The Term Structure of VIX

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Abstract

We extend the concept of CBOE constant 30-day VIX to other maturities and construct daily VIX term structure data from starting date available to August 2009. We propose a simple yet powerful two-factor stochastic volatility framework for VIXs. Our empirical analysis indicates that the framework is good at both capturing time-series dynamics of VIXs and generating rich cross-sectional shape of the term structure. In particular, we show that the two time-varying factors may be interpreted as factors corresponding to level and slope of the VIX term structure. Moreover, we explore the information content of VIXs relative to historical volatility in forecasting future realized volatility. Consistent with previous studies, we find that VIXs contain more information than historical volatility.

1 Introduction

In 1993, the Chicago Board Option Exchange (CBOE) introduced the VIX index, which quickly became the benchmark for stock market volatility. The VIX measures market expectations of near term volatility conveyed by equity-index options, and is often referred to as the "investor fear gauge". It is widely followed by theorists and practitioners, especially after financial turmoil during 2008. The index was originally computed as averaged Black-Scholes implied volatilities of near-the-money S&P100 index (OEX) American style option prices. On September 22, 2003, the CBOE revised the methodology of calculation[3], using theoretical results by Carr and Madan (1998) and Demeterfi et al (1999). The main difference are that the new VIX is model-free and is based on the S&P 500 index (SPX) European style options. It is able to incorporate information from the volatility smirk as noted in Zhang and Xiang (2008) and Chang, Ren, and Shi (2009), by using a wider range of strike prices. Now, the CBOE has created an identical record for the new VIX dating back to 1986, as well as the old index which under the new ticker symbol "VXO".

The popularity of the VIX has also induced a huge demand on VIX related products, due to increasingly importance of volatility/variance trading. VIX futures and options were introduced by the CBOE on March 26, 2004 and February 24, 2006, respectively. Meanwhile, academic research on the exchange listed volatility derivative market has also been growing rapidly in recent years. Zhang and Zhu (2006) is the first attempt to study the VIX and VIX futures. Zhu and Zhang (2007) extend Zhang and Zhu (2006) model by allowing long-term mean level of variance to be time-dependent. Lin (2007) applies affine jump-diffusion model with jumps in both index and volatility processes. Recently, Zhang, Shu, and Brenner (2009) provide an comprehensive analysis on VIX futures market. On the other hand, Sepp (2008a, b) and Lin and Chang (2009) focus on VIX options.

[3]See the CBOE 2003 whitepaper, which is further updated in 2009 with more detail examples.

Although the literature on the VIX and its derivatives is fast growing, only the VIX with a single fixed 30-day maturity is considered. There is no comprehensive study directly on the term structure of VIX, which is the focus of the current paper. Generally speaking, two important determinants of implied volatility surface are strike price and time to maturity. Recent studies, such as Zhang and Xiang (2008), Chang, Ren, and Shi (2009), have explored effect of strike price on option pricing by examining the phenomenon of implied volatility smile. We investigate characteristics of implied volatility along time to maturity direction, which should enhance our understanding of the valuation of option prices. Actually, the CBOE has noted the importance of the volatility term structure and lunched S&P 500 3-month volatility index under the ticker symbol "VXV" on November 12, 2007. The VXV uses the same methodology used to calculate VIX, but with a different set of SPX options with expiration dates that bracket a constant 93-day maturity. One related study is Mixon (2007), who tests the expectations hypothesis of the term structure of implied volatility for several national stock market indexes. However, the data used in Mixon (2007) are based on the bid side Black-Scholes implied volatilities for at-the-money calls, while we use model-free volatilities for a wider range of strike prices. Moreover, we are the first to provide an in-depth study on the VIX term structure data provided by the CBOE. Since the term structure of VIX reflects significant insight on the market's expectation of future realized volatilities of different maturities, our results should be valuable for investors to have a better understanding of the SPX option prices, VIX futures and options. which should enhance our understanding of the valuation of option prices.

In this paper, we construct daily VIX term structure data with six maturities, ranging from January 2, 1992 to August 31, 2009, where the former is the starting date available. We find that the term structure of VIX exhibits typical upward sloping, downward sloping, as well as hump and inverted hump shapes. In addition, we propose a novel two-factor stochastic volatility framework for the instantaneous variance, with the second factor to be the long term mean level of the instantaneous variance. As noted by Egloff, Leippold, and Wu (2009), Zhang and Huang (2010), and Zhang, Shu, and Brenner (2010), it is necessary to model long-term mean level of the instantaneous variance as a second factor. Besides, Li and Zhang (2008) find that, in addition to the index itself, two state variables are adequate for index options pricing. Moreover, our framework is much more general than previous studies on VIX and its derivatives in the sense that it contains any martingale specifications for the instantaneous variance, including Duan and Yeh (2007), Lin (2007), Lin and Chang (2009), Zhang and Huang (2010) and Zhang, Shu, and Brenner (2010).

We estimate parameters by using VIX information in both time series and cross section. Our empirical analysis indicates that the model is capable of replicating various dynamics of the VIX term structure. Furthermore, we find that the instantaneous volatility and the difference between the instantaneous volatility and its long term mean correspond to level and slope of the VIX term structure, respectively.

Our paper also relates to the literature on information content of implied volatility in forecasting future realized volatility. While early studies (See Canina and Figlewski (1993)) find that implied volatility does not contain information beyond that in historical volatility, recent research provides evidence that implied volatility is a more efficient forecast for future realized volatility. Christensen and Prabhala (1998), Christensen, Hansen, and Prabhala (2001), Ederinton and Guan (2002), Poon et al. (2004) and Jiang and Tian (2005) are prominent examples among others. Following the literature, we investigate the information content of VIX term structure. In particular, we explore the relation between VIX, historical, and realized volatilities. Consistent with previous studies, we find that VIXs contain more information than historical volatility.

The rest of this paper is organized as follows. Section 2 proposes models for VIXs. Section 3 describes data construction details. Section 4 provides estimation procedure and empirical results. Section 5 studies information content of the VIX term structure. Section

6 concludes the paper.

2 Models

In this section, we first provide necessary introduction for VIX and define our VIX term structure. We also demonstrate that the jump component in dynamic of the S&P 500 index is negligible in modeling the VIXs index. Then, we propose a novel two-factor stochastic volatility framework for the instantaneous variance. Some discussion related to the modeling of VIX and its derivatives are also provided.

2.1 Definitions

Before introducing the term structure of VIX, we first give a brief review of the CBOE 30-day VIX. Carr and Madan (1998) and Demeterfi et al. (1999) provide theoretical fundamental for the CBOE revised VIX. They show that realized variance can be replicated by a dynamic trading strategy and a log contract or by a static portfolio of out-of-the-money call and put options, which correspond to two methods for calculating VIX as demonstrated below. Although the revised VIX is model-free, it is better to consider specific model for illustration. Assume that the process for the S&P 500 index, S_t, in the risk-neutral measure Q, is given by

$$\frac{dS_t}{S_t} = rdt + \sqrt{v_t}dW_t^Q, \tag{1}$$

where r is the risk-free rate and v_t, is the instantaneous variance of the index. W_t^Q is a standard $Q-$Bronian motion. Applying Ito's lemma to Equation (1) gives a process of logarithmic index

$$d\ln S_t = \left[r - \frac{1}{2}v_t\right]dt + \sqrt{v_t}dW_t^Q. \tag{2}$$

In principle, the CBOE 30-day VIX index squared is defined as the variance swap rate over the next 30 calendar days. It is equal to the risk-neutral expectation of the future variance over the period of 30 days from t to $t + \tau_0$ with $\tau_0 = 30/365$. That is, the VIX can be calculated as

$$\begin{aligned} VIX^2_{t,\tau_0} &\equiv E_t^Q\left[\frac{1}{\tau_0}\int_t^{t+\tau_0} v_u du\right], \\ &= \frac{1}{\tau_0}\int_t^{t+\tau_0} E_t^Q(v_u)du. \end{aligned} \tag{3}$$

or equivalently, by using Equations (1) and (2),

$$\begin{aligned} VIX^2_{t,\tau_0} &\equiv \frac{2}{\tau_0}E_t^Q\left[\int_t^{t+\tau_0} \frac{dS_u}{S_u} - d(\ln S_u)\right], \\ &= \frac{2}{\tau_0}E_t^Q\left[\int_t^{t+\tau_0} \left(\frac{1}{2}v_u\right) du\right], \\ &= \frac{1}{\tau_0}\int_t^{t+\tau_0} E_t^Q(v_u)du. \end{aligned} \tag{4}$$

Obviously, the two VIX formulas are identical when there is no jump in the index. However, this is not the case when jump is considered, which will be discussed soon.

Now, we are ready to extend the CBOE single 30-day VIX to other maturities and introduce the term structure of VIX. Generally, the term structure of VIX, like traditional term structure of interest rates, display the relationship between VIXs and their term to maturity. For example, a VIX squared at time t, with maturity τ, is defined as

$$VIX^2_{t,\tau} \equiv E_t^Q\left[\frac{1}{\tau}\int_t^{t+\tau} V_u du\right], \tag{5}$$

where V_t, is the instantaneous variance of the index. Note that we use V_t to denote the instantaneous variance rather than v_t as before, in the sense that jump component also contributes to the total variance when dynamic of the index is given by jump-diffusion process. In addition, we employ the method in Equation (3), which is equivalent to the method in Equation (4) when there is no jump in the index. An natural question arises is that what is the difference between the two methods when the underlying index do have jumps? The answer is presented in the following Proposition 1.

Proposition 1: *The jump component in dynamic of the S&P 500 index is negligible in modeling the VIX index.*

Proof. See appendix.

In other words, the proposition provides supportive evidence for models in Zhang and Zhu (2006), Zhang and Huang (2010) and Zhang, Shu, and Brenner (2010), where the dynamic of the S&P 500 index is given by a diffusion process.

