Why Does New Hampshire Matter — Simultaneous v.s. Sequential Election with Multiple Candidates

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Abstract

I study and compare preference aggregation in a simultaneous and a sequential multicandidate election. Voters have perfect information about their own preference but do not know the median voter's preference. A voter has an incentive to vote for her second choice for fear that a tie between her second and third choice is more likely than she would like. Therefore, a voter may want to coordinate with supports of her second choice. I show that when voters' preference intensity for their first choice is moderate, in the limit as the electorate increases, there is a unique equilibrium in the voting game within one voting round exhibiting multi-candidate support. In such an equilibrium, the ex ante probability that a candidate wins increases in her supporters' preference intensity and decreases in her opponents' preference intensity. There is too much coordination with supporters of a voter's second choice in that sometimes the median voter's second choice wins the election. A sequential election allows later voters to coordinate with earlier voters. Therefore, in the last voting round, votes are split between the two front runners. The voting outcome in the first round affects the voting behavior of the second round. A victory of a voter's favorite candidate in the first round may change the outcome of the second round from the voter's second choice to her favorite candidate or from her last choice to her second choice. When preference intensity is moderate, voters vote more for their first choice if they vote first in a sequential election than in a simultaneous election, and the probability that the median voter's first choice does not win a voting round is smaller if voting takes place sequentially. Using this model, I show that in a sequential election with ex ante identical states, no matter who the median voter in New Hampshire is, voting first is better than voting second if preference intensity is small.

1 Introduction

The outcomes of early elections play an out-of-proportion role in the US Presidential primary. Adam (1987) reports that the 1984 New Hampshire primary got nearly 20% of the season's coverage in ABC,CBS, NBC and the New York

Times, even though New Hampshire accounts for only 0.4% of the US population, and only four votes out of 538 electoral votes in the presidential election. In the 1980 Republican primaries, George Bush and Ronald Reagan spent about 3/4 of their respective campaign budgets in early primary states, which account for much less than a fifth of the votes in the Republican convention in 1980 (Malbin, 1985). The emphasis on winning early primaries may come from the widely-held belief that early winners gain "momentum" due to the sequential nature of the election.

However, recent primaries have become more "front-loaded" into the early weeks. California has recently passed a legislation to move forward its primary to Feb. 5, 2008, only after 4 other primaries held in January. The media in general views this as "selfish" behavior on the part of those states. It has been argued that a more front-loaded primary system makes it more important for candidates to raise a lot of money early (William Schneider, 1997) and a more front-loaded 2008 primary gives well-established candidates an advantage. On what ground do these assertions stand? And if they are true, through what channel does the timing structure affect the voting outcomes?

Existing literature that study sequential elections has for the most part restricted attention to contests between two candidates. However, there are usually many candidates in a presidential primary. For example, Sen. Hillary Rodham Clinton of New York, Sen. Barack Obama of Illinois and former senator John Edwards of North Carolina, are all considered front runners in the 2008 primary for the Democratic party. With only two candidates, voters simply vote for their preferred candidate. In a multi-candidate contest, however, some voters have to vote strategically for their second choice if they believe their most preferred candidate has a smaller probability of being in a close race. Therefore, voters' beliefs about relative popularity of every candidate, and the relative likelihood of different pivotal events, play an important role in their decision.

Given this element of coordination in multicandidate contest under plurality rule, it is not surprising that with common knowledge assumption of the electoral situation, the voting outcome involves either a complete success or failure of coordination. Duverger's Law (see Riker 1982) asserts that "plurality rule brings about and maintains two-party competition", because only two candidates should be expected to get any vote. This represents complete success of coordination. Most of the literature focuses on these "Duvergerian" equilibria, but offers no formal theory as to which two candidates should be considered "serious" contenders. In addition, it cannot explain the incomplete coordination observed in many multicandidate election outcomes. For example, in the 1970 New York senatorial election, even the trailing candidate among the three got more than 24% of the votes, and the winner gets only 2% more votes than the second.

Moreover, common knowledge of the electoral situation seems a very strong assumption. The 1997 British Election Survey indicates that about two-third of voters who expected their preferred party to come second actually found that it came third (Fisher, 2000). There was clearly lack of common knowledge among voters as to the identities of the first and second place winner, which is inconsistent with that literature.

This paper presents a model of preference aggregation in a multi-candidate election that features a candidate who is "a common second choice" for supporters of the other two extreme candidates. Voters in the model only have imperfect information about the distribution of preferences in the electorate. Supporters of an extreme candidate have an incentive to coordinate with supporters of the "common second choice" against their least favorite candidate. Relaxing common knowledge assumption enables meaningful analysis of this coordination effect. I show that this coordination incentive among supporters of an extreme candidate is stronger when preference intensity for that candidate is smaller, when preference intensity for the opposing extreme candidate is higher, or when the prior belief of the share of supporters of the extreme candidate is smaller. In addition, in those situations, there is excess coordination in that the "common second choice" wins too often, i.e. sometimes "the common second choice" wins even though the median voter favors one of the extreme candidates. One interpretation of "the common second choice" is a candidate that's widely known and considered a "safe option".

I then study an election that involves voting in three states (electorates) in which the candidate winning the most states wins the election. This is close to a Republican primary system. I compare voting behavior and outcomes under simultaneous and sequential election. When preference intensity is not too big, in the last state, supporters of the extreme candidate that has not garnered any victory always vote for the "common second choice". Thus the equilibrium exhibit winnowing down of front runners. In addition, a victory by one extreme candidate in the first state boosts the morale of her supporters in the second state and results in more aggressive voting behavior by her supporters and higher chance of winning in the second state. I show that when preference intensity is moderately small, or when the ex ante share of moderate voters is big (eg. larger than $\frac{1}{2}$), a sequential election reduces excess coordination motive in the first state as compared to the outcome under simultaneous election and reduces the ex ante probability that the candidate winning that state is not the median voter's first choice.

In addition to comparing voting behavior, I can also compare voting outcome between simultaneous and sequential election. Even if sequential election does not make extreme voters in the first state more aggressive, if the median voter in the first state is extreme, then if preference intensity is moderate, she prefers a sequential election to a simultaneous election because she can affect voting outcome in other states toward her favorite candidate. If the median voter in the first state is moderate, then she also prefers sequential election if the probability that an extreme voter wins her state is at least 70% of that of the share of extreme voters.

I can also compare voting outcome across voting order in a sequential election. If the median voter is extreme, then she always prefers voting earlier, i.e. first rather than second. If the median voter is moderate, then she prefers that her state votes first if the other state that her state swaps voting order with is "moderate" state, one in which preference intensity of extreme voters is small, or ex ante share of extreme voters is small. This is because when the other state is "moderate", voting first makes its extreme voters more aggressive, which is bad from a moderate voter's point of view.

2 Literature Review

Dekel and Piccione (2000) and Ali and Kartik (2006) both study sequential elections between two candidates in which some voters have only imperfect information about their own preference over the two candidates. Dekel and Piccione (2000) show that any outcome of a voting equilibrium in a simultaneous election is also an equilibrium outcome of a sequential election with any timing structure. Ali and Kartik (2006), on the other hand, construct a Perfect Bayesian equilibrium in which "herding," i.e. voting according to the history of vote counts so far and disregarding one's own information, happens with positive probability. This suggests that in a race between two candidates, a simultaneous election can be (but is not necessarily due to multiplicity of equilibria) more efficient in gathering information than a sequential election.

Myerson and Weber (1993) and Myerson (2002) both assume common knowledge of the preference distribution of the electorate, and show that under plurality rule, for any pair of candidates in a "three-horse race", there exists an equilibrium in which only this pair are considered "serious" and get any vote. Myerson (2002) call these discriminary equilibria because labeling of the candidates matter as to whether they have positive probability of winning. They argue that "a large multiplicity of equilibria creates a wider scope for focal manipulation by political leaders."

Myerson and Weber (1993) also show via an example the existence of a "non-Duvergerian" equilibria in which a group of voters fail completely to coordinate to avoid the worst outcome, and the two losers exactly tie. They conjecture that some additional assumption of dynamic stability or persistence may be used to eliminate these "non-Duvergerian" equilibria.

This paper is most closely related to Myatt (2007), which studies simultaneous elections under plurality rule in which one candidate (the conservative status quo) has a commonly known fixed fraction ($< \frac{1}{2}$) of supporters, while the rest of the electorate share the distaste of the status quo but disagree on which of the other two (liberal) candidates is optimal. This assumption effectively reduces an election under plurality rule with three candidates to one under qualified-majority rule between two candidates. Essentially, the (liberal) voters have to coordinate behind the two (liberal) candidates. They relax the common knowledge assumption by assuming that each voter gets an imperfect signal about the preference distribution of the electorate (as evident in the UK General Election of 1997). They construct a uniqe symmetric equilibrium that is consistent with the 1970 New York Senatorial election, which displays limited strategic voting and incomplete coordination. However, the assumption of a fixed and commonly known support for one candidate does not seem to fit US

Presidential primaries.

It is difficult to characterize equilibria in a large election because probability ratios of close-race events between different pairs of candidates can be quite intractable. Myatt (2007) develops the solution concept of *strategic-voting equilibrium* for large elections, which can be viewed as a Bayesian Nash equilibrium with a continuum of voters. It facilitates the calculation through law of large numbers arguments. Myerson (2000), on the other hand, tackels this issue by assuming population uncertainty. They assume that voter turnout follows a Poisson process with a commonly known preference distribution. The feature of Poisson process that an individual voter's belief about the behavior of the electorate does not depend on his own preference type facilitates comparison of limiting probabilities of different pivotal events as the size of the electorate goes to infinity.

On relaxing common knowledge assumptions in voting situations, Feddersen and Pesendorfer (1997, 1998) use a common value model for jury decision making. In their model, each juror decides on one of two votes based on a private signal about the defendant's guilt and aims to convict the guilty and acquit the innocent. Thus other jurors' information matters even for a juror's own preference over outcomes. Each juror infers about the merits of his two actions from an assessment of the information possessed by others conditional on his vote being pivotal. Therefore, if other jurors respond a lot to their signals, a juror may have an incentive to disregard his own signal because the information contained in the pivotal event outweighs his own information. This is why bandwagon effects may arise in sequential elections with two candidates in Ali and Kartik (2006). However, since there are only two outcomes, the coordination effect in multicandidate contests is not present in these models.

3 A Multicandidate Contest in One State

3.1 The Model

Three candidates L, M, R compete in a simultaneous election. There are n voters in the electorate where n follows a Poisson distribution with mean N. Each voter has to voter for exactly one candidate. A voter can be of three preference types: a right wing voter, r, prefers candidate R to M to L, a left-wing voter, l, prefers candidate L to M to R, while a moderate voter m prefers candidate M the most and is indifferent between R and L. A voter of preference type i receives payoff U_{ij} when candidate $j \in \{L, M, R\}$ wins the election. Write $\phi_r = \frac{U_{rR} - U_{rM}}{U_{rM} - U_{rL}}$. It represents the preference intensity of a right wing voter for her favorite candidate. Define $u_r = \log(2\phi_r)$. ϕ_l and u_l are defined analogously.

A voter in the electorate is right-wing with probability $F(\eta - \theta)$, left-wing with probability $F(-\eta - \theta)$ and moderate with probability $1 - F(\eta - \theta) - F(-\eta - \theta)$, where F is the cumulative distribution function for Laplace distribution with mean 0 and variance 2. θ is an exogenously given parameter of the model and in a way measures the size of the moderate population. Because preferences are single-peaked, we can define median voter to be the supporter of the Condorcet winner. The median voter is moderate if $\eta \in (-\theta, \theta)$, right-wing if $\eta > \theta$ and left-wing if $\eta < -\theta$.

A voter does not know the ideology of the median voter in her electorate. That is, a voter in the electorate does not know η. She believes that $\eta \sim Laplace\,(0, \alpha)$. Let $G\,(.)$ and $g\,(.)$ denote the cumulative distribution function and the probability density function of the prior. In addition to the common prior about η, voter i gets some additional information about the preference of the electorate. She obtains a signal $\hat{\eta}_i \in Laplace\,(\eta, 1)$ independent of her preference type. Based on her information and the prior, she then forms an updated belief about η. Denote by $f\,(.|\hat{\eta}_i)$ the probability density function of voter i's posterior given her signal $\hat{\eta}_i$.

3.2 Sincere Voting and Coordination Failure

If every voter simply votes for his favorite candidate, then in a large election, vote share of candidate C, denoted by $p_c\,(\eta)$, is almost equal to the probability that voter is of type c. If $2F\,(-\eta) < \frac{2}{3}$, when the share of two exreme voters are equal to each other, it is smaller than the share of moderate voters. Because $p_R\,(\eta)$ increases with η, and $p_R\,(\theta) = \frac{1}{2}$, the median voter preferes R to M if and only if $\eta > \theta$. However, when η is close to θ but smaller than θ, R still gets almost half of the votes, while M and L share the other half. Thus R wins the election even though the median voter is moderate and the majority prefer M to R. This happens because left-wing voters and moderates fail to coordinate with each other and support M together against R. I call this cross-camp coordination failure.

3.3 Equilibria

3.3.1 Strategies and best responses

A voter's type is her ideology-information pair $(o_i, \hat{\eta}_i)$ where $o_i \in \{l, m, r\}$ and $\hat{\eta}_i \in \mathbb{R}$. A pure strategy for voter i is then a mapping from her type to the set of candidates $\{L, M, R\}$. A sincere voting strategy simply chooses the candidate that's most preferred according to voter i's ideology.

There are many equilibria in this game. For example, if every voter votes for candidate j, then a voter is never pivotal and thus she is indifferent between all candidates. Given any two candidates c_1, c_2, there is an equilibrium in which every voter votes for the one in $\{c_1, c_2\}$ that she prefers. In such an equilibrium, the election is reduced to a binary voting game. One can say that the two candidates c_1 and c_2 are the front-runners and the focal point of the election. However, the model cannot answer the question of how front runners are chosen.

For these reasons, we focus on Bayesian Nash equilibria in type-dependent strategies. In particular, we focus on equilibria in symmetric pure voting strategies where the same type-dependent voting strategy $s(o_i, \hat{\eta}_i)$ is used by every voter.

Consider a voter's payoff given that voting strategy s is adopted by all the other voters. Let x_j denote the number of votes candidate j gets from everyone other than voter 0. Then (x_R, x_M, x_L) is a vector of random variables whose distribution depend on the voting strategy v adopted by everyone else. If voter 0 is moderate, then it is her best response to vote for M regardless of her information because she is indifferent between R and L. It is a strict best response as long as $\Pr\{x_R = x_M \cup x_M = x_L|\hat{\eta}_i\} > 0$. If voter 0 is right-wing, then her best response is to vote for R if

$$\begin{aligned} & \left(\Pr\{x_R = x_M|\hat{\eta}_i\} + \frac{1}{2}\Pr\{|x_R - x_M| = 1|\hat{\eta}_i\} + \frac{1}{2}\Pr\{x_R = x_L|\hat{\eta}_i\}\right)(U_{rR} - U_{rM}) \\ \geq\ & \frac{1}{2}\left(\Pr\{x_M = x_L|\hat{\eta}_i\} + \Pr\{x_R = x_L|\hat{\eta}_i\}\right)(U_{rM} - U_{rL}), \end{aligned}$$

and to vote for M otherwise. A left-wing voter's strategy is analogous. Therefore, candidate R gets votes only from right-wing voters.

Denote by $p_j(\eta|v)$ the probability that a voter votes for candidate j conditional on η given that voting strategy v is adopted. Then

$$p_R(\eta|v) = F(\eta - \theta)\Pr\{\hat{\eta}_i : v(r, \hat{\eta}_i) = R|\eta\}.$$

3.3.2 Voting in Large Electorates

We assume that the turn-out, n, follows a Poisson distribution with mean N. Denote by $s_N(o_i, \hat{\eta}_i)$ an equilibrium voting strategy in such an electorate. We focus on the limit of the equilibrium voting strategy $s_N(o_i, \hat{\eta}_i)$ as $N \to \infty$.

