An Axiomatization of the Uniform Rule without the Pareto Principle

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Abstract

Our objective is to identify *strategy-proof* rules in the allocation problem à la Benassy (1982). Sprumont (1991) showed that the uniform rule is the only rule satisfying *strategy-proofness*, *efficiency*, and *anonymity* (or *no-envy*). While either *anonymity* or *no-envy* (both are fairness condition) in his characterizations can be relaxed to *weak symmetry* (Ching 1994), whether the characterizations can be strengthened along the Pareto dimension has remained open. This question is taken up by this paper. We show that *efficiency* in Ching's (1994) characterization can be relaxed to *continuity* and a *null-player* axiom ("irrelevant" agents are not allocated with any amount). The *null-player* axiom is implied by *efficiency*.

Keywords: Strategy-proofness, single-peakedness, uniform rule

Very Preliminary and Incomplete

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1 Introduction

Incentive compatibility is a fundamental principle in mechanism design. If manipulability of a mechanism is a concern, a stronger incentive-compatibility condition is preferred. One of the strongest incentive-compatibility conditions in the literature is **strategy-proofness**, which requires revelation of the true preference be a weakly dominant strategy for every agent. The requirement of a weakly dominant strategy is quite demanding, because weakly dominant strategies may not exist. Indeed, a general impossibility theorem was established by Gibbard (1973) and Satterthwaite (1975) for "nondictatorial" *strategy-proof* social choice functions.

Sprumont (1991) adopted the mechanism-design approach to study an allocation problem in the model à la Benassy (1982). He showed that the "uniform rule" (Benassy 1982) is the only (allocation) rule satisfying *strategy-proofness*, "efficiency", and "anonymity". **Efficiency** is the usual Pareto condition and **anonymity** simply rules out the dependency of a rule on the names of the agents.

The result in Sprumont's (1991) characterization of the uniform rule is threefold. First, it is an existence theorem for a rule satisfying *strategy-proofness*, *efficiency*, and *anonymity*. Second, it is a uniqueness theorem for such a rule. Third, it is a theorem identifying the uniform rule as a closed-form solution of such a unique rule. Additionally, Sprumont (1991) showed that *anonymity* in his characterization can be replaced by another fairness condition: **no-envy**, which says that no agent prefers to receive the amount allocated to any other agent.

It is uncommon to have a model allowing the co-existence of *strategy-proofness* (a strong incentive-compatibility condition), *efficiency* (a Pareto condition), and *anonymity* or *no-envy* (both are fairness condition). We consider the model a good platform for exploring the relationship of *strategy-proofness* and other axioms further. For instance, Ching (1994) showed that either *anonymity* or *no-envy* in Sprumont's (1991) characterizations can be relaxed to **weak symmetry**, which ensures that any agents with identical preference are indifferent to the amounts allocated to them.[1]

While both fairness conditions in Sprumont's (1991) characterizations can be relaxed (Ching 1994), whether the characterizations can be strengthened along the Pareto dimension has remained open. This question is taken up by this paper. We show that *efficieny* in Ching's (1994) characterization can be relaxed to "continuity"[2] and a "null-player" axiom. The idea of the "null-player"

[1] A stronger version of *weak symmetry* is **symmetry**, which requires any agents with identical preference be allocated the same amount. (*Symmetry* is weaker than *anonymity*.)

[2] "Continuity" is implied by *strategy-proofness* and *efficiency* together (see Sprumont 1991).

axiom is that an agent who is "irrelevant" to the problem is not allocated with any amount. The "null-player" axiom is implied by *efficiency*.

In the next section, we introduce the model and state the background results. The results are established in Section 3, where the "null-player" axiom is defined. In the last section, we discuss the extension of the main result to a model with free disposal.