2.2 Two-factor framework for VIX

Although it has advantages to calculate the VIX by using model-independent method, we do need specific models to study dynamics of the VIX and further explore information content of the VIX term structure. Previously, we discuss VIXs calculation by concentrating on the S&P 500 index process and do not require any specification of the variance dynamics. Recently, the importance of modeling long term mean of the variance as the second factor is well recognized in the literature on volatility/variance derivatives. Zhang and Huang (2010) study the CBOE S&P 500 three-month variance futures and suggest that a floating long-term mean level of variance is probably a good choice for the variance futures pricing. Zhang, Shu, and Brenner (2010) build a two-factor model for VIX futures, where long-term mean level of variance is treated as a pure Brownian motion. They find that the model produces good forecasts of VIX futures prices. Egloff, Leippold, and Wu (2009) show that two risk factors are needed to capture variance risk dynamics in variance swap markets.

In this paper, we propose a more general framework for modeling variance dynamics, which contains above models as special cases. We use F_t to denote the forward price of the S&P 500 index at time t. Since F_t is a martingale under the forward measure $\mathcal{F}$, we consider the following two-factor model for the variance V_t,

$$\begin{aligned} dV_t &= \kappa(\theta_t - V_t)dt + dM^{\mathcal{F}}_{1,t}, \\ d\theta_t &= dM^{\mathcal{F}}_{2,t}, \end{aligned} \tag{6}$$

where θ_t is the long-term mean level of the variance. κ is the mean-reverting speed of the variance. $dM_{1,t}^{\mathcal{F}}$ and $dM_{2,t}^{\mathcal{F}}$ are increments of two martingale processes. Then, the VIXs can be calculated as in the following proposition:

Proposition 2: *Under the framework described in Equation (6), the VIX index squared, at time t, with maturity τ, $VIX_{t,\tau}^2$, is given by*

$$VIX_{t,\tau}^2 = (1-\alpha_1)\theta_t + \alpha_1 V_t, \quad \alpha_1 = \frac{1-e^{-k\tau}}{k\tau}. \tag{7}$$

Proof: Since the dynamic of the variance is given by Equation (6), therefore,

$$E_t^Q(V_u) = \theta_t + (V_t - \theta_t)e^{-\kappa(u-t)}, \quad u > t. \tag{8}$$

By definition, the VIX squared is equal to the risk-neutral expectation of the variance over $[t, t+\tau]$, or

$$VIX_{t,\tau}^2 \equiv E_t^Q\left(\frac{1}{\tau}\int_t^{t+\tau} V_u du\right), \tag{9}$$

$$= \frac{1}{\tau}\int_t^{t+\tau} E_t^Q(V_u)du, \tag{10}$$

$$= (1-\alpha_1)\theta_t + \alpha_1 V_t, \quad \alpha_1 = \frac{1-e^{-k\tau}}{k\tau}. \tag{11}$$

Remark 1 *We do not specify the underlying dynamics, which means that the model is flexible to include existing models in index option pricing literature as special cases. As seen before, when jump is added into index process, the realized variance of the index is modified with an additional jump-related term (e.g., Duan and Yeh (2007) and Sepp (2008b)).*

Remark 2 *We directly model the total variance of the index rather than the diffusion variance in the literature. Meanwhile, the framework allows jump component in variance dynamics. More importantly, in contrast with previous studies (e.g., Lin (2007), Sepp (2008a), Lin and Chang (2009)), the martingale specification tremendously simplifies expression for VIX.*

Remark 3 *The current framework is general enough to contain any martingale specification for the random noises in the variance, such as Brownian motions, compensated jump processes, or a mixture of both. Actually, Zhang and Huang (2010) can be obtained with constant θ_t and Browmian motion innovation. Zhang, Shu, and Brenner (2010) and Egloff, Leippold, and Wu (2009) are special cases with Brownian motion innovations for two factors.*

Remark 4 *Since α_1 is a number between 0 and 1, $VIX^2_{t,\tau}$ is the weighted average between the instantaneous variance V_t and its long-term mean level θ_t with α_1 as the weight.*

3 Data

In this section we construct our VIX term structure data. The daily VIX term structure data provided by the CBOE are available since 2008 with historical data going back to January 2, 1992.[4] The VIX term structure is a collection of volatility values tied to particular SPX option expirations. They are calculated by applying the CBOE VIX formula to a single strip of options having the same expiration date. However, unlike the VIX index, VIX term structure data does not reflect constant-maturity volatility. Generally, the CBOE lists SPX option series in three near-term contract months plus at lest three additional contracts expiring on the March quarterly cycle; that is, on the third Friday of March, June, September and December. Therefore, for each day, there are different numbers of expiration dates and corresponding VIXs. For example, on January 2, 1992 and June 18, 1992, there are eight and seven VIXs, respectively.

Note that the CBOE calculate VIX term structure data using a "business day" convention to measure time to expiration, as well as the "calendar day" convention used in the VIX index itself. In particular, the generalized VIX formula has been modified to reflect

[4] http://www.cboe.com/micro/vix/vixtermstructure.aspx

business time to expiration as:

$$\sigma^2 = \frac{2}{T_{\text{Business}}} \sum_i \frac{\Delta K_i}{K_i^2} e^{RT_{\text{Calendar}}} Q(K_i) - \frac{1}{T_{\text{Business}}} \left[\frac{F}{K_0} - 1 \right]^2, \tag{12}$$

where the volatility σ times 100 gives the value of the VIX index level. T_{Business} is business time to expiration and T_{Calendar} is a calender day measure that is used to discount the option prices. K_i is the strike price of ith out-of-money options, ΔK_i is the interval between two strikes. R is the risk-free rate to expiration. $Q(K_i)$ is the midpoint of the bid-ask spread of each option with strike K_i. F is the implied forward index level derived from the nearest to the money index option prices by using put-call parity and K_0 is the first strike that is below the forward index level.[5]

Consistent with this modification, we use interpolation as in the CBOE VIX calculation procedure to construct VIX term structure data with constant maturities. For example, on Jan 2, 1992, we use implied volatility values of two SPX options with expiration dates March 21, 1992 (56 business days) and June 20, 1992 (121 business days) to compute the VIX with 63 trading days to expiration. That is,

$$VIX_{t,63} = \sqrt{\left\{ T_1 \sigma_1^2 \left[\frac{N_{T_2} - N_{63}}{N_{T_2} - N_{T_1}} \right] + T_2 \sigma_2^2 \left[\frac{N_{63} - N_{T_1}}{N_{T_2} - N_{T_1}} \right] \right\} \times \frac{N_{252}}{N_{63}}}, \tag{13}$$

where T_1 and T_2 are business days to expiration of two SPX options, and σ_1 and σ_2 are corresponding volatilities. We construct the daily VIX term structure data with fixed maturities 1, 3, 6, 9, 12 and 15 months, which corresponds to 22, 63, 126, 189, 252 and 315 business days, from January 2, 1992 to August 31, 2009. Note that the CBOE calculates three separate volatility values based on SPX option bid, offer and midpoint prices at each point. We will focus on midpoint data in the following sections.

Table 1 provides descriptive statistics for the daily VIX term structure data quoted in annualized percentage terms. The following stylized facts emerge: the average VIXs are

[5] Please refer to the VIX whitepaper and the VIX term structure description for more details.

not monotonic, rise from 19.7 percent for a 1-month VIX to 20.4 percent for a 6-month VIX and then decrease; both VIXs and VIXs spreads are quite volatile, which implies that there is substantial variation in both level and shape of the VIX term structure; the variation of VIXs is downward sloping as maturity increases, with long VIXs varies moderately relative to its mean; all VIXs are highly skewed and leptokurtic as might be expected, especially for the 1-month VIX. The principal component analysis in Table 2 shows that the main principal component explains around 97% of the total variation in the data, while the first two components explain more than 99%. It means that the convexity effect is negligible for the VIX term structure data. The eigenvectors indicate that the first and second principal components are related to level and slope factors in the VIX term structure cure, respectively. We will investigate this point further in later section.

Figure 1 shows a three-dimensional plot of the VIX term structure data and Figure 2 plots time series of three selected VIXs. On time series perspective, looking at VIX with maturity 1-month in Figure 2, the index is relatively low (less than 20 percent) during the period 1992 to 1996, and shifts to above 20 percent since 1997. It experiences a dramatic rise in late 1997, September 1998, November 2001 and August 2002. The 1-month VIX reverts to stay around 20 percent during the June 2003 to August 2008 period and reaches peak during the 2008 financial crisis. It takes about ten months to come back to normal level. On cross-sectional perspective, the term structure is almost upward sloping during the periods 1992 to 1995 and 2004 to 2006. It shifts between upward sloping and downward sloping, and exhibits hump and inverted hump shapes. Interestingly, the slope of VIX term structure is usually negative during turbulent periods, as expected.

4 Estimation

In this section, we use above VIX term structure data to estimate parameters of the model introduced in Section 2. Since the stochastic volatility is unobservable, we have to estimate model's parameters, κ, as well as the spot variances $\{V_t\}_{t=1,...,T}$ and its long term mean $\{\theta_t\}_{t=1,...,T}$, where T is the number of observations. We adopt an efficient iterative two-step procedure in Christoffersen, Heston, and Jacobs (2009), which is a modification of the approach by Bates (2000). The procedure starts from an initial value for κ.

Step 1: Obtain time series of $\{V_t, \theta_t\}$, $t = 1, ..., T$. In particular, for a given parameter set $\{\kappa\}$, we solve T optimization problems of the form:

$$\{\hat{V}_t, \hat{\theta}_t\} = \arg\min \sum_{j=1}^{N_t} \left(VIX_{t,\tau_j}^{Mkt} - VIX_{t,\tau_j} \right)^2, \quad t = 1, ..., T, \tag{14}$$

where VIX_{t,τ_j}^{Mkt} is the market value of VIX with maturity τ_j on day t and VIX_{t,τ_j} is the corresponding theoretical value given by Equation (7). N_t is the number of maturities used at day t.

Step 2: Estimate parameter set $\{\kappa\}$ with $\{V_t, \theta_t\}$ obtained in Step 1. That is, we minimize aggregate sum of squared errors

$$\{\hat{\kappa}\} = \arg\min \sum_{t=1}^{T} \sum_{j=1}^{N_t} \left(VIX_{t,\tau_j}^{Mkt} - VIX_{t,\tau_j} \right)^2. \tag{15}$$

Iteration between Step 1 and Step 2 is continued until there is no further significant improvement in the aggregate objective function in Step 2. Note that, the two-step procedure is well-behaved due to simple closed-form formula for VIX in the model. Moreover, only few iterations are required within each step and for overall convergence.

