# On holomorphic isometric embeddings of the unit $n$-ball into products of two unit $m$-balls

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**Abstract**

We study holomorphic isometric embeddings of the complex unit $n$-ball into products of two complex unit $m$-balls with respect to their Bergman metrics up to normalization constants (the isometric constant). There are two trivial holomorphic isometric embeddings for $m \geq n$, given by $F_1(\mathbf{z}) = (\mathbf{0}, I_{n;m}(\mathbf{z}))$ with the isometric constant equal to $(m+1)/(n+1)$ and $F_2(\mathbf{z}) = (I_{n;m}(\mathbf{z}), I_{n;m}(\mathbf{z}))$ with the isometric constant equal to $2(m+1)/(n+1)$. Here $I_{n;m} : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ is the canonical embedding. We prove that when $m < 2n$, these are the only holomorphic isometric embeddings up to unitary transformations.

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## 1 Introduction

Let $\Omega$ be an irreducible bounded symmetric domain equipped with its Bergman metric $ds^2_\Omega$. In relation to a problem in number theory, Clozel and Ullmo [1] studied the holomorphic isometric embeddings of $\Omega$ into its Cartesian products $\Omega^p$ up to normalizing constants, in which $\Omega^p$ is equipped with the product metric. By using the arguments in Hermitian metric rigidity (see Mok [2, 3]), they argued in their article that when $\text{rank}(\Omega) \geq 2$, any such embedding must be totally geodesic. When $\text{rank}(\Omega) = 1$, i.e. when $\Omega = \mathbb{B}^n$, the complex unit balls, Mok [4] showed that for $n \geq 2$, the embeddings must also be totally geodesic. While for dimension 1, he has constructed a non-totally geodesic holomorphic isometric embedding of the unit disk $\Delta$ into $\Delta^p$ for every $p \geq 2$. (see [5])

Let $m, n \geq 2$ be two integers. In this article, we consider holomorphic isometric embeddings of $\mathbb{B}^n$ into $\mathbb{B}^m \times \mathbb{B}^m$ up to normalization constants with respect to their Bergman metrics $ds^2_{\mathbb{B}^n}$ and $ds^2_{\mathbb{B}^m\times\mathbb{B}^m}$. More precisely, for a positive real number $\lambda$, $F : \mathbb{B}^n \longrightarrow \mathbb{B}^m \times \mathbb{B}^m$ is said to be a holomorphic isometric embedding with the isometric constant $\lambda$ if $F : (\mathbb{B}^n, \lambda ds^2_{\mathbb{B}^n}) \longrightarrow (\mathbb{B}^m \times \mathbb{B}^m, ds^2_{\mathbb{B}^m\times\mathbb{B}^m})$ is a holomorphic isometric embedding. If $m \geq n$ and $I_{n;m} : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ is the canonical embedding, then $F_1(\mathbf{z}) = (\mathbf{0}, I_{n;m}(\mathbf{z}))$ and $F_2(\mathbf{z}) = (I_{n;m}(\mathbf{z}), I_{n;m}(\mathbf{z}))$ are two holomorphic isometric embeddings with the isometric constant equal to $(m+1)/(n+1)$ and

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$2(m+1)/(n+1)$ respectively. The main purpose of this paper is to prove that for $m < 2n$, they are the only holomorphic isometric embeddings up to unitary transformations.

**Main theorem** *Let $m, n$ be positive integers with $m, n \geq 2$ and $m < 2n$. Let $F : \mathbb{B}^n \longrightarrow \mathbb{B}^m \times \mathbb{B}^m$ be a holomorphic isometric embedding with the isometric constant $\lambda$. Then $m \geq n$ and up to unitary transformations, either $F(\mathbf{z}) = (\mathbf{0}, I_{n;m}(\mathbf{z}))$ with $\lambda = (m+1)/(n+1)$, or $F(\mathbf{z}) = (I_{n;m}(\mathbf{z}), I_{n;m}(\mathbf{z}))$ with $\lambda = 2(m+1)/(n+1)$, where $I_{n;m} : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ is the canonical embedding.*