2 Model

The problem is to allocate a perfectly divisible commodity among $n \geq 2$ agents. The amount of the commodity to be divided is normalized to 1. Let $N = \{1, \ldots, n\}$ be the set of agents. The preference relation of agent $i \in N$ over $[0, 1]$ is denoted by R_i. Let P_i be the strict relation of R_i and I_i the indifferent relation. The preference relation R_i is **single-peaked** if there exists $p(R_i) \in [0, 1]$ such that for all $x, y \in [0, 1]$, if $x < y < p(R_i)$ or $p(R_i) < y < x$, then $p(R_i)\ P_i\ y\ P_i\ x$. Let $\mathcal{R}$ be the domain of single-peaked preferences. Let $S \subseteq N$ be a coalition[3] and $R_S = (R_i)_{i \in S}$ be a preference profile of the coalition.

An **allocation rule** is a function $\phi : \mathcal{R}^n \to [0, 1]^n$ such that

$$\sum \phi_i(R_N) = 1 \text{ for all } R_N \in \mathcal{R}^n. \tag{1}$$

The feasibility constraint (1) assumes no free disposal, i.e. $\sum \phi_i(R_N) = 1$ even when $\sum p(R_i) < 1$. The case $\sum p(R_i) < 1$ is interpreted as "excess supply", and $\sum p(R_i) > 1$ as "excess demand". Note that the problem of excess supply is not assumed away by no free disposal.[4]

The allocation rule mentioned in the introduction is the **uniform rule** (Benassy 1982), which is defined as follows. For all $R_N \in \mathcal{R}^n$ and all $i \in N$,

$$U_i(R_N) = \begin{cases} \max\{p(R_i), \lambda(R_N)\} & \text{if } \sum p(R_j) \leq 1 \\ \min\{p(R_i), \lambda(R_N)\} & \text{if } \sum p(R_j) \geq 1 \end{cases} \tag{2}$$

where $\lambda(R_N)$ solves $\sum U_j(R_N) = 1$.

In (2), either the max or min formula can be used for any $R_N \in \mathcal{R}^n$ such that $\sum p(R_j) = 1$. Let's consider the max formula first. Note that any $\lambda(R_N) \leq \min\{p(R_j)\}$ can be used to solve the feasibility constraint:

$$U_j(R_N) = \max\{p(R_j), \lambda(R_N)\} = p(R_j) \text{ for all } j \in N \text{ and}$$
$$\sum U_j(R_N) = \sum p(R_j) = 1$$

[3]We use the symbol $\subseteq$ for subset relationship and $\subset$ for *proper* subset relationship.

[4]Had there been free disposal, it is natural to assume that for all $R_N \in \mathcal{R}^n$, if $\sum p(R_i) \leq 1$, then $\phi_j(R_N) = p(R_j)$ for all $j \in N$, i.e. excess supply is no longer an issue.

and any $\lambda > \min\{p(R_j)\}$ results in a violation the feasibility constraint:

$$\sum U_j(R_N) = \sum \max\{p(R_j), \lambda\} > \sum p(R_j) = 1$$

If the min formula is used, the same argument shows that the feasibility constraint is satisfied if and only if $\lambda(R_N) \geq \max\{p(R_j)\}$ is used. To summarize, for any $R_N \in \mathcal{R}^n$ such that $\sum p(R_j) = 1$,

(1) $U_j(R_N) = p(R_j)$ for all $j \in N$, whether the max or min formula is used;

(2) $\lambda(R_N)$ is not be uniquely determined by the max formula, nor the min formula.

On the other hand, for any $R_N \in \mathcal{R}^n$ such that $\sum p(R_j) \neq 1$, $\lambda(R_N)$ is uniquely determined by (2). Suppose, without loss of generality, that $\sum p(R_j) < 1$ (excess supply).[5] To apply (2), $\lambda(R_N)$ is chosen to solve $\sum \max\{p(R_j), \lambda(R_N)\} = 1$, which implies that $\lambda(R_N) > p(R_k)$ for some $k \in N$. $\lambda(R_N)$ is unique, because any $\lambda' \neq \lambda(R_N)$ results in a violation of the feasibility constraint. If $\lambda' > \lambda(R_N)$, then

$$\begin{aligned} \max\{p(R_k), \lambda'\} = \lambda' > \lambda(R_N) = \max\{p(R_k), \lambda(R_N)\} \text{ and} \\ \max\{p(R_j), \lambda'\} \geq \max\{p(R_j), \lambda(R_N)\} \text{ for all } j \neq k \end{aligned}$$