We obtain a unique solution for parameter: $\kappa = 7.0655$ and daily values of V_t and θ_t. Figure 3 plots time series of estimated V_t and θ_t. The long term mean, θ_t, stayed at a level of about 3 percent before July 1997, and volatile at around 5 percent at most time during the period August 1997 to September 2008. It rose to the level of 20 percent in October and November 2008 and remains at 10 percent until now. These results are consistent with those obtained in Zhang, Shu, and Brenner (2010) by using daily VIX futures data. The instantaneous variance, V_t, is quite highly volatile relative to its long term mean, especially during the periods 1997-1998, 2001-2002, and October 2008 to February 2009. It even rose to 80 percent during the 2008 global financial crisis.

With these estimates, we are able to calculate daily fitted VIX term structure value by using formula (7) and compare them with market data. Figures 4-6 show time series of three selected VIXs with maturities of 1, 6 and 15 months. Figure 7 shows the term structure of VIX for some selected dates. It is obvious that our model fits to the market data very well. Furthermore, the model is capable of generating various term structure shapes: upward sloping, downward sloping, humped and inverted humped.

We can also compare model implied level and slope of the VIX term structure with market data implied level and slope. We define the model-based level as the long-term mean level of instantaneous volatility, $\sqrt{\theta_t}$, and model-based slope as the difference between the instantaneous volatility and its long term mean level, $\sqrt{\theta_t} - \sqrt{V_t}$. Moreover, the data-based level and slope are defined to be the 15-month VIX and the difference between the 15-month and the 1-month VIXs, respectively. Figure 8 plots time series of model-based level along with the data-based level. Figure 9 plots time series of model-based and data-based slopes. The figures mean that the two factors in our model correspond to level and slope, which is consistent with our previous principal component analysis in Table 2. Actually, the correlation coefficients are 0.9834 and 0.9881, respectively.

5 Information content of the VIX term structure

In this section, we explore the information content of the VIXs relative to historical volatility in forecasting future realized volatility.

5.1 Volatility indices data

We calculate the annualized realized volatility (RVol) over a period $[t, t+\tau]$ as in Zhang and Huang (2010):

$$RVol = \sqrt{\frac{252}{N_e - 1} \sum_{i=1}^{N_a - 1} R_i^2}, \tag{16}$$

where $R_i = \ln(S_{i+1}/S_i)$, N_e is the number of expected S&P 500 values needed to calculate daily returns during $[t, t+\tau]$, N_a is the actual number of S&P 500 values used.

We collect monthly realized volatility data observed on the Wednesday immediately following the expiry date of the month, as in Christensen and Prabhala (1998) and Jiang and Tian (2005). The main reason is that trading volume is relative large during the week following the expiration date and Wednesday has the fewest holidays among all weekdays. The following Thursday then the proceeding Tuesday will be used in case the Wednesday is not a trading day. To avoid the telescoping overlap problem described by Christensen, Hansen, and Prabhala (2001), we extract realized volatilities at fixed maturities of 22(1m), 63(3m), 126(6m), 189(9m), 252(12m) and 315(15m) trading days, which match our VIX term structure maturities. Following Canina and Figlewski (1993) and Christensen and Prabhala (1998), we calculate the monthly historical volatility over a matching period immediately proceeding the current observation date. For example, in order to calculate τ-month historical volatility at time t, we employ the formula in Equation (16) over the period $[t-\tau, t]$. The sample period is January 1992 to June 2008, totally 198 observations.

Tables 3 and 4 provide summary statistics for monthly volatility indices and their natural logarithms, respectively. As shown in Table 3, VIXs are on average higher than corresponding realized volatilities, which turn out to be higher than historical volatilities. This observation indicates that VIXs are likely up biased forecast for realized volatilities, while historical volatilities are down biased forecast for realized volatilities. It is consistent with negative market price of risk observed in the literature (See, e.g., Duan and Yeh (2007), Carr and Wu (2009), Egloff, Leippold, and Wu (2009) and Zhang and Huang (2010)).

5.2 Relation between VIXs and realized volatilities

Now, we explore relation between the VIX term structure and realized volatilities. Following Jiang and Tian (2005), we specify following encompassing regressions

$$\sigma_{t,\tau}^{RE} = \alpha_\tau + \beta_\tau^{VIX} VIX_{t,\tau} + \beta_\tau^{HIS} \sigma_{t,\tau}^{HIS} + \epsilon_{t,\tau}, \tag{17}$$

$$V_{t,\tau}^{RE} = \alpha_\tau + \beta_\tau^{VIX} VIX_{t,\tau}^2 + \beta_\tau^{HIS} V_{t,\tau}^{HIS} + \epsilon_{t,\tau}, \tag{18}$$

$$\ln \sigma_{t,\tau}^{RE} = \alpha_\tau + \beta_\tau^{VIX} \ln VIX_{t,\tau} + \beta_\tau^{HIS} \ln \sigma_{t,\tau}^{HIS} + \epsilon_{t,\tau}, \tag{19}$$

where $\sigma_{t,\tau}$ and $V_{t,\tau}$ are volatility and variance, respectively. The superscripts RE, VIX,and HIS stand for **RE**alized, **VIX**, and **HIS**torical, respectively. The subscripts t and τ are observation date and maturity, respectively. Univariate regressions are obtained if one of the two regressors are dropped. As noted in previous section, $t = 1, ..., 198$ and $\tau = 1, 3, 6, 9, 12$ and 15 months. We run OLS regressions for all six maturities.

Tables 5-7 show results from both univariate and encompassing regressions by using 1-, 6- and 15-month volatilities. Panel A, B and C present results from the three specifications, respectively. Numbers in brackets below the parameter estimates are the standard errors.

Some notable observations are in order. First, the VIXs explains more variations in future realized volatilities for the short and the long maturities than historical volatilities. The R^2 for the VIXs with maturities 1 and 15 months ranging from 50% to 65% and 21% to 42%, respectively, which are higher than those for historical volatilities across the three specifications. However, in case of 6-month maturity, historical volatility performs slightly better. Second, the Durbin-Watson statistics are not significantly different from two in most cases for 1-month maturity, indicating that the regression residuals are not autocorrelated. However, there are not the case for 6- and 15-month maturities. It should be related to our monthly data sampling procedure, which match 1-month maturity. We check it by sampling data for every 3 months and obtain 66 observations. The OLS regression by using 3-month volatilities in Table 8 confirms it.

6 Conclusion

The CBOE VIX has been publicly available since 1993. It is widely accepted as the premier measure of stock market volatility and investor sentiment, often interpreted as the "investor fear gauge". In fact, the VIX is only market expectation of future volatility in the following 30 calender days. We go a step further by studying the VIXs with other maturities as well, or the term structure of investor fear.

We demonstrate that the jump component in dynamic of the S&P 500 index is negligible in modeling VIXs. Thus, we provide supportive evidence for Zhang and Zhu (2006), Zhang and Huang (2010) and Zhang, Shu, and Brenner (2010). Moreover, we propose a simple yet powerful two-factor stochastic volatility framework for VIXs. The framework can be served as a platform for further modeling VIX futures and options in the future. We estimate model parameters by an efficient method with the constructed daily VIX term structure data. Our empirical analysis indicates that the framework is good at both capturing time-series dynamics of VIXs and generating rich cross-sectional shape of the term structure. More importantly, we show that the two time-varying factors may be interpreted as factors corresponding to level and slope of the VIX term structure, respectively.

We also investigate information content of the VIX term structure. Generally, we find the VIXs to be an informative, but upward biased, forecast of future realized volatility that tends to dominate historical volatility. These results are consistent with recent studies. Since the term structure of VIX conveys more insights than a single constant 30-day VIX on how the market views, our results should be valuable for investors to have a better understanding of the risks of SPX options, VIX futures and options of different maturities.

7 Appendix

Proof of Proposition 1. The idea is to compare VIX formulas in the two settings whether or not jump is added into the dynamic of the index. We consider the following general jump-diffusion process for the index, S_t,

$$\frac{dS_t}{S_{t^-}} = rdt + \sqrt{v_t}dW_t^Q + (e^x - 1)\ dN_t - \lambda\ E_t^Q(e^x - 1)\ dt, \tag{20}$$

where S_{t^-} is the value of S_t before a possible jump occurs. N_t is a pure jump process with intensity λ. x is the jump size of the logarithm index, $E_t^Q(e^x-1)$ stands for the expectation of $(e^x - 1)$, and the term, $\lambda E_t^Q(e^x - 1)dt$, compensates jump innovation. In addition, N_t is assumed to be independent of W_t^Q. Other symbols are the same as before. Applying Ito's lemma with jumps to Equation (20) gives a process of logarithmic index

$$d\ln S_t = \left[r - \frac{1}{2}v_t - \lambda E_t^Q(e^x - 1)\right] dt + \sqrt{v_t}dW_t^Q + xdN_t. \tag{21}$$

Since the jump component also affects variance of the index, the instantaneous total variance of the index, V_t, is different and becomes

$$V_t = v_t + E_t^Q(\lambda x^2), \tag{22}$$

where the first term is diffusion variance and the second term is jump variance. Then, according to definition in Equation (5), the VIX squared is given by

$$\begin{aligned} \widehat{VIX}^2_{t,\tau} &= E_t^Q\left[\frac{1}{\tau}\int_t^{t+\tau} V_u du\right], \\ &= \frac{1}{\tau}\int_t^{t+\tau} E_t^Q(V_u)du, \\ &= \frac{1}{\tau}\int_t^{t+\tau} E_t^Q(v_u + \lambda x^2)du, \\ &= \frac{1}{\tau}\int_t^{t+\tau} E_t^Q(v_u)du + E_t^Q(\lambda x^2), \end{aligned} \tag{23}$$

where we have used property of iterated expectations. On the other hand, the VIX squared can also be calculated, by using Equations (20) and (21), as following

$$\begin{aligned}\widetilde{VIX}^2_{t,\tau} &= \frac{2}{\tau}E^Q_t\left[\int_t^{t+\tau}\frac{dS_u}{S_u} - d(\ln S_u)\right], \\ &= \frac{2}{\tau}E^Q_t\left[\int_t^{t+\tau}\left(\frac{1}{2}v_u + \lambda(e^x - 1 - x)\right)du\right], \\ &= \frac{1}{\tau}\int_t^{t+\tau}E^Q_t(v_u)du + E^Q_t[2\lambda(e^x - 1 - x)]. \end{aligned} \tag{24}$$

Therefore, the difference between the two formulas in Equations (23) and (24) is

$$\begin{aligned}\Delta &= E^Q_t[2\lambda(e^x - 1 - x) - \lambda x^2], \\ &\approx E^Q_t\left(\frac{1}{3}\lambda x^3\right). \end{aligned} \tag{25}$$

When jump size, x, is assumed to be normally distributed with mean -0.1 and volatility 0.2, and $\lambda = 0.1$, then

$$E^Q_t[2\lambda(e^x - 1 - x)] = 2 * 0.1 * (e^{-0.1+0.5*0.2^2} - 1 + 0.1) = 0.0046, \tag{26}$$

$$E^Q_t[\lambda x^2] = 0.1 * [(-0.1)^2 + 0.2^2] = 0.005, \tag{27}$$

$$\Delta = -0.0004. \tag{28}$$

Thus, for general value of VIX at 20, we have

$$\widetilde{VIX}_{t,\tau} = 20, \quad \widehat{VIX}_{t,\tau} = 20.1, \tag{29}$$

which corresponds to 0.5% overvalue by using Equation (23) or our definition Equation (5). In other words, the jump component only contributes marginally to VIX index and hence negligible.

