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Stability Analysis of Markovian Jump Systems with Multiple Delay Components and Polytopic Uncertainties

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Abstract This paper investigates the stability problem of Markovian jump systems with multiple delay components and polytopic uncertainties. A new Lyapunov–Krasovskii functional is used for the stability analysis of Markovian jump systems with or without polytopic uncertainties. Two numerical examples are provided to demonstrate the applicability of the proposed approach.

Keywords Delay · Markovian jump system · Stability · Polytopic uncertainty

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1 Introduction

Markovian jump linear systems are hybrid systems with many operation models and each model corresponds to a deterministic dynamic system. The switching amongst the system modes is governed by a Markov process. This class of systems is often used to model systems whose structures are subject to abrupt changes and their extensive applications have been applied to many physical systems such as manufacturing systems, aircraft control, target tracking, robotics, solar receiver control, power systems, and so on [2, 18]. Considerable attention has recently been devoted to the study of Markovian jump linear systems such as controllability [16, 17], stability [5, 10, 37], H_∞ control [2, 7, 19, 23, 28], H_2 control [9], H_∞ filtering [11, 22, 26, 36, 39], guaranteed cost control [8], and model reduction [30, 42].

As time delay is very common in many physical, industrial and engineering systems [1, 24, 25], it is often introduced to model these systems; for example, multiple delays are required to model the RLC circuits with delayed elements [29], networked control systems [14, 20] and nonlinear stochastic systems [1]. However, since dynamic systems with time delays are of infinite dimensions, the control of these systems is fairly complicated and the desired performance of the closed-loop systems is difficult to achieve. Therefore, the presence of time delays substantially complicates the analytical and theoretical aspects of control system design. A survey of recent results developed to analyze the stability of delay systems is given in [38], which includes delay-independent and delay-dependent results. Recently, a new simplified and more efficient stability criterion for linear continuous system with multiple components has been used in [13] to improve the delay-dependent stability condition. As a result, the study of Markovian jump linear systems with time delay has received attention of many researchers in topics of stability [40], H_∞ control [4, 31, 34], and filtering [12, 32, 33].

To accurately characterize the physical systems, description of the uncertainty in the system parameters is often needed. Two types of uncertainties, norm bounded uncertainties and polytopic ones, are commonly used [6]. For jump systems with norm bounded uncertainties, the stability and stabilization [5, 21, 35, 41], H_∞ control [15], and robustness H_∞ filtering [36] have been addressed. The robust stability and H_∞ control of Markovian jump systems with constant delay and polytopic uncertainties is studied in [4].

In this paper, we are concerned with the robust stability of continuous-time Markovian jump systems with mode-dependent multiple time delays and time-varying polytopic uncertainties. The rest of the paper is organized as follows. Section 2 describes the continuous-time Markovian jump systems with mode-dependent multiple time delays and polytopic uncertainties, and presents preliminary results. The main results are given in Sect. 3, which includes the stability and robust stability of continuous-time Markovian jump system with mode-dependent multiple time delays without polytopic uncertainties or with polytopic uncertainties, respectively. Two numerical examples are used to illustrate the main results in Sect. 4, which is followed by the conclusion in Sect. 5.

Notation Throughout this paper, the notation $X > Y$ for real symmetric matrices X and Y means that the matrix $X - Y$ is positive definite. $\mathbb{R}^n$ is the set of all column vectors with n real entries. M^T represents the transpose of the matrix M. Identity matrices are invariably denoted by I when their dimensions are obvious and otherwise denoted by I_n to represent an $n \times n$ identity matrix, while zero matrices are invariably denoted by 0. The notation $\mathrm{col}\{\cdot\}$ denotes a matrix column with blocks given by the matrices in $\{\cdot\}$. For a matrix $M \in \mathbb{R}^{n\times m}$ with rank r, the orthogonal complement is defined as a (possibly non-unique) $m \times (m-r)$ matrix such that $MM^{\perp} = 0$ and $M^{\perp T} M^{\perp} > 0$.

2 Problem Formulation

Given is a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of subsets of the sample space and $\mathcal{P}$ is the probability measure defined on $\mathcal{F}$. On this probability space $(\Omega, \mathcal{F}, \mathcal{P})$, we consider the following class of continuous-time Markovian jump systems with mode-dependent multiple time delays and polytopic uncertainties:

$$\Sigma_p: \quad \begin{aligned} &\dot{x}(t) = A(r_t)x(t) + A_d(r_t)x\left(t - \sum_{k=1}^{r} d_k(r_t)\right), \\ &x(t) = \varphi(t), \quad t \in [-\mu, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $\{r_t\}$ is a continuous-time homogeneous Markov process with right-continuous trajectories and taking values in a finite set $\tilde{N} = \{1, \dots, N\}$, and stationary transition probability matrix $\Pi \triangleq [\pi_{ij}]_{i,j\in\tilde{N}}$ is given by

$$\Pr\{r_{t+h} = j | r_t = i\} = \begin{cases} \pi_{ij}h + o(h) & i \neq j \\ 1 + \pi_{ii}h + o(h) & i = j \end{cases}$$

where $h > 0$, $\lim_{h\to 0}(o(h)/h) = 0$, and $\pi_{ij} \geq 0$, for $i \neq j$, is the transition rate from mode i at time t to mode j at time $t + h$, and

$$\pi_{ii} = -\sum_{j=1, j\neq i}^{N} \pi_{ij}.$$

The $d_k(r_t)$, $k = 1, \dots, r$, denote the time delay components in the state and are assumed to satisfy the following conditions:

$$0 < d_k(r_t) = d_{ik} \leq \bar{d}_{ik} < \infty, \quad r_t = i \in \tilde{N},$$

and

$$\mu = \max_{i\in\tilde{N}} \left\{ \sum_{k=1}^{r} \bar{d}_{ik} \right\}.$$

In (1), $\varphi(t)$ is a vector-valued initial continuous function defined on the interval $[-\mu, 0]$. When system Σ_d is in mode r_t, $A(r_t)$ and $A_d(r_t)$ are matrix functions of r_t.

For each possible value $r_t = i$, $i \in \tilde{N}$, the matrices associated with the ith mode are unknown and assumed to belong to a convex compact set of polytopic type,

$$\begin{bmatrix} A(i) & A_d(i) \end{bmatrix} \in U_p(i),$$

where

$$U_p(i) \triangleq \left\{ \sum_{l=1}^{q} \alpha_l(t) \begin{bmatrix} A_{il} & A_{dil} \end{bmatrix},\ \forall \alpha_l(t) \geq 0,\ \sum_{l=1}^{q} \alpha_l(t) = 1 \right\} \tag{2}$$

with A_{il} and A_{dil} being given matrices with appropriate dimensions.