Summing up the left(most) hand side and right(most) hand side of the above,

$$\sum_{j \in N} \max\{p(R_j), \lambda'\} > \sum_{j \in N} \max\{p(R_j), \lambda(R_N)\} = 1$$

The above equation violates the feasibility constraint. Alternatively, if $\lambda' < \lambda(R_N)$, then

$$\begin{aligned} \max\{p(R_k), \lambda'\} < \lambda(R_N) = \max\{p(R_k), \lambda(R_N)\} \text{ and} \\ \max\{p(R_j), \lambda'\} \leq \max\{p(R_j), \lambda(R_N)\} \text{ for all } j \neq k \end{aligned}$$

Again, summing up the above left(most) hand side and right(most) hand side shows that the feasibility constraint is not met.

The uniform rule has occupied a central position in the literature. It was first characterized by Sprumont (1991), who showed that it is the only rule scoring well on three fronts: incentive compatibility, efficiency, and fairness. The incentive-compatibility axiom used by Sprumont (1991) is "strategy-proofness", which requires revelation of the true preference be a weakly dominant strategy for every agent. Its formal definition is stated below:

Strategy-proofness: For all $i \in N$, all $R_i, R'_i \in \mathcal{R}$, and all $R_{N\setminus\{i\}} \in \mathcal{R}^{n-1}$, $\phi_i(R_i, R_{N\setminus\{i\}})\ R_i\ \phi_i(R'_i, R_{N\setminus\{i\}})$.

[5] The case of excess demand ($\sum p(R_j) > 1$) can be handled by the same argument.

Strategy-proofness a demanding property, because a weakly dominant strategy may not exist. Indeed, a general impossibility theorem was established by Gibbard (1973) and Satterthwaite (1975) for *strategy-proof* social choice functions. It is uncommon to have a model allowing the co-existence of *strategy-proof*, *efficiency* and *anonymity* or *no-envy*. We consider the current model a good platform to explore the relationship of *strategy-proofness* and other axioms.

The other two axioms used by Sprumont (1991) are standard. The efficiency axiom is the usual Pareto condition and the fairness axiom is simply an anonymous requirement. To define these two axioms, some basic definitions are introduced first. Let the set of feasible allocations be $Z = \{z \in [0,1]^n | \sum z_i = 1\}$.

Efficiency: For all $R_N \in \mathcal{R}^n$, there is no $z \in Z$ such that $z_i \; R_i \; \phi_i(R_N)$ for all $i \in N$ and $z_i \; P_i \; \phi_i(R_N)$ for some $i \in N$

Let $\pi : N \to N$ be a permutation and $R_{\pi(N)} = (R_{\pi(i)})_{i \in N}$. Let Π be the collection of π.

Anonymity: For all $R_N \in \mathcal{R}^n$, all $i \in N$, and all $\pi \in \Pi$, $\phi_{\pi(i)}(R_{\pi(N)}) = \phi_i(R_N)$.

We are now ready to state Sprumont's (1991) characterization of the uniform rule.

Theorem 1 (Sprumont 1991)**.** *The uniform rule is the only rule satisfying strategy-proofness, efficiency, and anonymity.*

Anonymity in the above theorem can be replaced by **no-envy:** For all $R_N \in \mathcal{R}^n$ and all $i, j \in N$, $\phi_i(R_N) \; R_i \; \phi_j(R_N)$.

Theorem 2 (Sprumont 1991)**.** *The uniform rule is the only rule satisfying strategy-proofness, efficiency, and no-envy.*

Either *anonymity* in Theorem 1 or *no-envy* in Theorem 2 can be relaxed to **weak symmetry:** For all $R_N \in \mathcal{R}^n$ and all $i, j \in N$, if $R_i = R_j$, then $\phi_i(R_N) \; I_i \; \phi_j(R_N)$.