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Table 1: **Descriptive Statistics for Daily VIX Term Structure**

	Maturity	Mean	Std.dev.	Skewness	Kurtosis	Minimum	Maximum
Panel A: VIXs							
	1-m	19.696	8.650	2.100	10.058	9.212	80.352
	3-m	20.169	7.814	1.837	8.209	9.971	70.562
	6-m	20.405	7.114	1.603	6.739	5.746	61.956
	9-m	20.175	6.623	1.586	6.560	10.775	56.892
	12-m	20.153	6.332	1.466	6.049	7.730	53.410
	15-m	20.177	6.231	1.339	5.440	12.129	50.535
Panel B: VIX spreads							
	3-m	0.473	1.861	-2.724	20.937	-20.330	7.675
	6-m	0.710	2.843	-2.817	20.556	-29.540	16.215
	9-m	0.480	3.539	-2.512	20.824	-35.745	32.595
	12-m	0.458	4.025	-2.617	18.001	-41.130	25.591
	15-m	0.482	4.094	-2.576	15.752	-38.079	14.233

This table provides descriptive statistics for the daily VIX term structure data with maturities 1, 3, 6, 9, 12 and 15 months. Panel A and B present summary statistics for the VIXs levels and VIX spreads relative to the 1-month VIX, respectively. Reported are the mean, standard deviation, skewness, kurtosis, minimum and maximum. All VIXs are expressed in annualized percentage terms. The data consist of 4432 observations covering the period January 2, 1992 to August 31, 2009.

Table 2: **Principal Component Analysis of Daily VIX Term Structure**

	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}
Percent						
	96.56%	2.77%	0.29%	0.18%	0.13%	0.07%
Eigenvectors						
	0.4851	0.7138	-0.3395	-0.3231	-0.1880	0.0117
	0.4492	0.2224	0.3092	0.5796	0.4261	-0.3683
	0.4103	-0.1093	0.2548	0.1884	-0.0876	0.8435
	0.3783	-0.2698	0.5946	-0.4972	-0.2874	-0.3173
	0.3576	-0.4212	-0.3885	-0.3586	0.6433	0.0357
	0.3515	-0.4228	-0.4688	0.3852	-0.5281	-0.2251

This table provides principal component analysis of daily VIX term structure data with maturities 1, 3, 6, 9, 12 and 15 months. The data consist of 4432 observations covering the period January 2, 1992 to August 31, 2009.

Table 3: **Descriptive Statistics for Monthly Volatilities**

	Maturity	Mean	Std.dev.	Skewness	Kurtosis	Minimum	Maximum
Panel A: VIXs							
	1-m	18.052	6.079	0.801	3.008	9.424	37.517
	3-m	18.518	5.585	0.689	2.754	10.622	36.585
	6-m	19.162	5.270	0.639	2.530	12.027	35.389
	9-m	18.996	4.824	0.602	2.380	12.283	32.613
	12-m	18.879	4.593	0.498	2.157	12.126	30.815
	15-m	19.109	4.714	0.476	2.169	12.630	31.758
Panel B: Realized volatilities							
	1-m	14.518	7.025	1.377	5.179	5.275	43.176
	3-m	14.929	6.432	0.929	3.254	6.074	35.207
	6-m	15.548	7.180	1.675	8.070	6.832	54.395
	9-m	16.034	7.628	1.687	7.268	7.655	50.137
	12-m	16.400	7.864	1.546	6.041	7.909	45.550
	15-m	16.689	7.873	1.357	4.966	8.396	41.695
Panel C: Historical volatilities							
	1-m	14.421	6.874	1.351	5.247	4.905	43.259
	3-m	14.764	6.300	0.938	3.274	6.378	35.369
	6-m	14.895	5.951	0.735	2.578	6.754	31.994
	9-m	14.897	5.688	0.606	2.168	7.551	29.288
	12-m	14.900	5.524	0.514	1.887	7.891	27.450
	15-m	14.912	5.382	0.447	1.719	8.387	25.769

This table provides descriptive statistics for the monthly volatilities with maturities 1, 3, 6, 9, 12 and 15 months. Panel A, B and C show VIXs, realized volatilities and historical volatilities, respectively. Reported are the mean, standard deviation, skewness, kurtosis, minimum and maximum. All volatilities are expressed in annualized percentage terms. The data consist of 198 monthly observations covering the period January 1992 to June 2008.

Table 4: **Descriptive Statistics for Monthly Log Volatilities**

	Maturity	Mean	Std.dev.	Skewness	Kurtosis	Minimum	Maximum
Panel A: Log VIXs							
	1-m	2.840	0.324	0.247	2.113	2.243	3.625
	3-m	2.875	0.293	0.226	2.013	2.363	3.600
	6-m	2.917	0.267	0.268	1.909	2.487	3.566
	9-m	2.913	0.247	0.283	1.869	2.508	3.485
	12-m	2.909	0.239	0.209	1.806	2.495	3.428
	15-m	2.921	0.243	0.174	1.806	2.536	3.458
Panel B: Log realized volatilities							
	1-m	2.573	0.446	0.288	2.524	1.663	3.765
	3-m	2.617	0.412	0.239	2.109	1.804	3.561
	6-m	2.654	0.416	0.417	2.440	1.922	3.996
	9-m	2.681	0.423	0.471	2.493	2.035	3.915
	12-m	2.701	0.429	0.464	2.417	2.068	3.819
	15-m	2.719	0.430	0.414	2.278	2.128	3.730
Panel C: Log historical volatilities							
	1-m	2.569	0.443	0.244	2.511	1.590	3.767
	3-m	2.608	0.407	0.253	2.125	1.853	3.566
	6-m	2.625	0.387	0.257	1.843	1.910	3.466
	9-m	2.631	0.373	0.243	1.665	2.022	3.377
	12-m	2.634	0.365	0.222	1.563	2.066	3.312
	15-m	2.638	0.358	0.194	1.503	2.127	3.249

This table provides descriptive statistics for the monthly natural logarithms of volatilities with maturities 1, 3, 6, 9, 12 and 15 months. Panel A, B and C show natural logarithms of VIXs, realized volatilities and historical volatilities, respectively. Reported are the mean, standard deviation, skewness, kurtosis, minimum and maximum. The data consist of 198 monthly observations covering the period January 1992 to June 2008.

Table 5: **Information content of 1-month volatilities: Univariate and encompassing regressions (Monthly data)**

	α_{1-m}	β^{VIX}_{1-m}	$\beta^{Historical}_{1-m}$	Adj. R^2	DW
Panel A: $\sigma^{RE}_{t,1-m}$					
	-1.688	0.898		0.601	1.856
	(1.073)	(0.067)			
	4.152		0.719***	0.492	2.303
	(0.851)		(0.067)		
	-1.257	0.764*	0.137	0.604	2.010
	(1.160)	(0.127)	(0.100)		
Panel B: $V^{RE}_{t,1-m}$					
	-27.609	0.793**		0.503	1.882
	(26.094)	(0.095)			
	103.336		0.614***	0.348	2.215
	(21.615)		(0.105)		
	-24.084	0.741*	0.059	0.502	1.948
	(28.572)	(0.153)	(0.109)		
Panel C: $\ln \sigma^{RE}_{t,1-m}$					
	-0.568	1.106*		0.646	1.829
	(0.173)	(0.061)			
	0.637		0.754***	0.559	2.377
	(0.116)		(0.045)		
	-0.408	0.876	0.192**	0.653	2.067
	(0.191)	(0.124)	(0.086)		

This table presents the OLS regression results for specifications in Equations (17)-(19) in the content by using 1-month volatilities. The numbers in parentheses below the parameter estimates are the standard errors. *, ** and *** indicate that the leading term β coefficient is significantly different from one or the remaining term β coefficient is significantly different from zero at the 10%, 5%, and 1% level, respectively. The data consist of 198 monthly observations covering the period January 1992 to June 2008.

Table 6: **Information content of 6-month volatilities: Univariate and encompassing regressions (Monthly data)**

	α_{6-m}	β_{6-m}^{VIX}	$\beta_{6-m}^{Historical}$	Adj. R^2	DW
Panel A: $\sigma_{t,6-m}^{RE}$					
	-1.286	0.879*		0.413	0.268
	(1.142)	(0.070)			
	3.776		0.790***	0.426	0.206
	(0.845)		(0.070)		
	0.671	0.412***	0.468***	0.444	0.226
	(1.143)	(0.144)	(0.147)		
Panel B: $V_{t,6-m}^{RE}$					
	30.920	0.664***		0.198	0.229
	(23.739)	(0.083)			
	105.369		0.730**	0.208	0.198
	(18.804)		(0.113)		
	52.329	0.320***	0.444**	0.219	0.210
	(22.678)	(0.135)	(0.190)		
Panel C: $\ln \sigma_{t,6-m}^{RE}$					
	-0.767	1.173***		0.565	0.316
	(0.189)	(0.066)			
	0.508		0.817***	0.574	0.228
	(0.122)		(0.048)		
	-0.199	0.560***	0.465***	0.594	0.256
	(0.223)	(0.159)	(0.115)		

This table presents the OLS regression results for specifications in Equations (17)-(19) in the content by using 6-month volatilities. The numbers in parentheses below the parameter estimates are the standard errors. *, ** and *** indicate that the leading term β coefficient is significantly different from one or the remaining term β coefficient is significantly different from zero at the 10%, 5%, and 1% level, respectively. The data consist of 198 monthly observations covering the period January 1992 to June 2008.

