A Memory Saving Vector Fast Multipole Algorithm for Solving the Augmented EFIE

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Abstract—**An augmented EFIE (A-EFIE)[9], [10] has been proposed to separate the contributions of the vector potential and the scalar potential for avoiding the imbalance at low frequencies. The corresponding low frequency fast multipole algorithm (LF-FMA) [11] was also developed for solving the A-EFIE. Instead of the factorization of the scalar Green's function by using scalar addition theorem in the LF-FMA, we adopt the vector addition theorem for the factorization of the dyadic Green's function to realize memory savings. We are to develop a vector fast multipole algorithm for solving the A-EFIE.**

I. INTRODUCTION

The electric field integral equation (EFIE) solved by the method of moments (MoM) is one of the popular full-wave methods for numerically solving various electromagnetics problems. However, it is still difficult to handle low frequency problems by using this method, where the objects or parts are very small compared with the wavelength. The difficulty comes from the imbalance of the contributions of the vector potential and the scalar potential parts. In the low frequency, the contribution of the vector potential is much smaller than that of the scalar potential in the EFIE and the finite machine precision will make the contribution of the vector potential lost in the numerical simulation. Since the integral operator corresponding to the scalar potential part has a null space, the matrix system is nearly singular and is difficult to solve.

To deal with this issue, many methods have been discussed. Most popular methods are based on the loop-tree or loop-star decomposition [2], [3]. But it is difficult for the loop-tree based method to search for loops when geometries are very complicated. Another way is to apply the self-regularizing property of the EFIE for preconditioning [5], [6], which leads to a well conditioned matrix, independent of the frequency and discretization density. However, the square of the EFIE operator is complicated to discretize and it is difficult to apply to open structures. Moreover, some other methods based on the idea of separating the currents and charges are also been discussed [7], [8].

Recently, an augmented EFIE (A-EFIE) [9], [10] has been proposed to separate the contributions of the vector potential and the scalar potential for avoiding the imbalance at low frequencies. This method is free of low frequency breakdown and it is not necessary to search for the loop-tree basis. For large scale problems, a constraint preconditioner was designed and the corresponding fast multipole algorithm was developed for solving the preconditioned A-EFIE in [11].

In our method, instead of the factorization of the scalar Green's function by using scalar addition theorem in [11], we adopt the vector addition theorem for the factorization of the dyadic Green's function to realize memory savings. We are to develop a vector fast multipole algorithm for solving the A-EFIE. By applying the vector fast multipole algorithm to the A-EFIE, the main elements of the augmented matrix can be expressed by radiation patterns, translators and receiving patterns. In the calculation of far field interactions, which is the main parts of the whole simulation process for large scale problems, the storage of translators in the method is larger than that in the LF-FMA with scalar addition theorem for A-EFIE. Fortunately, it is independent of the number of unknowns. Meanwhile, the storage of radiation and receiving patterns is linearly dependent on the number of unknowns. Therefore it is worthwhile for large scale problems to reduce the storage of this part. In this method, the storage of radiation and receiving patterns can be reduced by 25 percent compared with the LF-FMA for solving AEFIE.

II. FORMULATION OF THE AUGMENTED EFIE

A brief introduction of the A-EFIE will be given in this section. Definitions of symbols in this section are similar to those in [11]. We first define three matrices $\overline{\mathbf{V}} \in \mathbb{C}^{e \times e}$, $\overline{\mathbf{S}} \in \mathbb{C}^{e \times e}$ and $\overline{\mathbf{P}} \in \mathbb{C}^{p \times p}$ as

$$\left[\overline{\mathbf{V}}\right]_{m,n} = \mu_r \cdot \int_{S_m} \mathbf{\Lambda}_m(\mathbf{r}) \cdot \int_{S_n} g(\mathbf{r}, \mathbf{r}')\mathbf{\Lambda}_n(\mathbf{r}')dS'dS$$

$$\left[\overline{\mathbf{S}}\right]_{m,n} = \epsilon_r^{-1} \cdot \int_{S_m} \nabla \cdot \mathbf{\Lambda}_m(\mathbf{r}) \int_{S_n} g(\mathbf{r}, \mathbf{r}')\nabla' \cdot \mathbf{\Lambda}_n(\mathbf{r}')dS'dS$$