Theorem 3 (Ching 1994)**.** *The uniform rule is the only rule satisfying strategy-proofness, efficiency, and weak symmetry.*

Both *anonymity* and *no-envy* are fairness condition. Hence, Theorem 3 strengthens both Sprumont's (1991) characterizations along the fairness dimension. One may wonder whether the same can be done along the Pareto dimension. The consequences of relaxing *efficiency* are investigated in this paper.

3 Results

It may not be desirable to drop *efficiency* altogether. For instance, the "equal-division rule"[6] is a "fair" allocation rule that satisfies *strategy-proofness* all the time. It is obviously "fair"[7] and trivially *strategy-proofness* (because it is a constant function). Hence, we should allow a rule to be responsive to preferences, but restrict it to be "continuous" with respect to preferences.[8]

For simplicity, we adopt the notion of "continuity"[9] formulated by Sprumont (1991). Our point of departure is the following intermediate result by Sprumont (1991), who showed that a one-person rule $f : \mathcal{R} \to [0,1]$ satisfies *strategy-proofness* and *continuity*[10] if and only if there exist two parameters $a, b \in [0,1]$ such that

$$f(R) = \text{med}\{p(R), a, b\} \ \forall R \in \mathcal{R} \tag{3}$$

Let ϕ be **continuous** if and only if for all $i \in N$ and all $R_{N\setminus\{i\}} \in \mathcal{R}^{n-1}$, ϕ_i is *continuous* (in $R_i \in \mathcal{R}$). Then the above single-person result can be applied to a (multi-person) rule ϕ as follows:

Lemma 1 (Sprumont 1991)**.** *A rule ϕ satisfies strategy-proofness and continuity if and only if for all $i \in N$, there exist two functions $a_i, b_i : \mathcal{R}^{n-1} \to [0,1]$ such that for all $R_{N\setminus\{i\}} \in \mathcal{R}^{n-1}$,*

$$\phi_i(R_i, R_{N\setminus\{i\}}) = \text{med}\{p(R_i), a_i(R_{N\setminus\{i\}}), b_i(R_{N\setminus\{i\}})\} \ \forall R_i \in \mathcal{R} \tag{4}$$

Lemma 1 does not give a closed-form characterization of the rules satisfying *stratgy-proofness* and *continuity*, because there are two unknown functions a_i, b_i in (4). To pin down these two unknown functions, "extreme preferences" $\underline{R}_i, \overline{R}_i \in \mathcal{R}$ such that $p(\underline{R}_i) = 0$ and $p(\overline{R}_i) = 1$ are introduced. They are used to link the functions a_i, b_i to ϕ_i in the next lemma.

Lemma 2. *A rule ϕ satisfies strategy-proofness and continuity if and only if for all $i \in N$ and all $R_{N\setminus\{i\}} \in \mathcal{R}^{n-1}$,*

$$\phi_i(R_i, R_{N\setminus\{i\}}) = \text{med}\{p(R_i), \phi_i(\underline{R}_i, R_{N\setminus\{i\}}), \phi_i(\overline{R}_i, R_{N\setminus\{i\}})\} \ \forall R_i \in \mathcal{R} \tag{5}$$

Proof. Setting $R_i = \underline{R}_i$ in Equation (4) gives

$$\phi_i(\underline{R}_i, R_{N\setminus\{i\}}) = \min\{a_i(R_{N\setminus\{i\}}), b_i(R_{N\setminus\{i\}})\}$$

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[6]**Equal-division rule** E**:** For all $R_N \in \mathcal{R}^n$, $E_i(R_N) = \frac{1}{n}$. The generalization of the equal-division rule to an m-dimensional allocation problem is straightforward: $E_i(R_N) = \frac{D}{n}$ for all $R_N \in \mathcal{R}^n$, where $D \in \mathbb{R}^m_+$ is the endowment.

[7]Obviously, it is (*weakly*) *symmetric*, *anonymous*, and *envy-free.*

[8]From a practical point of view, reporting preferences and receiving the information cannot be error-free. The robustness of a rule is guaranteed by "continuity", which ensures that a rule is not very sensitive to any such small errors.