Table 7: **Information content of 15-month volatilities: Univariate and encompassing regressions (Monthly data)**

	α_{15-m}	β_{15-m}^{VIX}	$\beta_{15-m}^{Historical}$	Adj. R^2	DW
Panel A: $\sigma_{t,15-m}^{RE}$					
	-1.824	0.969		0.333	0.077
	(1.681)	(0.100)			
	6.946		0.653***	0.195	0.024
	(1.183)		(0.082)		
	-2.657	1.194	-0.233	0.337	0.102
	(2.122)	(0.265)	(0.211)		
Panel B: $V_{t,15-m}^{RE}$					
	4.349	0.867		0.212	0.062
	(38.952)	(0.136)			
	196.973		0.570***	0.073	0.023
	(32.895)		(0.109)		
	-9.424	1.223	-0.494^*	0.230	0.095
	(45.987)	(0.303)	(0.266)		
Panel C: $\ln \sigma_{t,15-m}^{RE}$					
	-0.630	1.147*		0.417	0.086
	(0.243)	(0.085)			
	0.894		0.691***	0.327	0.024
	(0.162)		(0.063)		
	-0.623	1.139	0.006	0.414	0.085
	(0.357)	(0.274)	(0.186)		

This table presents the OLS regression results for specifications in Equations (17)-(19) in the content by using 15-month volatilities. The numbers in parentheses below the parameter estimates are the standard errors. *, ** and *** indicate that the leading term β coefficient is significantly different from one or the remaining term β coefficient is significantly different from zero at the 10%, 5%, and 1% level, respectively. The data consist of 198 monthly observations covering the period January 1992 to June 2008.

Table 8: **Information content of 3-month volatilities: Univariate and encompassing regressions (3-monthly data)**

	α_{3-m}	β_{3-m}^{VIX}	$\beta_{3-m}^{Historical}$	Adj. R^2	DW
Panel A: $\sigma_{t,3-m}^{RE}$					
	-1.182	0.876		0.505	2.123
	(1.962)	(0.120)			
	5.116		0.657***	0.417	2.311
	(1.436)		(0.111)		
	-0.816	0.797	0.073	0.498	2.176
	(2.012)	(0.218)	(0.188)		
Panel B: $V_{t,3-m}^{RE}$					
	7.700	0.701*		0.384	2.194
	(45.050)	(0.156)			
	120.247		0.542***	0.271	2.252
	(34.826)		(0.159)		
	2.230	0.761	−0.063	0.375	2.151
	(40.877)	(0.210)	(0.209)		
Panel C: $\ln \sigma_{t,3-m}^{RE}$					
	-0.587	1.116		0.566	2.070
	(0.334)	(0.116)			
	0.733		0.720***	0.509	2.354
	(0.212)		(0.082)		
	-0.350	0.850	0.201	0.567	2.219
	(0.387)	(0.243)	(0.159)		

This table presents the OLS regression results for specifications in Equations (17)-(19) in the content by using 3-month volatilities. The numbers in parentheses below the parameter estimates are the standard errors. *, ** and *** indicate that the leading term β coefficient is significantly different from one or the remaining term β coefficient is significantly different from zero at the 10%, 5%, and 1% level, respectively. The data consist of 66 every three months' observations covering the period January 1992 to June 2008.

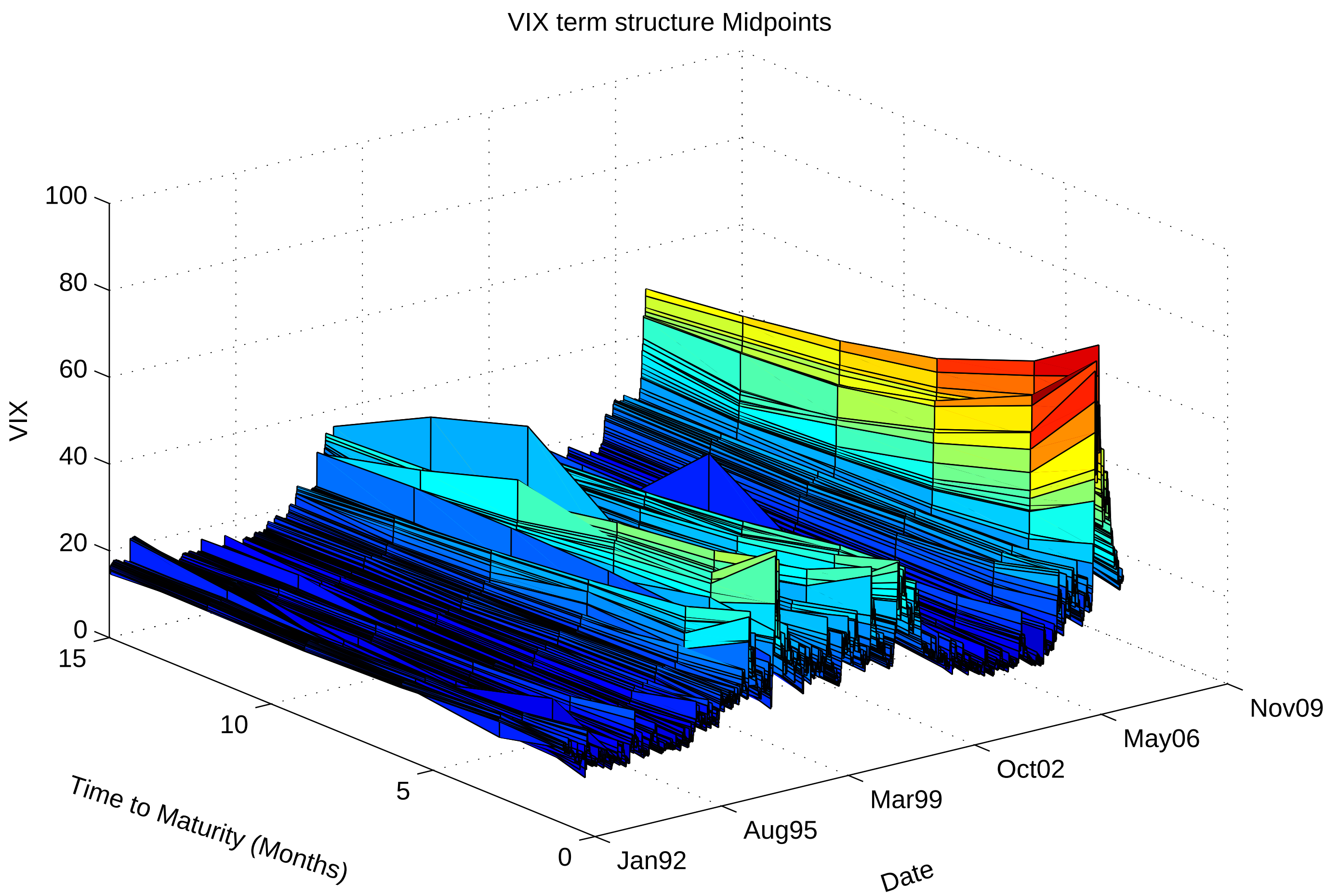

Figure 1: **VIX term structure from 1992 to 2009.** We show a three-dimensional plot of daily VIX term structure with maturities of 1, 3, 6, 9, 12 and 15 months. The sample period is January 2, 1992 to August 31, 2009 with 4432 observations. All volatilities are expressed in percentage terms.