Lemma 3.1 *If everyone else in the electorate adopts a voting strategy such that the probability that a voter votes for candidate c is equal to $p_c(\eta)$ when the state variable is η, and voter turn-out follows a Poisson process with mean N, then for any $d \in \{-1, 0, 1\}$,*

$$\begin{aligned} & \lim_{N\to\infty} \frac{\Pr\{|x_R - x_M| = d \text{ and } \min\{x_R, x_M\} > x_L|\hat{\eta}_i, p\}}{\Pr\{|x_L - x_M| = d \text{ and } \min\{x_M, x_L\} > x_R|\hat{\eta}_i, p\}} \\ =\ & \frac{f(\eta_R|\hat{\eta}_i)}{f(\eta_L|\hat{\eta}_i)} \frac{|p'_L(\eta_L) - p'_M(\eta_L)|}{|p'_R(\eta_R) - p'_M(\eta_R)|}, \end{aligned}$$

where η_R is the solution to $p_R(\eta) = p_M(\eta)$ and η_L is the solution to $p_L(\eta) = p_M(\eta)$. In addition, if $p_R(\hat{\eta}) < p_M(\hat{\eta})$ for all solution $\hat{\eta}$ to $p_R(\eta) = p_L(\eta)$, then

$$\lim_{N\to\infty} \frac{\Pr\{|x_R - x_L| = d \text{ and } \min\{x_R, x_L\} > x_M|\hat{\eta}_i, p\}}{\Pr\{|x_j - x_M| = d \text{ and } \min\{x_M, x_j\} > x_{-k}|\hat{\eta}_i, p\}} = 0$$

where $j, k \in \{R, L\}$ and $j \neq k$.

3.3.3 Equilibria Characterization

It follows that a right-wing voter votes for R if and only if

$$\log \frac{f\left(\eta_R|\hat{\eta}_i\right)}{f\left(\eta_L|\hat{\eta}_i\right)} \geq -\log 2\frac{U_{rR}-U_{rM}}{U_{rM}-U_{rL}} - \log \frac{\left|p_R'\left(\eta_R\right)-p_M'\left(\eta_R\right)\right|}{\left|p_L'\left(\eta_L\right)-p_M'\left(\eta_L\right)\right|}$$

where $f\left(.|\hat{\eta}_i\right)$ is a voter's posterior about η given her signal $\hat{\eta}_i$. Using Bayes update, we have

$$\log \frac{f\left(\eta_R|\hat{\eta}_i\right)}{f\left(\eta_L|\hat{\eta}_i\right)} = \begin{cases} \left(\eta_R-\eta_L\right)+2\alpha\left(\eta_0-\frac{\eta_R+\eta_L}{2}\right) & \text{if} \quad \hat{\eta}_i > \eta_R \\ 2\left(\hat{\eta}_i-\frac{\eta_R+\eta_L}{2}\right)+2\alpha\left(\eta_0-\frac{\eta_R+\eta_L}{2}\right) & \text{if} \quad \hat{\eta}_i \in \left(\eta_L,\eta_R\right) \\ -\left(\eta_R-\eta_L\right)+2\alpha\left(\eta_0-\frac{\eta_R+\eta_L}{2}\right) & \text{if} \quad \hat{\eta}_i < \eta_L \end{cases}.$$

Similarly, a left-wing voter votes for L if and only if

$$-\log \frac{f\left(\eta_R|\hat{\eta}_i\right)}{f\left(\eta_L|\hat{\eta}_i\right)} \geq -u_l + \log \frac{\left|p_R'\left(\eta_R\right)-p_M'\left(\eta_R\right)\right|}{\left|p_L'\left(\eta_L\right)-p_M'\left(\eta_L\right)\right|}.$$

Let $BR_N\left(v_N\right)\left(o_i,\hat{\eta}_i\right)$ be a voter's best response when everyone else adopts s_N when the mean of voter turnout is N. Write $u_R = \log 2\frac{U_{rR}-U_{rM}}{U_{rM}-U_{rL}}$, then $\lim_{N\to\infty} BR_N\left(v_N\right)\left(r,\hat{\eta}_i\right) = R$ if and only if

$$\max\left\{\min\left\{\hat{\eta}_i,\eta_R\right\},\eta_L\right\} \geq \left(1+\alpha\right)\frac{\eta_R+\eta_L}{2} - \alpha\eta_0 - \frac{1}{2}\log\frac{\left|p_R'\left(\eta_R\right)-p_M'\left(\eta_R\right)\right|}{\left|p_L'\left(\eta_L\right)-p_M'\left(\eta_L\right)\right|} - \frac{1}{2}u_R$$

where η_c is the such that $\lim_{N\to\infty}\left(p_R\left(\eta_c|v_N\right)-p_M\left(\eta_c|v_N\right)\right) = 0$, for $c \in \{R, L\}$. Let s^* be the limit of s_N. Then by continuity, $\lim_N BR_N\left(s^*\right)\left(r,\hat{\eta}_i\right) = R$ if and only if

$$\max\left\{\min\left\{\hat{\eta}_i,\eta_R\right\},\eta_L\right\} \geq \left(1+\alpha\right)\frac{\eta_R+\eta_L}{2} - \alpha\eta_0 - \frac{1}{2}\log\frac{\left|p_R'\left(\eta_R\right)-p_M'\left(\eta_R\right)\right|}{\left|p_L'\left(\eta_L\right)-p_M'\left(\eta_L\right)\right|} - \frac{1}{2}u_R$$

where η_c is the such that $p_c\left(\eta_c|s^*\right) = p_M\left(\eta_c|s^*\right) > p_{-c}\left(\eta_c|s^*\right)$, for $c \in \{R, L\}$. Therefore, if s^* is the limit of a symmetric equilibrium as $N \to \infty$, then it is a fixed point of the mapping $\lim_N BR_{N\to\infty}$.

A best response to any symmetric voting strategy profile is a cutoff strategy involving an information threshold: r votes for R if $\hat{\eta}_i \geq a - \frac{u_R}{2}$ and l votes for L if and only if $-\hat{\eta}_i \geq -a - \frac{u_L}{2}$. The information cutoff depends on the voter's preference intensity, but also on a systematic bias a. $a > 0$ represents a bias toward L because the information cutoff is higher than preference intensity for right wing voters, but lower than preference intensity for left-wing voters.

If everyone else adopts such a cutoff strategy indexed by a, then the probability that voter i votes for R is equal to

$$p_R\left(\eta;a\right) = F\left(\eta-\theta\right)F\left(\eta-a+\frac{u_R}{2}\right)$$

and the probability that voter i votes for L is

$$p_L(\eta; a) = F(-\eta - \theta) F\left(-\eta + a + \frac{u_L}{2}\right).$$

Because p_R is increasing in η and p_L is decreasing in η, there exists a unique solution $\tilde{\eta}$ to $p_R(\eta) = p_L(\eta)$. For $\theta > \frac{3}{2}$, $F(-\theta) < \frac{1}{3}$. Thus $p_R(\tilde{\eta}) = p_L(\tilde{\eta}) < p_M(\tilde{\eta})$. Therefore, the probability that R ties with L for the winner becomes infinitestimally small relatively to the probability R ties with M for the winner as $N \to \infty$. Define $\eta_R(a)$ to be the solution to $2p_R(\eta; a) + p_L(\eta; a) = 1$. Then if everyone adopts a cutoff strategy s_a indexed by a, the probability that voter i votes for M is equal to the probability that voter i votes for R when $\eta = \eta_R(a)$. When the electorate is large, R ties with M for the winner at η near $\eta_R(a)$. Define

$$\hat{a}(a) = (1+\alpha)\frac{\eta_R(a) + \eta_L(a)}{2} - \alpha\eta_0 - \frac{1}{2}\log\frac{|p_R'(\eta_R(a)) - p_M'(\eta_R(a))|}{|p_L'(\eta_L(a)) - p_M'(\eta_L(a))|}.$$

Then if $\hat{a}(a) \in \left(\eta_L(a) + \frac{u_r}{2}, \eta_R(a) + \frac{u_r}{2}\right)$,

$$\lim_{N\to\infty} BR_N(s_a)(r, \hat{\eta}_i) = \begin{cases} R & \text{if } \hat{\eta}_i \geq \hat{a}(a) - \frac{u_r}{2} \\ M & \text{otherwise} \end{cases}.$$

If $\hat{a}(a) \leq \eta_L(a) + \frac{u_r}{2}$, $\lim_{N\to\infty} BR_N(s_a)(r, \hat{\eta}_i) = R$ and if $\hat{a}(a) \geq \eta_R(a) + \frac{u_r}{2}$, $\lim_{N\to\infty} BR_N(s_a)(r, \hat{\eta}_i) = M$. Therefore, if s_N is a symmetric equilibrium where voters use their information in an electorate with mean N, $\lim_{N\to\infty} s_N$ is a cutoff strategy indexed by a^* where a^* is a fixed point of $\hat{a}$.

We first solve for $p_R(\eta, a) = p_M(\eta, a)$.

Lemma 3.2 *If $u_R + u_L < 0$ and $\theta > \frac{3}{2}$, then*

$$\eta_R(a) = \log\frac{e^{\theta} + e^{a - \frac{u_R}{2}} + \sqrt{e^{2\theta} + e^{2a - u_R} - e^{-\theta + a + \frac{u_L}{2}}}}{2}.$$

Proposition 1 *If $\theta > \frac{u_r + u_l}{4}$, , $\alpha < \frac{1}{4}$ and $\theta > \min\left\{-\frac{u_R + u_L}{4}, -\frac{\max\{u_R, u_L\}}{2} + \log 2\right\}$, then there exists a unique fixed point a^* for the mapping $\hat{a}$. In addition, $a^* \cdot (u_R - u_L) < 0$ and $\eta_R^*(u_r, u_l, \theta, \alpha) := \eta_R(a^*, u_R, u_l, \theta, \alpha) > \theta$, $\eta_L^*(u_r, u_l, \theta, \alpha) := \eta_L(a^*, u_r, u_l, \theta, \alpha) < -\theta$. If the fixed point a^* is in $\left(-\theta + \frac{u_r}{2}, \theta - \frac{u_l}{2}\right)$ or if $u_r + u_l < 0$, then it is a BNE for r to vote for R iff $\hat{\eta}_i \geq a^* - \frac{u_r}{2}$ and l to vote for L iff $-\hat{\eta}_i \geq -a^* - \frac{u_l}{2}$.*

Therefore, the game has a unique symmetric equilibrium with multi-candidate support when the average preference intensity is not too high and the stronger intensity is not too small and when the prior is sufficiently diffused. The equilibrium involves threshold a^* such that a right wing voter votes for R if and only if her information is more optimistic than the threshold $a^* - \frac{u_r}{2}$ and left wing voters vote for L if and only if her information is more optimistic than the threshold $-a^* - \frac{u_l}{2}$. Therefore, a positive a^* imposes a higher threshold for r voters than for l voters. a^* is positive, i.e. r voters behave more conservatively, if right wing preference intensity is weaker, or if prior probability of a right wing voter is smaller than that of a left-wing voter. In addition, the threshold decreases with preference intensity of one's own camp and increases with preference intensity of the opposing camp.

3.4 Comparative Statics

Let $\eta_c^*(u_r, u_l, \theta, \alpha) = \eta_c(a^*(u_r, u_l, \theta, \alpha))$ for $c \in \{R, L\}$.

3.4.1 Comparative Statics of Strategic Voting Equilibrium

Proposition 2 *If* $u_r + u_l < 0$, $\alpha < \frac{1}{4}$, *and* $\theta > \min\left\{-\frac{u_r+u_l}{4}, -\frac{\max\{u_r,u_l\}}{2} + \log 2, \frac{3}{2}\right\}$, *then* $\frac{\partial\left|\eta_j^*(u_r,u_l,\theta,\alpha)\right|}{\partial u_j} < 0$ *and* $\frac{\partial\left|\eta_j^*(u_r,u_l,\theta,\alpha)\right|}{\partial u_k} > 0$ *for* $j \neq k$ *and* $j, k \in \{L, R\}$.

η_R^* decreases with right wing voters' preference intensity u_r and increases with left-wing voters preference intensity u_l. In other words, the prior probability that R wins the election increases with u_r and decreases with u_l. This is true for preference intensities that are not very strong nor too weak.

When left-wing voters' preference intensity u_l goes up, there are two offsetting effects. First, this will increase the information threshold for right-wing voters and thus decrease the probability that a right wing voter votes for R by increasing the fixed point a^*. On the other hand, given the same a, this will decrease the information threshold for left-wing voters, and this will also decrease equilibrium a^*. A stronger left-wing force will eat into the voter base for M, and improves the prospect of R w.r.t. M. When u_l is not too big , the former force dominates.

$\eta_R^*(u, u, \theta, \alpha)$ is decreasing in u and increasing in θ. The ex ante probability that over-coordination occurs, i.e. the ex ante probability that $\eta \in (\theta, \eta_R^*)$ or $(-\eta_L^*, -\theta)$, decreases with θ. Here θ should be viewed as precision of private information.

4 Sequential v.s. Simultaneous Election

4.1 Model

The electorate consists of three states, state 1,2,3, or say, NH, MI and CA. The candidate that wins most states wins the election. In case of a tie between 2 or 3 candidates, the winner is determined by a random draw among those that tie for the first place. The winner within a state is determined also by plurality rule as described in the previous section. Voter i is state k is right-wing with probability $F(\eta_k - \theta_k)$ and left-wing with probability $F(-\eta_k - \theta_k)$. Every voter shares the same prior that η_k's follow *i.i.d. Laplace* $(0, \alpha_k)$. In addition to the common prior, voter i in state k obtains an additional signal $\hat{\eta}_i$ about η_k where $\hat{\eta}_i \sim$ *Laplace* $(\eta_k, 1)$. The independence of η_k's across states implies that there is no learning when voting takes place sequentially. This allows me to focus on the coordination effect of sequential voting. θ_k's and α_k's are common knowledge among voters in every state. Let G_k be the prior distribution of η_k.

Let v_{oc} denote the payoff to voter of ideology type o in state k when candidate c wins the election. We will look at the symmetric case where $v_{rR} = v_{lL} > v_{rM} = v_{lM} > v_{rL} = v_{lR}$ and $v_{mM} > v_{mL} = v_{mR}$. Define

$$\phi_k = \frac{v_{rRk} - v_{rMk}}{v_{rMk} - v_{rLk}}$$

and $u = \log 2\phi_k$. We call ϕ_k the extreme voters' preference intensity for their favorite candidate. Because right wing and left-wing voters both have preference intensity ϕ_k, the threshold a^* is 0 no matter how big ϕ_k is.

4.2 Sequential Election

This section analyzes equilibria in a sequential election and illustrate the coordination effect. We only look at the election where $\phi < \frac{1}{2}$. In such elections, coordination is important because the payoff difference between the second and the least favorite candidate is more than twice that of the first and the second favorite candidate.

4.2.1 Voting in the last state, CA

It is weakly dominant for a moderate voter to vote for M. Given any voting strategy in which m always votes for M, the probability that candidate R ties with L vanishes more quickly than the probability that candidate L ties with M. Therefore, voter i only weighs between the probability of an $R-M$ tie and the probability of an $M-L$ tie.

When candidate L and candidate M each wins one state, then a right wing voter's payoff when candidate c wins the third state is given by $U_{rR} = \frac{v_{rR}+v_{rM}+v_{rL}}{3}$, $U_{rM} = v_{rM}$ and $U_{rL} = v_{rL}$. When $\phi < 1$, $U_{rM} - U_{rL} = \frac{(v_{rR}-v_{rM})-(v_{rM}-v_{rL})}{3} < 0$. Therefore, in both an $R-M$ tie and an $M-L$ tie, a right wing voter prefers to vote for M. Therefore, in all weakly undominated equilibria, a right wing voter votes for M. Thus the last state is a runoff between L and M. L wins the last state and the election if $\eta_3 < -\theta$ and M wins the last state and the election if $\eta_3 > -\theta$.