[9]To be a bit more precisely, f is "continuous in $R \in \mathcal{R}$". See Sprumont (1991) for a formal definition of *continuity.*

[10]*Continuity* is implied by *strategy-proofness* and *efficiency*, as shown by Sprumont (1991).

and $R_i = \overline{R}_i$ in (4) gives

$$\phi_i(\overline{R}_i, R_{N\setminus\{i\}}) = \max\{a_i(R_{N\setminus\{i\}}), b_i(R_{N\setminus\{i\}})\}$$

Note that $\phi_i(\underline{R}_i, R_{N\setminus\{i\}}) \leq \phi_i(\overline{R}_i, R_{N\setminus\{i\}})$ for all $R_{N\setminus\{i\}} \in \mathcal{R}^{n-1}$. □

Lemma 2 is still not a closed-form characterization, because the function ϕ_i appears on both sides of (5). This problem can easily be tackled when $n = 2$, as illustrated in the next proposition.

Proposition 1. *Let $n = 2$. A rule ϕ satisfies strategy-proofness and continuity if and only if for $i \neq j$ and all $R_j \in \mathcal{R}$,*

$$\begin{aligned}\phi_i(R_i, R_j) = \text{med}\{&p(R_i), 1 - \text{med}\{p(R_j), \phi_j(\underline{R}_i, \underline{R}_j), \phi_j(\underline{R}_i, \overline{R}_j)\},\\ &1 - \text{med}\{p(R_j), \phi_j(\overline{R}_i, \underline{R}_j), \phi_j(\overline{R}_i, \overline{R}_j)\}\} \ \forall R_i \in \mathcal{R}\end{aligned} \tag{6}$$

Proof. By the feasibility constraint (1),

$$\phi_i(\underline{R}_i, R_j) = 1 - \phi_j(\underline{R}_i, R_j) \tag{7}$$

By (5),

$$\phi_j(\underline{R}_i, R_j) = \text{med}\{p(R_j), \phi_j(\underline{R}_i, \underline{R}_j), \phi_j(\underline{R}_i, \overline{R}_j)\} \tag{8}$$

Substituting (8) into (7) gives

$$\phi_i(\underline{R}_i, R_j) = 1 - \text{med}\{p(R_j), \phi_j(\underline{R}_i, \underline{R}_j), \phi_j(\underline{R}_i, \overline{R}_j)\} \tag{9}$$

Similarly,

$$\phi_i(\overline{R}_i, R_j) = 1 - \text{med}\{p(R_j), \phi_j(\overline{R}_i, \underline{R}_j), \phi_j(\overline{R}_i, \overline{R}_j)\} \tag{10}$$

Substituting (9) and (10) into (5) gives (6). □

Two remarks are in order. First, all $\{\phi_j(\underline{R}_{N\setminus S}, \overline{R}_S)\}_{j=1,2,S\subseteq N}$ in (6) are fixed for ϕ, so they are parameters of ϕ. For notational convenience, let $a_{j,S} = \phi_j(\underline{R}_{N\setminus S}, \overline{R}_S)$ for all $j = 1, 2$ and all $S \subseteq N$. Altogether, there are eight such parameters, which are subject to the following monotonicity constraint (see Lemma 2):

$$a_{j,S} \leq a_{j,S\cup\{j\}} \ \forall j = 1, 2, \forall S \subseteq N \tag{11}$$

and the feasibility constraint (1):

$$a_{1,S} + a_{2,S} = 1 \ \forall S \subseteq N \tag{12}$$

I.e. feasibility implies that only four parameters are independent.

Second, ϕ in Proposition 1 is a function of the two peaks $\{p(R_i)\}_{i=1,2}$ and four parameters, e.g. $\{a_{1,S}\}_{S\subseteq N}$, of which the functional form is the recursive median formula (6). Hence, Proposition 1 is a closed-form characterization of the rules satisfying *strategy-proofness* and *continuity* for $n = 2$.

It should be pointed out that the equal-division rule is *continuous*, because it is a constant rule and any constant rule is trivially *continuous*. To rule out constant rules, we require a rule be responsive to preferences and propose an axiom based on the concept of "null player". Agent $i \in N$ is a **null player** if $p(R_i) = 0$ (or $R_i = \underline{R}_i$). The idea of the "null-player" axiom is that nothing is given to a null player when he is considered to be "irrelevant" to the problem.