Definition 1 [4] System Σ_p is said to be robust stochastically stable if for any $A(i)$ and $A_d(i) \in U_p(i)$, there exists a constant $U_0(\varphi(\cdot), r_0)$, which is dependent on the initial condition $(\varphi(\cdot), r_0)$ and satisfies $U_0(0, 0) = 0$, such that

$$\mathbb{E}\left[\int_0^{\infty} \left\| x\big(t, \varphi(\cdot), r_0\big) \right\|^2 dt \right] \leq U_0\big(\varphi(\cdot), r_0\big),$$

where $\mathbb{E}[\cdot]$ denotes the expectation and $x(t, \varphi(\cdot), r_0)$ denotes the solution of system Σ_p at time t under the initial conditions $\varphi(\cdot)$ and r_0.

Lemma 1 [13] *Let $Y \in \mathbb{R}^{n \times n}$ and the bidiagonal upper triangular block matrix be*

$$J_K(Y) = \begin{bmatrix} I_n & -Y & 0 & \cdots & 0 \\ 0 & I_n & -Y & \cdots & 0 \\ \cdots & \cdots & \ddots & \ddots & \cdots \\ 0 & 0 & 0 & \ddots & -Y \\ 0 & 0 & 0 & \cdots & I_n \end{bmatrix} \in \mathbb{R}^{Kn \times Kn}.$$

If $Z = [J_K(Y)\ \ S] \in \mathbb{R}^{Kn \times (Kn+m)}$, where $S = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_{K-1} \\ S_K \end{bmatrix} \in \mathbb{R}^{Kn \times m}$ with $S_i \in \mathbb{R}^{n \times n}$ $(i = 1, \ldots, k)$, then

$$Z^{\perp} = \operatorname{col}\left\{ -\sum_{i=1}^{K} Y^{i-1} S_i, -\sum_{i=2}^{K} Y^{i-2} S_i, \ldots, -S_K, I_m \right\}.$$

Lemma 2 [27] (Finsler's Lemma) *Consider real matrices B and M such that B has full row rank and $M = M^T$. The following statements are equivalent*:

1. *There exists a vector x such that $x^T M x < 0$ and $Bx = 0$;*
2. *There exists a scalar $l \in \mathbb{R}$ such that*

$$l B^T B - M > 0;$$

3. *The following condition holds*:

$$B^{\perp T} M B^{\perp} < 0.$$

3 Main Results

3.1 Stability

If $A(r_t)$ and $A_d(r_t)$ are known and do not have polytopic uncertainties, then system Σ_p becomes

$$\Sigma:\qquad x(t) = A(r_t)x(t) + A_d(r_t)x\left(t - \sum_{k=1}^{r} d_k(r_t)\right), \tag{3}$$

$$x(t) = \varphi(t), \quad t \in [-\mu, 0], \tag{4}$$

where for each possible value $r_t = i$, $i \in \tilde{N}$, hold $A_i = A(i)$ and $A_{di} = A_{di}(i)$. Define

$$\eta = \max_{i \in \tilde{N}}\{|\pi_{ii}|\}, \qquad \lambda_{ip} = \sum_{k=1}^{p} d_{ik}, \qquad \bar{\lambda}_{ip} = \sum_{k=1}^{p} \bar{d}_{ik}, \quad i \in \tilde{N},$$

with $\lambda_{i0} = 0$ and $\bar{\lambda}_{i0} = 0$, which leads to

$$\lambda_{ip} \le \bar{\lambda}_{ip} \le \mu, \quad i \in \tilde{N},\ l = 0, 1, \dots, r. \tag{5}$$

Then we have the following theorem by applying the methodology used in [13].

Theorem 1 *If there exist $P_i > 0$, $i \in \tilde{N}$, $Q_m > 0$, $m = 1, \dots, r$, $X > 0$ and $Y > 0$ such that*

$$Q_{m+1} - Q_m \le 0, \quad m = 1, \dots, r-1, \tag{6}$$

$$T^{\perp T}\begin{bmatrix} \Gamma_{i1} + \Gamma_{i2} & 0 & 0 \\ 0 & \Gamma_{i3} & 0 \\ 0 & 0 & \Gamma_{i4} \end{bmatrix} T^{\perp} < 0, \tag{7}$$

for each $i \in \tilde{N}$, then the system Σ is stochastically stable, where

$$\Gamma_{i1} = \begin{bmatrix} \begin{matrix}\sum_{j=1}^{N}\pi_{ij}P_j + X + (1+\eta\mu)Q_1 \\ +P_iA_i + A_i^TP_i\end{matrix} & 0 & \cdots & 0 & P_iA_{di} \\ 0 & Q_2 - Q_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & & & Q_r - Q_{r-1} & 0 \\ A_{di}^TP_i & 0 & & 0 & -Q_r \end{bmatrix} \in \mathbb{R}^{(r+1)n\times(r+1)n}, \tag{8}$$

$$\Gamma_{i2} = \begin{bmatrix} A_i^T & 0 & \cdots & 0 & A_{di}^T \end{bmatrix}\mu Y \begin{bmatrix} A_i \\ 0 \\ \vdots \\ 0 \\ A_{di} \end{bmatrix} \in \mathbb{R}^{(r+1)n\times(r+1)n}, \tag{9}$$

$$\Gamma_{i3} = \begin{bmatrix} -X & 0 & \cdots & 0 & 0 \\ 0 & -\bar{d}_{i1}^{-1}Y & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\bar{d}_{i(r-1)}^{-1}Y & 0 \\ 0 & 0 & \cdots & 0 & -(\mu - \lambda_{i(r-1)})^{-1}Y \end{bmatrix}$$

$$\in \mathbb{R}^{(r+1)n\times(r+1)n}, \tag{10}$$

$$\Gamma_{i4} = \begin{bmatrix} \pi_{ii}d_{i1}^{-1}Q_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \pi_{ii}d_{i(r-1)}^{-1}Q_{r-1} & 0 \\ 0 & \cdots & 0 & \pi_{ii}d_{ir}^{-1}Q_r \end{bmatrix} \in \mathbb{R}^{rn\times rn}, \tag{11}$$

$$T^{\perp} = \mathrm{col}\left\{-\sum_{i=1}^{r} S_i, -\sum_{i=2}^{r} S_i, \ldots, -S_r, I_{(2r+2)n}\right\} \in \mathbb{R}^{(3r+2)n\times 2(r+1)n}, \tag{12}$$

$$\begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_{r-1} \\ S_r \end{bmatrix} = \begin{bmatrix} 0 & -I_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & -I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -I_n & 0 \\ -I_n & 0 & 0 & \cdots & 0 & -I_n \end{bmatrix}.$$