When candidate L and R each wins one state, $U^{LR}_{rR} - U^{LR}_{rM} = \frac{u_{rR}-u_{rM}+u_{rR}-u_{rL}}{3}$ and $U_{rM} - U_{rL} = \frac{u_{rR}-u_{rL}+u_{rM}-u_{rL}}{3}$. Therefore, the preference intensity for the last-state election, denoted by ϕ^{RL}, is equal to 1. Thus, the equilibrium in the subgame after R and L split the first two states gives rise to the two cutoff points $\eta^*_R(1,1,\alpha,\theta)$ and $\eta^*_L(1,1,\alpha,\theta)$. Because $\theta > \frac{1+1}{4}$, $\eta^*_R(1,1,\alpha,\theta) > \theta$.

4.2.2 Voting in the second state, MI.

In this section we will show how the cutoff points on η_{MI} for different voting outcomes in state 2 depends on the voting outcome in New Hampshire, the first primary. In particular, we will show that when preference intensity for the overall election is moderate, probability that candidate R wins Michigan increases as the winner of New Hampshire changes from L to M to R. In particular, we will analyze how η_R^h changes with h, where $h \in \{R, M, L\}$ is winner in New Hampshire and η_R^h is the lower bound on η_2 for candidate R to win the second state.

Given the voting outcome $h \in \{R, M, L\}$ of state 1, the final election outcome depends on the voting outcome of state 2 and the electoral preference of state 3, η_3. Figure illustrates how the election outcome depends on the voting outcomes of the first two states and η_3.

Consider the voting game in state 2 after candidate R wins the first state. State 2's voting outcome is pivotal only when M will win state 3, i.e. $\eta_3 < \theta$. Therefore, we get $U_{rR} - U_{rM} = G_{CA}(\theta_{CA})(v_{rR} - v_{rM})$, where G_{CA} is the cumulative distribution function of the prior on η_{CA}. But the payoff difference when M wins state 2 v.s. when L wins state 2 gets even smaller. Therefore, we get

$$\begin{aligned}\phi_r^R &= \frac{G(\theta)\,\phi}{\frac{1}{2} - \frac{G(-\theta,\theta)}{2}\phi - \frac{G(-\theta,\theta)}{6}(1-\phi) - \frac{G\left(\theta,\eta_R^*(1,1)\right)}{3}(1-\phi)} \\ &= ,\end{aligned}$$

where $\phi = \phi_{MI}$ is the inherent preference intensity of extreme voters in Michigan and $G = G_{CA}$, $\theta = \theta_{CA}$. So a win by R boosts the preference intensity of right-wing voters in the second state. The ratio $\frac{\phi_r^R}{\phi}$ is higher the weaker the general preference intensity is, and the less likely an extreme candidate will win state 3. Because the game is symmetric, $\phi_l^L = \phi_r^R$. ϕ_r^R is different from the payoff difference ratio in a simultaneous election conditional on one state being taken by candidate R. Conditional on one state being R, a R-win or an M-win makes a difference when state 3 is taken by either M or L. But in a sequential election, L never wins state 3 if R wins state 1 and state 2 is taken by either R or M. In other words, voting outcome in the first two states can change a left-wing state from being taken by L to being taken by M.

Consider the voting game in state 2 after L wins the first state. We get that

$$\phi_r^L = \frac{\frac{1}{2}\phi - \frac{G(-\theta,\theta)}{2} + \frac{G(-\theta,\theta)}{6}(1-\phi) + \frac{G(\theta,\eta_R^*(1,1))}{3}(1-\phi)}{G(\theta)}.$$

If M wins the first state, $\phi_r^M = \phi$. This is because L will not win California unless L wins Michigan. Thus when comparing expected payoff from R being the winner in Michigan and expected payoff from M being the winner, a voter does not need to consider a $M - L$ tie in other states. In other words, when Michigan's vote matters, winner in Michigan is winner of the election. Because whether R or M wins Michigan matters when at least half of the population in California is left-wing, and whether L or M wins MI matters when California is right-wing. That the prior probability of California being left-wing or right-wing implies that within-state preference intensity is equal to inherent pereference intensity.

Therefore, we see that ϕ_r^h increases as h changes from L to M to R. Right wing voters' preference intensity for the voting outcome in the second state is higher the closer the voting outcome in the first state is to their preferred choice. This is not surprising because when $\phi < 1$, $U_R - U_M$ is highest when there is an $R-L$ tie, second when there is an $R-M$ tie, but negative conditional on an $M-L$ tie. Conditional on R winning NH, an $M-L$ tie between NH and CA is ruled out, therefore boosting the payoff difference between a victory by R and a victory by M, while reducing the payoff difference between a victory by M and a victory by L. If L wins the first state, then voting for R is very risky: it is good in an $R-L$ tie but bad in an $M-L$ tie. So $\eta_R^L = \infty$ if

$$\phi \leq \frac{G(-\theta, \theta) - G(\theta, \eta_R^*(1,1))}{1 + G(\theta, \infty) - G(\theta, \eta_R^*(1,1))}$$

This term is increasing in θ. It is sufficient if $\phi < G_k(-\theta_k, \theta_k)$, the prior probability that a voter in California is moderate.

Proposition 3 *If $u < 0, \alpha < \frac{1}{4}$, and $\theta > \min\left\{-\frac{u}{2} + \log 2, \frac{3}{2}\right\}$, then $\eta_R^R(u, \theta, \alpha) < \eta_R^M(u, \theta, \alpha) < \eta_R^L(u, \theta, \alpha)$.*

This follows immediately from Proposition 2 because right wing voters' within-state preference intensity increases while left-wing voters' within-state preference intensity decreases as the winner of New Hampshire changes from L to M to R.

Note that η_R^R is still greater than θ. So the within-camp coordination problem still exists in the primary of Michigan. But this problem is less severe when the the camp's favorite candidate wins New Hampshire and more severe when the camp's worst enemy wins New Hampshire. If we define $\frac{\phi^h}{\phi}$ as the degree of sensitivity of Michigan's within-state preference intensity w.r.t. history, then the following proposition says that the larger moderate population in California is, the more sensitive MI's preference intensity is to history. On the other hand, the stronger MI extreme voters' inherent preference intensity is, the more sensitive their within-state preference intensity is to good news, i.e. to the history where their favorite candidate wins NH, but the less sensitive their within-state preference intensity is to bad news, i.e. to the history where their worst enemy wins N.

Lemma 4.1 $\log \frac{\phi^R}{\phi}$ *increases with both $G_{CA}(\theta_{CA})$ and ϕ, while $\left|\log \frac{\phi^L}{\phi}\right|$ increases with $G_{CA}(\theta_{CA})$ but decreases with ϕ.*

4.2.3 Voting in the first state, NH

Because the final outcome depends on the voting result in MI and CA, eg. when M and L splits MI and CA, victory by R in NH results in a random draw between all three candidates, while victory by M in NH results in a solid victory by M in the final election, the expected payoff difference between a victory by R and M in NH is a linear combination of the payoff difference from final election outcome between R and M and M and L, i.e. $v_R - v_M$ and $v_M - v_L$. Roughly speaking, a victory by R instead of M may change the final winner from R to M, from M to L, from L to M. Because R and L are symmetric in every state,

$$\phi_r^{\emptyset} = \frac{\phi - c^{\emptyset}}{1 - c^{\emptyset}\phi}$$

where

$$\begin{aligned} c^{\emptyset} &= \frac{\frac{1}{3}P(m)F\left(-\infty,\eta_L^R\right) - \frac{1}{3}F\left(\theta,\eta_R^*(1,1)\right)F\left(\eta_L^R\right) - F(-\theta)F\left(\eta_L^R,\eta_L^M\right)}{\begin{array}{c} P(m)F\left(\eta_R^R,\infty\right) + P(m)P(r) + 2P(r)P(l) + \frac{1}{3}P(m)F\left(\eta_L^R\right) \\ -\frac{1}{3}F\left(\theta,\eta_R^*(1,1)\right)F\left(\eta_L^R\right) + F\left(\eta_R^R,\eta_R^M\right)P(r) \end{array}} \\ &= \frac{1}{2}\frac{\frac{2}{3} - \frac{2}{3}\frac{F(\theta,\eta_R^*(1,1))}{P(m)} - 2\frac{F\left(\eta_R^M,\eta_R^L\right)}{P^L(R)}\frac{P(r)}{P(m)}}{\frac{P^R(R)}{P^L(R)} + \frac{P(r)}{P^L(R)} + \frac{1}{2} - \frac{1}{6} - \frac{1}{3}\frac{F\left(\theta,\eta_R^*(1,1)\right)}{P(m)} + 2\frac{P(r)}{P^L(R)}\frac{P(r)}{P(m)} + \frac{F\left(\eta_R^R,\eta_R^M\right)}{P^L(R)}\frac{P(r)}{P(m)}}. \end{aligned}$$

Because $\phi < 1, \phi_r^{\emptyset}$ is decreasing in $c^{\emptyset}$.

Outcome in the first state can change outcome in the second state and/or outcome in state 3. The reason a right-wing voter may strategically vote for M instead of her favorite candidate R is for fear of a tie between M and L and getting L elected instead of M in that situation. Roughly speaking, M and L tie in the overall election when one of the other two states is moderate and the other is left-wing. But when R wins the first state, and M wins the second state, no one votes for L in the third state and M will win the third state and the final election even if the median voter in CA is left-wing.

When $\phi < 1$, an extreme voter worry quite a lot about failing to coordinate with a moderate state and letting L win the election. Note that for the second state, after one victory by M, a victory by R ensures that L cannot win the election. Therefore, for NH, if the effect from changing MI from L to M is small, then the preference intensity for NH voters is smaller than that for MI voters when M wins NH. That is, $\eta_R^{\emptyset} > \eta_R^M$. But if the effect of changing MI from L to M is big, then $\eta_R^{\emptyset} < \eta_R^M$.

4.2.4 Why Does New Hampshire want to vote first?

Does a median voter in NH prefer to vote first or second in a sequential primary? That is, does the median voter in NH prefer to vote first or to switch order with Michigan? This depends on the distribution of preferences in NH and MI. If the median voter in NH and MI are both right-wing, then NH's median voter weakly prefers the more aggressive state to vote first. If NH and MI are ex ante identical, conditional on the super majority in both states being of the same camp, payoff does not depend on whether NH swaps order with MI. If NH is only mildly right-wing and MI is moderate, then whether NH votes first or after knowing that M wins MI may change the identity of the winner in

NH. More specifically, if η^{NH} is between η_R^M and $\eta_R^\emptyset$ and M wins MI regardless of order, then voting first makes right-wing voters behave more aggressively if $\eta_R^\emptyset < \eta_R^M$, while voting after being assured that M wins MI makes them behave more aggressively if $\eta_R^\emptyset > \eta_R^M$. Thus, conditional on NH being mildly right-wing and MI being moderate, median voter in NH prefers to vote first if and only if $\eta_R^\emptyset < \eta_R^M$.

However, if MI is of the opposite camp from NH, then the median voter in NH definitely prefers to vote first. This is because the winner in MI will be M instead of L if NH votes first and R wins NH, which makes the final election outcome more favorable to NH's median voter, or because NH is not right wing enough and thus voting after MI implies voting after knowing that L has won MI, which makes right wing voters in NH more conservative and results in a victory by M instead of R in NH even though the super majority in NH prefers R.

Thus if $\eta_R^\emptyset > \eta_R^M$, conditional on the median voter in NH being right wing, whether she prefers that NH votes first or MI votes first depends on the relative probability of $\left\{\left(\eta^{NH}, \eta^{MI}\right) \in \left(\eta_R^M, \eta_R^\emptyset\right) \times \left(\eta_L^\emptyset, \eta_R^R\right)\right\}$ and $\left\{\left(\eta^{NH}, \eta^{MI}\right) \in \left(\eta_R^\emptyset, \infty\right) \times \left(\eta_L^\emptyset, \eta_L^R\right) \cup \left(\eta_R^\emptyset, \eta_R^L\right) \times \left(\eta_L^R,\right.$

If the median voter in NH is moderate, then they prefer to vote first if and only if knowing that M wins NH tampers the behavior of extreme voters in MI and makes a victory by M more likely. That is, if median voter in NH is moderate, she prefers to vote first if and only $\eta_R^\emptyset < \eta_R^M$.

Thus we can conclude that if $\eta_R^\emptyset < \eta_R^M$, expected payoff from voting first is weakly higher than that from voting second for any η^{NH}. That is, voting first is unambigiously better for voters in New Hampshire when $\eta_R^\emptyset < \eta_R^M$. $\eta_R^\emptyset < \eta_R^M$ when the effect of influencing the second state's behavior is sufficiently large. Thus voting first is unambigiously better than voting second if inherent preference intensity is small. For example, when ϕ is smaller than the probability of a moderate voter in California, winnowing happens in the second state and thus winning the first state is necessary for an extreme candidate to win the election. Therefore, voting first is better than second.

4.3 Simultaneous (Front-loaded) Election

The payoff difference to voter i in state k when candidate c wins state k v.s. candidate c' depends on how the voting outcome in state k affects the election outcome. We will focus on symmetric equilibria in which every voter in very state use the same voting strategy. Suppose voters in the other two states use voting strategy s such that R wins state k if $\eta_k > \tilde{\eta}$ and L wins state k if $\eta_k < -\tilde{\eta}$. Then the probability that R wins state k is $G(-\tilde{\eta})$. Denote by $p^F(c)$ the probability that candidate c wins a state. This vector of probabilities depend on the voting strategy s employed and is determined by $\tilde{\eta}$.

$$\begin{aligned}
U_R^F - U_M^F &= P^F(R) P^F(M)(v_{cR} - v_{cM}) + P^F(R) P^F(L) \frac{(v_{cR} - v_{cM}) + (v_{cR} - v_{cL})}{3} \\
&\quad + P^F(M) P^F(L) \frac{(v_{cR} - v_{cM}) - (v_{cM} - v_{cL})}{3} \\
&= \left(P^F(R) P(M) + \frac{2}{3} P(R) P(L) + \frac{1}{3} P(M) P(L) \right)(v_{cR} - v_{cM}) \\
&\quad - P^F(L)\left(P^F(M) - P^F(R)\right)(v_{cM} - v_{cL}).
\end{aligned}$$

Because the game is symmetric and we are looking for symmetric equilibria, $P^F(R) = P^F(L)$ and we get

$$\begin{aligned}
\phi^F &:= \frac{U_R^F - U_M^F}{U_M^F - U_L^F} \\
&= \frac{\phi - c^F}{1 - c^F \phi}
\end{aligned}$$

where

$$\begin{aligned}
c^F &= \frac{\frac{2}{3} P^F(M) P^F(L) - \frac{2}{3} P^F(R) P^F(L)}{2P^F(R) P^F(M) + 2 \times \frac{2}{3} P^F(R) P^F(L) + 2 \times \frac{1}{3} P^F(M) P^F(L)} \\
&= \frac{\frac{1}{3}\left(P^F(M) - P^F(R)\right)}{\frac{4}{3} P^F(M) + \frac{2}{3} P^F(L)} \\
&= \frac{1}{2} \frac{1 - \frac{P^F(R)}{P^F(M)}}{2 + \frac{P^F(L)}{P^F(M)}}.
\end{aligned}$$

Note that c^F is a function of $\tilde{\eta}$, and thus u^F is a function of u and $\tilde{\eta}$.

Given that voters in the other two states use symmetric voting strategy v characterized by $\tilde{\eta}$, preference intensity for voting outcome of the state is given by $u^F(u, \tilde{\eta})$. Because the game within the state is symmetric, $a^* = 0$. In this equilibrium, an extreme voter votes for her favorite candidate if her signal $\hat{\eta}_i > -\frac{u^F(u,\tilde{\eta})}{2}$. Note that when $\phi < 1$, $\phi^F(\phi, \tilde{\eta}) < \phi$ if and only if $\frac{P^F(R)}{P^F(M)} < 1$. Therefore, in a symmetric equilibrium, the cutoff for R to win a state is $\eta_R^F(u, \theta, \alpha) > \eta_R^*(u, u, \theta, \alpha)$. Define $\eta^F(\tilde{\eta}; u, \theta, \alpha) = \eta_R^*\left(u^F(u, \tilde{\eta}), u^F(u, \tilde{\eta}), \theta, \alpha\right)$. $\eta^F(\tilde{\eta})$ is increasing for $\tilde{\eta} \geq \eta_R^*(u, u, \theta, \alpha)$ and $\eta^F(\eta_R^*(u, u, \theta, \alpha)) > \eta_R^*$. Define the fixed point to be ∞ when $\eta^F(\tilde{\eta}) > \tilde{\eta}$ for all $\tilde{\eta} > \eta_R^*(u, u, \theta, \alpha)$. Then η_R^F is a fixed point of the function. $\eta_R^F = \infty$ is a simultaneous voting equilibrium in which all voters vote for M.