A null player may or may not be "relevant" to a problem, depending on whether there is excess supply or excess demand. A null player, who has zero demand, is clearly not responsible for creating excess demand. Hence, we consider a null player to be "irrelevant" when there is excess demand. On the other hand, a null player cannot be considered "irrelevant" when there is excess supply, which is due to insufficient demand. This discussion leads to the following formulation of the "null-player" axiom. A piece of notation is introduced first.

For all $R_N \in \mathcal{R}^n$, let the set of null players be $\underline{N}(R_N) = \{i \in N | R_i = \underline{R}_i\}$ (which can be empty).

Null-player axiom: For all $R_N \in \mathcal{R}^n$, if $\sum p(R_j) \geq 1$, then $\phi_i(R_N) = 0$ for all $i \in \underline{N}(R_N)$.[11]

A simple argument shows that the *null-player* axiom is implied by *efficiency*. Let ϕ be *efficient*. By contradiction, suppose that the *null-player* axiom does not hold for ϕ, i.e. there exist $R_N \in \mathcal{R}^n$ such that $\sum p(R_j) \geq 1$ and $i \in \underline{N}(R_N)$ such that $\phi_i(R_N) > 0$. By feasibility, there exists $k \in N$ such that $\phi_k(R_N) < p(R_k)$. Let z be such that

$$\begin{aligned} z_i &= \phi_i(R_N) - \min\{\phi_i(R_N), p(R_k) - \phi_k(R_N)\} \\ z_k &= \phi_k(R_N) + \min\{\phi_i(R_N), p(R_k) - \phi_k(R_N)\} \\ z_j &= \phi_j(R_N) \text{ for all } j \neq i, k \end{aligned}$$

Clearly, $z_i \; P_i \; \phi_i(R_N)$, $z_k \; P_k \; \phi_k(R_N)$, $z_j \; I_j \; \phi_j(R_N)$ for all $j \neq i, k$, and $z \in Z$, contradicting *efficiency*.

Our result is stated below, which shows that *efficiency* in Theorem 3 can be relaxed to *continuity* and the *null-player* axiom. Hence, it further strengthens Theorems 1 and 2 along the Pareto dimension (in addition to relaxing either *anonymity* or *no-envy* to *weak symmetry*).

Theorem 4. *The uniform rule is the only rule satisfying strategy-proofness, continuity, the null-player axiom, and weak symmetry.*

[11] The *null-player* axiom has no bite in case of excess supply.

The following notation is handy for proving Theorem 4. For all $R_N \in \mathcal{R}^n$, let $\overline{N}(R_N) = \{i \in N | R_i = \overline{R}_i\}$ (which may be called a "full-player" set).

Proof. Let ϕ satisfy *strategy-proofness*, *continuity*, the *null-player* axiom, and *weak symmetry*. Let $R_N \in \mathcal{R}^n$ and $i \in N$. By Lemma 2,

$$\phi_i(R_i, R_{N\setminus\{i\}}) = \text{med}\{p(R_i), \phi_i(\underline{R}_i, R_{N\setminus\{i\}}), \phi_i(\overline{R}_i, R_{N\setminus\{i\}})\}$$

To obtain a closed-form characterization of ϕ, we need to pin down the two functions $\phi_i(\underline{R}_i, R_{N\setminus\{i\}})$ and $\phi_i(\overline{R}_i, R_{N\setminus\{i\}})$ in the above equation. By the feasibility constraint and *weak symmetry* together,

$$\phi_i(\underline{R}_i, R_{N\setminus\{i\}}) = \frac{1 - \sum_{j \notin \underline{N}(\underline{R}_i, R_{N\setminus\{i\}})} \phi_j(\underline{R}_i, R_{N\setminus\{i\}})}{|\underline{N}(\underline{R}_i, R_{N\setminus\{i\}})|} \tag{13}$$

$$\phi_i(\overline{R}_i, R_{N\setminus\{i\}}) = \frac{1 - \sum_{j \notin \overline{N}(\overline{R}_i, R_{N\setminus\{i\}})} \phi_j(\overline{R}_i, R_{N\setminus\{i\}})}{|\overline{N}(\overline{R}_i, R_{N\setminus\{i\}})|} \tag{14}$$