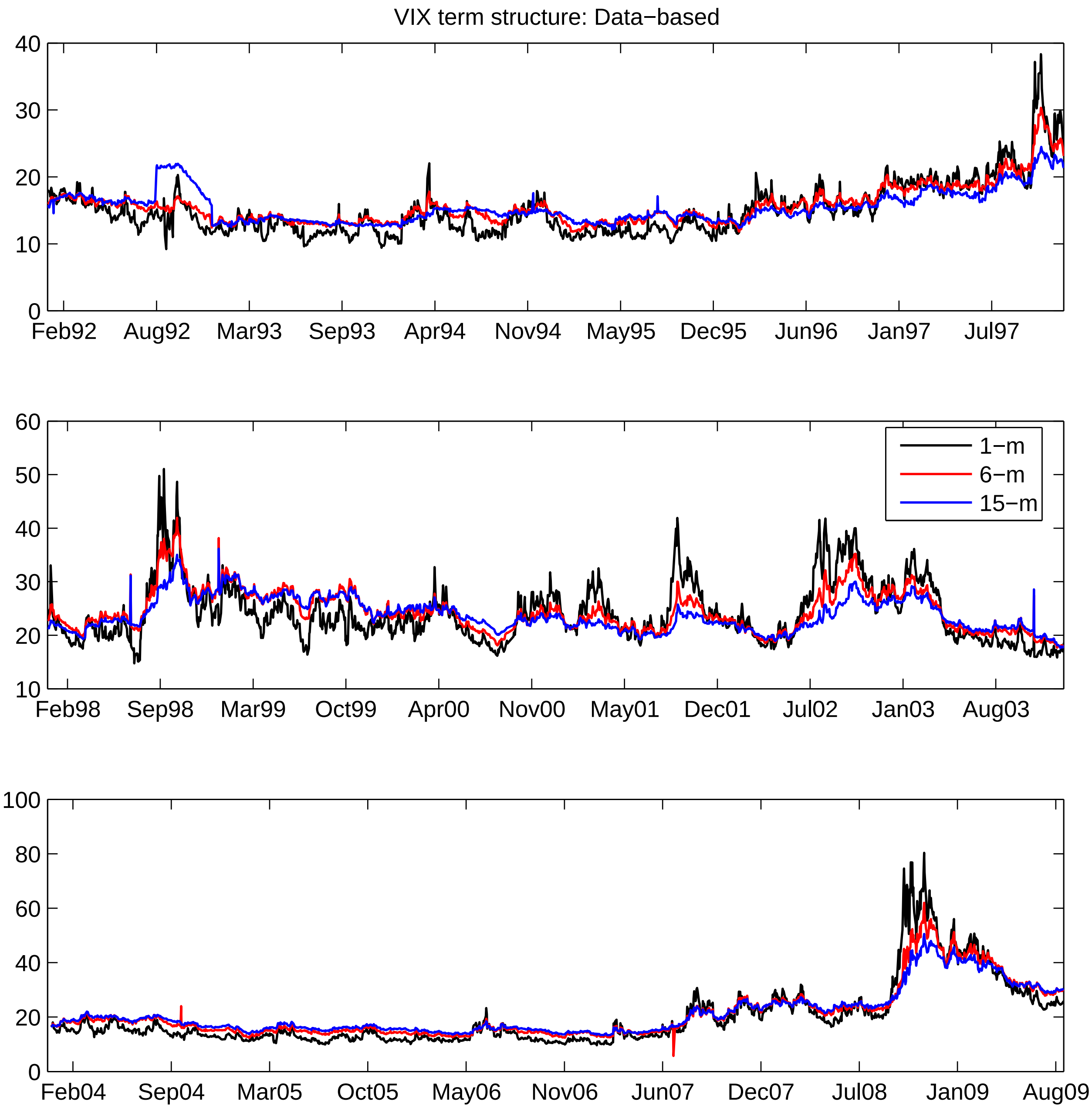

Figure 2: **Time series of VIXs with maturities of 1, 6 and 15 months.** We show time series of daily VIXs with maturities of 1 (black lines), 6 (red lines) and 15 (blue lines) months from January 2, 1992 to August 31, 2009 with 4432 observations. All volatilities are expressed in percentage terms.

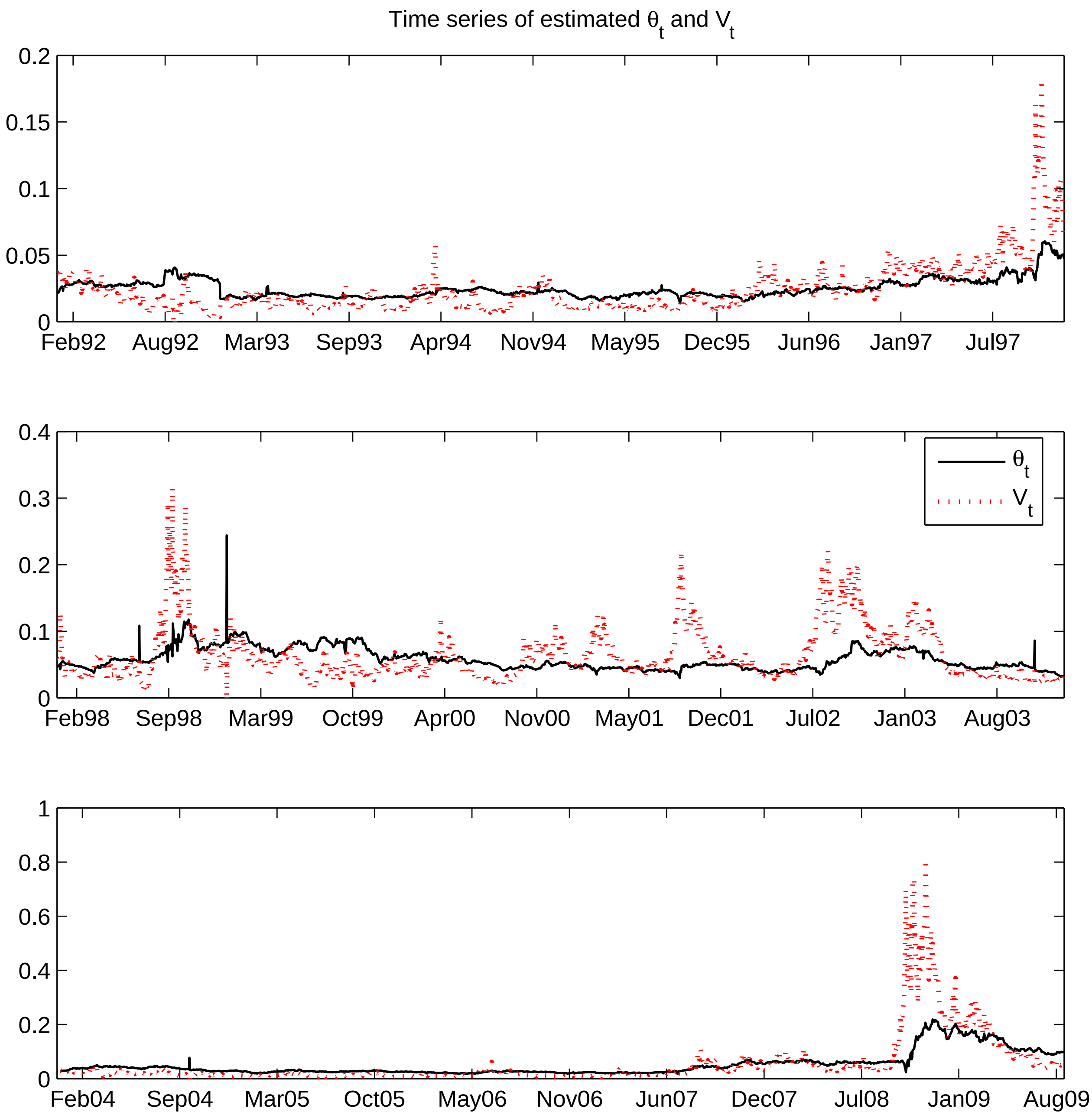

Figure 3: **Time series of the estimated instantaneous variance and its long term mean level.** We show time series of the daily estimated instantaneous variance (dotted red lines), V_t, and its long term mean level (black lines), θ_t, from January 2, 1992 to August 31, 2009 with 4432 observations.

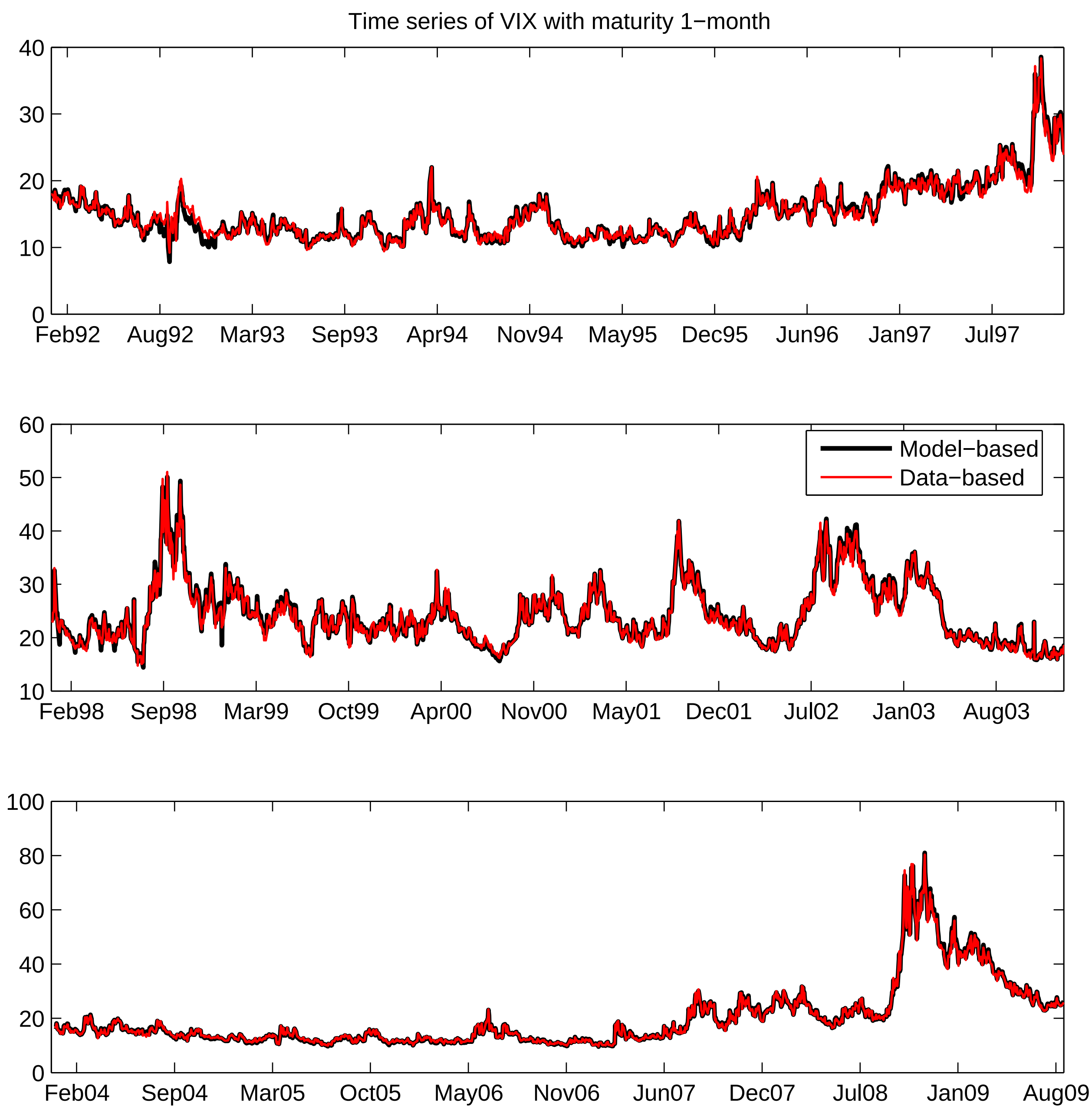

Figure 4: **Time series of model-based and data-based VIXs with maturity 1-month.** We show time series of model-based (black lines) and market-based (red lines) VIXs with maturity 1 month from January 2, 1992 to August 31, 2009 with 4432 observations. All volatilities are expressed in percentage terms.