Proof Note that $\{(x(t), r_t), t \geq 0\}$ is not a Markov process. From [4], we define a new process $\{(x_t, r_t), t \geq 0\}$ by

$$x_t(s) = x(t+s), \quad t - \mu \leq s \leq t,$$

which leads to conclusion that $\{(x_t, r_t), t \geq 0\}$ is a Markov process with initial state $(\varphi(\cdot), r_0)$. Define a new stochastic Lyapunov functional candidate of the system Σ as

$$V(x_t, r_t) = \sum_{p=1}^{5} V_p(x_t, r_t), \tag{13}$$

where

$$V_1(x_t, r_t) = x^T(t)P(r_t)x(t), \qquad V_2(x_t, r_t) = \int_{t-\mu}^{t} x^T(s)Xx(s)\,ds,$$

$$V_3(x_t, r_t) = \int_{-\mu}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Y\dot{x}(s)\,ds\,d\theta,$$

$$V_4(x_t, r_t) = \sum_{l=1}^{r}\int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}} x^T(s)Q_l x(s)\,ds,$$

$$V_5(x_t, r_t) = \eta\int_{-\mu}^{0}\int_{t+\theta}^{t} x^T(s)Q_1 x(s)\,ds\,d\theta.$$

Let $\mathcal{A}$ denote the weak infinitesimal generator of the random process $\{x_t, r_t\}$. Then, for each $r_t = i$, $i \in \tilde{N}$, it can be verified that

$$\mathcal{A}\big[V_1(x_t,i)\big] = x^T(t)\Bigg(\sum_{j=1}^{N}\pi_{ij}P_j\Bigg)x(t) + 2x^T(t)P_i\big(A_i x(t) + A_{di}x(t-\lambda_{ir})\big),$$

$$\mathcal{A}\big[V_2(x_t,i)\big] = x^T(t)Xx(t) - x^T(t-\mu)Xx(t-\mu).$$

By Jensen's integral inequality and $0 \le \lambda_{i(r-1)} \le \mu$, $i \in \tilde{N}$, we have

$$\begin{aligned}
\mathcal{A}\big[V_3(x_t,i)\big] &= \mu\dot{x}^T(t)Y\dot{x}(t) - \int_{t-\mu}^{t}\dot{x}^T(s)Y\dot{x}(s)\,ds \\
&= \mu\dot{x}^T(t)Y\dot{x}(t) - \int_{t-\mu}^{t-\lambda_{i(r-1)}}\dot{x}^T(s)Y\dot{x}(s)\,ds \\
&\quad - \sum_{l=1}^{r-1}\int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}}\dot{x}^T(s)Y\dot{x}(s)\,ds \\
&\le \mu\dot{x}^T(t)Y\dot{x}(t) - (\mu-\lambda_{i(r-1)})^{-1}\bigg(\int_{t-\mu}^{t-\lambda_{i(r-1)}}\dot{x}(s)\,ds\bigg)^T Y \\
&\quad \times \bigg(\int_{t-\mu}^{t-\lambda_{i(r-1)}}\dot{x}(s)\,ds\bigg) \\
&\quad - \sum_{l=1}^{r-1} d_{il}^{-1}\bigg(\int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}}\dot{x}(s)\,ds\bigg)^T Y\bigg(\int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}}\dot{x}(s)\,ds\bigg) \\
&\le \mu\dot{x}^T(t)Y\dot{x}(t) - (\mu-\lambda_{i(r-1)})^{-1}\bigg(\int_{t-\mu}^{t-\lambda_{i(l-1)}}\dot{x}(s)\,ds\bigg)^T Y \\
&\quad \times \bigg(\int_{t-\mu}^{t-\lambda_{i(l-1)}}\dot{x}(s)\,ds\bigg) \\
&\quad - \sum_{l=1}^{r-1} \bar{d}_{il}^{-1}\bigg(\int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}}\dot{x}(s)\,ds\bigg)^T Y\bigg(\int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}}\dot{x}(s)\,ds\bigg).
\end{aligned}$$

From the proof of Theorem 3.1 in [3] and

$$V_4(x_t,i) = \sum_{l=1}^{r}\bigg(\int_{t-\lambda_{il}}^{t}x^T(s)Q_l x(s)\,ds - \int_{t-\lambda_{i(l-1)}}^{t}x^T(s)Q_l x(s)\,ds\bigg),$$

we have

$$\mathbb{E}\big[V_4(x_{t+\Delta}, r_{t+\Delta})|(x_t, r_t = i)\big]$$

$$
\begin{aligned}
&= \sum_{j\neq i} \mathbb{E}\biggl[I_{(r_{t+\Delta}=j)} \biggl(\sum_{l=1}^{r} \biggl(\int_{t}^{t+\Delta} x^T(s) Q_l x(s)\, ds \\
&\quad + \int_{t+\Delta-\lambda_{jl}}^{t} x^T(s) Q_l x(s)\, ds \biggr) \biggr) \bigg| (x_t, r_t = i) \biggr] \\
&\quad + \mathbb{E}\biggl[I_{(r_{t+\Delta}=i)} \biggl(\sum_{l=1}^{r} \biggl(\int_{t+\Delta-\lambda_{il}}^{t+\Delta} x^T(s) Q_l x(s)\, ds \biggr) \biggr) \bigg| (x_t, r_t = i) \biggr] \\
&\quad - \sum_{j\neq i} \mathbb{E}\biggl[I_{(r_{t+\Delta}=j)} \biggl(\sum_{l=1}^{r} \biggl(\int_{t}^{t+\Delta} x^T(s) Q_l x(s)\, ds \\
&\quad + \int_{t+\Delta-\lambda_{j(l-1)}}^{t} x^T(s) Q_l x(s)\, ds \biggr) \biggr) \bigg| (x_t, r_t = i) \biggr] \\
&\quad - \mathbb{E}\biggl[I_{(r_{t+\Delta}=i)} \biggl(\sum_{l=1}^{r} \biggl(\int_{t+\Delta-\lambda_{i(l-1)}}^{t+\Delta} x^T(s) Q_l x(s)\, ds \biggr) \biggr) \bigg| (x_t, r_t = i) \biggr] \\
&= W_1 - W_2,
\end{aligned}
$$