For (13), if $\overline{N}(\underline{R}_i, R_{N\setminus\{i\}}) \neq \emptyset$, then $\sum_{j\neq i} p(R_j) \geq 1$, so the *null-player* axiom implies that

$$\phi_i(\underline{R}_i, R_{N\setminus\{i\}}) = 0 \text{ if } \overline{N}(\underline{R}_i, R_{N\setminus\{i\}}) \neq \emptyset \tag{13$'$}$$

Note that $j \neq i$ in (13) and (14). By Lemma 2,

$$\phi_j(\underline{R}_i, R_{N\setminus\{i\}}) = \text{med}\{p(R_j), \phi_j(\underline{R}_i, \underline{R}_j, R_{N\setminus\{i,j\}}), \phi_j(\underline{R}_i, \overline{R}_j, R_{N\setminus\{i,j\}})\} \tag{15}$$

$$\phi_j(\overline{R}_i, R_{N\setminus\{i\}}) = \text{med}\{p(R_j), \phi_j(\overline{R}_i, \underline{R}_j, R_{N\setminus\{i,j\}}), \phi_j(\overline{R}_i, \overline{R}_j, R_{N\setminus\{i,j\}})\} \tag{16}$$

Repeating the above argument,

$$\phi_j(\underline{R}_i, \underline{R}_j, R_{N\setminus\{i,j\}}) = \frac{1 - \sum_{k \notin \underline{N}(\underline{R}_i, \underline{R}_j, R_{N\setminus\{i,j\}})} \phi_k(\underline{R}_i, \underline{R}_j, R_{N\setminus\{i,j\}})}{|\underline{N}(\underline{R}_i, \underline{R}_j, R_{N\setminus\{i,j\}})|} \tag{17}$$

$$= 0 \text{ if } \overline{N}(\underline{R}_i, \underline{R}_j, R_{N\setminus\{i,j\}}) \neq \emptyset \tag{17$'$}$$

$$\phi_j(\underline{R}_i, \overline{R}_j, R_{N\setminus\{i,j\}}) = \frac{1 - \sum_{k \notin \overline{N}(\underline{R}_i, \overline{R}_j, R_{N\setminus\{i,j\}})} \phi_k(\underline{R}_i, \overline{R}_j, R_{N\setminus\{i,j\}})}{|\overline{N}(\underline{R}_i, \overline{R}_j, R_{N\setminus\{i,j\}})|} \tag{18}$$

$$\phi_j(\overline{R}_i, \underline{R}_j, R_{N\setminus\{i,j\}}) = 0 \tag{19}$$

$$\phi_j(\overline{R}_i, \overline{R}_j, R_{N\setminus\{i,j\}}) = \frac{1 - \sum_{k \notin \overline{N}(\overline{R}_i, \overline{R}_j, R_{N\setminus\{i,j\}})} \phi_k(\overline{R}_i, \overline{R}_j, R_{N\setminus\{i,j\}})}{|\overline{N}(\overline{R}_i, \overline{R}_j, R_{N\setminus\{i,j\}})|} \tag{20}$$

Obviously, $k \neq i, j$ in (17), (20), and $k \neq j$ in (18). Furthermore, it is without loss of generality to assume that $k \neq i$ in (18), because $\phi_i(\underline{R}_i, \overline{R}_j, R_{N\setminus\{i,j\}}) = 0$ (by the *null-player* axiom), so

$$\begin{aligned} &\sum_{k \notin \overline{N}(\underline{R}_i, \overline{R}_j, R_{N\setminus\{i,j\}})} \phi_k(\underline{R}_i, \overline{R}_j, R_{N\setminus\{i,j\}}) \\ = &\sum_{k \notin \overline{N}(\underline{R}_i, \overline{R}_j, R_{N\setminus\{i,j\}}) \cup \{i\}} \phi_k(\underline{R}_i, \overline{R}_j, R_{N\setminus\{i,j\}}) \end{aligned}$$