4.4 Comparison between Sequential and Simultaneous Election

4.4.1 Voting Behavior in State 1 (NH) under sequential and front-loaded election

Comparing $\eta_R^{\emptyset}$ and η_R^F is equivalent to comparing $c^{\emptyset}$ and c^F. When $\phi < 1$, $\eta_R^{\emptyset} < \eta_R^F$ if and only if $c^{\emptyset} < c^F$.

Proposition 4 *For θ big enough (for example when $P(r) < \frac{1}{4}$), or u small enough, voters in state 1 behave more aggressively under a sequential election than under a simultaneous election.*

It amounts to finding conditions under which $c^{\emptyset} < c^F$. If we ignore the effect of affecting other states voting behavior, we will be comparing c^F with

$$\frac{\frac{1}{3}P(m)\,F\left(-\infty,\eta_L^R\right)}{P(m)\,F\left(\eta_R^R,\infty\right)+P(m)\,P(r)+2P(r)\,P(l)+\frac{1}{3}P(m)\,F\left(\eta_L^R\right)}.$$

Because NH voters choose without knowing the voting results of MI and CA in both systems, expected payoff difference between R-victory and M-victory depends on probability of $M-L$ tie, $R-L$ tie and $R-M$ tie in MI and CA. Payoff difference between an R-NH and M-NH is largest when R and L split MI and CA. More importantly, an $R-L$ tie offset worries about an $M-L$ tie. The more likely an $R-L$ tie is relatively to an $M-L$ tie, the higher preference intensity is. Because every state is ex ante identical, the more likely an extreme voter will win a state, the higher ϕ^F is. When NH votes in a sequential primary, an R-L tie does not happen. If MI is R and CA is won by L in a simultaneous election, then in a sequential election, left-wing voters in CA will coordinate with moderates and thus M will win CA instead of L, and thus an $R-L$ tie in simultaneous primary turns into an $R-M$ tie in a sequential primary. If MI is won by L and CA by R, then right-wing voters in CA coordinate with moderates and ensure a victory by M in CA instead of R if M wins NH. Again an $R-L$ tie turns into an $R-M$ tie. In other words, victory by M in NH forces voters in CA to coordinate with moderates in CA and in a sense with moderates in NH so that M wins the election instead of a random draw. If this channel is important, $\phi^F > \phi^{\emptyset}$. This channel is important if the probability that an extreme candidate wins a state in a simultaneous election is low.

r voters in NH worry about $M-L$ tie. If the probability of an $M-L$ tie is smaller, then preference intensity is bigger. An $M-L$ tie happens in a sequential primary with half of the probability of that in a simultaneous primary. This is because if MI is M and CA is L, a victory by R in NH forces left-wing voters in CA to coordinate with moderates, which result in a sure victory by M instead of an $M-L$ tie. This channel increases preference intensity under a sequential primary relative to that under a simultaneous primary.

Which channel is more important depends on whether an $R-L$ tie or an $L-R$ is more likely or a $L-M$ tie. By symmetry, $\phi^{\emptyset}$ is higher if $P^F(R) < \frac{1}{2}P^F(M)$.

Because voters behave too conservatively in a strategic voting equilibrium, this is true whenever ex ante share of an extreme voter is no bigger than half of that of a moderate voter. This explains why $\eta_R^{\emptyset} < \eta_R^F$ when θ is large.

In a sequential primary, who wins NH affects voting behavior in MI and CA. In particular, it affects voting behavior in MI. In particular, victory by R in NH makes it harder for L to win MI, thus making an $L-M$ tie less likely. This effect increases NH's preference intensity in a sequential primary. This effect is larger when intrinsic preference intensity in MI is smaller, or when the ex ante share of extreme preference voters is smaller.

4.4.2 Comparing election winner between Simultaneous and Sequential Primary

I will consider parameters such that extreme voters in New Hampshire behave more aggressively in a sequential election than in a simultaneous election, i.e. $\theta < \eta_R^{\emptyset} < \eta_R^F$. In this situation, $\eta_R^h > \theta$ for any history in both election systems. Therefore, M always wins a state whenever M is the condorcet winner in that state. Thus, if M is the condorcet winner in at least two states, M will win the election regardless of primary system. I then need to discuss only cases where either an extreme candidate is the Condorcet winner in at least two states, or the Condorcet winner is different in every state.

rrr: it seems straightforward that the a better system selects candidate R more often. The Condorcet winner in state k is R if and only if $\eta^k > \theta$, but the winner in state k is R if and only if $\eta^k > \eta_R^h$ where h is either F which indicates the simultaneous system or a history in the sequential system. R wins the election if R wins at least two states. Because $h \in \{\emptyset, R, M, RR, RM, MM\}$ given that $\eta^k > \theta$ for every state k, $\eta_R^h > \eta_R^F$. Therefore, if R wins in a simultaneous system, R wins in a sequential system, and there are $\left(\eta^{NH}, \eta^{MI}, \eta^{CA}\right)$ such that R wins in a sequential system but M wins in a simultaneous system.

rrm or rmr or mrr: $h \in \{\emptyset, R, M, RR, RM, MR, MM\}$.

rrl: R wins if and only if R win both $r-$states. If R wins only one r-state and M wins the $l-$state, or if R wins no r-states, then M wins the election. If R wins an r-state in simultaneous election, then R wins that state in sequential election because $\eta_R^h < \eta_R^F$ for all possible histories an r state faces in a sequential election in this case. Therefore, whenever R wins a simultaneous election, R would win a sequential election. If R wins only one r-state and L wins the l-state, then all three candidates tie in the election and the outcome is a random draw among the three. Because $\phi < 1$, all voters prefer a sure victoyr by M to a random draw among every candidate. Thus this is the worst election outcome in this case. Because $\eta_L^h = -\infty$ for $h \in \{RM, MR\}$, when California's vote matters, left-wing voters there will not vote for L and L will not win California if the first two states are both right-wing. Therefore, a sequential election never produces the worst outcome, a three-way tie, while a simultaneous election may. So sequential election produces better outcome conditional on rrl.

rlr: R wins the election if and only if R win both r-states. The set of histories that an $r-$state may face in this case is $\{\emptyset, RM, RL, MM, ML\}$. Therefore, if an R wins an r-state in simultaneous election, R would win in a sequential election as well. Therefore, R wins a simultanoues election only if R wins a sequential election. If a sequential primary results in a three-way split, then the winner of each state in the order of voting must be RLM because R will never win California in a sequential primary after history ML. The outcome under sequential is worse than that under simtulaneous election if and only if the winner in order under sequential is RLM while that under simultaneous is MLM. The election outcome would change from $\frac{1}{3}R+\frac{1}{3}M+\frac{1}{3}L$, a random draw among all, to M when the system becomes simultaneous if $\left(\eta^{NH},\eta^{MI},\eta^{CA}\right)\in$

$$\left(\eta_R^{\emptyset},\eta_R^F\right)\times\left(-\infty,\eta_L^R\right)\times\left(\theta,\eta_R^*\left(1\right)\right).$$

This is the only parameter range in this case where the outcome under sequential is worse than that under simultaneous. On the other hand, the outcome would change from R to $\frac{1}{3}R+\frac{1}{3}M+\frac{1}{3}L$ when the primary becomes simultaneous if $\left(\eta^{NH},\eta^{MI},\eta^{CA}\right)\in$

$$\left(\eta_R^{\emptyset},\eta_R^F\right)\times\left(-\infty,\eta_L^R\right)\times\left(\eta_R^F,\infty\right).$$

Because R is the best outcome and a random draw among all is the worst, the second effect more than cancels out the first if $G_{CA}\left(\eta_R^F,\infty\right)<G_{CA}\left(\theta,\eta_R^*\left(1\right)\right)$. Because $G_{CA}\left(-\theta,\theta\right)>\frac{1}{3}$, this is true only if the probability that R will win California in a simultaneous election is less than $\frac{1}{6}$ or only if $G_{CA}\left(\theta,\eta_R^*\left(1\right)\right)>\frac{1}{6}$. But then, $MLM\rightarrow MLR$

$$\left(\theta,\eta_R^{\emptyset}\right)\times\left(-\infty,\eta_L^M\right)\times\left(\eta_R^F,\infty\right).$$

This effect cancels out the first if $G_{CA}\left(\eta_R^{\emptyset},\eta_R^F\right)<G_{CA}\left(\eta_R^F,\infty\right)$.

The only problem is when $G_{CA}\left(\eta_R^F,\infty\right)<\min\left\{G_{CA}\left(\theta,\eta_R^*\left(1\right)\right),G_{CA}\left(\eta_R^{\emptyset},\eta_R^F\right)\right\}$. Then $G_{CA}\left(\theta,\eta_R^F\right)>2G_{CA}\left(\eta_R^F,\infty\right)$.

Suppose the parameters are such that extreme voters in New Hampshire behave more aggressively in a sequential primary than a simultaneous primary. Because M always wins a state whenever M is the condorcet winner in that state, while there is always too much coordination cross camp other than in the last primary in a sequential election after the camp's favorite candidate splits with M, conditional on the median voter in every state supports the same candidate, the universally favored candidate is the winner with higher probability in a sequential election than in a simultaneous election. In addition, sequential primary facilitates coordination across camp across states and thus a three-way split between all candidates is less likely to happen under a sequentail primary. When $\phi<1$, every voter prefers a sure victory by M than a random draw among all candidates. Thus every voter prefers a sequential primary conditional on the simultanoues primary outcome being a three-way split. In general, if the median voter in two states support the same candidate, then sequential primary is preferred unless the candidate most preferred by median voters in the last two states, MI and CA, is an extreme candidate, say R, and the median voter in the NH supports the other extreme candidate, say L. In the latter situation, the eventual winner may be M instead of R if MI is the probability of right-wing voter is not high enough. This is due to the disproportionate impact of the winner in NH in a sequential primary.

4.4.3 For NH — Sequential or Simultaneous?

If voters in NH behave more aggresively in a sequential election, then if the median voter in NH is extreme, he must prefer sequential primary to simultaneous primary. In fact, even if voting behavior is less aggressive in a sequential election, as long as the difference is small, if median voter in NH is extreme, he still prefers sequential primary. This is because the voting outcome in NH may change voting outcome in MI and/or CA toward NH's median voter's preferred candidate. For example, if $\eta_1 > \max\left\{\eta_R^F, \eta_R^\emptyset\right\}$, then the voting outcome in NH is R regardless of primary system. This makes it harder for L to win MI than if the primary system is simultaneous. If $\eta^{MI} < \eta_L^R$, then moving to a sequential primary changes the voting outcome in MI from L to M, which futher changes voting outcome in CA from L to M thus final winner from L to M if $\eta^{CA} < \eta_L^F$. Suppose η_2 is such that outcome in MI does not depend on primary system either. Then it changes CA's voting outcome from L to M and final election outcome from a random draw among all three to M if $\eta^{CA} < \eta_L^F$ and the primary system is sequential intead of simultaneous.

If in addition to the effect of changing voting outcome in MI and/or CA from L to M or from M to R, voting behavior in NH is more aggressive in a sequential primary, then moving to a sequential primary changes winner in NH from M to R if $\eta^{NH} \in \left(\eta_R^\emptyset, \eta_R^F\right)$. That r voters in NH vote for R with positive probability in equilibrium indicates that expected payoff if R wins NH is higher than that if M wins NH.

If the median voter in NH is moderate, then the median voter prefers sequential primary if and only if

$$F\left(\eta_L^F\right)^2 - F\left(\eta_L^F, -\theta\right) F\left(\eta_L^M\right) - F\left(\eta_L^F, \eta_L^M\right) F\left(\eta_L^F\right) > 0.$$

This holds if

$$F\left(\eta_L^F\right) > \frac{\left(1+\sqrt{2}\right)}{2+\sqrt{2}} F\left(-\theta\right).$$

So if voters don't behave too conservatively in a simultaneous election, then if NH's median voter is moderate, he prefers sequential election. In a sequential election, the effect of forcing forcing left-wing voters to coordinate with NH's moderates when R wins MI makes sequential election preferable to a moderate voter in NH. However, if both MI and CA are extreme on the same side, eg. both left-wing, then left-wing voters in CA are much more aggressive in a sequential election because they are now sure of an $M-L$ tie. In addition, extreme voters in MI behave more aggressively when then know that M wins NH. This increases the probability of a final victory by an extreme candidate if the primary system is sequential. Which one is better for a moderate median voter depends on which happens with higher probability.

5 Conclusion

This paper studies preference aggregation in a multi-candidate contest when the preference of the electorate is not common knowledge. In a multi-candidate contest, voters have an incentive to coordinate with supporters of their second choice to avoid a victory by the least favorite candidate. I show that the coordination incentive is stronger when preference intensity is weaker. I then use this model as cornerstone to compare a simultaneous election in which several states vote at the same time and a sequential election in which each state votes one by one after observing outcomes of previous states. I show that when the prior probability of extreme voters is small or when the preference intensity of extreme voters is small, coordination incentives are stronger for extreme voters and thus they vote more aggressively in a sequential election than in a simultaneous election. As a result, the prior probability that the winner in a state is not the first choice of the median voter is smaller in a sequential election.

6 Appendix

6.1 Proof for lemma 3.1.

Proof. It suffices to show that

$$\lim_{N\to\infty} N \Pr\{V_R = V_M > V_L|\hat{\eta}_i, p\} = \frac{f(\eta_R|\hat{\eta}_i)}{|p'_R(\eta_R) - p'_M(\eta_R)|}.$$

Let

$$H^u = \{(V_R, V_M, V_L)\,|V_R = V_M > V_L \text{ where } V_c \geq 0 \text{ for } c = R, M, L\}$$

Then

$$\Pr\{V_R = V_M > V_L|\hat{\eta}_i, p\} = \int_{\eta=-\infty}^{\infty} P(H^u|N, p(\eta))\, f(\eta|\hat{\eta}_i)\, d\eta.$$

Let

$$H = \{(V_R, V_M, V_L)\,|V_R = V_M \text{ where } V_c \geq 0 \text{ for } c = R, M, L\}$$

and $H^* = \{(V_R, V_M, V_L)\,|V_R = V_M \text{ where } V_c \geq 0 \text{ for } c = R, M, L\}$. Then H is a hyperplane in $(N \cup \{0\})^3$ spanned by $w_1 = (1, 1, 0)$ and $w_2 = (0, 0, 1)$.