where

$$
\begin{aligned}
W_1 &= \sum_{j\neq i} \mathbb{E}\biggl[I_{(r_{t+\Delta}=j)} \biggl(\sum_{l=1}^{r} \biggl(\int_{t}^{t+\Delta} x^T(s) Q_l x(s)\, ds \\
&\quad + \int_{t+\Delta-\lambda_{jl}}^{t} x^T(s) Q_l x(s)\, ds \biggr) \biggr) \bigg| (x_t, r_t = i) \biggr] \\
&\quad + \mathbb{E}\biggl[I_{(r_{t+\Delta}=i)} \biggl(\sum_{l=1}^{r} \biggl(\int_{t+\Delta-\lambda_{il}}^{t+\Delta} x^T(s) Q_l x(s)\, ds \biggr) \biggr) \bigg| (x_t, r_t = i) \biggr], \\
W_2 &= \sum_{j\neq i} \mathbb{E}\biggl[I_{(r_{t+\Delta}=j)} \sum_{l=1}^{r} \biggl(\int_{t}^{t+\Delta} x^T(s) Q_l x(s)\, ds \\
&\quad + \int_{t+\Delta-\lambda_{j(l-1)}}^{t} x^T(s) Q_l x(s)\, ds \biggr) \bigg| (x_t, r_t = i) \biggr] \\
&\quad + \mathbb{E}\biggl[I_{(r_{t+\Delta}=i)} \biggl(\sum_{l=1}^{r} \biggl(\int_{t+\Delta-\lambda_{i(l-1)}}^{t+\Delta} x^T(s) Q_l x(s)\, ds \biggr) \biggr) \bigg| (x_t, r_t = i) \biggr].
\end{aligned}
$$

The first part can be expressed as

$$
W_1 = \sum_{j\neq i} \mathbb{E}\biggl[I_{(r_{t+\Delta}=j)} \biggl(\sum_{l=1}^{r} \biggl(\int_{t}^{t+\Delta} x^T(s) Q_l x(s)\, ds
$$

$$
\begin{aligned}
&+\int_{t+\Delta-\lambda_{jl}}^{t} x^T(s)Q_l x(s)\,ds\Big)\Big)\Big|(x_t, r_t=i)\Big] \\
&+\mathbb{E}\Big[I_{(r_{t+\Delta}=i)}\Big(\sum_{l=1}^{r}\Big(\int_{t+\Delta-\lambda_{il}}^{t+\Delta} x^T(s)Q_l x(s)\,ds\Big)\Big)\Big|(x_t, r_t=i)\Big] \\
=\ &\mathcal{F}+\mathcal{FF}+\mathcal{FFF},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{F} &= \sum_{j\neq i}\mathbb{E}\Big[I_{(r_{t+\Delta}=j)}\Big(\sum_{l=1}^{r}\int_{t}^{t+\Delta} x^T(s)Q_l x(s)\,ds\Big)\Big|(x_t, r_t=i)\Big], \\
\mathcal{FF} &= \sum_{j\neq i}\mathbb{E}\Big[I_{(r_{t+\Delta}=j)}\Big(\sum_{l=1}^{r}\int_{t+\Delta-\lambda_{jl}}^{t} x^T(s)Q_l x(s)\,ds\Big)\Big|(x_t, r_t=i)\Big], \\
\mathcal{FFF} &= \mathbb{E}\Big[I_{(r_{t+\Delta}=i)}\Big(\sum_{l=1}^{r}\Big(\int_{t+\Delta-\lambda_{il}}^{t+\Delta} x^T(s)Q_l x(s)\,ds\Big)\Big)\Big|(x_t, r_t=i)\Big].
\end{aligned}
$$

Note that

$$
\begin{aligned}
&0\le\mathcal{F}\le o\big(\Delta^2\big), \\
\mathcal{FF} &= \sum_{j\neq i}P[r_{t+\Delta}=j|r_t=i]\sum_{l=1}^{r}\Big(\int_{t+\Delta-\lambda_{jl}}^{t} x^T(s)Q_l x(s)\,ds\Big) \\
&= \sum_{j\neq i}\big(\pi_{ij}\Delta+o(\Delta)\big)\sum_{l=1}^{r}\Big(\int_{t+\Delta-\lambda_{jl}}^{t} x^T(s)Q_l x(s)\,ds\Big), \\
\mathcal{FFF} &= P[r_{t+\Delta}=i|r_t=i]\times\mathbb{E}\Big[\sum_{l=1}^{r}\Big(\int_{t+\Delta-\lambda_{il}}^{t+\Delta} x^T(s)Q_l x(s)\,ds\Big)\Big|(x_t, r_t=i)\Big] \\
&= \big(1+\pi_{ii}\Delta+o(\Delta)\big)\sum_{l=1}^{r}\Big(\int_{t+\Delta-\lambda_{il}}^{t+\Delta} x^T(s)Q_l x(s)\,ds\Big).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
W_2 = &\sum_{j\neq i}\mathbb{E}\Big[I_{(r_{t+\Delta}=j)}\Big(\sum_{l=1}^{r}\Big(\int_{t}^{t+\Delta} x^T(s)Q_l x(s)\,ds\Big)\Big)\Big|(x_t, r_t=i)\Big] \\
&+\sum_{j\neq i}\big(\pi_{ij}\Delta+o(\Delta)\big)\sum_{l=1}^{r}\Big(\int_{t+\Delta-\lambda_{j(l-1)}}^{t} x^T(s)Q_l x(s)\,ds\Big) \\
&+\big(1+\pi_{ii}\Delta+o(\Delta)\big)\sum_{l=1}^{r}\Big(\int_{t+\Delta-\lambda_{j(l-1)}}^{t+\Delta} x^T(s)Q_l x(s)\,ds\Big).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\left(\frac{1}{\Delta}\right)\bigl\{\mathbb{E}\bigl[V_4(x_{t+\Delta},r_{t+\Delta})\big|(x_t,r_t=i)\bigr]-V_4(x_t,r_t=i)\bigr\}\\
&\quad=\left(\frac{1}{\Delta}\right)\left(\sum_{l=1}^{r}\left(\int_{t+\Delta-\lambda_{il}}^{t+\Delta}x^T(s)Q_lx(s)\,ds\right)-\sum_{l=1}^{r}\left(\int_{t-\lambda_{il}}^{t}x^T(s)Q_lx(s)\,ds\right)\right)\\
&\qquad+\pi_{ii}\sum_{l=1}^{r}\left(\int_{t+\Delta-\lambda_{il}}^{t+\Delta}x^T(s)Q_lx(s)\,ds\right)\\
&\qquad+\sum_{j\neq i}\pi_{ij}\sum_{l=1}^{r}\left(\int_{t+\Delta-\lambda_{jl}}^{t}x^T(s)Q_lx(s)\,ds\right)\\
&\qquad-\left(\frac{1}{\Delta}\right)\left(\sum_{l=1}^{r}\left(\int_{t+\Delta-\lambda_{il}}^{t+\Delta}x^T(s)Q_lx(s)\,ds\right)-\sum_{l=1}^{r}\left(\int_{t-\lambda_{il}}^{t}x^T(s)Q_lx(s)\,ds\right)\right)\\
&\qquad-\pi_{ii}\sum_{l=1}^{r}\left(\int_{t+\Delta-\lambda_{j(l-1)}}^{t+\Delta}x^T(s)Q_lx(s)\,ds\right)\\
&\qquad-\sum_{j\neq i}\pi_{ij}\sum_{l=1}^{r}\left(\int_{t+\Delta-\lambda_{j(l-1)}}^{t}x^T(s)Q_lx(s)\,ds\right).
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\mathcal{A}\bigl[V_4(x_t,i)\bigr]&=\sum_{l=1}^{r}\mathcal{A}\left[\int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}}x^T(s)Q_lx(s)\,ds\right]\\
&=\lim_{\Delta\to 0}\frac{1}{\Delta}\bigl\{\mathbb{E}\bigl[V_4(x_{t+\Delta},r_{t+\Delta})\big|(x_t,r_t=i)\bigr]-V_4(x_t,r_t=i)\bigr\}\\
&=\sum_{l=1}^{r}\left(\mathcal{A}\left[\int_{t-\lambda_{il}}^{t}x^T(s)Q_lx(s)\,ds\right]-\mathcal{A}\left[\int_{t-\lambda_{i(l-1)}}^{t}x^T(s)Q_lx(s)\,ds\right]\right)\\
&=\sum_{l=1}^{r}\begin{pmatrix} x^T(t)Q_lx(t)-x^T(t-\lambda_{il})Q_lx(t-\lambda_{il})\\ +\sum_{j=1}^{N}\pi_{ij}\int_{t-\lambda_{il}}^{t}x^T(s)Q_lx(s)\,ds\\ -x^T(t)Q_lx(t)+x^T(t-\lambda_{i(l-1)})Q_lx(t-\lambda_{i(l-1)})\\ -\sum_{j=1}^{N}\pi_{ij}\int_{t-\lambda_{il}}^{t}x^T(s)Q_lx(s)\,ds\end{pmatrix}\\
&=\sum_{l=1}^{r}\bigl(x^T(t-\lambda_{i(l-1)})Q_lx(t-\lambda_{i(l-1)})-x^T(t-\lambda_{il})Q_lx(t-\lambda_{il})\bigr)\\
&\quad+\sum_{j=1}^{N}\pi_{ij}\sum_{l=1}^{r}\int_{t-\lambda_{jl}}^{t-\lambda_{j(l-1)}}x^T(s)Q_lx(s)\,ds.
\end{aligned}
$$

