Evaluation of Union Bounds for Space-Time Codes based on a Common Function

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Abstract- **Error-rate evaluation of Space-Time codes using Union bounds sometimes requires very heavy computational loads and so is impractical to use. In this paper, a Common function shared by different Union bounds is derived and used to develop a modified Union bound (MUB) for error-rate evaluation. Results of numerical evaluations and Monte-Carlo simulation on two 2x2 rotation-based S-T codes show that the MUB provides a good compromise between the required computational load and the accuracy for error-rate evaluation.**

***Index Terms*- Common function, Space-Time codes, Union bound**

I. INTRODUCTION

For numerical evaluation of bit-error rates (BERs) of space-time (S-T) codes [1], the Union bound, which is a function of Pair-wise Error Probabilities (PEPs), is known to be accurate [2~8]. For S-T codes with large codebook sizes, numerical evaluations of bit-error rates (BERs) using the Union bound sometimes are impractical due to the heavy computational load required. In this paper, a function which is common to different Union bounds is derived and then used to develop a modified Union bound (MUB) for BER evaluation of S-T codes. Results show that the MUB provides a good compromise between the computational load and accuracy for BER evaluations. Numerical evaluation results on the BERs of a D code show that, at BER=10^{-4}, the difference between the MUB and exact Union bound (EUB) is about 1 dB. While for a H code, at BER=10^{-3}, the difference between the MUB and EUB is less than 0.5 dB. However, the computational time for the EUB is about 5 times longer than that of the MUB.

The remainder of this paper is organized as follows. Section II describes the system model used for the study. Different PEPs and Union bounds for S-T codes are introduced in section III. A Common function used to express the different Union bounds and the MUB are derived in section IV. Results and discussions of the 2-by-2 rotation based S-T codes (D and H code) are presented in section V. Section VI is the conclusions.

II. SYSTEM MODEL

The S-T coded system with transmit matrix **X** and receive matrix **R** considered here can be modeled as:

$$\mathbf{R} = \mathbf{H}\mathbf{X} + \mathbf{Y} \tag{1}$$

where $\mathbf{H}$ is a $M \times N$ channel matrix with N and M being the number of transmit and receive antennas, respectively. Each of the elements $h_{m,n}$ in $\mathbf{H}$ is the channel transfer function from the *n*-th transmit antenna to the *m*-th receive antenna. For flat Rayleigh fading channels, all elements $h_{m,n}$ in $\mathbf{H}$ are independent identically distributed (i.i.d) complex-Gaussian variables with zero mean and variance 0.5 for both the real and imaginary parts. The channel is assumed to be static within the transmission interval of a block of coded symbols. In (1), $\mathbf{X}$ is a $N \times L$ coded symbol matrix where L is the number of time intervals to transmit a complete coded-symbol block. Each of the elements $x_{n,t}$ in $\mathbf{X}$ is a coded symbol transmitted from the *n*-th transmit antenna in the *t*-th transmission interval and having average bit energy E_b. The received signal matrix **R** is a $M \times L$ matrix with element $r_{m,t}$ being the signal received from the *m*-th antenna in the *t*-th transmission interval. For simplicity, transmission delay is neglected here. Additive white Gaussian noise (AWGN) in the channel is modeled by a $M \times L$ matrix **Y** with all its elements being i.i.d complex Gaussian variables with zero-mean and variance $N_0/2$ for both the real and imaginary parts. At the receiver, it is assumed that the channel matrix **H** is perfectly estimated and Maximum-Likelihood (ML) detection is used.

III Different PEPs and Union bounds for S-T codes

3.1 Exact PEP:

The PEP is denoted here as $P_e(\mathbf{X}_i \to \mathbf{X}_j)$ and is defined as the error probability when codeword $\mathbf{X}_i$ is transmitted but is falsely detected as $\mathbf{X}_j$ with binary detection assumption. With the use of ML detection, the exact PEP can be written in an integral form as [3, 5]:

$$P_e(\mathbf{X}_i \to \mathbf{X}_j) = \frac{1}{\pi}\int_0^{\pi/2} \left\{ \prod_{k=1}^{K_{i,j}} \left(\frac{\sin^2\theta}{\sin^2\theta + \frac{\lambda_{k,i,j} SNR}{4}} \right)^M \right\} d\theta \tag{2}$$

where $SNR = E_b / N_0$ and $K_{i,j}$ is the number of nonzero eigenvalues of the matrix $(\mathbf{X}_i - \mathbf{X}_j)(\mathbf{X}_i - \mathbf{X}_j)^*$ with * denoting the operation of transpose conjugate. The closedform exact PEP depends on whether all these eigenvalues $\{\lambda_{1,i,j},..,\lambda_{K_{i,j},i,j}\}$ are equal or not, as described below.