Given η, we first show that $y_N := \left(\left[N\sqrt{p_R(\eta)\, p_M(\eta)}\right], \left[N\sqrt{p_R(\eta)\, p_M(\eta)}\right], [Np_L(\eta)]\right)$ is a near maximizer $\sum_c p_c \psi\left(\frac{x(c)}{Np_c}\right)$ over x in H^* where $\psi(\theta) = \theta(1 - \log\theta) - 1$. $H^* = \{\gamma(1, 1, 0) + j(0, 0, 1)\,|\gamma \geq 0 \text{ and } j \geq 0\}$. Let

$$(\gamma^*, j^*) \in \arg\max_{\gamma\geq 0, j\geq 0} \left(p_R\psi\left(\frac{\gamma}{Np_R}\right) + p_M\psi\left(\frac{\gamma}{Np_M}\right) + p_L\psi\left(\frac{j}{Np_L}\right)\right).$$

Because the derivative is ∞ for $\gamma = 0$ or $j = 0$ and the function goes to 0 as γ or $j \to \infty$, the solution must be interior of H^*. Thus γ^*, j^* satisfy the first order condition:

$$\begin{aligned} 0 &= -\log \frac{\gamma}{Np_R} - \log \frac{\gamma}{Np_M} \\ 0 &= -\log \frac{j}{Np_L}. \end{aligned}$$

So $\gamma^* = N\sqrt{p_R p_M}$ and $j^* = Np_L$. Then y_N as defined is a near maximizer.

$$\begin{aligned} & p_R \psi\left(\frac{\gamma^*}{Np_R}\right) + p_M \psi\left(\frac{\gamma^*}{Np_M}\right) + p_L \psi\left(\frac{j^*}{Np_L}\right) \\ = \;& p_R\left(\frac{\gamma^*}{Np_R}\left(1 - \log\left(\frac{\gamma^*}{Np_R}\right)\right) - 1\right) \\ & + p_M\left(\frac{\gamma^*}{Np_M}\left(1 - \log\left(\frac{\gamma^*}{Np_M}\right)\right) - 1\right) \\ & + p_L\left(\frac{j^*}{Np_L}\left(1 - \log\left(\frac{j^*}{Np_L}\right)\right) - 1\right) \\ = \;& -1 + \frac{\gamma^*}{N}\left(1 - \log\left(\frac{\gamma^*}{Np_R}\right) + 1 - \log\left(\frac{\gamma^*}{Np_M}\right)\right) \\ & + \frac{j^*}{N}\left(1 - \log\left(\frac{j^*}{Np_L}\right)\right) \\ = \;& -1 + 2\frac{\gamma^*}{N} + \frac{j^*}{N} \\ = \;& 2\sqrt{p_R p_M} - (1 - p_L) \\ = \;& 2\sqrt{p_R p_M} - p_R - p_M \\ = \;& -\left(\sqrt{p_R} - \sqrt{p_M}\right)^2. \end{aligned}$$

Then using theorem 3 in Myerson (2000),

$$\lim_{N\to\infty} \frac{\Pr\{H|Np(\eta)\}}{\Pr\{y_N|Np(\eta)\}(2\pi)\left(\det\left(M\left(y_N\right)\right)\right)^{-0.5}} = 1$$

where $M\left(y_N(\eta)\right) = \begin{bmatrix} \frac{2}{\left[N\sqrt{P_R(\eta)p_M(\eta)}\right]} & 0 \\ 0 & \frac{1}{[Np_L(\eta)]} \end{bmatrix}$ and $\lim_{N\to\infty} N * M\left(y_N\right) =$

$\begin{bmatrix} \frac{2}{\sqrt{p_R(\eta)p_M(\eta)}} & 0 \\ 0 & \frac{1}{p_L(\eta)} \end{bmatrix}$. By Myerson (2000),

$$\begin{aligned}
\Pr\{y_N|Np(\eta)\} &\approx \frac{e^{N*\left(p_R\psi\left(\frac{\gamma^*}{Np_R}\right)+p_M\psi\left(\frac{\gamma^*}{Np_M}\right)+p_L\psi\left(\frac{j^*}{Np_L}\right)\right)}}{\Pi_{c\in\{R,M,L\}}\sqrt{2\pi y_N(c)}} \\
&= \frac{e^{-N\left(\sqrt{p_R}-\sqrt{p_M}\right)^2}}{(2\pi)^{\frac{3}{2}}\sqrt{(\gamma^*)^2 j^*}} \\
&= \frac{e^{-N\left(\sqrt{p_R}-\sqrt{p_M}\right)^2}}{(2N\pi)^{\frac{3}{2}}\sqrt{p_R p_M p_L}}.
\end{aligned}$$

$$\begin{aligned}
(\det(M(y_N)))^{-0.5} &\approx \left(\frac{1}{N^2\sqrt{p_R p_M}p_L}\right)^{-0.5} \\
&= N\sqrt{\sqrt{p_R p_M}p_L}.
\end{aligned}$$

So

$$\begin{aligned}
\Pr\{H^*|Np(\eta)\} &\approx \Pr\{y_N|Np(\eta)\}(2\pi)(\det(M(y_N)))^{-0.5} \\
&\approx N\sqrt{\sqrt{p_R p_M}p_L}\,(2\pi)\,\frac{e^{-N\left(\sqrt{p_R}-\sqrt{p_M}\right)^2}}{(2N\pi)^{\frac{3}{2}}\sqrt{p_R p_M p_L}} \\
&= \frac{e^{-N\left(\sqrt{p_R}-\sqrt{p_M}\right)^2}}{\sqrt{2\pi N}\sqrt{\sqrt{p_R p_M}}}.
\end{aligned}$$

Given $\varepsilon > 0$, let δ be such that $|p_R(\eta) - p_M(\eta)| \geq \varepsilon$ for all η such that $|\eta - \eta_R| \geq \delta$. Define $\Lambda_\delta := \{\eta : |\eta - \eta_R| < \delta\}$. Then want to show that $\lim_{N\to\infty} \frac{\Pr\{H|Np(\eta)\}}{\Pr(H^*|Np(\eta))} = 1$ for $\eta \in \Lambda_\delta$. Then show that $\lim_{N\to\infty} N\Pr\{H^*|Np(\eta)\} = 0$ for $\eta \notin \Lambda_\delta$. Then

$$\begin{aligned}
&\lim_{N\to\infty} N\Pr\{V_R = V_M > V_L|\hat{\eta}_i, p\} \\
= &\lim_{N\to\infty} N\int_\eta \Pr\{H|Np(\eta)\} f(\eta|\hat{\eta}_i)\,d\eta \\
= &\lim_{N\to\infty}\left(N\int_{\eta\in\Lambda_\delta}\Pr\{H|Np(\eta)\} f(\eta|\hat{\eta}_i)\,d\eta + N\int_{\eta\notin\Lambda_\delta}\Pr\{H|Np(\eta)\} f(\eta|\hat{\eta}_i)\,d\eta\right) \\
= &\lim_{N\to\infty} N\int_{\eta\in\Lambda_\delta}\Pr\{H|Np(\eta)\} f(\eta|\hat{\eta}_i)\,d\eta + \lim_{N\to\infty} N\int_{\eta\notin\Lambda_\delta}\Pr\{H|Np(\eta)\} f(\eta|\hat{\eta}_i)\,d\eta.
\end{aligned}$$

$$\begin{aligned}
&\lim_{N\to\infty} N\int_{\eta\notin\Lambda_\delta}\Pr\{H|Np(\eta)\} f(\eta|\hat{\eta}_i)\,d\eta \\
\leq &\lim_{N\to\infty}\int_{\eta\notin\Lambda_\delta} N\Pr\{H|Np(\eta)\} f(\eta|\hat{\eta}_i)\,d\eta \\
= &\ 0.
\end{aligned}$$

And

$$
\begin{aligned}
& N\int_{\eta\in\Lambda_\delta} \Pr\left\{H|Np(\eta)\right\} f(\eta|\hat{\eta}_i)\, d\eta \\
\in\quad & \left[\frac{\sqrt{2}f(\eta_R|\hat{\eta}_i)}{\sqrt{\sqrt{\frac{p_M(\eta_R)}{p_R(\eta_R)}}p_R'(\eta_R)-\sqrt{\sqrt{\frac{p_R(\eta_R)}{p_M(\eta_R)}}p_M'(\eta_R)}}-\zeta,\frac{\sqrt{2}f(\eta_R|\hat{\eta}_i)}{\sqrt{\sqrt{\frac{p_M(\eta_R)}{p_R(\eta_R)}}p_R'(\eta_R)-\sqrt{\sqrt{\frac{p_R(\eta_R)}{p_M(\eta_R)}}p_M'(\eta_R)}}+\zeta\right] \\
& *\int_{\eta\in\Lambda_\delta} N\left(\frac{\sqrt{2}}{\sqrt{\sqrt{\frac{p_M}{p_R}}p_R'(\eta)-\sqrt{\sqrt{\frac{p_R}{p_M}}p_M'(\eta)}}}\right)^{-1} \Pr\left\{H|Np(\eta)\right\} d\eta \\
=\quad & \left[\frac{\sqrt{2}f(\eta_R|\hat{\eta}_i)}{p_R'(\eta_R)-p_M'(\eta_R)}-\zeta,\frac{\sqrt{2}f(\eta_R|\hat{\eta}_i)}{p_R'(\eta_R)-p_M'(\eta_R)}+\zeta\right] \\
& *\int_{\eta\in\Lambda_\delta} N\left(\frac{\sqrt{2}}{\sqrt{\sqrt{\frac{p_M}{p_R}}p_R'(\eta)-\sqrt{\sqrt{\frac{p_R}{p_M}}p_M'(\eta)}}}\right)^{-1} \Pr\left\{H|Np(\eta)\right\} d\eta
\end{aligned}
$$

$$
\begin{aligned}
& \int_{\eta\in\Lambda_\delta} N\Pr\left\{H|Np(\eta)\right\} d\eta \\
=\quad & \int_{\eta=\eta_R-\varepsilon}^{\eta_R+\varepsilon} \frac{\sqrt{N}e^{-N\left(\sqrt{p_R(\eta)}-\sqrt{p_M(\eta)}\right)^2}}{\sqrt{2\pi}\sqrt{\sqrt{p_R(\eta)\,p_M(\eta)}}} d\eta.
\end{aligned}
$$

Write $x=\sqrt{2N}\left(\sqrt{p_R(\eta)}-\sqrt{p_M(\eta)}\right)$. Then

$$
\begin{aligned}
dx &= \sqrt{2N}\frac{\sqrt{p_M(\eta)}p_R'(\eta)-\sqrt{p_R(\eta)}p_M'(\eta)}{2\sqrt{p_Rp_M}}d\eta \\
&= \sqrt{N}\frac{\sqrt{\sqrt{\frac{p_M}{p_R}}p_R'(\eta)-\sqrt{\sqrt{\frac{p_R}{p_M}}p_M'(\eta)}}}{\sqrt{2}\sqrt{\sqrt{p_Rp_M}}}d\eta
\end{aligned}
$$

$$
\begin{aligned}
& \int_{\eta\in\Lambda_\delta} N\Pr\left\{H|Np(\eta)\right\} d\eta \\
=\quad & \int_{x=\sqrt{2N}\left(\sqrt{p_R(\eta_R-\varepsilon)}-\sqrt{p_M(\eta_R-\varepsilon)}\right)}^{\sqrt{2N}\left(\sqrt{p_R(\eta_R+\varepsilon)}-\sqrt{p_M(\eta_R+\varepsilon)}\right)} \frac{\sqrt{2}}{\sqrt{\sqrt{\frac{p_M}{p_R}}p_R'(\eta)-\sqrt{\sqrt{\frac{p_R}{p_M}}p_M'(\eta)}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx.
\end{aligned}
$$

Then

$$\lim_{N\to\infty}\int_{\eta\in\Lambda_\delta} N\left(\frac{\sqrt{2}}{\sqrt{\sqrt{\frac{p_M}{p_R}}p_R'(\eta)}-\sqrt{\sqrt{\frac{p_R}{p_M}}p_M'(\eta)}}\right)^{-1}\Pr\{H|Np(\eta)\}\,d\eta$$

$$= \lim_{N\to\infty}\int_{x=\sqrt{2N}\left(\sqrt{p_R(\eta_R-\varepsilon)}-\sqrt{p_M(\eta_R-\varepsilon)}\right)}^{\sqrt{2N}\left(\sqrt{p_R(\eta_R+\varepsilon)}-\sqrt{p_M(\eta_R+\varepsilon)}\right)}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx$$

$$= 1.$$

Let $\zeta\to 0$. Then we get $\lim_{N\to\infty}N\Pr\{V_R=V_M>V_L|\hat{\eta}_i,p\}=\frac{\sqrt{2}f(\eta_R|\hat{\eta}_i)}{p_R'(\eta_R)-p_M'(\eta_R)}$. Need $\frac{\sqrt{2}f(\eta|\hat{\eta}_i)}{\sqrt{\sqrt{\frac{p_M(\eta)}{p_R(\eta)}}p_R'(\eta)}-\sqrt{\sqrt{\frac{p_R(\eta)}{p_M(\eta)}}p_M'(\eta)}}$ to be absolutely continuous. ■

6.2 Additional proofs and lemmas for Section3.3.3

Lemma 6.1 *If $\theta>\log\frac{3}{2}$, then $p_R(\tilde{\eta};s)<p_M(\tilde{\eta};s)$ if $p_R(\tilde{\eta};s)=p_M(\tilde{\eta};s)$ where $p_c(\eta;s)$ is the probability that a voter using strategy s votes for candidate c.*

Proof. Let $F_c(\eta)$ denote the probability that a voter's favorite candidate is c. Then $F_M(0)=1-2F(-\theta)=1-e^{-\theta}>\frac{1}{3}$ and $F_R(0)=F_L(0)=\frac{1}{3}$. In addition, $F_M(\eta)-F_L(\eta)$ first increases and then decreases on $(-\infty,0]$ and $F_M(0)>F_L(0)$ because for $\eta<(-\theta,0)$,

$$\begin{aligned}\frac{\partial(F_M(\eta)-F_L(\eta))}{\partial\eta} &= \frac{\partial}{\partial\eta}\left(1-e^{-\theta-\eta}-\frac{1}{2}e^{-(\theta-\eta)}\right)\\ &= e^{-\theta}\left(e^{-\eta}-\frac{1}{2}e^{\eta}\right)\end{aligned}$$

which is positive iff $\eta<-\frac{\log 2}{2}$. For $\eta<-\theta$,

$$F_M(\eta)-F_R(\eta)=\frac{1}{2}e^{\eta+\theta}-e^{\eta-\theta}=e^{\eta-\theta}\left(\frac{1}{2}e^{2\theta}-1\right)$$

which is positive because $e^\theta>\frac{3}{2}$. So for all η, $F_M(\eta)>\min\{F_R(\eta),F_L(\eta)\}$.

Because an extreme candidate can only get votes from its supporters, given any voting strategy, $p_c(\eta;s)\le F_c(\eta)$ for $c=R,L$ and $p_M(\eta;s)\ge F_M(\eta)$. If $p_R(\tilde{\eta};s)=p_L(\tilde{\eta};s)$, then $p_R(\tilde{\eta};s)=\min\{p_R(\tilde{\eta};s),p_L(\tilde{\eta};s)\}\le\min\{F_R(\tilde{\eta}),F_L(\tilde{\eta})\}<F_M(\tilde{\eta})$. ■

Lemma 6.2 *If $\theta>\frac{u_R+u_L}{4}$, then $\eta_R(a)>\max\left\{\theta,a-\frac{u_R}{2}\right\}$ and $\eta_L(a)<\min\left\{-\theta,a+\frac{u_L}{2}\right\}$.*

Proof. We first observe that $2p_R(\eta,a)+p_L(\eta,a)$ is increasing in η.