$$\left[\overline{\mathbf{P}}\right]_{m,n} = \epsilon_r^{-1} \cdot \int_{T_m} h_m(\mathbf{r}) \int_{T_n} g(\mathbf{r}, \mathbf{r}')h_n(\mathbf{r}')dS'dS$$

where $\mathbf{\Lambda}_i(\mathbf{r})$ is i-th RWG basis function, which is defined on the domain S_i, $h_i(\mathbf{r})$ is i-th pulse basis function, which is defined on the triangle T_i, $g(\mathbf{r}, \mathbf{r}')$ is the scalar Green's function in homogeneous medium, μ_r is the relative permeability and ϵ_r is the relative permittivity. Moreover, here p and e are the number of patches and the number of inner edges respectively.

With these notations, the conventional RWG based EFIE can be alternatively written as

$$\left(ik_0\eta_0\overline{\mathbf{V}} + \frac{\eta_0}{ik_0}\overline{\mathbf{S}} \right) \cdot \mathbf{J} = \mathbf{b} \tag{1}$$

where k_0 and η_0 are wave number and wave impedance, respectively. The vector $\mathbf{J}$ contains the current coefficients and the vector $\mathbf{b}$ is the excitation. It can be seen that the vector potential is represented by the matrix $\overline{\mathbf{V}}$ and the scalar potential by the matrix $\overline{\mathbf{S}}$.

The scalar potential matrix can be expressed as

$$\overline{\mathbf{S}} = \overline{\mathbf{D}}^T \cdot \overline{\mathbf{P}} \cdot \overline{\mathbf{D}} \tag{2}$$

where $\overline{\mathbf{D}}$ gives the relationship between the RWG basis and the triangle pair. Moreover, the current continuity condition yields

$$\overline{\mathbf{D}} \cdot \mathbf{J} = ik_0c_0\rho \tag{3}$$

where c_0 is the speed of light.

With (1), (2) and (3), we get the A-EFIE as

$$\begin{bmatrix} \overline{\mathbf{V}} & \overline{\mathbf{D}}^T \cdot \overline{\mathbf{P}} \\ \overline{\mathbf{D}} & k_0^2\overline{\mathbf{I}} \end{bmatrix} \cdot \begin{bmatrix} ik_0\mathbf{J} \\ c_0\rho \end{bmatrix} = \begin{bmatrix} \eta_0^{-1}\mathbf{b} \\ \mathbf{0} \end{bmatrix} \tag{4}$$

Considering the charge neutrality, the vector of charge is reduced to be $\rho_r \in \mathbb{C}^{(p-t)\times 1}$ and two mapping matrices $\overline{\mathbf{F}} \in \mathbb{R}^{(p-t)\times p}$ and $\overline{\mathbf{B}} \in \mathbb{R}^{p\times(p-t)}$ are defined

$$\rho_r = \overline{\mathbf{F}} \cdot \rho, \quad \rho = \overline{\mathbf{B}} \cdot \rho_r. \tag{5}$$

Then the A-EFIE (4) is modified as

$$\begin{bmatrix} \overline{\mathbf{V}} & \overline{\mathbf{D}}^T \cdot \overline{\mathbf{P}} \cdot \overline{\mathbf{B}} \\ \overline{\mathbf{F}} \cdot \overline{\mathbf{D}} & k_0^2\overline{\mathbf{I}}_r \end{bmatrix} \cdot \begin{bmatrix} ik_0\mathbf{J} \\ c_0\rho_r \end{bmatrix} = \begin{bmatrix} \eta_0^{-1}\mathbf{b} \\ \mathbf{0} \end{bmatrix} \tag{6}$$

A constraint preconditioner for (6) has been constructed to improve the convergence for large scalar problems and the efficiency of the fast multipole algorithm for solving the preconditioned system has been shown in [11].