From (5) and (6) it can be further shown that

$$
\begin{aligned}
\mathcal{A}\big[V_4(x_t,i)\big] &= x^T(t)Q_1x(t) - x^T(t-\lambda_{ir})Q_rx(t-\lambda_{ir}) \\
&\quad - \sum_{l=1}^{r-1} x^T(t-\lambda_{il})(Q_l - Q_{l+1})x(t-\lambda_{il}) \\
&\quad + \sum_{j=1, j\neq i}^{N} \pi_{ij} \sum_{l=1}^{r} \int_{t-\lambda_{jl}}^{t-\lambda_{j(l-1)}} x^T(s)Q_lx(s)\,ds \\
&\quad - |\pi_{ii}| \sum_{l=1}^{r} \int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}} x^T(s)Q_lx(s)\,ds \\
&\le x^T(t)Q_1x(t) - x^T(t-\lambda_{ir})Q_rx(t-\lambda_{ir}) \\
&\quad - \sum_{l=1}^{r-1} x^T(t-\lambda_{il})(Q_l - Q_{l+1})x(t-\lambda_{il}) \\
&\quad + \sum_{j=1, j\neq i}^{N} \pi_{ij} \sum_{l=1}^{r} \int_{t-\lambda_{jl}}^{t-\lambda_{j(l-1)}} x^T(s)Q_lx(s)\,ds \\
&\quad - |\pi_{ii}| \sum_{l=1}^{r} d_{il}^{-1} \int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}} x^T(s)\,ds\, Q_l \int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}} x(s)\,ds \\
&\le x^T(t)Q_1x(t) - x^T(t-\lambda_{ir})Q_rx(t-\lambda_{ir}) \\
&\quad - \sum_{l=1}^{r-1} x^T(t-\lambda_{il})(Q_l - Q_{l+1})x(t-\lambda_{il}) \\
&\quad + \sum_{j=1, j\neq i}^{N} \pi_{ij} \sum_{l=1}^{r} \int_{t-\lambda_{jl}}^{t-\lambda_{j(l-1)}} x^T(s)Q_1x(s)\,ds \\
&\quad - |\pi_{ii}| \sum_{l=1}^{r} d_{il}^{-1} \int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}} x^T(s)\,ds\, Q_l \int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}} x(s)\,ds \\
&\le x^T(t)Q_1x(t) - x^T(t-\lambda_{ir})Q_rx(t-\lambda_{ir}) \\
&\quad - \sum_{l=1}^{r-1} x^T(t-\lambda_{il})(Q_l - Q_{l+1})x(t-\lambda_{il}) \\
&\quad + \sum_{j=1, j\neq i}^{N} \pi_{ij} \int_{t-\mu}^{t} x^T(s)Q_1x(s)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&- |\pi_{ii}| \sum_{l=1}^{r} d_{il}^{-1} \int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}} x^T(s)\,ds\, Q_l \int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}} x(s)\,ds \\
={}& x^T(t) Q_1 x(t) - x^T(t-\lambda_{ir}) Q_r x(t-\lambda_{ir}) \\
&- \sum_{l=1}^{r-1} x^T(t-\lambda_{il})(Q_l - Q_{l+1}) x(t-\lambda_{il}) \\
&- \pi_{ii} \int_{t-\mu}^{t} x^T(s) Q_1 x(s)\,ds \\
&+ \pi_{ii} d_{il}^{-1} \int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}} x^T(s)\,ds\, Q_l \int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}} x(s)\,ds \\
\le{}& x^T(t) Q_1 x(t) - x^T(t-\lambda_{ir}) Q_r x(t-\lambda_{ir}) \\
&- \sum_{l=1}^{r-1} x^T(t-\lambda_{il})(Q_l - Q_{l+1}) x(t-\lambda_{il}) \\
&+ \eta \int_{t-\mu}^{t} x^T(s) Q_1 x(s)\,ds \\
&+ \pi_{ii} \sum_{l=1}^{r} d_{il}^{-1} \int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}} x^T(s)\,ds\, Q_l \int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}} x(s)\,ds,
\end{aligned}
$$

and

$$
\mathcal{A}\big[V_5(x_t,i)\big] = \eta\mu x^T(t) Q_1 x(t) - \eta \int_{t-\mu}^{t} x^T(s) Q_1 x(s)\,ds.
$$