## 2 Functional equation

Let $m, n \geq 2$ be two integers and $F : \mathbb{B}^n \longrightarrow \mathbb{B}^m \times \mathbb{B}^m$, $F(\mathbf{z}) = (A(\mathbf{z}), B(\mathbf{z}))$ be a holomorphic isometric embedding with the isometric constant $\lambda$. Without loss of generality, we may assume that $F(\mathbf{0}) = (\mathbf{0}, \mathbf{0})$. The Bergman metric on $\mathbb{B}^n$ is given by $ds^2_{\mathbb{B}^n} = 2\mathrm{Re} \sum g_{i\bar{j}} dz^i \otimes d\bar{z}^j$, where $g_{i\bar{j}} = -(n+1) \dfrac{\partial^2}{\partial z_i \partial \bar{z}_j} \log(1 - \|\mathbf{z}\|^2)$. We write $(\mathbf{z}_1, \mathbf{z}_2)$ for a point in $\mathbb{B}^m \times \mathbb{B}^m$. We can take as Kähler potentials for $ds^2_{\mathbb{B}^n}$ and $ds^2_{\mathbb{B}^m \times \mathbb{B}^m}$ the real analytic functions $-(n+1)\log(1 - \|\mathbf{z}\|^2)$ and $-(m+1)\log(1 - \|\mathbf{z}_1\|^2)(1 - \|\mathbf{z}_2\|^2)$ respectively. By the assumption that $F^* ds^2_{\mathbb{B}^m \times \mathbb{B}^m} = \lambda ds^2_{\mathbb{B}^n}$ it follows that

$$-(m+1)\sqrt{-1}\partial\bar{\partial} \log(1 - \|A\|^2)(1 - \|B\|^2) = -\lambda(n+1)\sqrt{-1}\partial\bar{\partial} \log(1 - \|\mathbf{z}\|^2),$$

hence,

$$(m+1)\log(1 - \|A\|^2)(1 - \|B\|^2) = \lambda(n+1)\log(1 - \|\mathbf{z}\|^2) + \mathrm{Re}\, h$$

for some holomorphic function $h$ on $\mathbb{B}^n$. Since $F(\mathbf{0}) = (\mathbf{0}, \mathbf{0})$, by comparing Taylor expansions we conclude that $h \equiv 0$. Therefore we obtain

$$(m+1)\log(1 - \|A\|^2)(1 - \|B\|^2) = \lambda(n+1)\log(1 - \|\mathbf{z}\|^2). \tag{2.1}$$

i.e.

$$(1 - \|A\|^2)(1 - \|B\|^2) = (1 - \|\mathbf{z}\|^2)^{\lambda(n+1)/(m+1)}. \tag{2.2}$$

Eq.(2.2) is a real-analytic equation and we can consider an associated *polarized* functional equation. In general, given two power series $\sum a_{i\bar{j}} z^i \bar{z}^j$ and $\sum b_{i\bar{j}} z^i \bar{z}^j$, they are equal if and only if $a_{i\bar{j}} = b_{i\bar{j}}$, $\forall i, j$. Therefore their equality will also imply the polarized equation $\sum a_{i\bar{j}} z^i \bar{w}^j = \sum b_{i\bar{j}} z^i \bar{w}^j$. Since we can polarize each variable separately, the polarized equation of Eq.(2.1) is

$$(m+1)\log(1 - \langle A(\mathbf{z}), A(\mathbf{w})\rangle)(1 - \langle B(\mathbf{z}), B(\mathbf{w})\rangle) = \lambda(n+1)\log(1 - \langle \mathbf{z}, \mathbf{w}\rangle)$$

for $\|\mathbf{z}\|, \|\mathbf{w}\| < 1$. Here log denotes the principal branch of the logarithm and $\langle\,,\,\rangle$ is the complex Euclidean inner product. We can rewrite it as

$$(1 - \langle A(\mathbf{z}), A(\mathbf{w})\rangle)(1 - \langle B(\mathbf{z}), B(\mathbf{w})\rangle) = (1 - \langle \mathbf{z}, \mathbf{w}\rangle)^{\lambda(n+1)/(m+1)}, \tag{2.3}$$

where

$$(1 - \langle \mathbf{z}, \mathbf{w}\rangle)^{\lambda(n+1)/(m+1)} \equiv e^{[\lambda(n+1)/(m+1)]\log(1 - \langle \mathbf{z}, \mathbf{w}\rangle)}.$$

# 3 Algebraic extension

In [5], Mok has established the following extension result.

