LMI Conditions for Time-Varying Uncertain Systems Can Be Non-Conservative

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Abstract

Establishing robust asymptotical stability of uncertain systems affected by time-varying uncertainty is a key problem. LMI sufficient conditions have been proposed in the literature for addressing this problem based on homogeneous polynomial Lyapunov functions. Unfortunately, till now it has been unclear whether these conditions are also necessary. This paper proposes a proof in order to show that one of these conditions is not only sufficient but also necessary for a sufficiently large degree of the Lyapunov function.

Keywords: Uncertain systems; Time-varying; Robustness; LMI.

1 Introduction

It is well-known that polytopic systems, i.e. uncertain linear systems affected linearly by an uncertain vector constrained in a polytope, are a fundamental area of automatic control. As such, numerous contributions have been proposed in the past in order to address various problems, first of all the essential problem of ensuring asymptotical stability for all admissible uncertainties. These contributions have mainly considered two cases depending on the relation between uncertainty and time.

One case considers time-invariant uncertainty, and amounts to establishing if a polytope of matrices contains only stable matrices. Most methods proposed to address this problem make use of linear matrix inequalities (LMIs), e.g. looking for a common quadratic Lyapunov function (see e.g. [5]), or looking for a parameter-dependent quadratic Lyapunov function with

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various types of dependencies on the uncertainty such as linear (see e.g. [15]), polynomial (see e.g. [4, 18]), and homogeneous polynomial (see e.g. [16, 12]). It is worth mentioning also non-LMI methods such as [14] which proposes a branch-and-bound technique.

Another case considers time-varying uncertainty, and amounts to establishing if the origin of the system is asymptotically stable for all possible functions of the uncertainty with respect to the time provided that they remain confined in a given polytope. For this problem it has been proposed the use of Lyapunov functions that are quadratic (see e.g. [5]), piecewise quadratic (see e.g. [19, 1]), polyhedral (see e.g. [3]), and homogeneous polynomial (see e.g. [6]). In particular, for the class of homogeneous polynomial Lyapunov functions, sufficient conditions based on LMIs have been proposed, firstly in [20] and successively improved in [11], which have the benefit of requiring the solution of a convex optimization problem. It is worth observing that homogeneous polynomial Lyapunov functions have been exploited also in the case of uncertain systems with rational dependence on the uncertainty, e.g. [9].

While non-conservative conditions have been obtained in terms of LMIs for the case of time-invariant uncertainty (see e.g. [7, 10]), till now it has been unclear whether and how sufficient conditions that are also necessary can be obtained for the case of time-varying uncertainty through LMIs. This paper provides an answer to this question, in particular showing that the LMI condition proposed in [11, 12] through the use of homogeneous polynomial Lyapunov functions and the square matrix representation (SMR), is not only sufficient, but also necessary for a sufficiently large degree of the function. This proof is derived by showing that, if the system is robustly asymptotically stable, then it is possible to construct a homogeneous polynomial Lyapunov function that satisfies the LMI condition in [11, 12] by exploiting recent results in the theory of positive polynomials and Hilbert's 17th problem.

The organization of the paper is as follows. Section 2 introduces some preliminaries. Section 3 derives the proposed result. Lastly, Section 4 concludes the paper with some final remarks.

2 Preliminaries

The notation is as follows: 0_n: null vector $n \times 1$; I_n: identity matrix $n \times n$; $\mathbb{N}, \mathbb{R}$: natural and real number spaces; $\mathbb{R}_0^n$: $\mathbb{R}^n \setminus \{0_n\}$; A': transpose of matrix A; $A > 0$ (resp., $A \geq 0$): symmetric positive definite (resp., semidefinite) matrix A; $A \otimes B$: Kronecker's product of matrices A and B; $\mathrm{conv}(a, b, \ldots)$: convex hull of vectors $a, b, \ldots$; $\|a\|_p$: p-norm of vector a, i.e. $\|a\|_p = (|a_1|^p + |a_2|^p + ...)^{1/p}$.

Let us consider the time-varying uncertain system

$$\dot{x}(t) = A(u(t))x(t) \quad \forall t \geq 0 \tag{1}$$

where $t \in \mathbb{R}$ is the time, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^q$ is the time-varying uncertain parameter, and $A : \mathbb{R}^q \times \mathbb{R}^{n \times n}$ is an affine linear function expressed as

$$A(u) = A_0 + \sum_{i=1}^{q} u_i A_i \tag{2}$$

for some matrices $A_1, \ldots, A_q \in \mathbb{R}^{n \times n}$. It is supposed that $u(t)$ satisfies the constraint

$$u(t) \in \mathcal{U} \quad \forall t \geq 0 \tag{3}$$

where $\mathcal{U}$ is the bounded convex polytope defined as

$$\mathcal{U} = \mathrm{conv}\left\{u^{(1)}, \ldots, u^{(r)}\right\} \tag{4}$$

for some vectors $u^{(1)}, \ldots, u^{(r)} \in \mathbb{R}^q$. Moreover, it is supposed that $u(t)$ is regular enough to guarantee that the solution $x(t)$ of the differential equation (1) exists.