Case 1 $\theta < a - \frac{\tilde{u}_R}{2}$. *Suppose to the contrary that* $\eta_R^* \in \left[\theta, a - \frac{\tilde{u}_R}{2}\right]$, *then*

$$\begin{aligned}
& 2p_R'(\eta) + p_L'(\eta) \\
= \;& \frac{1}{2}e^{-\eta+\theta}e^{\eta-a+\frac{\tilde{u}_R}{2}} + \left(1 - \frac{1}{2}e^{-(\eta-\theta)}\right)e^{\left(\eta-a+\frac{\tilde{u}_R}{2}\right)} \\
& -\frac{1}{2}e^{-\eta-\theta}\left(F\left(-\eta+a+\frac{\tilde{u}_L}{2}\right) + f\left(-\eta+a+\frac{u_L}{2}\right)\right) \\
> \;& e^{\eta-a+\frac{\tilde{u}_R}{2}} - \frac{1}{2}e^{-\eta-\theta} \\
(\quad \because \;& F(x) + f(x) \le 1 \text{ for all } x)
\end{aligned}$$

and $2p_R''(\eta) + p_L''(\eta) > 0$ *for all* $\eta \in \left(\theta, a - \frac{u_R}{2}\right)$. *Thus the maximum of* $2p_R(\eta) + p_L(\eta)$ *on* $\left[\theta, a - \frac{u_R}{2}\right]$ *is attained at either* $\eta = \theta$ *or* $\theta = a - \frac{u_R}{2}$. *But*

$$\begin{aligned}
& 2p_R\left(a - \frac{\tilde{u}_R}{2}\right) + p_L\left(a - \frac{\tilde{u}_R}{2}\right) - 1 \\
\le \;& -\frac{1}{2}e^{\theta-a+\frac{\tilde{u}_R}{2}} + \frac{1}{2}e^{-\theta-a+\frac{\tilde{u}_R}{2}} < 0,
\end{aligned}$$

and

$$\begin{aligned}
& 2p_R(\theta) + p_L(\theta) - 1 \\
= \;& \frac{1}{2}e^{\theta-a+\frac{u_R}{2}} + \frac{1}{2}e^{-2\theta}F\left(-\theta+a+\frac{u_L}{2}\right) - 1 \\
< \;& \frac{1}{2} + \frac{1}{2}e^{-2\theta} - 1 < 0.
\end{aligned}$$

Thus $2p_R(\eta_R) + p_L(\eta_R) - 1 < 0$, *contradiction.*

Case 2 $a - \frac{u_R}{2} \le \theta$. *In this case,*

$$\begin{aligned}
& 2p_R'(\eta) + p_L'(\eta) \\
= \;& e^{\eta-\theta} - \frac{1}{2}e^{-\eta-\theta}\left(F\left(-\eta+a+\frac{\tilde{u}_L}{2}\right) + f\left(-\eta+a+\frac{\tilde{u}_L}{2}\right)\right) \\
> \;& e^{\eta-\theta} - \frac{1}{2}e^{-\eta-\theta} > 0
\end{aligned}$$

for all $\eta > 0$ *because* $F(x) + f(x) \le 1$ *for all* x *(*

$$F(x) + f(x) = \begin{cases} 1 - \frac{1}{2}e^{-x} + \frac{1}{2}e^{-x} = 1 & \text{if } x > 0 \\ \frac{1}{2}e^{x} + \frac{1}{2}e^{x} = e^{x} < 1 & \text{if } x < 0 \end{cases}$$

. *Suppose to the contrary that* $\eta_R^* \in \left[a - \frac{\tilde{u}_R}{2}, \theta\right]$. *Because* $\eta_R^* > 0 > -\theta$, *then*

$$\begin{aligned}
0 &= 2p_R(\eta_R) + p_L(\eta_R) - 1 \\
&= \left(1 - \frac{1}{2}e^{-\eta + a - \frac{\tilde{u}_R}{2}}\right)e^{\eta - \theta} + \frac{1}{2}e^{-\eta_R - \theta}F\left(-\eta + a + \frac{\tilde{u}_L}{2}\right) - 1 \\
&\leq 2p_R(\theta) + p_L(\theta) - 1 \\
&\leq -\frac{1}{2}e^{-\theta + a - \frac{\tilde{u}_R}{2}} + \frac{1}{2}e^{-\theta - \theta}\frac{1}{2}e^{-\theta + a + \frac{u_L}{2}} \\
&\quad \textit{(this is because } 1 - \frac{1}{2}e^{-x} \textit{ is smaller than} \\
&\quad \frac{1}{2}e^{x} \textit{ for } x \textit{ greater than } 0\textit{)} \\
&< 0
\end{aligned}$$

if $a - \frac{u_R}{2} > -2\theta + a + \frac{u_L}{2} - \log 2$, *i.e.* $\theta > \frac{u_R + u_L}{4} - \log 2$. *Thus a contradiction because we assume* $\theta > \frac{u_R + u_L}{4}$.

■

Lemma 6.3 *If* $\theta > \frac{u_R + u_L}{4}$, *then* $\sqrt{e^{2\theta} + e^{2a - u_R} - e^{-\theta + a + \frac{1}{2}u_L}} > \max\left\{\frac{e^{\theta} + e^{a - \frac{u_R}{2}}}{2}, \left|e^{\theta} - e^{a - \frac{u_R}{2}}\right|\right\}$

Proof.

$$\begin{aligned}
& 4\sqrt{e^{2\theta} + e^{2a - \tilde{u}_R} - e^{a - \theta + \frac{\tilde{u}_L}{2}}}^2 - \left(e^{a - \frac{u_R}{2}} + e^{\theta}\right)^2 \\
&= 3\left(e^{2\theta} + e^{2a - \tilde{u}_R}\right) - 4e^{a - \theta + \frac{\tilde{u}_L}{2}} - 2e^{a - \frac{u_R}{2} + \theta} \\
&= 3\left(e^{a - \frac{u_R}{2}} - e^{\theta}\right)^2 + 4\left(e^{a - \frac{u_R}{2} + \theta} - e^{a - \theta + \frac{\tilde{u}_L}{2}}\right) \\
&\geq 0
\end{aligned}$$

if $\frac{u_R + u_L}{2} < 2\theta$. ■

Lemma 6.4 *For a such that* $\max\left\{\theta, a - \frac{u_R}{2}\right\} > a + \frac{u_L}{2}$ *and* $\max\left\{\theta, -a - \frac{u_L}{2}\right\} > -a + \frac{u_R}{2}$,

$$\hat{\eta}_R(a) = \log\left(\frac{1}{2}e^{\theta} + \frac{1}{2}e^{a - \frac{1}{2}u_R} + \frac{1}{2}\sqrt{e^{2\theta} + e^{2a - u_R} - e^{-\theta + a + \frac{1}{2}u_L}}\right),$$

and

$$\hat{\eta}_L(a) = -\log\left(\frac{1}{2}e^{\theta} + \frac{1}{2}e^{-a - \frac{1}{2}u_L} + \frac{1}{2}\sqrt{e^{2\theta} + e^{-2a - u_L} - e^{-\theta - a + \frac{1}{2}u_R}}\right),$$

and

$$\begin{aligned}
& \hat{a}(a) \\
= & \left(1+\frac{\alpha}{2}\right)\left[\begin{array}{c} \log\left(e^{\theta}+e^{a-\frac{1}{2}u_R}+\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}\right) \\ -\log\left(e^{\theta}+e^{-a-\frac{1}{2}u_L}+\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}\right) \end{array}\right] \\
& -\frac{1}{2}\left[\begin{array}{c} \log\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}} \\ -\log\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}} \end{array}\right]-\alpha\eta_0
\end{aligned}$$

Proof. Suppose $\eta_R > \max\left\{\theta, a-\frac{1}{2}\tilde{u}_R, a+\frac{\tilde{u}_L}{2}\right\}$ and $\eta_L < \min\left\{-\theta, a+\frac{u_L}{2}, -a-\frac{u_R}{2}\right\}$, then η_R is the solution to

$$1=2\left(1-\frac{1}{2}e^{-(\eta_R-\theta)}\right)\left(1-\frac{1}{2}e^{-\left(\eta_R-a+\frac{\tilde{u}_R}{2}\right)}\right)+\frac{1}{2}e^{-\eta_R-\theta}\frac{1}{2}e^{-\eta_R+a+\frac{\tilde{u}_L}{2}}.$$

So

$$e^{\eta_R}=\frac{1}{2}e^{\theta}+\frac{1}{2}e^{a-\frac{1}{2}u_R}+\frac{1}{2}\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}.$$

By symmetry,

$$e^{-\eta_L}=\frac{1}{2}e^{\theta}+\frac{1}{2}e^{-a-\frac{1}{2}u_L}+\frac{1}{2}\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}.$$

If $\theta > \frac{u_R+u_L}{4}$, then for $a < \theta-\frac{u_L}{2}$,

$$\begin{aligned}
\eta_R(a) &= \log\left(\frac{1}{2}e^{\theta}+\frac{1}{2}e^{a-\frac{1}{2}u_R}+\frac{1}{2}\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}\right) \\
&> \max\left\{\theta, a-\frac{u_R}{2}, a+\frac{u_L}{2}\right\}.
\end{aligned}$$

Suppose $\eta_R > \max\left\{\theta, a-\frac{1}{2\beta}\tilde{u}_R\right\}$. Then

$$\begin{aligned}
p_R'(\eta_R) &= (1-F(\eta_R-\theta))F\left(\eta_R-a+\frac{\tilde{u}_R}{2}\right)+F(\eta_R-\theta)\left(1-F\left(\eta_R-a+\frac{1}{2}\tilde{u}_R\right)\right) \\
&= F\left(\eta_R-a+\frac{\tilde{u}_R}{2}\right)+F(\eta_R-\theta)-2p_R(\eta_R).
\end{aligned}$$

If $\eta_R > \max\left\{-\theta, a+\frac{\tilde{u}_L}{2}\right\}$, then

$$p_L'(\eta_R)=-2p_L(\eta_R).$$

So if $\eta_R > \max\left\{\theta, a - \frac{1}{2}\tilde{u}_R, a + \frac{\tilde{u}_L}{2}\right\}$, then

$$\begin{aligned} & p'_R(\eta_R) - p'_M(\eta_R) \\ = \;& 2p'_R(\eta_R) + p'_L(\eta_R) \\ = \;& 2F\left(\eta_R - a + \frac{\tilde{u}_R}{2}\right) + 2F(\eta_R - \theta) - 4p_R(\eta_R) - 2p_L(\eta_R) \\ = \;& 2F\left(\eta_R - a + \frac{\tilde{u}_R}{2}\right) + 2F(\eta_R - \theta) - 2 \\ = \;& 1 - e^{-\left(\eta_R - a + \frac{\tilde{u}_R}{2}\right)} + 1 - e^{-(\eta_R - \theta)} \in (0, 2) \\ = \;& e^{-\eta_R}\left(2e^{\eta_R} - e^{\theta} - e^{a - \frac{u_R}{2}}\right) \\ = \;& e^{-\eta_R}\sqrt{e^{2\theta} + e^{2a - u_R} - e^{-\theta + a + \frac{1}{2}u_L}}. \end{aligned}$$

So

$$\begin{aligned} \frac{\left|p'_R(\eta^*_R) - p'_M(\eta^*_R)\right|}{\left|p'_L(\eta^*_L) - p'_M(\eta^*_L)\right|} &= \frac{1 - e^{-\eta_R + a - \frac{\tilde{u}_R}{2}} + 1 - e^{-(\eta_R - \theta)}}{1 - e^{\eta_L - a - \frac{1}{2}\tilde{u}_L} + 1 - e^{\eta_L - \theta}} \\ &= \frac{\frac{2e^{\eta_R} - \left(e^{a - \frac{u_R}{2}} + e^{\theta}\right)}{e^{\eta_R}}}{\frac{2e^{-\eta_L} - \left(e^{-a - \frac{u_L}{2}} + e^{\theta}\right)}{e^{-\eta_L}}} \\ &= e^{-\eta_R - \eta_L}\frac{\sqrt{e^{2\theta} + e^{2a - u_R} - e^{-\theta + a + \frac{1}{2}u_L}}}{\sqrt{e^{2\theta} + e^{-2a - u_L} - e^{-\theta - a + \frac{1}{2}u_R}}}. \end{aligned}$$

We thus get $\hat{a}(a)$ but substituting these expressions into

$$\hat{a}(a) = (1 + \alpha)\frac{\eta_R + \eta_L}{2} - \alpha\eta_0 - \frac{1}{2}\log\frac{\left|p'_R(\eta^*_R) - p'_M(\eta^*_R)\right|}{\left|p'_L(\eta^*_L) - p'_M(\eta^*_L)\right|}.$$

■

Lemma 6.5 *If $\theta > \frac{u_R + u_L}{4}$, then for a such that* $\max\left\{\theta, a - \frac{u_R}{2}\right\} > a + \frac{u_L}{2}$ *and* $\max\left\{\theta, -a - \frac{u_L}{2}\right\} > -a + \frac{u_R}{2}$,

$$\begin{aligned} \hat{a}'(a) \;=\;& \frac{1}{2} + \frac{1}{2}\frac{e^{a - \frac{u_R}{2}} - e^{\theta}}{\sqrt{e^{2\theta} + e^{2a - u_R} - e^{-\theta + a + \frac{1}{2}u_L}}}\left(1 - \frac{\frac{e^{\theta} + e^{a - \frac{u_R}{2}}}{2}}{\sqrt{e^{2\theta} + e^{2a - u_R} - e^{-\theta + a + \frac{1}{2}u_L}}}\right) \\ & + \frac{1}{2}\frac{e^{-a - \frac{u_L}{2}} - e^{\theta}}{\sqrt{e^{2\theta} + e^{-2a - u_L} - e^{-\theta - a + \frac{1}{2}u_R}}}\left(1 - \frac{\frac{e^{\theta} + e^{-a - \frac{u_L}{2}}}{2}}{\sqrt{e^{2\theta} + e^{-2a - u_L} - e^{-\theta - a + \frac{1}{2}u_R}}}\right) \\ & + \frac{\alpha}{2}\left(1 + \frac{1}{2}\frac{e^{a - \frac{u_R}{2}} - e^{\theta}}{\sqrt{e^{2\theta} + e^{2a - u_R} - e^{-\theta + a + \frac{1}{2}u_L}}} + \frac{1}{2}\frac{e^{-a - \frac{u_L}{2}} - e^{\theta}}{\sqrt{e^{2\theta} + e^{-2a - u_L} - e^{-\theta - a + \frac{1}{2}u_R}}}\right). \end{aligned}$$

Proof. From lemma, because $\theta > \frac{u_R+u_L}{4}$, we have $\eta_R(a) > \max\left\{\theta, a - \frac{u_r}{2}\right\}$ and $-\eta_L(a) < -\max\left\{\theta, -a - \frac{u_l}{2}\right\}$.

Again, if $\eta_R > \left\{\theta, a - \frac{1}{2\beta}\tilde{u}_R, a + \frac{\tilde{u}_L}{2\beta}\right\}$, then

$$\begin{aligned}\frac{\partial p_R(\eta_R; a)}{\partial a} &= -F(\eta_R - \theta) + p_R(\eta_R)\\ \frac{\partial p_L(\eta_R; a)}{\partial a} &= p_L(\eta_R).\end{aligned}$$

So

$$\begin{aligned}\frac{\partial \eta_R(a)}{\partial a} &= -\frac{2\frac{\partial p_R(\eta_R;a)}{\partial a} + \frac{\partial p_L(\eta_R;a)}{\partial a}}{2p'_R(\eta_R) + p'_L(\eta_R)} = \frac{2F(\eta_R - \theta) - 1}{1 - e^{-\eta_R + a - \frac{\tilde{u}_R}{2}} + 1 - e^{-(\eta_R - \theta)}}\\ &= \frac{1 - e^{-(\eta_R - \theta)}}{1 - e^{-\eta_R + a - \frac{\tilde{u}_R}{2}} + 1 - e^{-(\eta_R - \theta)}} \in (0,1)\\ &= \frac{1}{2}\frac{e^{a - \frac{u_R}{2}} - e^{\theta} + \sqrt{e^{2\theta} + e^{2a - u_R} - e^{-\theta + a + \frac{1}{2}u_L}}}{\sqrt{e^{2\theta} + e^{2a - u_R} - e^{-\theta + a + \frac{1}{2}u_L}}}\\ &= \frac{1}{2}\left(1 + \frac{e^{a - \frac{u_R}{2}} - e^{\theta}}{\sqrt{e^{2\theta} + e^{2a - u_R} - e^{-\theta + a + \frac{1}{2}u_L}}}\right)\end{aligned}$$

because $\eta_R(a) > \max\left\{a - \frac{u_R}{2}, \theta\right\}$. I have checked that this expression is equal to

$$\frac{\partial \eta_R(a)}{\partial a} = \frac{e^{a - \frac{u_R}{2}} + \frac{2e^{2a - u_R} - e^{-\theta + a + \frac{u_L}{2}}}{2\sqrt{e^{2\theta} + e^{2a - u_R} - e^{-\theta + a + \frac{1}{2}u_L}}}}{e^{a - \frac{u_R}{2}} - e^{\theta} + \sqrt{e^{2\theta} + e^{2a - u_R} - e^{-\theta + a + \frac{1}{2}u_L}}}.$$