Now we can repeat the argument for $\phi_k(\underline{R}_i, \underline{R}_j, R_{N\setminus\{i,j\}})$ in (17), $\phi_k(\underline{R}_i, \overline{R}_j, R_{N\setminus\{i,j\}})$ in (18) and $\phi_k(\overline{R}_i, \overline{R}_j, R_{N\setminus\{i,j\}})$ in (20), and keep doing so until $\phi_i(R_i, R_{N\setminus\{i\}})$ becomes a recursive median formula of n peaks $\{p(R_j)\}_{j\in N}$, some zeroes, and some parameters from the list $\{\phi_j(\underline{R}_{N\setminus S}, \overline{R}_S)\}_{j\in N, S\subseteq N}$. With an explicit functional form derived for $\phi_i(R_i, R_{N\setminus\{i\}})$ (the recursive median formula), $\phi_i(R_i, R_{N\setminus\{i\}})$ is unique if and only if the parameters from $\{\phi_j(\underline{R}_{N\setminus S}, \overline{R}_S)\}_{j\in N, S\subseteq N}$ are unique.

The next step is to show uniqueness of all the parameters in $\{\phi_j(\underline{R}_{N\setminus S}, \overline{R}_S)\}_{j\in N, S\subseteq N}$. By *weak symmetry*,

$$\phi_j(\underline{R}_N) = \phi_j(\overline{R}_N) = \frac{1}{n} \text{ for all } j \in N$$

For all $S \subset N$ and all $j \in N$, the *null-player* axiom implies that

$$\phi_j(\underline{R}_{N\setminus S}, \overline{R}_S) = 0 \text{ if } j \notin S$$

and feasibility and *weak symmetry* together imply that

$$\phi_j(\underline{R}_{N\setminus S}, \overline{R}_S) = \frac{1}{s} \text{ if } j \in S$$

Hence, all the parameters $\{\phi_j(\underline{R}_{N\setminus S}, \overline{R}_S)\}_{j\in N, S\subseteq N}$ are unique.

The unique rule is the uniform rule, because the uniform rule satisfies *strategy-proofness*, *continuity*, the *null-player* axiom, and *weak symmetry.* □

4 Discussion

One variation of the model is to assume free disposal. When free disposal is assumed, the following concepts need to be modified.

An **allocation rule** is a function $\phi : \mathcal{R}^n \to [0,1]^n$ such that,

$$\sum \phi_i(R_N) = \min\{\sum p(R_i), 1\} \text{ for all } R_N \in \mathcal{R}^n \text{ and} \tag{1'}$$

$$\phi_j(R_N) = p(R_j) \text{ for all } j \in N \text{ if } \sum p(R_i) \leq 1 \tag{1''}$$

The **uniform rule** is redefined accordingly. For all $R_N \in \mathcal{R}^n$ and all $i \in N$,

$$U_i(R_N) = \begin{cases} p(R_i) & \text{if } \sum p(R_j) \leq 1 \\ \min\{p(R_i), \lambda(R_N)\} & \text{if } \sum p(R_j) \geq 1 \end{cases} \tag{2'}$$

where $\lambda(R_N)$ solves $\sum U_j(R_N) = 1$ when $\sum p(R_j) \geq 1$.

The **null-player** axiom also needs to be modified as follows: For all $R_N \in \mathcal{R}^n$, $\phi_i(R_N) = 0$ for all $i \in \underline{N}(R_N)$.

Note that the above *null-player* axiom can be interpreted as individual rationality for the null player. There is no need to modify other axioms, e.g. *strategy-proofness*, *continuity*, and *weak symmetry*, so we are ready to state the new version of Theorem 4.

Theorem 4′. *The uniform rule is the only rule satisfying strategy-proofness, continuity, the null-player axiom, and weak symmetry.*

The proof of Theorem 4′ is essentially the same as the proof of Theorem 4.

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