The problem here considered is to establish whether the system (1)–(4) is robustly asymptotically stable, i.e.

$$\left\{ \begin{array}{l} \forall \varepsilon > 0 \ \exists \delta > 0 : \ \|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \\ \qquad \forall t \geq 0 \ \forall u(\cdot) \in \mathcal{U} \\ \lim\limits_{t \to \infty} x(t) = 0_n \ \forall x(0) \in \mathbb{R}^n \ \forall u(\cdot) \in \mathcal{U}. \end{array} \right. \tag{5}$$

Before proceeding let us summarize the square matrix representation (SMR), also known as Gram matrix method. Specifically, let $f(x)$ be a homogeneous polynomial of degree $2m$ in $x \in \mathbb{R}^n$, and let $x^{\{m\}} \in \mathbb{R}^{\sigma(n,m)}$ be a vector containing all monomials of degree equal to m in x, where $\sigma(n, m)$ is the number of these monomials given by

$$\sigma(n, m) = \frac{(n + m - 1)!}{(n - 1)!m!}. \tag{6}$$

Then, $f(x)$ can be written according to the SMR as

$$f(x) = x^{\{m\}'} \left(F + L(\alpha)\right) x^{\{m\}} \tag{7}$$

where $F = F' \in \mathbb{R}^{\sigma(n,m)\times\sigma(n,m)}$ is such that

$$f(x) = x^{\{m\}'} F x^{\{m\}}, \tag{8}$$

$L : \mathbb{R}^{\tau(n,m)} \to \mathbb{R}^{\sigma(n,m)\times\sigma(n,m)}$ is a linear parametrization of the linear subspace

$$\mathcal{L}(n,m) = \left\{ L = L' : \ x^{\{m\}'} L x^{\{m\}} = 0 \ \forall x \in \mathbb{R}^n \right\} \tag{9}$$

and $\alpha \in \mathbb{R}^{\tau(n,m)}$ is a free vector, where $\tau(n,m)$ is the dimension of $\mathcal{L}(n,m)$ given by

$$\tau(n,m) = \frac{1}{2}\sigma(n,m)\left(\sigma(n,m)+1\right) - \sigma(n,2m). \tag{10}$$

The SMR allows one to establish whether a homogeneous polynomial $f(x)$ is a sum of squares of polynomials (SOS), i.e. $f(x) = \sum_{i=1}^{k} f_i(x)^2$ for some polynomials $f_1(x), \ldots, f_k(x)$. Indeed, $f(x)$ is SOS if and only if there exists $\alpha \in \mathbb{R}^{\tau(n,m)}$ satisfying the following LMI introduced in [13]:

$$F + L(\alpha) \geq 0. \tag{11}$$

The reader is referred to [12, 8] for further details and for algorithms about the SMR and SOS polynomials.

3 Non-Conservative LMI Condition

For $m \in \mathbb{N}$, let $B_i \in \mathbb{R}^{\sigma(n,m)\times\sigma(n,m)}$ be defined as

$$\nabla x^{\{m\}} A\left(u^{(i)}\right) x = B_i x^{\{m\}}, \tag{12}$$

which can be computed with the formula given in [12].

Let us start by recalling the following result, which provides a sufficient condition for establishing whether the system (1)–(4) is robustly asymptotically stable in terms of an LMI feasibility test built by exploiting homogeneous polynomial Lyapunov functions and the SMR.

Theorem 1 ([11, 12]) *Let $m \in \mathbb{N}$ be given. Suppose that there exist a matrix $V = V' \in \mathbb{R}^{\sigma(n,m)\times\sigma(n,m)}$ and vectors $\alpha^{(1)}, \ldots, \alpha^{(r)} \in \mathbb{R}^{\tau}(n,m)$ such that the following LMIs hold:*

$$\left\{ \begin{array}{ccl} 0 & < & V \\ 0 & > & VB_i + B_i'V + L(\alpha^{(i)}) \ \ \forall i = 1, \ldots, r. \end{array} \right. \tag{13}$$

Then, the system (1)–(4) is robustly asymptotically stable.

The following result explains that the condition provided in Theorem 1 is not only sufficient but also necessary.

Theorem 2 *Suppose that the system* (1)–(4) *is robustly asymptotically stable. Then, there exists a sufficiently large* m *such that the LMIs in* (13) *hold for some matrix* $V = V' \in \mathbb{R}^{\sigma(n,m)\times\sigma(n,m)}$ *and vectors* $\alpha^{(1)}, \ldots, \alpha^{(r)} \in \mathbb{R}^{\tau(n,m)}$.