A. Equal-eigenvalue case

If eigenvalues are all equal: $\lambda_{k,i,j}=\lambda_{i,j}$, the exact PEP can be calculated as [9]:

$$P_e(\mathbf{X}_i \to \mathbf{X}_j) = F(MK_{i,j}, u_{i,j}) \quad (3a)$$

where $$u_{i,j} = \sqrt{\frac{0.25\lambda_{i,j}SNR}{1+0.25\lambda_{i,j}SNR}} \quad (3b)$$

and $F(N, u)$ is given by [11]:

$$F(N,u) := [0.5(1-u)]^N \sum_{k=0}^{N-1}\left[\binom{N-1+k}{k}(0.5(1+u))^k\right] \quad (4)$$

with N and u being positive integer and real number, respectively.

B. Unequal-eigenvalue case

Assume that among the $K_{i,j}$ nonzero eigenvalues some of them are equal, so there are only $R_{i,j}$ different eigenvalues, i.e., $\left\{\tilde{\lambda}_{1,i,j} \;\; \ldots \;\; \tilde{\lambda}_{R_{i,j},i,j}\right\}$, where $K_{i,j} \geq R_{i,j} > 1$. Denoting $W_{r,i,j}$, for $r=1,.., R_{i,j}$, as the number of eigenvalues having the same value $\tilde{\lambda}_{r,i,j}$, the exact PEP can be calculated as [3, 5~7]:

$$P_e(\mathbf{X}_i \to \mathbf{X}_j) = \sum_{r=1}^{R_{i,j}} \sum_{s=1}^{MW_{r,i,j}} A_{r,s,i,j} F(s, u_{r,i,j}) \quad (5a)$$

where $$u_{r,i,j} = \sqrt{\frac{0.25\tilde{\lambda}_{r,i,j}SNR}{1+0.25\tilde{\lambda}_{r,i,j}SNR}} \quad (5b)$$

and $A_{r,s,i,j}$ is determined by the high order derivative [3,5] :

$$A_{r,s,i,j} = \frac{(-1)^{MW_{r,i,j}-s}}{\left(\tilde{\lambda}_{r,s,i,j}\right)^{MW_r-s}(MW_{r,i,j}-s)!} \left\{\frac{d^{MW_{r,i,j}-s}}{d(x)^{MW_{r,i,j}-s}}\left[\prod_{k=1,k\neq r}^{R} \frac{1}{\left(1-x\tilde{\lambda}_{r,i,j}\right)^{MW_{r,i,j}}}\right]\right\}\Bigg|_{x=\frac{1}{\tilde{\lambda}_{r,i,j}}} \quad (6)$$

Noted that the exact PEP expressions in both the equal-eigenvalue and unequal-eigenvalue cases consist of the same mathematical function (4) in which the variable u is a non-linear function of SNR as indicated in (3b) or (5b).

3.2 Other PEPs based on different bounds:

Based on Chernoff bound

Using the Chernoff bound for the Q function, the PEP is bounded by [1]:

$$P_e(\mathbf{X}_i \to \mathbf{X}_j) \leq \left(\prod_{k=1}^{K_{i,j}}\left(1+0.25\lambda_{k,i,j}SNR\right)\right)^{-M} \quad (7)$$

Based on Asymptotic bound

At high SNR, the PEP in (7) is further upper-bounded by the Asymptotic bound [1]:

$$P_e(\mathbf{X}_i \to \mathbf{X}_j) \leq \left(\prod_{k=1}^{K_{i,j}}\lambda_{k,i,j}\right)^{-M}\left(0.25SNR\right)^{-K_{i,j}M} \quad (8)$$

3.3 Different Union bounds

The Union bound is defined as [2~3]:

$$UB = \sum_{i=1}^{C}\sum_{j=1,j\neq i}^{C} P(\mathbf{X}_i)\frac{e_{i,j}}{B}P_e(\mathbf{X}_i \to \mathbf{X}_j) \quad (9)$$

where C is the size of the codebook, B is the number of bits per codeword, $P(\mathbf{X}_i)$ is the probability of $\mathbf{X}_i$ being sent and $e_{i,j}$ is the number of bits error due to the error event $(\mathbf{X}_i \to \mathbf{X}_j)$. Different PEPs used in (9) produce different Union bounds. For example, substituting the exact PEP of (3a) and (5a) into (9) gives the exact Union bound (EUB), while substituting the PEPs of (7) and (8) into (9) give the Chernoff Union bound (CUB) and Asymptotic Union bound (AUB), respectively.