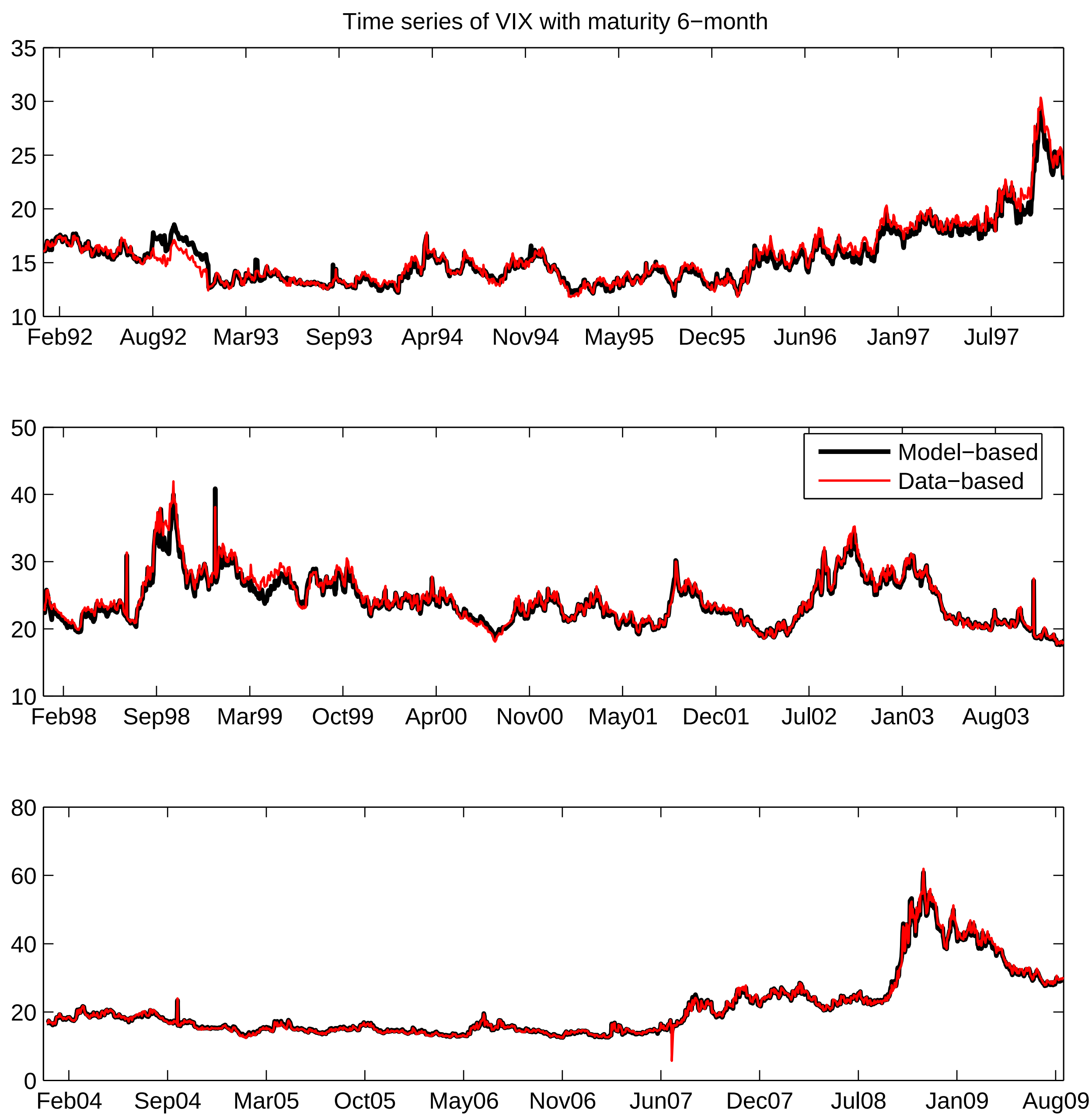

Figure 5: **Time series of model-based and data-based VIXs with maturity 6 months.** We show time series of model-based (black lines) and data-based (red lines) VIXs with maturity 6-month from January 2, 1992 to August 31, 2009 with 4432 observations. All volatilities are expressed in percentage terms.

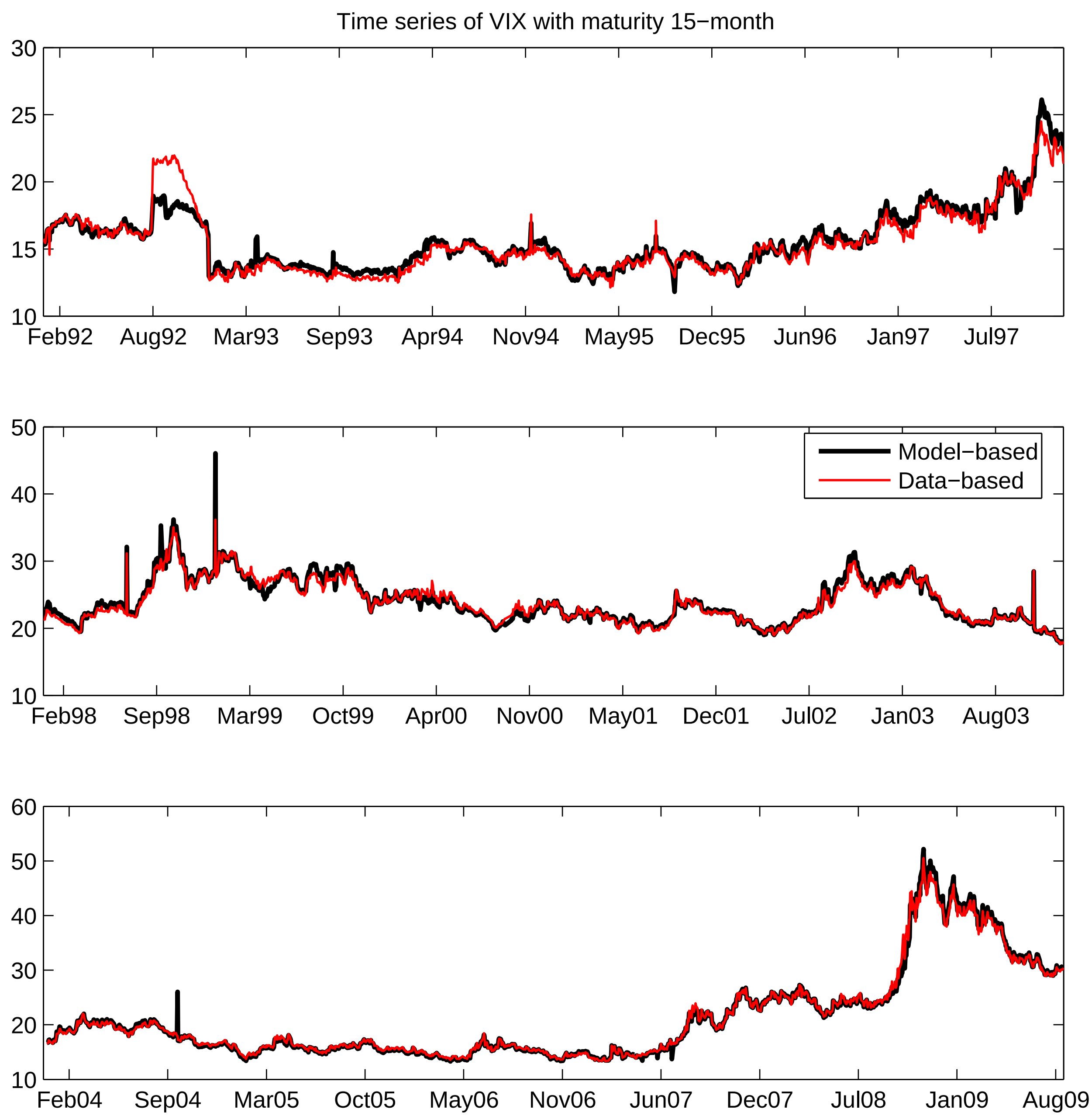

Figure 6: **Time series of model-based and data-based VIXs with maturity 15 months.** We show time series of model-based (black lines) and market-based (red lines) VIXs with maturity 15 months from January 2, 1992 to August 31, 2009 with 4432 observations. All volatilities are expressed in percentage terms.