**Theorem 3.1** (Mok)**.** *Let $\Omega \Subset \mathbb{C}^n$ and $\Omega' \Subset \mathbb{C}^N$ be bounded symmetric domains in their Harish-Chandra realizations. Let $\lambda$ be any positive real number and $f : (\Omega, \lambda ds^2_{\Omega}) \longrightarrow (\Omega', ds^2_{\Omega'})$ be a germ of holomorphic isometry at $0 \in \Omega$ with $f(0) = 0$. Then, the germ of the graph of $f$ extends to an affine algebraic variety $S^{\#} \subset \mathbb{C}^n \times \mathbb{C}^N$ such that $S = S^{\#} \cap (\Omega \times \Omega')$ is the graph of a holomorphic isometric embedding $F : \Omega \longrightarrow \Omega'$ extending the germ of the holomorphic map $f$.*

From the existence of algebraic extension, we can prove

**Proposition 3.2.** *Let $(\mathbb{B}^n, \lambda ds^2_{\Delta}) \longrightarrow (\mathbb{B}^m \times \mathbb{B}^m, ds^2_{\mathbb{B}^m \times \mathbb{B}^m})$ be a holomorphic isometric embedding. Then $\dfrac{\lambda(n+1)}{(m+1)}$ is a positive integer.*

*Proof.* By Theorem 3.1, we know that the embedding can be extended across a general point on the unit sphere $\partial\mathbb{B}^n$. Let $\mathbf{z_0}$ be a point on $\partial\mathbb{B}^n$ at which the embedding can be extended across in a neighborhood. By unitary transformations, we may assume that $\mathbf{z_0} = (z_0, 0, \ldots, 0)$. Consider the restriction of $F$ on the disk $\Delta = \{(z, 0, \ldots, 0), |z| < 1\} \subset \mathbb{B}^n$, denote by $f(z) = (a(z), b(z))$, where $a(z)$, $b(z) \in \mathbb{B}^m$. Then by Eq.(2.3), $f(z)$ satisfies

$$(1 - \langle a(z), a(w)\rangle)\,(1 - \langle b(z), b(w)\rangle) = (1 - z\overline{w})^{\lambda(n+1)/(m+1)}. \tag{3.1}$$

If we consider Eq.(3.1) and substitute $w = z_0$, then because each factor on the L.H.S. can only vanish with an integral order at $z = z_0$ and therefore $\dfrac{\lambda(n+1)}{(m+1)}$ on the R.H.S. must be a positive integer. □

Write $k = \dfrac{\lambda(n+1)}{(m+1)}$. By Eq.(2.2) and Schwarz's lemma on holomorphic maps, we have $k \le 2$ and hence $k = 1, 2$. When $k = 2$, by Schwarz's lemma again, we must have $\|\mathbf{z}\| = \|A\| = \|B\|$ and therefore $m \ge n$ and up to unitary transformations, $A(\mathbf{z}) = B(\mathbf{z}) = I_{n;m}(\mathbf{z})$, where $I_{n;m} : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ is the canonical embedding. Thus, it remains to consider the case when $k = 1$, i.e. $\lambda = (m+1)/(n+1)$.

We first state a well known lemma of holomorphic maps.

**Lemma 3.3.** *Let $f : U \subset \mathbb{C}^n \longrightarrow \mathbb{C}^m, f = (f_1, \ldots, f_n)$ be a holomorphic map defined on some open set $U$ and write $\|f\|^2 = \sum_{i=1}^n |f_i|^2$. If $g : U \longrightarrow \mathbb{C}^m$ is another holomorphic map with $\|f\|^2 = \|g\|^2$, then there exists a unitary transformation $\mathbf{U}$ in $\mathbb{C}^m$ such that $\mathbf{U} \circ f = g$.*

Let $F : \mathbb{B}^n \longrightarrow \mathbb{B}^m \times \mathbb{B}^m$, $F(\mathbf{z}) = (A(\mathbf{z}), B(\mathbf{z}))$ be an isometric embedding with the isometric constant $\lambda = (m+1)/(n+1)$. Then the functional equation Eq.(2.2) satisfied by $F$ reduces to

$$(1 - \|A(\mathbf{z})\|^2)(1 - \|B(\mathbf{z})\|^2) = 1 - \|\mathbf{z}\|^2. \tag{3.2}$$