Then we have

$$
\begin{aligned}
\mathcal{A}\big[V(x_t,i)\big] ={}& \sum_{p=1}^{5} \mathcal{A}\big[V_p(x_t,i)\big] \\
\le{}& x^T(t)\left(\sum_{j=1}^{N} \pi_{ij} P_j\right) x(t) + 2x^T(t) P_i \big(A_i x(t) + A_{di} x(t-\lambda_{ir})\big) \\
&+ x^T(t) X x(t) - x^T(t-\mu) X x(t-\mu) \\
&+ \mu \dot{x}^T(t) Y \dot{x}(t) - (\mu - \lambda_{i(r-1)})^{-1} \left(\int_{t-\mu}^{t-\lambda_{i(r-1)}} \dot{x}(s)\,ds\right)^T Y \\
&\times \left(\int_{t-\mu}^{t-\lambda_{i(r-1)}} \dot{x}(s)\,ds\right) \\
&- \sum_{l=1}^{r-1} \left(\int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}} \dot{x}(s)\,ds\right)^T \big(\bar{d}_{il}^{-1} Y\big) \left(\int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}} \dot{x}(s)\,ds\right)
\end{aligned}
$$

$$
\begin{aligned}
&+x^T(t)Q_1x(t)-x^T(t-\lambda_{ir})Q_rx(t-\lambda_{ir})\\
&+\eta\mu x^T(t)Q_1x(t)-\sum_{l=1}^{r-1}x^T(t-\lambda_{il})(Q_l-Q_{l+1})x(t-\lambda_{il})\\
&+\pi_{ii}\sum_{l=1}^{r}d_{il}^{-1}\left(\int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}}x^T(s)\,ds\right)Q_l\left(\int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}}x(s)\,ds\right),
\end{aligned}
$$

which equals

$$
\mathcal{A}\big[V(x_t,i)\big]\le\zeta^T(t)\begin{bmatrix}\varGamma_{i1}+\varGamma_{i2} & 0 & 0\\ 0 & \varGamma_{i3} & 0\\ 0 & 0 & \varGamma_{i4}\end{bmatrix}\zeta(t),
$$

where $\varGamma_{ij}$, $i\in\tilde{N}$, $j=1,2,3$, are defined in (8)–(11) and

$$
\begin{aligned}
\zeta(t)=\Big[&x(t)\quad x(t-\lambda_{i1})\quad\cdots\quad x(t-\lambda_{ir})\\
&x(t-\mu)\quad\int_{t-\lambda_{i1}}^{t}\dot{x}(s)\,ds\quad\cdots\quad\int_{t-\lambda_{i(r-1)}}^{t-\lambda_{i(r-2)}}\dot{x}(s)\,ds\quad\int_{t-\mu}^{t-\lambda_{i(r-1)}}\dot{x}(s)\,ds\\
&\int_{t-\lambda_{i1}}^{t}x(s)\,ds\quad\cdots\quad\int_{t-\lambda_{i(r-1)}}^{t-\lambda_{i(r-2)}}x(s)\,ds\quad\int_{t-\lambda_{ir}}^{t-\lambda_{i(r-1)}}x(s)\,ds\Big]^T.
\end{aligned}
\tag{14}
$$

From the Newton–Leibniz formula we have

$$
\begin{aligned}
x(t)-x(t-\lambda_{i1})-\int_{t-\lambda_{i1}}^{t}\dot{x}(s)\,ds&=0,\\
x(t-\lambda_{i1})-x(t-\lambda_{i2})-\int_{t-\lambda_{i2}}^{t-\lambda_{i1}}\dot{x}(s)\,ds&=0,\\
&\vdots\\
x(t-\lambda_{i(r-2)})-x(t-\lambda_{i(r-1)})-\int_{t-\lambda_{i(r-1)}}^{t-\lambda_{i(r-2)}}\dot{x}(s)\,ds&=0,\\
x(t-\lambda_{i(r-1)})-x(t-\mu)-\int_{t-\mu}^{t-\lambda_{i(r-1)}}\dot{x}(s)\,ds&=0,
\end{aligned}
$$

which is equivalent to

$$
T\zeta(t)=\left[\begin{array}{ccccccc|cccccc|ccccccc}
I_n & -I_n & 0 & \cdots & 0 & 0 & 0 & 0 & -I_n & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & I_n & -I_n & \cdots & 0 & 0 & 0 & 0 & 0 & -I_n & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & I_n & -I_n & 0 & 0 & 0 & 0 & \cdots & -I_n & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & I_n & 0 & -I_n & 0 & 0 & \cdots & 0 & -I_n & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{array}\right]\zeta(t)=0.
$$

Therefore, from Lemma 2, $\mathcal{A}[V(x_t, i)] < 0$ if

$$\zeta^T(t)\left(T^T T - \begin{bmatrix} \Gamma_{i1} + \Gamma_{i2} & 0 & 0 \\ 0 & \Gamma_{i3} & 0 \\ 0 & 0 & \Gamma_{i4} \end{bmatrix}\right)\zeta(t) > 0,$$

which is equivalent to (7) from Lemma 2, where $T^{\perp}$ can be computed from Lemma 1 and given in (12). This completes the proof of Theorem 1. □

Remark 1 Sufficient condition with reduced conservatism is obtained for the stability analysis of Markovian jump system Σ with mode-dependent multiple time delays. The conservatism can be reduced further by introducing more delay components as shown in Remarks 4, 5 and 6 in [13] at the expense of more matrix variables and LMIs in larger dimension in Theorem 1.

3.2 Robust Stability

Now we consider the robust stochastic stability of system Σ_p.