IV A Common function for Union bounds

4.1 Function common to different Union bounds

Although the EUB, CUB and AUB can be derived using different PEPs, as described in the previous section, here we show that all these Union bounds can be expressed in terms of a common function which can be used for BER evaluation of S-T codes.

Assume all matrices $\{(\mathbf{X}_i-\mathbf{X}_j)(\mathbf{X}_i-\mathbf{X}_j)^*\}$ considered here have rank K and all codewords are equally likely to be transmitted, i.e. $P(\mathbf{X}_i)=\frac{1}{C}$, then the EUB can be readily proved (in Appendix) to be upper bounded by:

$$EUB \leq (C-1)F(MK, u_\Omega) \quad (10a)$$

where $$u_\Omega = \sqrt{\frac{0.25\Omega SNR}{1+0.25\Omega SNR}} \tag{10b}$$

and $$\Omega = \left\{ \frac{1}{C(C-1)} \sum_{i=1}^{C} \sum_{j=1, j\neq i}^{C} \frac{e_{i,j}}{B} \left(\prod_{k=1}^{K} \lambda_{k,i,j} \right)^{-M} \right\}^{-\frac{1}{KM}} \tag{10c}$$

Similarly, the CUB is proved to be upper bounded, in terms of Ω, by:

$$CUB \le (C-1)(1+0.25\Omega SNR)^{-KM} \tag{11}$$

These new Union bounds in (10a) and (11) are called the modified Union bound (MUB) and the modified Chernoff Union bound (MCUB), respectively. Furthermore, the AUB can also be expressed in terms of Ω as:

$$AUB = (C-1)\Omega^{-KM}(0.25SNR)^{-KM} \tag{12}$$

Thus the function Ω is common to all the Union bounds studied here and is named as the "Common function" which, as given by (10c), is not a function of SNR.

If all the matrices $\{ (\mathbf{X}_i - \mathbf{X}_j)(\mathbf{X}_i - \mathbf{X}_j)^* \}$ have different ranks $\{K_{i,j}\}$, applying the upper bounds in (10a) and (11) to the matrices with identical ranks gives the general results for the MUB and MCUB. If the matrices $\{ (\mathbf{X}_i - \mathbf{X}_j)(\mathbf{X}_i - \mathbf{X}_j)^* \}$ only have v different ranks $\{K_v\}$, for $v = 1...V$, and there are U_v matrixes having rank K_v, then the MUB of the general case can be written as:

$$EUB \le \frac{1}{C}\sum_{v=1}^{V} U_v F(MK_v, u_{\Omega_v}) \tag{13a}$$

where

$$\Omega_v = \left\{ \frac{1}{U_v} \sum_{i=1}^{C} \sum_{j=1, j\neq i,}^{C} \frac{e_{i,j}}{B} \left(\prod_{k=1}^{K_v} \lambda_{k,i,j} \right)^{-M} \right\}^{-\frac{1}{K_v M}} \Bigg|_{Rank((\mathbf{X}_i-\mathbf{X}_j)(\mathbf{X}_i-\mathbf{X}_j)^*)=K_v} \tag{13b}$$

while the MCUB of the general case can be written as:

$$CUB \le \frac{1}{C}\sum_{v=1}^{V} U_v (1+0.25\Omega_v SNR)^{-K_v M} \tag{14}$$

and the AUB can also be rewritten, in terms of Ω_v, as:

$$AUB = \frac{1}{C}\sum_{v=1}^{V} U_v \Omega_v^{-K_v M}(0.25SNR)^{-K_v M} \tag{15}$$

4.2 Error rate evaluation based on the Common function for S-T codes

To compute the error rates of S-T codes using Union bounds, the EUB is most complicated. Since the function $F(s, u_{r,i,j})$ in the exact PEP expressions in (3a) or (5a) is SNR and code-pair $(\mathbf{X}_i, \mathbf{X}_j)$ dependent, calculating the EUB needs to evaluate the exact PEP $C(C$-1) times for each SNR. Furthermore, for the unequal eigenvalues case, the calculation of the partial fraction expansion $A_{r,s,i,j}$ in (6) is time consuming.

If the CUB is used, the computational load required is relatively less, but it still needs the repeated calculations for different SNRs. For large values of *C,* the computation load is also heavy. Moreover, the accuracy of the CUB is sometimes not very good, as will be shown in the next section

The computational load for using the MUB is significantly less because the expression consists of the function $F(MK, u_\Omega)$ in which Ω or Ω_v needs to be computed only once for all SNRs. If all matrices $\{ (\mathbf{X}_i - \mathbf{X}_j)(\mathbf{X}_i - \mathbf{X}_j)^* \}$ have a full rank, one can use the determinant to compute Ω instead of the eigenvalues which are more complicated to compute. Of course, the computation load can be further reduced if the AUB is used for error rate evaluation, but at the expense of poorer accuracy as shown in the next section.