Proof. Suppose that (5) holds. Then, from Theorem 3.2 in [2] one has that there exist $p, s \in \mathbb{N}$ and $W \in \mathbb{R}^{s\times n}$ such that the function

$$v(x) = \|Wx\|_{2p} \tag{14}$$

is a homogeneous Lyapunov function for (1), i.e. satisfies

$$\left\{ \begin{array}{lcl} 0 & < & v(x) \ \ \forall x \in \mathbb{R}^n_0 \\ 0 & < & d_i(x) \ \ \forall x \in \mathbb{R}^n_0 \ \forall i = 1, \ldots, r \end{array} \right. \tag{15}$$

where

$$d_i(x) = -\nabla v(x) A\left(u^{(i)}\right) x. \tag{16}$$

In fact, observe that since $A(u)$ is affine linear in u, it is sufficient to establish that the derivative of $v(x)$ is negative definite at the vertices of $\mathcal{U}$ in (15), moreover $v(x)$ is homogeneous since it is a norm and hence satisfies $v(\gamma x) = \gamma v(x)$ for all $\gamma \geq 0$.

First, let us observe that the function $\bar{v}(x)$ defined as

$$\bar{v}(x) = v(x)^{2p} \tag{17}$$

is a Lyapunov function for (1). In fact, $\bar{v}(x)$ is positive definite, moreover the time derivative of $-\bar{v}(x)$ for $u = u^{(i)}$, denoted by $\bar{d}_i(x)$, is positive definite as well being given by $\bar{d}_i(x) = (2p)v(x)^{2p-1}d_i(x)$. In addition, one has that $\bar{v}(x)$ can be written as

$$\bar{v}(x) = \sum_{i=1}^{s} \left((w_i' x)^p\right)^2, \tag{18}$$

i.e. $\bar{v}(x)$ is a SOS homogeneous polynomial. Let us observe, however, that the homogeneous polynomials $\bar{d}_i(x)$ are not guaranteed to be SOS, though they are positive definite.

Now, as stated in Section 2.1.8 of [17], if a homogeneous polynomial $f(x)$ of degree $2a$ is positive definite, then

$$\begin{array}{l} \forall g(x), \text{ with } g(x) \text{ SOS homogeneous polynomial,} \\ \text{positive definite and of degree } 2b \text{ such that } a/b \in \mathbb{N}, \\ \exists\, k \in \mathbb{N}: \;\; f(x)g(x)^k \text{ is SOS.} \end{array} \tag{19}$$

Let us observe that (19) can be viewed as an extension of Artin's result which states that a homogeneous polynomial is positive semidefinite if and only if it is the ratio of two SOS polynomial: indeed, if a polynomial is positive definite, then the denominator of this representation can be an arbitrary positive definite SOS polynomial (of suitable degree) according to (19).

Hence, let us define the new function

$$\hat{v}(x) = \bar{v}(x)^k \tag{20}$$

where $k \in \mathbb{N}$ has to be selected. One has that $\hat{v}(x)$ is positive definite, moreover the time derivative of $-\hat{v}(x)$ for $u = u^{(i)}$, denoted by $\hat{d}_i(x)$, is positive definite being given by $\hat{d}_i(x) = k\bar{v}(x)^{k-1}\bar{d}_i(x)$. Also, $\hat{v}_i(x)$ is a SOS homogeneous polynomial. Moreover, by setting $f(x) = \bar{d}_i(x)$ and $g(x) = \bar{v}(x)$, it follows from (19) that there exists k, denoted by k_i, such that $\hat{d}_i(x)$ is a SOS homogeneous polynomial as well (in this case, (19) holds with $b = 1$ since the degrees of $\bar{d}_i(x)$ and $\bar{v}(x)$ are equal).

Therefore, let us select k as

$$k = \max_{i=1,\ldots,r} k_i \tag{21}$$

and let us express $\hat{v}(x)$ and $\hat{d}_i(x)$ through the SMR as

$$\begin{array}{rcl} \hat{v}(x) & = & x^{\{kp\}'}\hat{V}x^{\{kp\}} \\ \hat{d}_i(x) & = & x^{\{kp\}'}\hat{D}_i x^{\{kp\}}. \end{array} \tag{22}$$

Since $\hat{v}(x)$ and $\hat{d}_i(x)$ are SOS, it follows that $\hat{V}$ and $\hat{D}_i$ can be chosen positive semidefinite from Section 2.