**Proposition 3.4.** *Let $V$ be the irreducible $n$-dimensional algebraic subvariety in $\mathbb{C}^n \times (\mathbb{C}^m)^2$ extending the graph of $F$ and $\pi$ be the projection map from $V$ to the first factor. There exists a proper algebraic subvariety $W \subset \mathbb{C}^n$ such that the restriction $\pi : V \setminus \pi^{-1}(W) \longrightarrow \mathbb{C}^n \setminus W$ is a finite unbranched covering map.*

*Proof.* From Eq.(3.2),

$$\|A\|^2 + \|B\|^2 = \|A\|^2\|B\|^2 + \|\mathbf{z}\|^2.$$

$$\iff \sum_{i=1}^{m} |a_i|^2 + \sum_{i=1}^{m} |b_i|^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} |a_i b_j|^2 + \sum_{i=1}^{n} |z_i|^2.$$

By Lemma 3.3, (because $m^2 + n > 2m$) there exists an $(m^2+n) \times (m^2+n)$ unitary matrix $\mathbf{U}$ such that

$$\begin{bmatrix} a_1 \\ \vdots \\ a_m \\ b_1 \\ \vdots \\ b_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{U} \begin{bmatrix} a_1 b_1 \\ \vdots \\ a_1 b_m \\ \vdots \\ a_m b_1 \\ \vdots \\ a_m b_m \\ z_1 \\ \vdots \\ z_n \end{bmatrix}. \tag{3.3}$$

Consider the first $2m$ equations above, they are

$$\begin{aligned} a_1 &= L_1^a(a_1b_1, \ldots, a_mb_m, z_1, \ldots, z_n); \\ \vdots &\quad \vdots \\ a_m &= L_m^a(a_1b_1, \ldots, a_mb_m, z_1, \ldots, z_n); \\ b_1 &= L_1^b(a_1b_1, \ldots, a_mb_m, z_1, \ldots, z_n); \\ \vdots &\quad \vdots \\ b_m &= L_m^b(a_1b_1, \ldots, a_mb_m, z_1, \ldots, z_n), \end{aligned}$$

where $L_i^a, L_j^b$ are some linear functions.

By applying the Implicit Function Theorem, we see that the algebraic subvariety defined by these $2m$ equations is smooth at the origin. Therefore $V$ is the irreducible component of this algebraic subvariety containing the origin. Let $\overline{V}$ be the closure of $V$ in $\mathbb{P}^n \times (\mathbb{P}^m)^2$. $\overline{V}$ is obtained by replacing the inhomogeneous coordinates of the algebraic equations defining $V$ by homogeneous coordinates and $\overline{V}$ is a proper analytic subvariety of $\mathbb{P}^n \times (\mathbb{P}^m)^2$.

The singular part of $\overline{V}$ is a proper analytic subvariety $S$ of $\overline{V}$. By Proper Mapping Theorem, $\pi(S)$ is a proper analytic subvariety of $\mathbb{P}^n$. Thus, when restricting on $\overline{V}' = \overline{V} \setminus \pi^{-1}(\pi(S))$, $\pi$ is a proper holomorphic map between complex manifolds and let us denote by $R$ the ramification locus of $\pi$. Let $\overline{R}$ be the closure of $R$ in $\overline{V}$. We are going to show that $\overline{R}$ is a proper analytic subvariety of $\overline{V}$. Take a point $v \in \overline{R}$ and let $U$ be a small coordinate open ball in $\mathbb{P}^n \times (\mathbb{P}^m)^2$ containing $v$ such that $\overline{V}$ is defined by $h_1 = \cdots = h_{2m} = 0$ for some holomorphic functions $h_j$, $1 \le j \le 2m$, in $U$. Let $(u_1, \ldots, u_{n+2m})$ be a coordinate system of $U$. Write $\pi = (p_1, \ldots, p_n)$, where $p_i$ are holomorphic in $U$. Then $R$ is defined by the equation $dp_1 \wedge \cdots \wedge dp_n|_{\overline{V}'} = 0$. Take $y$ be a point in $\overline{V}' \setminus R$. By doing a linear change of coordinates, we may assume that $\dfrac{\partial}{\partial u_j}$, $1 \le j \le n$ are tangent to $\overline{V}$ at the point $y$, and hence $\left(\dfrac{\partial p_i}{\partial u_j}\right)_{1 \le i,j \le n}$ is non-singular at $y$.