III. FACTORIZATION OF DYADIC GREEN'S FUNCTION

Based on the vector addition theorem, we can factorize the dyadic Green's function with vector addition theorem as

$$\overline{\mathbf{G}}(\mathbf{r}_1, \mathbf{r}_2) = ik \sum_{J,M} \sum_{J',M'} \Re g\overline{\overline{\mathbf{\Psi}}}_{J',M'}^t(\mathbf{r}_{13})$$
$$\cdot\overline{\overline{\alpha}}_{J'M',JM}(\mathbf{r}_{34}) \cdot \Re g\overline{\overline{\mathbf{\Psi}}}_{JM}^*(\mathbf{r}_{24}) - ik \sum_{J,M} \sum_{J',M'} \Re g\mathbf{L}_{J',M'}(\mathbf{r}_{13}) \cdot \alpha_{J'M',JM}(\mathbf{r}_{34}) \cdot \Re g\mathbf{L}_{J,M}^{*t}(\mathbf{r}_{24}) \tag{7}$$

and the tensorial Green's function $\overline{\mathbf{I}}g(\mathbf{r}_1, \mathbf{r}_2)$ can be expressed as

$$\overline{\mathbf{I}}g(\mathbf{r}_1, \mathbf{r}_2) = ik \sum_{J,M} \sum_{J',M'} \Re g\overline{\overline{\mathbf{\Psi}}}_{J',M'}^t(\mathbf{r}_{13})$$
$$\cdot\overline{\overline{\alpha}}_{J'M',JM}(\mathbf{r}_{34}) \cdot \Re g\overline{\overline{\mathbf{\Psi}}}_{JM}^*(\mathbf{r}_{24}) \tag{8}$$

where $\overline{\overline{\alpha}}_{J'M',JM}$ is the vector translator, the $\Re g$ operator implies taking the regular part of the function,

$$\overline{\overline{\mathbf{\Psi}}}_{J,M}^t(\mathbf{r}) = [\mathbf{F}_{J,M}(\mathbf{r}), \mathbf{H}_{J,M}(\mathbf{r}), \mathbf{T}_{J,M}(\mathbf{r})] \in \mathbb{C}_{3\times 3}, \tag{9}$$

and

$$\mathbf{L}_{JM}(\mathbf{r}_1) = \frac{\sqrt{J+1}}{\sqrt{2J+1}}\mathbf{H}_{JM}(\mathbf{r}_1) + \frac{\sqrt{J}}{\sqrt{2J+1}}\mathbf{T}_{JM}(\mathbf{r}_1). \tag{10}$$

Here

$$\mathbf{F}_{JM}(\mathbf{r}) = \frac{iz_J(kr)}{\sqrt{J(J+1)}}\nabla_s Y_{J,M}(\theta, \phi) \times \mathbf{e}_r$$
$$\mathbf{H}_{JM}(\mathbf{r}) = z_{J+1}(kr)\left[\frac{\nabla_s Y_{J,M} - (J+1)Y_{J,M}\mathbf{e}_r}{\sqrt{(2J+1)(J+1)}}\right] \tag{11}$$
$$\mathbf{T}_{JM}(\mathbf{r}) = z_{J-1}(kr)\left[\frac{JY_{J,M}\mathbf{e}_r + \nabla_s Y_{J,M}}{\sqrt{2J+1}\sqrt{J}}\right]$$

where $Y_{J,M}$ is the spherical harmonics [1] ($J = 0, 1, 2, \ldots$, and $M = -J, \ldots, -1, 0, 1, \ldots, J$). Here, $z_J(x)$ is the first kind spherical Hankel function of order J, and $\mathbf{e}_r$ is the unit vector of $\mathbf{r}$.

IV. VECTOR FAST MULTIPOLE ALGORITHM FOR A-EFIE

Based on the factorizations of the dyadic Green's function and the tensorial Green's function, we will present the vector fast multipole algorithm (VFMA) for solving the A-EFIE at low frequencies in this section. We will also show the memory efficiency of this method for large scale problems.

For large scale problems, the iteration solvers are often adopted. In the iterative process for solving preconditioned A-EFIE (6), the key step is the multiplication of the matrix on the left hand side of Equation (6) and a vector. From the construction of the matrix, we can see that $\overline{\mathbf{V}}$ and $\overline{\mathbf{P}}$ are main submatrices. Therefore the storage and calculation of the two submatrices should be specially treated in the fast multipole algorithm.