Then, in order to prove that $\hat{V}$ can be chosen positive definite, let us observe that, since $v(x)$ is positive definite, the function $\bar{v}(x)$ can be expressed as

$$\bar{v}(x) = \left\| \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} x \right\|_{2p}^{2p} \tag{23}$$

for some matrices $W_1 \in \mathbb{R}^{n\times n}$ and $W_2 \in \mathbb{R}^{(s-n)\times n}$ with W_1 non-singular. This implies that $\bar{v}(x) \geq \bar{v}_1(x)$ where

$$\bar{v}_1(x) = \|W_1 x\|_{2p}^{2p}. \tag{24}$$

Then, let us observe that

$$\bar{v}_1(W_1^{-1}x) = \sum_{i=1}^{n} x_i^{2p} \tag{25}$$

which can be expressed as $\bar{v}_1(W_1^{-1}x) = x^{\{p\}'} H x^{\{p\}}$ for some $H > 0$. This implies that $\bar{v}_1(x) = x^{\{p\}'} \bar{V}_1 x^{\{p\}}$ with $\bar{V}_1 = J' H J$ and $J = (K_p' K_p)^{-1} K_p' W_1^{-\otimes p} K_p$, where K_p is the matrix satisfying

$$x^{\otimes p} = K_p x^{\{p\}}. \tag{26}$$

Since K_p has full column rank, it follows that $\bar{V}_1 > 0$, which implies that $\bar{v}(x)$ can be expressed as $\bar{v}(x) = x^{\{p\}'} \bar{V} x^{\{p\}}$ for some $\bar{V} > 0$. Hence, one has that $\hat{V}$ can be chosen as

$$\hat{V} = K_{kp}' \bar{V}^{\otimes k} K_{kp} \tag{27}$$

which is positive definite since $\bar{V} > 0$ and K_{kp} has full column rank.

Lastly, consider $\hat{D}_i$. Since $\hat{d}_i(x) = k\bar{v}(x)^{k-1}\bar{d}_i(x)$ is positive definite, it follows that there exists $\varepsilon > 0$ such that the homogeneous polynomial defined as

$$y_i(x) = \hat{d}_i(x) - \varepsilon \|x\|^{2kp} \tag{28}$$

is positive definite. From (19) there exists l_i such that $z_i = y_i(x)\bar{v}(x)^{l_i}$ is a SOS homogeneous polynomial, i.e. it can be expressed as $z_i(x) = x^{\{(k+l_i)p\}'} Z_i x^{\{(k+l_i)p\}}$ for some $Z_i > 0$. This implies that one can write $\hat{d}_i(x)\bar{v}(x)^{l_i} = x^{\{(k+l_i)p\}'} E_i x^{\{(k+l_i)p\}}$ where E_i satisfies

$$E_i = Z_i + \varepsilon N'(M \otimes \bar{V})N \tag{29}$$

where $M > 0$ is such that $\|x\|^{2kp} = x^{\{kp\}'} M x^{\{kp\}}$ and N is such that

$$x^{\{kp\}} \otimes x^{\{l_i p\}} = N x^{\{(k+l_i)p\}}. \tag{30}$$

Since $Z_i \geq 0$, ε, $M > 0$, $\bar{V} > 0$, and N has full column rank, it follows that $E_i > 0$. Hence, by redefining k as $k + l$ where

$$l = \max_{i=1,\ldots,r} l_i, \tag{31}$$

it follows that $\hat{D}_i$ can be chosen as E_i.

Therefore, (13) holds by selecting $m = kp$, $V = \hat{V}$, and $\alpha^{(i)}$ such that $-\hat{D}_i = VB_i + B_i'V + L(\alpha^{(i)})$ (such $\alpha^{(i)}$ exist since $-\hat{D}_i$ and $VB_i + B_i'V$ represent the same homogeneous polynomial). $\square$

Theorem 2 is important as it states that robust stability is equivalent to the existence of a homogeneous polynomial Lyapunov function that can be computed via a convex optimization problem with LMIs. This extends previous works in this area, in particular [2] where it is proved that robust stability is equivalent to the existence of a polyhedral Lyapunov function.

Let us observe that it would be useful to provide a bound on the value of m for which the necessity is achieved in Theorem 2, however this bound depends on how the functions $d_i(x)$ in (15) are positive definite, i.e. on a sort of robust stability margin, which is unknown (if known, then one knows already whether the system is robustly stable).

Lastly, it is useful to observe that, while robust stability analysis via homogeneous polynomial Lyapunov functions amounts to solving convex optimization problems with LMIs as stated in Theorems 1 and 2, robust control synthesis via such Lyapunov functions requires the solution of bilinear matrix inequalities (BMIs) which lead to nonconvex optimization problems.

4 Conclusion

The question whether LMI conditions can be non-conservative for establishing robust asymptotical stability of uncertain systems affected by time-varying uncertainty has remained unanswered till now. This paper has finally provided a proof that an existing LMI condition based on homogeneous polynomial Lyapunov functions is not only sufficient but also necessary for a sufficiently large degree of such a function.

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