**Claim:** There exist holomorphic functions $f_1, \ldots, f_{2m}$ in $U$ such that for $1 \le k \le 2m$, $f_k|_{\overline{V}} = 0$ and $df_k(y) = du_{n+k}(y)$.

Let us assume the claim for the moment. Denote by $\mathcal{R}$ the analytic subvariety of $U$ defined by $dp_1 \wedge \cdots \wedge dp_n \wedge df_1 \wedge \cdots \wedge df_{2m} = 0$. $\mathcal{R}$ is a proper subvariety because it does not contain $y$ by our construction. Let $\widetilde{R} = \mathcal{R} \cap \overline{V}$. $\widetilde{R}$ is then a subvariety in $\overline{V} \cap U$ of codimension 1 and $\overline{R} \cap U \subset \widetilde{R}$ by our construction. $\widetilde{R}$ has only a finite number of irreducible components and let $\widetilde{R}_l$, $1 \le l \le q$ be those having non-empty intersections with $R \cap U$. Since both $R \cap U$ and $\widetilde{R}$ are divisors in $U$ and $(R \cap U) \subset \widetilde{R}$, we must have $\overline{R} \cap U = \bigcup_{l=1}^{q} \widetilde{R}_l$. Thus, $\overline{R}$ is an analytic subvariety of $\overline{V}$.

Now Proper Mapping Theorem says that $\pi(\overline{R})$ is an analytic subvariety of $\mathbb{P}^n$. If we let $\overline{W} = \pi(S) \cup \pi(\overline{R})$, then $\pi : \overline{V} \setminus \pi^{-1}(\overline{W}) \longrightarrow \mathbb{P}^n \setminus \overline{W}$ is a proper holomorphic covering map. It is finite because $\pi$ is proper and discrete on $\overline{V} \setminus \pi^{-1}(\overline{W})$. We can obtain the conclusion of the proposition by just restricting $\pi$ to the finite part of $\mathbb{P}^n \times (\mathbb{P}^m)^2$.

**Proof of the claim:** It is an extension problem with a prescribed first order derivative at $y$. We will use Cartan's Theorem B. Assume that the coordinates of $y$ are $u_1 = \cdots = u_{n+2m} = 0$. Let $\mathcal{O} = \mathcal{O}_U$ be the sheaf of holomorphic functions on $U$ and $\mathcal{I}$ the ideal sheaf in $\mathcal{O}$ generated by $h_j u_i$, $1 \le j \le 2m$, $1 \le i \le (n + 2m)$. $\mathcal{I}$ defines a coherent sheaf on the Stein manifold $U$ and $H^1(U, \mathcal{I}) = 0$ by Cartan's Theorem B. Thus, for the short exact sequence $0 \to \mathcal{I} \to \mathcal{O} \to \mathcal{O}/\mathcal{I} \to 0$, we have surjectivity for $H^0(U, \mathcal{O}) \longrightarrow H^0(U, \mathcal{O}/\mathcal{I})$ in the induced long exact sequence. Since $h_j u_i$ vanishes to the second order at the point $y$, an element on the stalk $\mathcal{O}/\mathcal{I}$ at $y$ corresponds to an equivalence class of germs of holomorphic functions in $U$, where $g_1, g_2 \in \mathcal{O}_{U;y}$ are equivalent if and only if $g_1|_{\overline{V}} = g_2|_{\overline{V}}$ and $dg_1(y) = dg_2(y)$. In any sufficiently small open neighborhood $\mathcal{W}_y$ of $y$ we can always construct for $1 \le k \le 2m$, a holomorphic function $f_{\mathcal{W}_y;k}$ in $\mathcal{W}_y$ vanishing on $\overline{V} \cap \mathcal{W}_y$ and $df_{\mathcal{W}_y;k}(y) = du_{n+k}(y)$. $f_{\mathcal{W}_y;k}$ induces a section of $\mathcal{O}/\mathcal{I}$ over $\mathcal{W}_y$ which is 0 except at $y$, thus defining a global section $s_k \in H^0(U, \mathcal{O}/\mathcal{I})$, where $s_k$ is taken to be 0 outside $\mathcal{W}_y$. Hence, the surjectivity above provides us the function $f_k$ on $U$ satisfying the desired properties in the claim. □