Because

$$\frac{|p'_R(\eta^*_R) - p'_M(\eta^*_R)|}{|p'_L(\eta^*_L) - p'_M(\eta^*_L)|} = \frac{1 - e^{-\eta_R + a - \frac{\tilde{u}_R}{2}} + 1 - e^{-(\eta_R - \theta)}}{1 - e^{\eta_L - a - \frac{1}{2}\tilde{u}_L} + 1 - e^{\eta_L - \theta}},$$

$$\begin{aligned}\frac{\partial}{\partial a}\log\frac{|p'_R(\eta^*_R) - p'_M(\eta^*_R)|}{|p'_L(\eta^*_L) - p'_M(\eta^*_L)|} &= \frac{\left(e^{-\eta_R + a - \frac{\tilde{u}_R}{2}} + e^{-(\eta_R - \theta)}\right)\frac{\partial \eta_R}{\partial a} - e^{-\eta_R + a - \frac{\tilde{u}_R}{2}}}{1 - e^{-\eta_R + a - \frac{\tilde{u}_R}{2}} + 1 - e^{-(\eta_R - \theta)}}\\ &\quad - \frac{-\left(e^{\eta_L - a - \frac{1}{2}\tilde{u}_L} + e^{\eta_L + \theta}\right)\frac{\partial \eta_L}{\partial a} + e^{\eta_L - a - \frac{1}{2}\tilde{u}_L}}{1 - e^{\eta_L - a - \frac{1}{2}\tilde{u}_L} + 1 - e^{\eta_L - \theta}}\\ &= \frac{\left(e^{\theta} + e^{a - \frac{u_R}{2}}\right)\frac{\partial \eta_R}{\partial a} - e^{a - \frac{u_R}{2}}}{\sqrt{e^{2\theta} + e^{2a - u_R} - e^{-\theta + a + \frac{1}{2}u_L}}}\\ &\quad + \frac{\left(e^{\theta} + e^{-a - \frac{u_L}{2}}\right)\frac{\partial \eta_L}{\partial a} - e^{-a - \frac{u_L}{2}}}{\sqrt{e^{2\theta} + e^{-2a - u_L} - e^{-\theta - a + \frac{1}{2}u_R}}}.\end{aligned}$$

So

$$
\begin{aligned}
\hat{a}'(a) &= \frac{1}{2}(1+\alpha)\left(\frac{\partial \eta_R}{\partial a}+\frac{\partial \eta_L}{\partial a}\right)-\frac{1}{2}\frac{\partial}{\partial a}\log\frac{\left|p_R'(\eta_R^*)-p_M'(\eta_R^*)\right|}{\left|p_L'(\eta_L^*)-p_M'(\eta_L^*)\right|} \\
&= \frac{1}{2}\left[\frac{e^{a-\frac{u_R}{2}}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}+\left(1+\alpha-\frac{\left(e^{\theta}+e^{a-\frac{u_R}{2}}\right)}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\right)\frac{\partial \eta_R}{\partial a}\right] \\
&\quad +\frac{1}{2}\left[\frac{e^{-a-\frac{u_L}{2}}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}+\left(1+\alpha-\frac{\left(e^{\theta}+e^{-a-\frac{u_L}{2}}\right)}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right)\frac{\partial \eta_L}{\partial a}\right] \\
&= \frac{1}{2}\left[\begin{array}{c}\frac{e^{a-\frac{u_R}{2}}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}+ \\ \left(1+\alpha-\frac{\left(e^{\theta}+e^{a-\frac{u_R}{2}}\right)}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\right)\frac{1}{2}\left(1+\frac{e^{a-\frac{u_R}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\right)\end{array}\right] \\
&\quad +\frac{1}{2}[]_L \\
&= \frac{1}{2}\left[\begin{array}{c}\frac{1+\alpha}{2} \\ +\frac{e^{a-\frac{u_R}{2}}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}-\frac{1}{2}\frac{\left(e^{\theta}+e^{a-\frac{u_R}{2}}\right)}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}} \\ +\frac{1}{2}\frac{e^{a-\frac{u_R}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\left(1+\alpha-\frac{\left(e^{\theta}+e^{a-\frac{u_R}{2}}\right)}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\right)\end{array}\right]+[]_L \\
&= \frac{1}{2}\left[\begin{array}{c}\frac{1+\alpha}{2}+\frac{1}{2}\frac{e^{a-\frac{u_R}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}} \\ +\frac{1}{2}\frac{e^{a-\frac{u_R}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\left(1+\alpha-\frac{\left(e^{\theta}+e^{a-\frac{u_R}{2}}\right)}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\right)\end{array}\right]+[]_L \\
&= \frac{1}{2}\left[\begin{array}{c}\frac{1+\alpha}{2} \\ +\frac{1}{2}\frac{e^{a-\frac{u_R}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\left(2+\alpha-\frac{\left(e^{\theta}+e^{a-\frac{u_R}{2}}\right)}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\right)\end{array}\right]+[]_L \\
&= \frac{1}{2}\left[\begin{array}{c}\frac{1+\alpha}{2} \\ +\frac{e^{a-\frac{u_R}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\left(1+\frac{\alpha}{2}-\frac{\frac{e^{\theta}+e^{a-\frac{u_R}{2}}}{2}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\right)\end{array}\right]+[]_L \\
&= \frac{1+\alpha}{2}+\frac{1}{2}\frac{e^{a-\frac{u_R}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\left(1+\frac{\alpha}{2}-\frac{\frac{e^{\theta}+e^{a-\frac{u_R}{2}}}{2}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\right) \\
&\quad +\frac{1}{2}\frac{e^{-a-\frac{u_L}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\left(1+\frac{\alpha}{2}-\frac{\frac{e^{\theta}+e^{-a-\frac{u_L}{2}}}{2}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right) \\
&= \frac{1}{2}+\frac{1}{2}\frac{e^{a-\frac{u_R}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\left(1-\frac{\frac{e^{\theta}+e^{a-\frac{u_R}{2}}}{2}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\right) \\
&\quad +\frac{1}{2}\frac{e^{-a-\frac{u_L}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\left(1-\frac{\frac{e^{\theta}+e^{-a-\frac{u_L}{2}}}{2}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right) \\
&\quad +\frac{\alpha}{2}\left(1+\frac{1}{2}\frac{e^{a-\frac{u_R}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}+\frac{1}{2}\frac{e^{-a-\frac{u_L}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right)
\end{aligned}
$$

Because $\max\left\{\left|e^{a-\frac{u_R}{2}}-e^{\theta}\right|, \frac{e^{a-\frac{u_R}{2}}+e^{\theta}}{2}\right\}<\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}$ and $\max\left\{\left|e^{-a-\frac{u_L}{2}}-e^{\theta}\right|, \frac{e^{-a-\frac{u_L}{2}}+e^{\theta}}{2}\right\}<\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}$, $\hat{a}'(a)>0$ for all a and $\hat{a}'(a)<1$ for α sufficiently small. ■

Therefore, for α sufficiently small, $\hat{a}'(a)<1$ if $u_R+u_L<0$ or for all $a\in\left(-\theta+\frac{u_R}{2},\theta-\frac{u_L}{2}\right)$. Let a^* denote a fixed point of $\hat{a}$.

Observation $\log\frac{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\log\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)}{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)}>0$ if $\theta>\frac{3}{4}(u_R+u_L)-\log 2$.

Proof.

$$\begin{aligned}
&\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}-\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}\\
=\ &\frac{e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}-\left(e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}\right)}{\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}+\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}}\\
=\ &\frac{e^{2a-\tilde{u}_R}-e^{-2a-\tilde{u}_L}-\left(e^{a-\theta+\frac{\tilde{u}_L}{2}}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}\right)}{\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}+\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}}\\
=\ &\frac{\left(e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}\right)\left(e^{a-\frac{u_R}{2}}+e^{-a-\frac{u_L}{2}}\right)-\left(e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}\right)e^{-\theta+\frac{u_R+u_L}{2}}}{\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}+\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}}\\
=\ &\left(e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}\right)\frac{e^{a-\frac{u_R}{2}}+e^{-a-\frac{u_L}{2}}-e^{-\theta+\frac{u_R+u_L}{2}}}{\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}+\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}}.
\end{aligned}$$

And

$$\begin{aligned}
&\frac{e^{a-\frac{u_R}{2}}+e^{-a-\frac{u_L}{2}}-e^{-\theta+\frac{u_R+u_L}{2}}}{\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}+\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}}\\
\geq\ &\frac{2e^{-\frac{u_R+u_L}{4}}-e^{-\theta+\frac{u_R+u_L}{2}}}{\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}+\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}}>0
\end{aligned}$$

if $\theta>\frac{3}{4}(u_R+u_L)-\log 2$. ■

Observation If $\theta<-\frac{u_R+u_L}{4}$, then $\left|\log\frac{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right|>\left|\log\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)}{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)}\right|$ and $\log\frac{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\log\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)}{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)}>0$. If $\theta>\max\left\{-\frac{u_R+u_L}{4},\frac{3}{4}(u_R+u_L)\right\}$, then $\left|\log \frac{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right| < \left|\log \frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)}{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)}\right|$.

Proof.

$$\begin{aligned}
&\left(\frac{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right)^2 - \frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)^2}{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)^2} \\
= \quad & \frac{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}} - \frac{e^{2a-u_R}+e^{2\theta}+2e^{a-\frac{u_R}{2}+\theta}}{e^{-2a-u_L}+e^{2\theta}+2e^{-a-\frac{u_L}{2}+\theta}} \\
= \quad & \frac{\left(e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}\right)\left(1-e^{2\theta+\frac{u_R+u_L}{2}}\right)\left(2e^{\theta-\frac{u_R+u_L}{2}}+e^{-\theta}\right)}{\left(e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}\right)\left(e^{-2a-u_L}+e^{2\theta}+2e^{-a-\frac{u_L}{2}+\theta}\right)}.
\end{aligned}$$

If $\theta < -\frac{u_R+u_L}{4}$, then either $\left(\frac{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right)^2 > \frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)^2}{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)^2} > 1$ because $e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}} > 0$ or $\left(\frac{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right)^2 < \frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)^2}{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)^2} < 1$ because $e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}} < 0$. So $\left|\log \frac{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right| > \left|\log \frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)}{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)}\right|$ and $\log \frac{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}} \log \frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)}{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)} > 0$. If $\theta > \max\left\{-\frac{u_R+u_L}{4}, \frac{3}{4}\left(u_R+u_L\right)\right\}$, then either $\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)^2}{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)^2} > \left(\frac{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right)^2 > 1$ when $e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}} > 0$ or $\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)^2}{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)^2} < \left(\frac{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right)^2 < 1$ when $e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}} < 0$. ■

Observation $\left(\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)^2}{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}} - \frac{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)^2}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right)\left(e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}\right) > 0$ if $\theta > \log\frac{3}{2}$.

Proof.

$$
\begin{aligned}
&\left(\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)^2}{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}\right)^2-\left(\frac{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)^2}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right)^2\\
=\;&\frac{1}{\left(e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}\right)\left(e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}\right)}\\
&\times\left\{\begin{array}{l}\left(e^{2a-u_R}+e^{2\theta}+2e^{\theta+a-\frac{u_R}{2}}\right)^2\left(e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}\right)\\ -\left(e^{-2a-u_L}+e^{2\theta}+2e^{\theta-a-\frac{u_L}{2}}\right)^2\left(e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}\right)\end{array}\right\}\\
\geq\;&0
\end{aligned}
$$

after some algebra.

$$
\begin{aligned}
&\left(\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)^2}{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}\right)^2-\left(\frac{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)^2}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right)^2\\
=\;&\frac{1}{\left(e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}\right)\left(e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}\right)}\\
&\times\left\{\begin{array}{l}\left(e^{2a-u_R}+e^{2\theta}+2e^{\theta+a-\frac{u_R}{2}}\right)^2\left(e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}\right)\\ -\left(e^{-2a-u_L}+e^{2\theta}+2e^{\theta-a-\frac{u_L}{2}}\right)^2\left(e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}\right)\end{array}\right\}.
\end{aligned}
$$

$$
\left\{\begin{array}{c}
\left(e^{2a-u_R}+e^{2\theta}+2e^{\theta+a-\frac{u_R}{2}}\right)^2\left(e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}\right) \\
-\left(e^{-2a-u_L}+e^{2\theta}+2e^{\theta-a-\frac{u_L}{2}}\right)^2\left(e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}\right)
\end{array}\right\}
$$

$$
\begin{aligned}
= \quad & e^{2\theta}\left\{\begin{array}{c}
e^{2\theta}\left(\begin{array}{c}
\left(e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}\right)-\left(e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}\right) \\
+\left(e^{2a-u_R}++e^{2\theta}+2e^{\theta+a-\frac{u_R}{2}}\right)-\left(e^{-2a-u_L}++e^{2\theta}+2e^{\theta-a-\frac{u_L}{2}}\right)
\end{array}\right) \\
+e^{-u_R-u_L}-e^{-u_R-u_L} \\
-e^{a-u_R+\frac{u_R}{2}-\theta}+e^{-a-u_L+\frac{u_L}{2}-\theta} \\
+2e^{\theta-a-\frac{u_R}{2}-u_L}-2e^{\theta+a-\frac{u_L}{2}-u_R} \\
-2+2
\end{array}\right\} \\
& +\left\{\begin{array}{c}
e^{2\theta}\left(\begin{array}{c}
\left(e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}\right)e^{2a-u_R}-\left(e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}\right)e^{-2a-u_L} \\
+\left(e^{2a-u_R}++e^{2\theta}+2e^{\theta+a-\frac{u_R}{2}}\right)e^{2a-u_R}-\left(e^{-2a-u_L}++e^{2\theta}+2e^{\theta-a-\frac{u_L}{2}}\right)e^{-2a-u_L}
\end{array}\right) \\
+e^{-u_R-u_L}e^{2a-u_R}-e^{-u_R-u_L}e^{-2a-u_L} \\
-e^{a-u_R+\frac{u_R}{2}-\theta}e^{2a-u_R}+e^{-a-u_L+\frac{u_L}{2}-\theta}e^{-2a-u_L} \\
+2e^{\theta-a-\frac{u_R}{2}-u_L}e^{2a-u_R}-2e^{\theta+a-\frac{u_L}{2}-u_R}e^{-2a-u_L} \\
-2e^{2a-u_R}+2e^{-2a-u_L}
\end{array}\right\} \\
& +\left\{\begin{array}{c}
e^{2\theta}\left(\begin{array}{c}
\left(e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}\right)2e^{a-\frac{u_R}{2}+\theta}-\left(e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}\right)2e^{-a-\frac{u_L}{2}+\theta} \\
+\left(e^{2a-u_R}++e^{2\theta}+2e^{\theta+a-\frac{u_R}{2}}\right)2e^{a-\frac{u_R}{2}+\theta}-\left(e^{-2a-u_L}++e^{2\theta}+2e^{\theta-a-\frac{u_L}{2}}\right)2e^{-a-\frac{u_L}{2}+\theta}
\end{array}\right) \\
+e^{-u_R-u_L}2e^{a-\frac{u_R}{2}+\theta}-e^{-u_R-u_L}2e^{-a-\frac{u_L}{2}+\theta} \\
-e^{a-u_R+\frac{u_R}{2}-\theta}2e^{a-\frac{u_R}{2}+\theta}+e^{-a-u_L+\frac{u_L}{2}-\theta}2e^{-a-\frac{u_L}{2}+\theta} \\
+2e^{\theta-a-\frac{u_R}{2}-u_L}2e^{a-\frac{u_R}{2}+\theta}-2e^{\theta+a-\frac{u_L}{2}-u_R}2e^{-a-\frac{u_L}{2}+\theta} \\
-2*2e^{a-\frac{u_R}{2}+\theta}+2*2e^{-a-\frac{u_L}{2}+\theta}
\end{array}\right.
\end{aligned}
$$