V Results and Discussions

5.1 2×2 Rotation-based S-T Code

In 2×2 rotation code construction, four independent information symbols, $s_{1,1}, s_{1,2}, s_{2,1}, s_{2,2}$, are coded and placed in a matrix **X** diagonally [10] as:

$$\mathbf{X} = \begin{bmatrix} \cos(\theta_1)s_{1,1} - \sin(\theta_1)s_{1,2} & \sin(\theta_2)s_{2,1} + \cos(\theta_2)s_{2,2} \\ \cos(\theta_2)s_{2,1} - \sin(\theta_2)s_{2,2} & \sin(\theta_1)s_{1,1} + \cos(\theta_1)s_{1,2} \end{bmatrix} \tag{16}$$

or horizontally [11] as:

$$\mathbf{X} = \begin{bmatrix} \cos(\theta_1)s_{1,1} - \sin(\theta_1)s_{1,2} & \sin(\theta_1)s_{1,1} + \cos(\theta_1)s_{1,2} \\ \cos(\theta_2)s_{2,1} - \sin(\theta_2)s_{2,2} & \sin(\theta_2)s_{2,1} + \cos(\theta_2)s_{2,2} \end{bmatrix} \tag{17}$$

where the values of the angle pair (θ_1, θ_2) in (16) and (17) are to be optimized for minimum BER. The code constructed using (16) is called the rotation-based diagonal space-time code or the D code and using (17) is called the rotation-based horizontal space-time code or the H code.

5.2 Results on the D code

The numerical calculations on the BER performances of the D code, using the EUB, MUB, CUB and AUB with the optimum angle pair, 2.96 rad and 1.05 rad, obtained by the methods proposed in [6], are shown in Fig 1. QPSK (i.e., C = 256) has been used in these calculations. For comparison purpose, the Monte Carlo simulation result of the same system is also shown in the same figure. It can be seen that the EUB is most accurate for BER evaluation. At BER=10^{-3}, the difference between the MUB and EUB is about 1.5 dB, between the CUB and EUB is about 3 dB, and between the AUB and EUB is more than 4 dB. At BER=10^{-4}, the difference between the MUB and EUB is only about 1 dB.

As far as computation load is concerned, the EUB and CUB require computing the PEP $C(C\text{-}1)$ times for each SNR, while the MUB requires computing the functions $F(N, u)$ and Ω which can be obtained more easily by using the determinant instead of the more complicated eigenvalue approach. Computer programs written in C language have been used to compare their required computation loads. Results have shown that the time taken for using the EUB to evaluate a BER for a particular SNR is about 5 times longer than that for using the MUB. To evaluate the BERs over a range of SNRs, e.g., from SNR = 9 to 19 dB at a step of one dB, using the EUB takes about 55 times longer than that using the MUB simply because Ω in the MUB expressions of (10) is SNR independent. The time required for using the CUB is slightly less than that of EUB, but is still about 3 times longer than that using the MUB, so the required computation load for the MUB is much less than those of the EUB and CUB. Although the AUB requires the least computation load, it is however the least accurate. Thus the MUB is a good compromise between the required computational load and accuracy for error-rate evaluation.

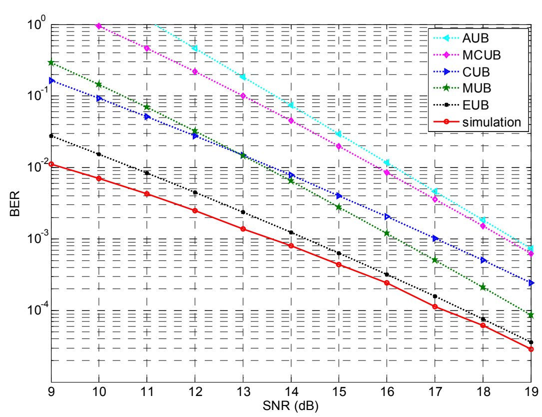

Fig 1: BER of D code with 2 receive antennas using different Union bounds

5.3 Results on the H code

For the H code, the determinant of matrix **X** in (17) is relating only to the angle difference $\theta_1 - \theta_2 = \Delta\theta$. From [6], the optimum angle difference for H code using QPSK is $\Delta\theta_{opt} = 2.81$. With this optimum angle difference, the numerical calculations on the BER performances of the H code using the EUB, MUB, CUB and AUB are shown in Fig 2. Again, for comparison, the Monte Carlo simulation result is also shown in the same figure. Observations similar to those of the D code are obtained here. The EUB is most accurate. At BER=10^{-2}, the difference between the MUB and EUB is about 1 dB, between the CUB and EUB is about 3.5 dB, and between the AUB and EUB is about 4 dB. At BER=10^{-3}, the difference between the MUB and EUB begins to vanish, while the difference between the AUB and CUB is still more than 3 dB.