For the vector potential part, any element $\left[\overline{\mathbf{V}}\right]_{m,n}$ can be expressed as

$$ik\mu_r \sum_{J,M} \sum_{J',M'} \left\langle \mathbf{\Lambda}_m(\mathbf{r}_1), \Re g\overline{\overline{\mathbf{\Psi}}}_{J',M'}^t(\mathbf{r}_{13}) \right\rangle$$
$$\cdot\overline{\overline{\alpha}}_{J'M',JM}(\mathbf{r}_{34}) \cdot \left\langle \Re g\overline{\overline{\mathbf{\Psi}}}_{J,M}^*(\mathbf{r}_{24}), \mathbf{\Lambda}_n^t(\mathbf{r}_2) \right\rangle \tag{12}$$

where $\left\langle \mathbf{\Lambda}_m(\mathbf{r}_1), \Re g\overline{\overline{\mathbf{\Psi}}}_{J',M'}^t(\mathbf{r}_{13}) \right\rangle$ and $\left\langle \Re g\overline{\overline{\mathbf{\Psi}}}_{J,M}^*(\mathbf{r}_{24}), \mathbf{\Lambda}_n^t(\mathbf{r}_2) \right\rangle$ are the receiving and radiation patterns respectively.

As for the scalar potential part $\overline{\mathbf{S}}$, according to (2), we can consider the matrix $\overline{\mathbf{P}}$, whose element in the far field part can

be written as

$$\left[\overline{\mathbf{P}}\right]_{m,n} = \epsilon^{-1} \langle h_m(\mathbf{r}_1), \ g(\mathbf{r}_1, \mathbf{r}_2), \ h_n(\mathbf{r}_2) \rangle$$

$$= ik\epsilon^{-1} \sum_{J,M} \sum_{J',M'} \langle h_m(\mathbf{r}_1), \Re g\psi_{J',M'}(\mathbf{r}_1 - \mathbf{r}_{l'}) \rangle \cdot \quad (13)$$

$$\cdot \alpha_{J'M',JM}(\mathbf{r}_{l'} - \mathbf{r}_l) \langle \Re g\psi^*_{JM}(\mathbf{r}_2 - \mathbf{r}_l), h_n(\mathbf{r}_2) \rangle$$

where ψ_{JM} is the scalar wave function, $\alpha_{J'M',JM}$ is the scalar translator, and $\mathbf{r}_{l'}$ denotes the center of the box to which the pulse basis h_m belongs.

After obtaining the factorization of elements of the matrices $\overline{\mathbf{V}}$ and $\overline{\mathbf{P}}$, the implementation process of the multiplication of the matrices and a vector is similar to the traditional LF-FMA [4].

Since radiation and receiving patterns are similar, we only discuss receiving patterns in the following to show the memory savings of the VFMA. By using integration by parts, we have

$$\langle \boldsymbol{\Lambda}_m(\mathbf{r}_1), \ \Re g\mathbf{L}_{J',M'}(\mathbf{r}_1 - \mathbf{r}_l) \rangle$$

$$= -\frac{1}{k} \langle \nabla \cdot \boldsymbol{\Lambda}_m(\mathbf{r}_1), \Re g\psi_{J',M'}(\mathbf{r}_1 - \mathbf{r}_l) \rangle$$

$$= -\frac{1}{k} (\langle h_{t_1}(\mathbf{r}_1), \Re g\psi_{J',M'}(\mathbf{r}_1 - \mathbf{r}_l) \rangle \qquad (14)$$

$$- \langle h_{t_2}(\mathbf{r}_1), \Re g\psi_{J',M'}(\mathbf{r}_1 - \mathbf{r}_l) \rangle)$$

where $\mathbf{r}_l$ is the center of the box to which the RWG basis $\boldsymbol{\Lambda}_m$ belongs. Let the center of the box to which the pulse basis h_{t_i} belongs be $\mathbf{r}_{t_i}$, $i = 1, \ 2$. It should be noted that $\mathbf{r}_{t_1}$ or $\mathbf{r}_{t_2}$ is not always equal to $\mathbf{r}_l$ [4]. If $\mathbf{r}_{t_1} \neq \mathbf{r}_l$, a translator is needed to transform $\langle h_{t_1}(\mathbf{r}_1), \Re g\psi_{J',M'}(\mathbf{r}_1 - \mathbf{r}_{t_1}) \rangle$ to $\langle h_{t_1}(\mathbf{r}_1), \Re g\psi_{J',M'}(\mathbf{r}_1 - \mathbf{r}_l) \rangle$.