# 4 Total geodesy

Recall the notation in Proposition 3.4. Let $V$ be the irreducible algebraic subvariety extending the graph of $F$ and $W \subset \mathbb{C}^n$ be a proper algebraic subvariety such that if we let $Z = \mathbb{C}^n \setminus W$ and $X = V \setminus \pi^{-1}(W)$, then $\pi : X \longrightarrow Z$ is a finite unbranched covering map. We start with a lemma.

**Lemma 4.1.** *If a component function is degenerate everywhere in $\mathbb{B}^n$, i.e. the tangent map is not injective anywhere, then it is constant.*

*Proof.* Let $A$ be the component function degenerate everywhere. Consider $A$ as a multi-valued map on $Z$ and let $Y$ be the set of points $\mathbf{z} \in Z$ such that $\|A(\mathbf{z})\| = 1$ on some branch. Since the functional equation Eq.(3.2) is satisfied on the whole algebraic subvariety $V$, we see that $Y \subset Z \cap \partial\mathbb{B}^n$.

Define $Z' = Z \setminus Y$. We first argue that by the degeneracy of $A$, $Z'$ is connected. Suppose on the contrary $Z'$ is not connected. Because $Y \subset Z \cap \partial\mathbb{B}^n$ and $Y$ is closed in $Z$, $Z'$ is not connected only if $Y = Z \cap \partial\mathbb{B}^n$. Hence for every point $\mathbf{z_0} \in Z \cap \partial\mathbb{B}^n$, there is some branch of

$A$ on which we have $A(\mathbf{z_0}) = \mathbf{a_0}$ with $\|\mathbf{a_0}\| = 1$. Because $A$ is degenerate everywhere, for a generic choice of $\mathbf{z_0}$, the set defined by $A(\mathbf{z}) = \mathbf{a_0}$ contains a non-constant complex analytic curve $\Gamma : \Delta \longrightarrow \mathbb{C}^n$ with $\Gamma(0) = \mathbf{z_0}$. Note that for all open set $U \subset \Delta$, $\Gamma(U)$ cannot be completely contained in $\partial\mathbb{B}^n$ and from the functional equation we see that $\Gamma(U) \setminus \partial\mathbb{B}^n$ must be contained in $W$. This is true for arbitrary $U$ and this implies that $\mathbf{z_0} = \Gamma(0) \in W$. So $W$ contains almost every point of $\partial\mathbb{B}^n$ and hence the whole $\partial\mathbb{B}^n$ which is not possible.

We now show that the connectedness of $Z'$ implies that $A$ is constant. It is clear that $\pi^{-1}(Z') \subset X$ can only have a finite number of connected components, therefore each connected component is open in $X$ and when $\pi$ is restricted to any one connected component it is still a covering map over $Z'$. Since $Z'$ is connected, on each connected component we have either $\|A\| < 1$ or $\|A\| > 1$ on the whole component. We choose one with $\|A\| < 1$, of which the existence is guaranteed because we started with an isometric embedding germ $F$ of $\mathbb{B}^n$ into a product of unit balls. We can then form elementary symmetric functions of $A$ with respect to this covering map and they are bounded holomorphic functions on $Z'$. Since $W$ is a proper subvariety, we can extend them separately throughout the two domains $\mathbb{B}^n$ and $\mathbb{C}^n \setminus \overline{\mathbb{B}^n}$. As $n \geq 2$, the symmetric functions in $\mathbb{C}^n \setminus \overline{\mathbb{B}^n}$ can be extended to the whole $\mathbb{C}^n$ by Hartog's extension and the extension must agree with the symmetric functions originally defined on $\mathbb{B}^n$ as $Z'$ is connected. Hence, the symmetric functions are bounded holomorphic functions on $\mathbf{C}^n$ and therefore constant. This implies that $A$ is constant. $\square$

We can now prove the main theorem of this article.

