n_{\mathfrak{g}})$ of index 2 acts within the spaces $H^0(A, \mathcal{O}(\mathcal{L}))$ and $H^0(LA, \mathcal{O}(L\mathcal{L}))$ while \tilde{S} exchanges them. More abstractly, we define a representation of $\Gamma^*(n_{\mathfrak{g}})$ on $\mathbb{C}[A^*/\check{\Lambda}] \oplus \mathbb{C}[(LA)^*/(LA)^\vee] = \text{span}_{\mathbb{C}}\{|u\rangle, |\mu\rangle \mid u \in A^*/\check{\Lambda}, \mu \in (LA)^*/(LA)^\vee\}$ such that \tilde{S} and T act as (using the same notations)

$$\begin{aligned} T|u\rangle &= e^{-\frac{\pi\sqrt{-1}}{12}n_{\mathfrak{g}}\frac{L\check{h}}{h}r+\pi\sqrt{-1}\langle u|u\rangle}|u\rangle, \\ T|\mu\rangle &= e^{-\frac{\pi\sqrt{-1}}{12}n_{\mathfrak{g}}\frac{\check{h}}{h}r+\pi\sqrt{-1}\langle\mu|\mu\rangle}|\mu\rangle, \\ \tilde{S}|u\rangle &= \frac{1}{\sqrt{|Z(\check{G})|}} \sum_{\mu \in (LA)^*/(LA)^\vee} e^{-2\pi\sqrt{-1}\frac{\langle\mu,u\rangle}{\sqrt{n_{\mathfrak{g}}}}}| \mu\rangle, \\ \tilde{S}|\mu\rangle &= \frac{1}{\sqrt{|Z(\check{G})|}} \sum_{u \in A^*/\check{\Lambda}} e^{-2\pi\sqrt{-1}\frac{\langle\mu,u\rangle}{\sqrt{n_{\mathfrak{g}}}}}|u\rangle. \end{aligned} \quad (4.14)$$

It is easy to see that \tilde{S} and T are unitary and that $\tilde{S}^2|u\rangle = |-u\rangle$, $\tilde{S}^2|\mu\rangle = |-\mu\rangle$. Because of the concrete realisation by the theta functions, the above formulae are consistent with the relations (4.13). We can check $(\tilde{S}T)^{2n_{\mathfrak{g}}} = \text{id}$ explicitly in all non-simply laced cases. Since the centre is either \mathbb{Z}_2 or trivial, the action of S^2 is the identity. The contribution of the new phase factors $e^{-\frac{\pi\sqrt{-1}}{12}n_{\mathfrak{g}}\frac{L\check{h}}{h}r}$ and $e^{-\frac{\pi\sqrt{-1}}{12}n_{\mathfrak{g}}\frac{\check{h}}{h}r}$ to $(\tilde{S}T)^{2n_{\mathfrak{g}}}$ is

$$\left(e^{-\frac{\pi\sqrt{-1}}{12}n_{\mathfrak{g}}\frac{L\check{h}}{h}r} e^{-\frac{\pi\sqrt{-1}}{12}n_{\mathfrak{g}}\frac{\check{h}}{h}r} \right)^{n_{\mathfrak{g}}} = e^{-\frac{\pi\sqrt{-1}}{12}n_{\mathfrak{g}}(n_{\mathfrak{g}}+1)r} \quad (4.15)$$

by using (4.3). Thus $(\tilde{S}T)^{2n_{\mathfrak{g}}}$ can be calculated as follows.

B_r or C_r	$(\tilde{S}T)^4 = e^{-\frac{\pi\sqrt{-1}}{12}2\cdot 3\cdot r} \left(e^{\pi\sqrt{-1}\frac{r}{2}} e^{\pi\sqrt{-1}\frac{r}{2}} \right) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$
F_4	$(\tilde{S}T)^4 = e^{-\frac{\pi\sqrt{-1}}{12}2\cdot 3\cdot 4} = 1$
G_2	$(\tilde{S}T)^6 = e^{-\frac{\pi\sqrt{-1}}{12}3\cdot 4\cdot 2} = 1$

If we define

$$\begin{aligned} |G\rangle &= \frac{1}{\sqrt{|\pi_1(G)|}} \sum_{u \in \pi_1(G)} |u\rangle \in \mathbb{C}[A^*/\check{\Lambda}], \\ |{}^L G\rangle &= \frac{1}{\sqrt{|\pi_1({}^L G)|}} \sum_{\mu \in \pi_1({}^L G)} |\mu\rangle \in \mathbb{C}[(LA)^*/(LA)^\vee], \end{aligned} \quad (4.16)$$

then $\tilde{S}|G\rangle = |{}^L G\rangle$ and $\tilde{S}|{}^L G\rangle = |G\rangle$ as before.

We summarize the results in the following

Theorem 4.1 *If \mathfrak{g} is non-simply laced, (4.14) defines a unitary representation of the Hecke group $\Gamma^*(n_{\mathfrak{g}}) \cong G(\sqrt{n_{\mathfrak{g}}})$ on $\mathbb{C}[A^*/\check{\Lambda}] \oplus \mathbb{C}[(LA)^*/(LA)^\vee]$. Moreover \tilde{S} interchanges $|G\rangle$ and $|{}^L G\rangle$ in (4.16).*

Finally, we note some new features of the non-simply laced case. The transformations (4.9) of the theta functions do not coincide with those of the character of affine Lie algebras. It would be interesting to find an alternative relation and to establish an analog of the vector-valued modular forms [12] for Hecke groups. Unlike the simply laced case, the new phase factors in the transformation of T do not come from the central charges of conformal field theory. In a forthcoming work, we will explore the consequences of this in S -duality.

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