Again, it should be noted here that the MUB requires much less computation load than those of the CUB and EUB because, for each SNR, there are $C(C\text{-}1)$ repeated calculations of the PEP for the EUB and CUB.

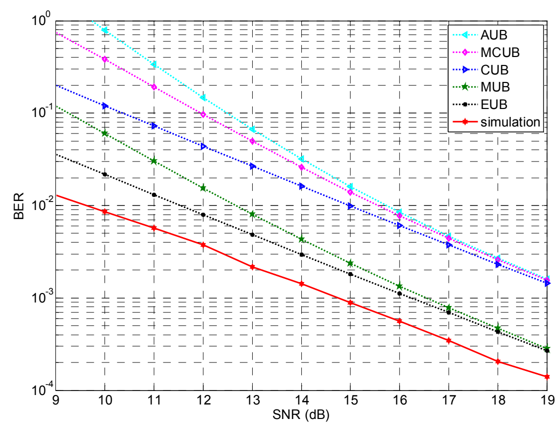

Fig 2: BER of H code with 2 receive antennas using different Union bounds

VI CONCLUSIONS

In this paper, a Common function of different Union bounds has been derived and subsequently used to develop the MUB for error-rate evaluations of S-T codes. At the BER of 10^{-4}, the difference between the MUB and EUB for the D code is about 1 dB. At the BER of 10^{-3}, the difference between the MUB and EUB for the H code is less than 0.5 dB. However, the MUB requires much less computational load than that of the EUB and is therefore a good compromise between the required computational load and accuracy for error-rate evaluation of S-T codes.

APPENDIX

The proof begins with the following inequality [8]:

$$P_e(\mathbf{X}_i \to \mathbf{X}_j) \le \frac{1}{\pi}\int_0^{\pi/2}\left(\frac{\sin^2\theta}{\sin^2\theta + 0.25\overline{\lambda}_{i,j}SNR}\right)^{KM} d\theta \quad \text{(A1)}$$

where $$\overline{\lambda}_{i,j} = \left(\prod_{k=1}^{K}\lambda_{k,i,j}\right)^{\frac{1}{K}} \quad \text{(A2)}$$

Substituting (A1) into (9), and since that $\frac{e_{i,j}}{B} \le 1$, the EUB is bounded by:

$$EUB \le \frac{1}{C\pi}\sum_{i=1}^{C}\sum_{j=1,j\ne i}^{C}\frac{e_{i,j}}{B}\int_0^{\pi/2}\left(\frac{\sin^2\theta}{\sin^2\theta + 0.25SNR\overline{\lambda}_{i,j}}\right)^{KM} d\theta$$

$$\le \frac{1}{C\pi}\sum_{i=1}^{C}\sum_{j=1,j\ne i}^{C}\int_0^{\pi/2}\left(\frac{\sin^2\theta}{\sin^2\theta + 0.25SNR\left(\overline{\lambda}_{i,j}{}^{-KM}\frac{e_{i,j}}{B}\right)^{-\frac{1}{KM}}}\right)^{KM} d\theta$$

(A3)

It can easily to verify that the function $\left(a + bx^{-\frac{1}{N}}\right)^{-N}$ is a concave function in terms of x (the second derivative is negative) for $a>0$, $b>0$, $N>0$ and $x>0$. Using Jensen's inequality, (A3) can be further bounded by:

$$EUB \le \frac{1}{C\pi}\sum_{i=1}^{C}\sum_{j=1,j\ne i}^{C}\int_0^{\pi/2}\left(\frac{\sin^2\theta}{\sin^2\theta + 0.25SNR\left(\overline{\lambda}_{i,j}{}^{-KM}\frac{e_{i,j}}{B}\right)^{-\frac{1}{KM}}}\right)^{KM} d\theta$$

$$\le \frac{(C-1)}{\pi}\int_0^{\pi/2}\left(\frac{\sin^2\theta}{\sin^2\theta + 0.25SNR\Omega}\right)^{KM} d\theta$$

$$= (C-1)F(MK, u_\Omega) = MUB$$

(A4)

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