*Proof.* (of the Main Theorem)

As explained after Proposition 3.2, it remains to prove the total geodesy of a holomorphic isometric embedding $F : \mathbb{B}^n \longrightarrow \mathbb{B}^m \times \mathbb{B}^m$, $F(\mathbf{z}) = (A(\mathbf{z}), B(\mathbf{z}))$ with the isometric constant $\lambda = (m+1)/(n+1)$.

If $m < n$, we certainly have degeneracy for both component functions and by Lemma 4.1 they are constant which is impossible. Therefore $m \geq n$.

By reducing the dimension of the target, we can always assume that the image of one of the component functions, say $B$, does not lie in any proper linear subspace of $\mathbb{B}^m$. If the other component $A$ is degenerate everywhere, then $A$ is constant by Lemma 4.1 and hence $A(\mathbf{z}) \equiv \mathbf{0}$. Therefore, up to unitary transformations, we have $F(\mathbf{z}) = (\mathbf{0}, I_{n;m}(\mathbf{z}))$, where $I_{n;m} : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ is the canonical embedding.

Now suppose $F$ is a holomorphic isometric embedding of $\mathbb{B}^n$ into $\mathbb{B}^m \times \mathbb{B}^m$ with $2n > m \geq n$, such that $A$ is non-degenerate at a generic point and the image of $B$ does not lie in any proper linear subspace of $\mathbb{C}^m$. We are going to show that it will lead to a contradiction.

Since the image $B$ do not lie in any proper linear subspace, in particular, it is non-constant and is non-degenerate at a generic point by Lemma 4.1. Therefore we may assume that both $A$ and $B$ are non-degenerate at the origin.

Denote the elements of the unitary matrix $\mathbf{U}$ in Eq.(3.3) by $u_{rs}$, $1 \leq r, s \leq (m^2+n)$. Since $a_i(0) = b_j(0) = 0$, $\forall i, j$ by assumption, if we consider the power series expansions of the last $(m^2+n-2m)$ equations in Eq.(3.3), we see that $u_{rs} = 0$ for $(2m+1) \leq r \leq (m^2+n)$ and $(m^2+1) \leq s \leq (m^2+n)$. Hence, if we let

$$\mathcal{X} = (a_1 b_1, \ldots, a_1 b_m, \ldots, a_m b_1, \ldots, a_m b_m) = (a_1 B, \ldots, a_m B) \tag{4.1}$$

be a $\mathbb{C}^{m^2}$-valued vector function, then the last $(m^2+n-2m)$ equations in Eq.(3.3) amounts to saying that there exist $(m^2+n-2m)$ constant orthonormal vectors $\{\mathcal{U}_j \in \mathbb{C}^{m^2} : 1 \leq j \leq (m^2+n-2m)\}$ such that

$$\mathcal{X} \perp Span\{\mathcal{U}_j\}.$$

If we let $X = Span\{\mathcal{U}_j\}^{\perp}$, then $Dim(X) = m^2 - (m^2+n-2m) = (2m-n)$ and $\forall \mathbf{z} \in \mathbb{B}^n$, $\mathcal{X}(\mathbf{z}) \in X$.

Let $\mathbf{u}$ be a vector in $\mathbb{C}^n$. If $\mathbf{u} = (u_1, \ldots, u_n)$, define the first directional derivative of a function $g$ along $\mathbf{u}$ as $\dfrac{\partial g}{\partial \mathbf{u}} := \displaystyle\sum_{i=1}^{n} u_i \frac{\partial g}{\partial z_i}$ and second directional derivative along $\mathbf{u}$ as $\dfrac{\partial^2 g}{\partial \mathbf{u}^2} := \dfrac{\partial}{\partial \mathbf{u}} \dfrac{\partial g}{\partial \mathbf{u}}$.