Theorem 2 *If there exist $P_i > 0$, $i \in \tilde{N}$, $Q_m > 0$, $m = 1, \ldots, r$, $X > 0$ and $Y > 0$ such that*

$$Q_{m+1} - Q_m \le 0, \quad m = 1, \ldots, r-1, \tag{15}$$

$$T^{\perp T}\begin{bmatrix} \Gamma'_{il1} + \Gamma'_{il2} & 0 & 0 \\ 0 & \Gamma_{i3} & 0 \\ 0 & 0 & \Gamma_{i4} \end{bmatrix} T^{\perp} < 0, \tag{16}$$

for each $i \in \tilde{N}$, $l = 1, 2, \ldots, q$, then the system Σ_p is robust stochastically stable, where

$$\Gamma'_{il1} = \begin{bmatrix} \sum_{j=1}^{N}\pi_{ij}P_j + X + (1+\eta\mu)Q_1 \\ \quad + P_iA_{il} + A_{il}^TP_i & 0 & \cdots & 0 & P_iA_{dil} \\ 0 & Q_2 - Q_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & & & Q_r - Q_{r-1} & 0 \\ A_{dil}^TP_i & 0 & & 0 & -Q_r \end{bmatrix},$$

$$\Gamma'_{il2} = \begin{bmatrix} A_{il}^T & 0 & \cdots & 0 & A_{dil}^T \end{bmatrix}\mu Y \begin{bmatrix} A_{il} \\ 0 \\ \vdots \\ 0 \\ A_{dil} \end{bmatrix}.$$

Proof From the proof of Theorem 1, it is easy to show that $\mathcal{A}[V(x_t, i)]$ in (13) satisfies

$$\mathcal{A}\big[V(x_t, i)\big] \le x^T(t)\left(\sum_{j=1}^{N}\pi_{ij}P_j + X + (1+\eta\mu)Q_1\right)x(t)$$

$$
\begin{aligned}
&+ 2x^T(t)P_i\big(A(i)x(t) + A_d(i)x(t-\lambda_{ir})\big) \\
&- x^T(t-\mu)Xx(t-\mu) - x^T(t-\lambda_{ir})Q_r x(t-\lambda_{ir}) \\
&+ \mu\dot{x}^T(t)Y\dot{x}(t) - (\mu-\lambda_{i(r-1)})^{-1}\bigg(\int_{t-\mu}^{t-\lambda_{i(r-1)}}\dot{x}(s)\,ds\bigg)^T Y \\
&\times\bigg(\int_{t-\mu}^{t-\lambda_{i(r-1)}}\dot{x}(s)\,ds\bigg) \\
&- \sum_{l=1}^{r-1}\bar{d}_{il}^{-1}\bigg(\int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}}\dot{x}(s)\,ds\bigg)^T Y\bigg(\int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}}\dot{x}(s)\,ds\bigg) \\
&- \sum_{l=1}^{r-1}x^T(t-\lambda_{il})(Q_l - Q_{l+1})x(t-\lambda_{il}) \\
&+ \pi_{ii}\sum_{l=1}^{r}d_{il}^{-1}\int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}}x^T(s)\,ds\,Q_l\int_{t-\lambda_{il}}^{t-\lambda_{i(l-1)}}x(s)\,ds \\
= {}& \zeta^T(t)\begin{bmatrix} \tilde{\Gamma}_{i1}+\tilde{\Gamma}_{i2} & 0 & 0 \\ 0 & \Gamma_{i3} & 0 \\ 0 & 0 & \Gamma_{i4}\end{bmatrix}\zeta(t),
\end{aligned}
\tag{17}
$$

where

$$
\tilde{\Gamma}_{i1} = \begin{bmatrix}
\begin{array}{c}\sum_{j=1}^{N}\pi_{ij}P_j + X + (1+\eta\mu)Q_1 \\ + P_iA(i) + A^T(i)P_i\end{array} & 0 & \cdots & 0 & P_iA_d(i) \\
0 & Q_2 - Q_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & & & Q_r - Q_{r-1} & 0 \\
A_d^T(i)P_i & 0 & & 0 & -Q_r
\end{bmatrix},
$$

$$
\tilde{\Gamma}_{i2} = \begin{bmatrix} A^T(i) & 0 & \cdots & 0 & A_d^T(i)\end{bmatrix}\mu Y\begin{bmatrix} A(i) \\ 0 \\ \vdots \\ 0 \\ A_d(i)\end{bmatrix},
$$

and $\zeta(t)$ is defined in (14), Γ_{i3} and Γ_{i4} are given in (10)–(11), and $A(i)$ and $A_d(i)$ are defined in (2). From the proof of Theorem 1,

$$
\zeta^T(t)\begin{bmatrix} \tilde{\Gamma}_{i1}+\tilde{\Gamma}_{i2} & 0 & 0 \\ 0 & \Gamma_{i3} & 0 \\ 0 & 0 & \Gamma_{i4}\end{bmatrix}\zeta(t) < 0
$$

if and only if

$$
T^{\perp T}\begin{bmatrix} \tilde{\Gamma}_{i1}+\tilde{\Gamma}_{i2} & 0 & 0 \\ 0 & \Gamma_{i3} & 0 \\ 0 & 0 & \Gamma_{i4}\end{bmatrix}T^{\perp} < 0,
$$

which, by Schur complement equivalence, is equivalent to

$$\begin{bmatrix} T^{\perp T}\begin{bmatrix} \tilde{\Gamma}_{i1} & 0 & 0 \\ 0 & \Gamma_{i3} & 0 \\ 0 & 0 & \Gamma_{i4} \end{bmatrix} T^{\perp} & T^{\perp T}\begin{bmatrix} \begin{bmatrix} A(i) \\ 0 \\ \vdots \\ 0 \\ A_d(i) \end{bmatrix} \\ 0 \\ 0 \end{bmatrix} \mu Y \\ \mu Y \begin{bmatrix} \begin{bmatrix} A^T(i) \; 0 \cdots 0 \; A_d^T(i) \end{bmatrix} \; 0 \; 0 \end{bmatrix} T^{\perp} & -\mu Y \end{bmatrix} < 0.$$

According to (16) and by Schur complement equivalence again, we have

$$\begin{bmatrix} T^{\perp T}\begin{bmatrix} \Gamma'_{il1} & 0 & 0 \\ 0 & \Gamma_{i3} & 0 \\ 0 & 0 & \Gamma_{i4} \end{bmatrix} T^{\perp} & T^{\perp T}\begin{bmatrix} \begin{bmatrix} A_{il} \\ 0 \\ \vdots \\ 0 \\ A_{dil} \end{bmatrix} \\ 0 \\ 0 \end{bmatrix} \mu Y \\ \mu Y \begin{bmatrix} \begin{bmatrix} A_{il}^T \; 0 \cdots 0 \; A_{dil}^T \end{bmatrix} \; 0 \; 0 \end{bmatrix} T^{\perp} & -\mu Y \end{bmatrix} < 0,$$