By using the scalar addition theorem [4], we know

$$\Re g\psi_{J,M}(\mathbf{r}) = \sum_{J',M'} \Re g\psi_{J',M'}(\mathbf{r}') \cdot \beta_{J'M',JM}(\mathbf{r}''), \quad (15)$$

where $\mathbf{r} = \mathbf{r}' + \mathbf{r}''$. So it can be obtained that

$$\langle h_{t_1}(\mathbf{r}_1), \Re g\psi_{J'M'}(\mathbf{r}_1 - \mathbf{r}_l) \rangle = \sum_{J,M} \langle h_{t_1}(\mathbf{r}_1),$$

$$\Re g\psi_{JM}(\mathbf{r}_1 - \mathbf{r}_{t_1}) \rangle \cdot \beta_{JM,J'M'}(\mathbf{r}_{t_1} - \mathbf{r}_l) \qquad (16)$$

Since the box to which a patch triangle of an RWG basis belongs is adjacent to or the same as the box to which the RWG basis belongs, the vector $\mathbf{r}_{t_1} - \mathbf{r}_l$ only has 27 possibilities at most. So the storage of translators $\beta_{J'M',JM}$ is small and remains constant.

In addition, by applying (10), we can express the third component of receiving patterns (12) in the vector potential part of the VFMA as

$$\langle \boldsymbol{\Lambda}_i(\mathbf{r}_1), \Re g\mathbf{T}_{JM}(\mathbf{r}_1 - \mathbf{r}_{l_i}) \rangle$$

$$= \frac{\sqrt{2J+1}}{\sqrt{J}} \langle \boldsymbol{\Lambda}_i(\mathbf{r}_1), \Re g\mathbf{L}_{JM}(\mathbf{r}_1 - \mathbf{r}_{l_i}) \rangle \qquad (17)$$

$$- \frac{\sqrt{J+1}}{\sqrt{J}} \langle \boldsymbol{\Lambda}_i(\mathbf{r}_1), \Re g\mathbf{H}_{JM}(\mathbf{r}_1 - \mathbf{r}_{l_i}) \rangle$$

Since the subtraction on the right hand side of (17) is between two terms with different highest orders, the real value of $\langle \boldsymbol{\Lambda}_i(\mathbf{r}_1), \Re g\mathbf{T}_{JM}(\mathbf{r}_1 - \mathbf{r}_{l_i}) \rangle$ will not be swamped by the subtraction error.

Therefore, the total storage of receiving patterns contains three parts: One is receiving patterns with the pulse basis in the scalar potential part, $\langle h_m(\mathbf{r}_1), \Re g\psi_{J',M'}(\mathbf{r}_1 - \mathbf{r}_{l'}) \rangle$, and the other two are the first two components of receiving patterns in the vector potential part. By using (14), (16) and (17), the third component can be obtained with the second component of receiving patterns in the vector potential part, receiving patterns of the scalar potential part and translators introduced in (15). Therefore, the total storage of radiation and receiving patterns of the LF-VFMA can be reduced by 25 percent compared with that of the LF-FMA.

As for the storage for vector translators, although elements of vector translators can be calculated with scalar translators of the LF-FMA and theoretically we can only store scalar translators for generating vector translators, some elements of vector translators should be calculated and stored alone to avoid numerical error due to the subtraction of two high order terms. Therefore, the storage for vector translators is larger than that for scalar translators. In addition, we should store translators used in (16), whose storage is small and remains constant.

Fortunately, the storage for translators used in this method is independent of the number of unknowns. With the increase in the number of unknowns, the storage of radiation and receiving patterns, which depends on the number of unknowns, will become the main part of the total storage. It is reduced by 25 percent in our method compared with that in the LF-FMA. The analysis of memory requirement implies that the method is superior in saving memory for solving large scale problems.

V. NUMERICAL EXAMPLE

We are developing some numerical examples, which will be shown on the conference.

VI. CONCLUSIONS

In this work, the vector addition theorem is adopted for the factorization of the dyadic Green's function. Then the factorization is used to develop the vector fast multipole algorithm for solving the A-EFIE at low frequencies. By analyzing the storage requirement of translators and radiation and receiving patterns, it is shown that memory savings can be realized for large scale problems.

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