Now, the second directional derivative of $\mathcal{X}$ along $\mathbf{u}$ at $\mathbf{0}$ is

$$\frac{\partial^2 \mathcal{X}}{\partial \mathbf{u}^2}(\mathbf{0}) = \left( 2\frac{\partial a_1}{\partial \mathbf{u}}\frac{\partial B}{\partial \mathbf{u}}, \ldots, 2\frac{\partial a_m}{\partial \mathbf{u}}\frac{\partial B}{\partial \mathbf{u}} \right)\Bigg|_{\mathbf{z}=\mathbf{0}}.$$

By doing unitary transformations in the target, we can assume that the tangent space of the image of $A$ at the origin of $\mathbb{C}^m$ is the linear subspace defined by $z_{n+1} = z_{n+2} = \cdots = z_m = 0$. Therefore we can find $n$ tangent vectors $\mathbf{u_1}, \ldots, \mathbf{u_n}$ such that

$$\left( \frac{\partial a_1}{\partial \mathbf{u}_i}, \ldots, \frac{\partial a_m}{\partial \mathbf{u}_i} \right)\Bigg|_{\mathbf{z}=\mathbf{0}} = E_i, \quad 1 \leq i \leq n,$$

where $E_i$ are the standard unit vectors in $\mathbb{C}^m$. Then

$$\begin{array}{rcl} \dfrac{\partial^2 \mathcal{X}}{\partial \mathbf{u_1}^2}(\mathbf{0}) & = & ( \;\; 2\dfrac{\partial B}{\partial \mathbf{u_1}}(\mathbf{0}), \quad 0, \quad 0, \; \cdots \quad 0, \quad 0, \; \cdots \; 0, \;\; ) \\ \dfrac{\partial^2 \mathcal{X}}{\partial \mathbf{u_2}^2}(\mathbf{0}) & = & ( \quad 0, \quad 2\dfrac{\partial B}{\partial \mathbf{u_2}}(\mathbf{0}), \quad 0, \; \cdots \quad 0, \quad 0, \; \cdots \; 0, \;\; ) \\ \vdots & & \qquad\qquad\qquad \vdots \\ \dfrac{\partial^2 \mathcal{X}}{\partial \mathbf{u_n}^2}(\mathbf{0}) & = & ( \quad 0, \quad 0, \quad 0, \; \cdots \; 2\dfrac{\partial B}{\partial \mathbf{u_n}}(\mathbf{0}), \quad 0, \; \cdots \; 0, \;\; ) \end{array} \tag{4.2}$$

Note that for all $i$, $\dfrac{\partial^2 \mathcal{X}}{\partial \mathbf{u}_i^2}(\mathbf{0}) \in X$. They are linearly independent because $\forall i \; \dfrac{\partial B}{\partial \mathbf{u}_i} \neq 0$ for $B$ is non-degenerate at the origin. Since $Dim(X) = (2m-n)$, we can complete $\{\dfrac{\partial^2 \mathcal{X}}{\partial \mathbf{u}_1^2}(\mathbf{0}), \ldots, \dfrac{\partial^2 \mathcal{X}}{\partial \mathbf{u}_n^2}(\mathbf{0})\}$ to a basis of $X$ by adding certain $(2m-2n)$ vectors in $\mathbb{C}^{m^2}$, denoted by $\{\mathcal{P}_1, \ldots, \mathcal{P}_{2m-2n}\}$. For each $j$, write $\mathcal{P}_j = (P_j^1, \ldots, P_j^m)$, where $P_j^i \in \mathbb{C}^m$.

Since $\mathcal{X}(\mathbf{z}) \in X = Span\{\dfrac{\partial^2 \mathcal{X}}{\partial \mathbf{u}_1^2}(\mathbf{0}), \ldots, \dfrac{\partial^2 \mathcal{X}}{\partial \mathbf{u}_n^2}(\mathbf{0}), \mathcal{P}_1, \ldots, \mathcal{P}_{2m-2n}\}$, by Eq.(4.1) and Eq.(4.2), we see from considering the last $m$ coordinates that for $m = n$, $B(\mathbf{z}) \in Span\{\dfrac{\partial B}{\partial \mathbf{u}_n}(\mathbf{0})\}$ and for $m > n$, $B(\mathbf{z}) \in Span\{P_1^m, \ldots, P_{2m-2n}^m\}$. In the first case $(m = n)$, the image of $B$ lies in a subspace of dimension 1 while in the second case $(m > n)$ in a subspace of dimension $2m-2n$ which is less than $m$ because $m < 2n$ and therefore in both cases the image of $B$ lies in a proper linear subspace of $\mathbb{C}^m$ and this contradicts our initial assumption and the proof is complete. □

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