which further leads to

$$\begin{aligned} &\sum_{l=1}^{q} \alpha_l(t) \begin{bmatrix} T^{\perp T}\begin{bmatrix} \Gamma'_{il1} & 0 & 0 \\ 0 & \Gamma_{i3} & 0 \\ 0 & 0 & \Gamma_{i4} \end{bmatrix} T^{\perp} & T^{\perp T}\begin{bmatrix} \begin{bmatrix} A_{il} \\ 0 \\ \vdots \\ 0 \\ A_{dil} \end{bmatrix} \\ 0 \\ 0 \end{bmatrix} \mu Y \\ \mu Y \begin{bmatrix} \begin{bmatrix} A_{il}^T \; 0 \cdots 0 \; A_{dil}^T \end{bmatrix} \; 0 \; 0 \end{bmatrix} T^{\perp} & -\mu Y \end{bmatrix} \\ &= \begin{bmatrix} T^{\perp T}\begin{bmatrix} \tilde{\Gamma}_{i1} & 0 & 0 \\ 0 & \Gamma_{i3} & 0 \\ 0 & 0 & \Gamma_{i4} \end{bmatrix} T^{\perp} & T^{\perp T}\begin{bmatrix} \begin{bmatrix} A(i) \\ 0 \\ \vdots \\ 0 \\ A_d(i) \end{bmatrix} \\ 0 \\ 0 \end{bmatrix} \mu Y \\ \mu Y \begin{bmatrix} \begin{bmatrix} A^T(i) \; 0 \cdots 0 \; A_d^T(i) \end{bmatrix} \; 0 \; 0 \end{bmatrix} T^{\perp} & -\mu Y \end{bmatrix} < 0. \end{aligned} \tag{18}$$

By Schur complement equivalence again, it is easy to obtain that (18) holds if and only if (16) is satisfied, which implies

$$\mathcal{A}\big[V(x_t,i)\big] \le \zeta^T(t)\begin{bmatrix} \tilde{\Gamma}_{i1}+\tilde{\Gamma}_{i2} & 0 & 0 \\ 0 & \Gamma_{i3} & 0 \\ 0 & 0 & \Gamma_{i4} \end{bmatrix}\zeta(t) < 0.$$

This completes the proof of Theorem 2. □

4 Numerical Examples

Next, two numerical examples will be given to illustrate the effectiveness of the proposed approach.

Example 1 The following matrices,

$$A_{11}=\begin{bmatrix} -2.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad A_{12}=\begin{bmatrix} -2 & 0 \\ 0.2 & -1.6 \end{bmatrix}, \qquad A_{21}=\begin{bmatrix} -2.1 & 0 \\ -0.1 & 0.9 \end{bmatrix},$$
$$A_{22}=\begin{bmatrix} -1.4 & 0.1 \\ 0.2 & -1.9 \end{bmatrix}, \qquad A_{d11}=\begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}, \qquad A_{d12}=\begin{bmatrix} -0.1 & 0 \\ 0 & 0.2 \end{bmatrix},$$
$$A_{d21}=\begin{bmatrix} -0.2 & 0.1 \\ 0.2 & 0 \end{bmatrix}, \qquad A_{d22}=\begin{bmatrix} 0.1 & 0 \\ 0.2 & 0.1 \end{bmatrix},$$

are the system matrices of a Markovian jump system Σ_p with mode-dependent delays and polytopic uncertainties in the form of (1). The delays are given by

$$d_{11}=0.8, \qquad d_{12}=0.26, \qquad d_{21}=0.26, \qquad d_{22}=0.8,$$

and the transition probability matrix is

$$M=\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}.$$

The solutions of (15) and (16) are given by

$$X=\begin{bmatrix} 114.2080 & 0.0058 \\ 0.0058 & 0.0043 \end{bmatrix} > 0, \qquad Y=\begin{bmatrix} 655.4847 & 0.4929 \\ 0.4929 & 0.0207 \end{bmatrix} > 0,$$
$$P_1=10^3\times\begin{bmatrix} 1.9823 & 0.0274 \\ 0.0274 & 0.1872 \end{bmatrix} > 0, \qquad P_2=10^3\times\begin{bmatrix} 2.0173 & 0.0246 \\ 0.0246 & 0.0836 \end{bmatrix} > 0,$$
$$Q_1=10^3\times\begin{bmatrix} 1.3046 & 0.0179 \\ 0.0179 & 0.0104 \end{bmatrix} > 0, \qquad Q_2=10^3\times\begin{bmatrix} 1.2973 & 0.0179 \\ 0.0179 & 0.0104 \end{bmatrix} > 0.$$

So the overall stability bound on delay is the sum of all parts, i.e. 1.06.

Example 2 The system matrices of a Markovian jump system Σ_p with mode-dependent delays and polytopic uncertainties in the form of (1) are given by

$$A_{11} = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} -3 & -0.1 \\ -0.1 & -4 \end{bmatrix}, \qquad A_{21} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix},$$
$$A_{22} = \begin{bmatrix} -4 & 0.1 \\ 0.1 & -3 \end{bmatrix}, \qquad A_{d11} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \qquad A_{d12} = \begin{bmatrix} -0.8 & 0.1 \\ 0.9 & 0 \end{bmatrix},$$
$$A_{d21} = \begin{bmatrix} -0.9 & 0 \\ 0 & 0.7 \end{bmatrix}, \qquad A_{d22} = \begin{bmatrix} 1 & 0 \\ 0.9 & 0.6 \end{bmatrix},$$

with

$$d_{11} = 1.22, \qquad d_{12} = 0.5, \qquad d_{21} = 1.21, \qquad d_{22} = 0.51.$$

The transition probability matrix is

$$M = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}.$$

The solutions of (15) and (16) are given by

$$X = \begin{bmatrix} 0.0306 & -0.0165 \\ -0.0165 & 0.0658 \end{bmatrix} > 0, \qquad Y = \begin{bmatrix} 0.3397 & 0.0958 \\ 0.0958 & 0.0306 \end{bmatrix} > 0,$$
$$P_1 = \begin{bmatrix} 17.3223 & 0.0878 \\ 0.0878 & 0.5715 \end{bmatrix} > 0, \qquad P_2 = \begin{bmatrix} 22.7518 & -0.0839 \\ -0.0839 & 0.4851 \end{bmatrix} > 0,$$
$$Q_1 = \begin{bmatrix} 7.9497 & -0.0401 \\ -0.0401 & 0.3088 \end{bmatrix} > 0, \qquad Q_2 = \begin{bmatrix} 7.9479 & -0.0380 \\ -0.0380 & 0.3019 \end{bmatrix} > 0.$$

So the overall stability bound on delay is the sum of all parts, i.e. 1.72. By the method in [4] without partitioning the delay, the upper bound of the time delay is 0.57, which shows that our method has improved the result in [4].

5 Conclusion

We have presented solutions to the stability analysis for Markovian jump systems with multiple delay components and possible polytopic uncertainties. A delay-dependent sufficient condition in terms of the LMI framework has been obtained by using a new form of stochastic Lyapunov functionals. Two examples have been given to demonstrate the effectiveness of the proposed result.

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