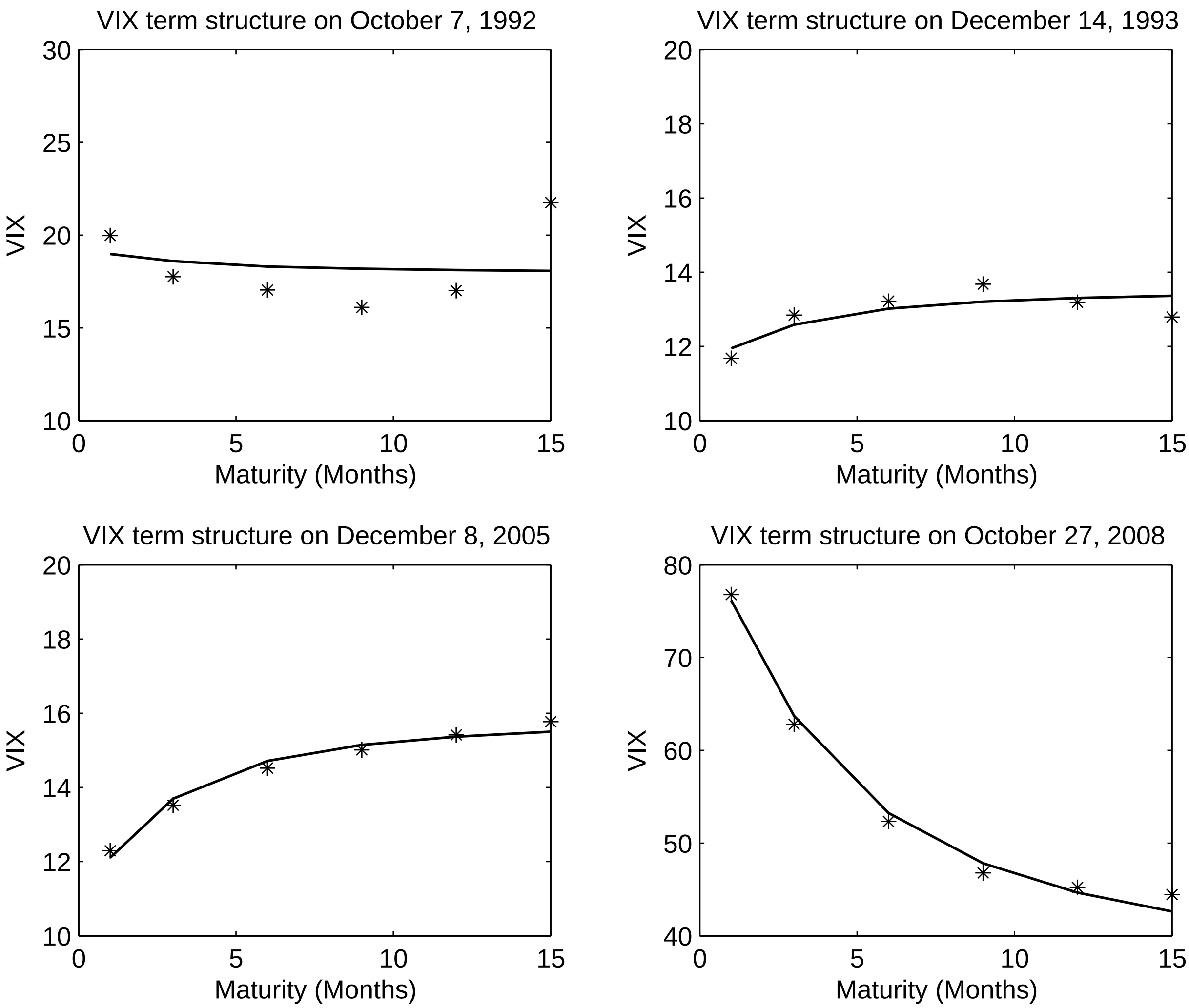

Figure 7: **Representative term structure shapes at different dates.** We plot some model-based (lines) and data-based (asterisks) representative term structure shapes at different dates. All volatilities are expressed in percentage terms.

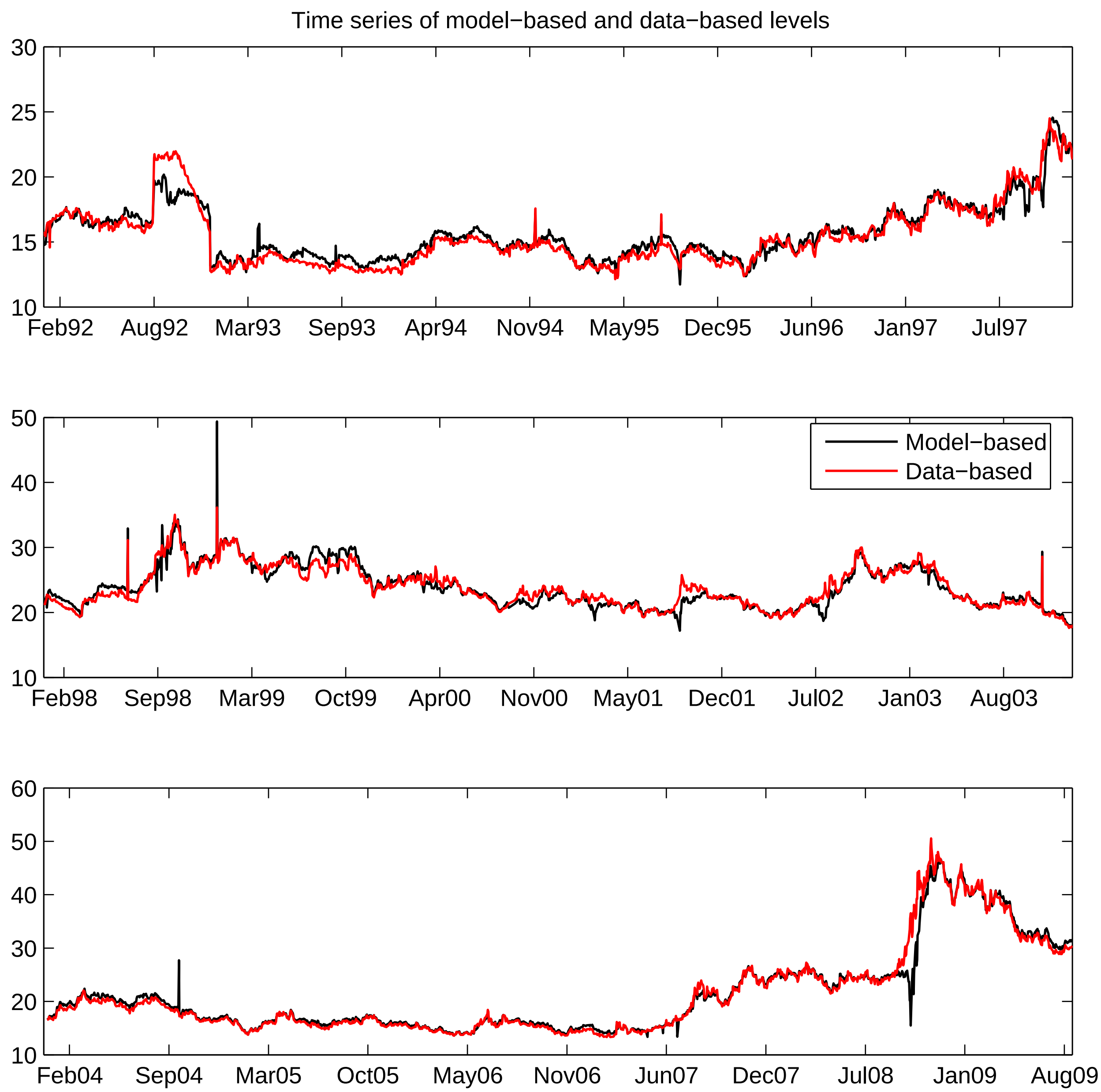

Figure 8: **Time series of model-based and data-based levels.** We show time series of model-based (black lines) and data-based (red lines) VIX term structure levels from January 2, 1992 to August 31, 2009. We define the data-based level as the 15-month VIX, and the model-based level as the estimated long term mean volatility, that is $\sqrt{\theta_t}$. All volatilities are expressed in percentage terms.

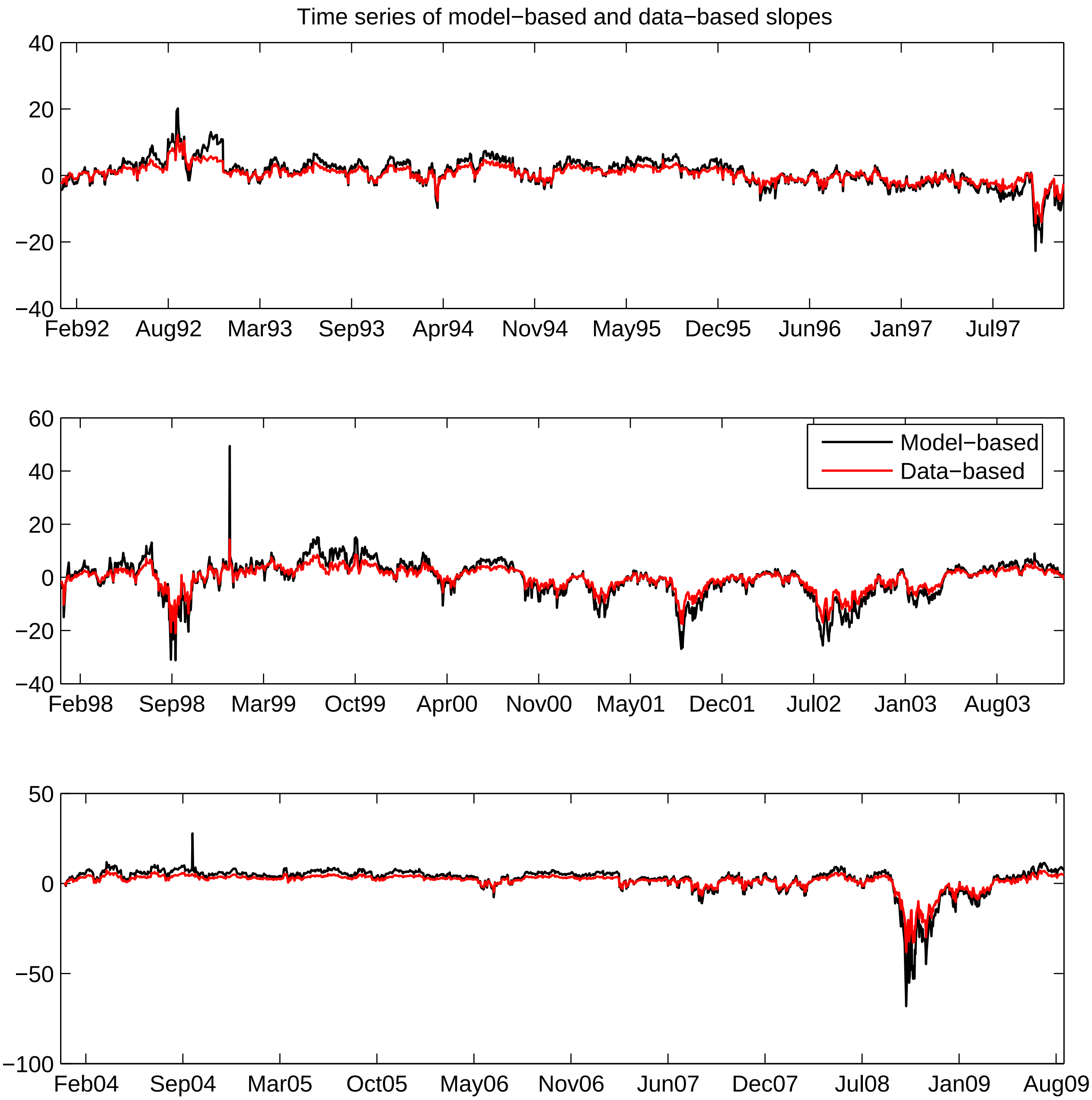

Figure 9: **Time series of model-based and data-based slopes.** We show time series of model-based (black lines) and data-based (red lines) VIX term structure slopes from January 2, 1992 to August 31, 2009. We define the market-based slope as the difference between the 15-month and the 1-month VIXs, and the data-based slope as the difference between the estimated long term mean and the instantaneous volatility, that is $\sqrt{\theta_t} - \sqrt{V_t}$.