first bracket

$$
\begin{aligned}
= \quad & e^{2\theta}\left\{\begin{array}{c}
e^{2\theta}\left(\begin{array}{c}
\left(e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}\right)-\left(e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}\right) \\
+\left(e^{2a-u_R}++e^{2\theta}+2e^{\theta+a-\frac{u_R}{2}}\right)-\left(e^{-2a-u_L}++e^{2\theta}+2e^{\theta-a-\frac{u_L}{2}}\right)
\end{array}\right) \\
+e^{-u_R-u_L}-e^{-u_R-u_L} \\
-e^{a-u_R+\frac{u_R}{2}-\theta}+e^{-a-u_L+\frac{u_L}{2}-\theta} \\
+2e^{\theta-a-\frac{u_R}{2}-u_L}-2e^{\theta+a-\frac{u_L}{2}-u_R} \\
-2+2
\end{array}\right\} \\
= \quad & e^{2\theta}\left\{\begin{array}{c}
e^{2\theta}\left(2e^{\theta}\left(e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}\right)+e^{-\theta+\frac{u_R+u_L}{2}}\left(e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}\right)\right) \\
-e^{-\theta}\left(e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}\right) \\
-2e^{\theta-\frac{u_R+u_L}{2}}\left(e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}\right)
\end{array}\right\} \\
= \quad & e^{2\theta}\left(e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}\right)\left(2e^{3\theta}+e^{\theta+\frac{u_R+u_L}{2}}-e^{-\theta}-2e^{\theta-\frac{u_R+u_L}{2}}\right).
\end{aligned}
$$

second bracket

$$
\begin{aligned}
&= \left\{ \begin{array}{c}
e^{2\theta} \left(\begin{array}{c}
\left(e^{-2a-u_L} + e^{2\theta} - e^{-a+\frac{u_R}{2}-\theta}\right) e^{2a-u_R} - \left(e^{2a-u_R} + e^{2\theta} - e^{a+\frac{u_L}{2}-\theta}\right) e^{-2a-u_L} \\
+ \left(e^{2a-u_R} + e^{2\theta} + 2e^{\theta+a-\frac{u_R}{2}}\right) e^{2a-u_R} - \left(e^{-2a-u_L} + e^{2\theta} + 2e^{\theta-a-\frac{u_L}{2}}\right) e^{-2a-u_L}
\end{array} \right) \\
+e^{-u_R-u_L} e^{2a-u_R} - e^{-u_R-u_L} e^{-2a-u_L} \\
-e^{a-u_R+\frac{u_R}{2}-\theta} e^{2a-u_R} + e^{-a-u_L+\frac{u_L}{2}-\theta} e^{-2a-u_L} \\
+2e^{\theta-a-\frac{u_R}{2}-u_L} e^{2a-u_R} - 2e^{\theta+a-\frac{u_L}{2}-u_R} e^{-2a-u_L} \\
-2e^{2a-u_R} + 2e^{-2a-u_L}
\end{array} \right\} \\
&= e^{2\theta} \left(\begin{array}{c}
2e^{2\theta} \left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) \left(e^{a-\frac{u_R}{2}} + e^{-a-\frac{u_L}{2}}\right) \\
-e^{-\theta} \left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) \\
+ \left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) \left(\left(e^{a-\frac{u_R}{2}}\right)^3 + \left(e^{a-\frac{u_R}{2}}\right)^2 e^{-a-\frac{u_L}{2}} + \left(e^{a-\frac{u_R}{2}}\right) \left(e^{-a-\frac{u_L}{2}}\right)^2 + \left(e^{-a-\frac{u_L}{2}}\right)^3 \right) \\
+2e^{\theta} \left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) \left(e^{2a-u_R} + e^{-\frac{u_R+u_L}{2}} + e^{-2a-u_L}\right)
\end{array} \right) \\
&\quad + \begin{array}{c}
+e^{-u_R-u_L} \left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) \left(e^{a-\frac{u_R}{2}} + e^{-a-\frac{u_L}{2}}\right) \\
-e^{-\theta} \left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) \left(e^{2a-u_R} + e^{-\frac{u_R+u_L}{2}} + e^{-2a-u_L}\right) \\
+2e^{\theta-u_R-u_L} \left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) \\
-2 \left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) \left(e^{a-\frac{u_R}{2}} + e^{-a-\frac{u_L}{2}}\right)
\end{array} \\
&= \left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) \left\{ \begin{array}{c}
2e^{4\theta} \left(e^{a-\frac{u_R}{2}} + e^{-a-\frac{u_L}{2}}\right) - e^{\theta} \\
+e^{2\theta} \left(\left(e^{a-\frac{u_R}{2}}\right)^3 + \left(e^{a-\frac{u_R}{2}}\right)^2 e^{-a-\frac{u_L}{2}} + \left(e^{a-\frac{u_R}{2}}\right) \left(e^{-a-\frac{u_L}{2}}\right)^2 + \left(e^{-a-\frac{u_L}{2}}\right)^3 \right) \\
+2e^{3\theta} \left(e^{2a-u_R} + e^{-\frac{u_R+u_L}{2}} + e^{-2a-u_L}\right) \\
+e^{-u_R-u_L} \left(e^{a-\frac{u_R}{2}} + e^{-a-\frac{u_L}{2}}\right) \\
-e^{-\theta} \left(e^{2a-u_R} + e^{-\frac{u_R+u_L}{2}} + e^{-2a-u_L}\right) \\
+2e^{\theta-u_R-u_L} - 2 \left(e^{a-\frac{u_R}{2}} + e^{-a-\frac{u_L}{2}}\right)
\end{array} \right\} \\
&= \left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) \left\{ \begin{array}{c}
e^{2\theta} \left(\left(e^{a-\frac{u_R}{2}}\right)^3 + \left(e^{-a-\frac{u_L}{2}}\right)^3 \right) \\
+ \left(e^{2a-u_R} + e^{-\frac{u_R+u_L}{2}} + e^{-2a-u_L}\right) \left(2e^{3\theta} - e^{-\theta}\right) \\
+ \left(e^{a-\frac{u_R}{2}} + e^{-a-\frac{u_L}{2}}\right) \left(2e^{4\theta} + 2e^{2\theta-\frac{u_R+u_L}{2}} + e^{-(u_R+u_L)} - 2\right) \\
+2e^{\theta-u_R-u_L} - e^{\theta}
\end{array} \right\}.
\end{aligned}
$$

third bracket

$$
\begin{aligned}
&= \left\{ \begin{array}{c}
e^{2\theta} \left(\begin{array}{c}
\left(e^{-2a-u_L} + e^{2\theta} - e^{-a+\frac{u_R}{2}-\theta}\right) 2e^{a-\frac{u_R}{2}+\theta} - \left(e^{2a-u_R} + e^{2\theta} - e^{a+\frac{u_L}{2}-\theta}\right) 2e^{-a-\frac{u_L}{2}+\theta} \\
+ \left(e^{2a-u_R} + e^{2\theta} + 2e^{\theta+a-\frac{u_R}{2}}\right) 2e^{a-\frac{u_R}{2}+\theta} - \left(e^{-2a-u_L} + e^{2\theta} + 2e^{\theta-a-\frac{u_L}{2}}\right) 2e^{-a-\frac{u_L}{2}+\theta}
\end{array} \right) \\
+e^{-u_R-u_L} 2e^{a-\frac{u_R}{2}+\theta} - e^{-u_R-u_L} 2e^{-a-\frac{u_L}{2}+\theta} \\
-e^{a-u_R+\frac{u_R}{2}-\theta} 2e^{a-\frac{u_R}{2}+\theta} + e^{-a-u_L+\frac{u_L}{2}-\theta} 2e^{-a-\frac{u_L}{2}+\theta} \\
+2e^{\theta-a-\frac{u_R}{2}-u_L} 2e^{a-\frac{u_R}{2}+\theta} - 2e^{\theta+a-\frac{u_L}{2}-u_R} 2e^{-a-\frac{u_L}{2}+\theta} \\
-2 * 2e^{a-\frac{u_R}{2}+\theta} + 2 * 2e^{-a-\frac{u_L}{2}+\theta}
\end{array} \right\} \\
&= 2e^{2\theta} \left\{ \begin{array}{c}
2e^{3\theta} \left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) \\
-e^{\theta-\frac{u_R+u_L}{2}} \left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) \\
-1+1 \\
+e^{\theta} \left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) \left(e^{2a-u_R} + e^{-\frac{u_R+u_L}{2}} + e^{-2a-u_L}\right) \\
+2e^{2\theta} \left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) \left(e^{a-\frac{u_R}{2}} + e^{-a-\frac{u_L}{2}}\right)
\end{array} \right\} \\
&\quad +2 \left(\begin{array}{c}
e^{\theta-u_R-u_L} \left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) \\
-\left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) \left(e^{a-\frac{u_R}{2}} + e^{-a-\frac{u_L}{2}}\right) \\
+2e^{2\theta} \left(e^{-u_R-u_L} - e^{-u_R-u_L}\right) \\
-2e^{\theta} \left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right)
\end{array} \right) \\
&= 2\left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) \left\{ \begin{array}{c}
e^{\theta} \left(e^{2a-u_R} + e^{-\frac{u_R+u_L}{2}} + e^{-2a-u_L}\right) \\
+ \left(e^{a-\frac{u_R}{2}} + e^{-a-\frac{u_L}{2}}\right) \left(2e^{\theta} - 1\right) \\
2e^{3\theta} + 2e^{2\theta} - 2e^{\theta} - e^{\theta-\frac{u_R+u_L}{2}} + e^{\theta-u_R-u_L}
\end{array} \right\}.
\end{aligned}
$$

So

$$
\begin{aligned}
& \frac{\text{the sum}}{\left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right)} \\
= \quad & e^{2\theta}\left(2e^{3\theta} + e^{\theta+\frac{u_R+u_L}{2}} - e^{-\theta} - 2e^{\theta-\frac{u_R+u_L}{2}}\right) \\
& + \left\{ \begin{array}{c} e^{2\theta}\left(\left(e^{a-\frac{u_R}{2}}\right)^3 + \left(e^{-a-\frac{u_L}{2}}\right)^3\right) \\ +\left(e^{2a-u_R}+e^{-\frac{u_R+u_L}{2}}+e^{-2a-u_L}\right)\left(2e^{3\theta}-e^{-\theta}\right) \\ +\left(e^{a-\frac{u_R}{2}}+e^{-a-\frac{u_L}{2}}\right)\left(2e^{4\theta}+2e^{2\theta-\frac{u_R+u_L}{2}}+e^{-(u_R+u_L)}-2\right) \\ +2e^{\theta-u_R-u_L}-e^{\theta} \end{array}\right\} \\
& +2\left\{ \begin{array}{c} e^{\theta}\left(e^{2a-u_R}+e^{-\frac{u_R+u_L}{2}}+e^{-2a-u_L}\right) \\ +\left(e^{a-\frac{u_R}{2}}+e^{-a-\frac{u_L}{2}}\right)\left(2e^{\theta}-1\right) \\ 2e^{3\theta}+2e^{2\theta}-2e^{\theta}-e^{\theta-\frac{u_R+u_L}{2}}+e^{\theta-u_R-u_L} \end{array}\right\} \\
= \quad & e^{2\theta}\left(\left(e^{a-\frac{u_R}{2}}\right)^3 + \left(e^{-a-\frac{u_L}{2}}\right)^3\right) \\
& +\left(e^{2a-u_R}+e^{-\frac{u_R+u_L}{2}}+e^{-2a-u_L}\right)\left(2e^{3\theta}-e^{-\theta}+2e^{\theta}\right) \\
& +\left(e^{a-\frac{u_R}{2}}+e^{-a-\frac{u_L}{2}}\right)\left(2e^{4\theta}+2e^{2\theta-\frac{u_R+u_L}{2}}+2e^{\theta}+e^{-(u_R+u_L)}-3\right) \\
& +e^{2\theta}\left(2e^{3\theta} + e^{\theta+\frac{u_R+u_L}{2}} - e^{-\theta} - 2e^{\theta-\frac{u_R+u_L}{2}}\right) + 2e^{\theta-u_R-u_L} - e^{\theta} \\
& +2\left(2e^{3\theta}+2e^{2\theta}-2e^{\theta}-e^{\theta-\frac{u_R+u_L}{2}}+e^{\theta-u_R-u_L}\right) \\
= \quad & e^{2\theta}\left(\left(e^{a-\frac{u_R}{2}}\right)^3 + \left(e^{-a-\frac{u_L}{2}}\right)^3\right) \\
& +\left(e^{2a-u_R}+e^{-\frac{u_R+u_L}{2}}+e^{-2a-u_L}\right)\left(2e^{3\theta}-e^{-\theta}+2e^{\theta}\right) \\
& +\left(e^{a-\frac{u_R}{2}}+e^{-a-\frac{u_L}{2}}\right)\left(2e^{4\theta}+2e^{2\theta-\frac{u_R+u_L}{2}}+2e^{\theta}+e^{-(u_R+u_L)}-3\right) \\
& +2e^{5\theta}+e^{3\theta+\frac{u_R+u_L}{2}}-2e^{3\theta-\frac{u_R+u_L}{2}}+4e^{3\theta}+4e^{2\theta}-6e^{\theta} \\
& +4e^{\theta-u_R-u_L}-2e^{\theta-\frac{u_R+u_L}{2}} \\
= \quad & e^{2\theta}\left(\left(e^{a-\frac{u_R}{2}}\right)^3 + \left(e^{-a-\frac{u_L}{2}}\right)^3\right) \\
& +\left(e^{2a-u_R}+e^{-\frac{u_R+u_L}{2}}+e^{-2a-u_L}\right)\left(2e^{3\theta}-e^{-\theta}+2e^{\theta}\right)-2e^{3\theta-\frac{u_R+u_L}{2}}-2e^{\theta-\frac{u_R+u_L}{2}} \\
& +\left(e^{a-\frac{u_R}{2}}+e^{-a-\frac{u_L}{2}}\right)\left(2e^{4\theta}+2e^{2\theta-\frac{u_R+u_L}{2}}+2e^{\theta}+e^{-(u_R+u_L)}-3\right) \\
& +2e^{5\theta}+e^{3\theta+\frac{u_R+u_L}{2}}+4e^{3\theta}+4e^{2\theta}-6e^{\theta}+4e^{\theta-u_R-u_L} \\
= \quad & e^{2\theta}\left(\left(e^{a-\frac{u_R}{2}}\right)^3 + \left(e^{-a-\frac{u_L}{2}}\right)^3\right) \\
& +\left(e^{2a-u_R}+e^{-\frac{u_R+u_L}{2}}+e^{-2a-u_L}\right)\left(-e^{-\theta}+e^{\theta}\right) \\
& +2e^{3\theta}\left\{\left(e^{2a-u_R}+e^{-\frac{u_R+u_L}{2}}+e^{-2a-u_L}\right)-e^{-\frac{u_R+u_L}{2}}\right\} \\
& +e^{\theta}\left\{\left(e^{2a-u_R}+e^{-\frac{u_R+u_L}{2}}+e^{-2a-u_L}\right)-2e^{-\frac{u_R+u_L}{2}}\right\} \\
& +\left(e^{a-\frac{u_R}{2}}+e^{-a-\frac{u_L}{2}}\right)\left(2e^{4\theta}+2e^{2\theta-\frac{u_R+u_L}{2}}+2e^{\theta}+e^{-(u_R+u_L)}-3\right) \\
& +2e^{5\theta}+e^{3\theta+\frac{u_R+u_L}{2}}+4e^{3\theta}+4e^{2\theta}-6e^{\theta}+4e^{\theta-u_R-u_L} \\
\geq \quad & 0
\end{aligned}
$$

if$-e^{-\theta}+e^{\theta}>0$ and $2e^{4\theta}+2e^{\theta}+e^{-(u_R+u_L)}-3>0$ and

$$2e^{4\theta}+e^{2\theta+\frac{u_R+u_L}{2}}+4e^{2\theta}+4e^{\theta}-6+4e^{-u_R-u_L}>0.$$

This holds if $e^{-\theta}<\frac{2}{3}$. ■

Lemma 6.6 $\left(\hat{a}\left(a\right)+\alpha\eta_0\right)\left(2a-\frac{u_R-u_L}{2}\right)<0$ *if* $\theta>\max\left\{\frac{3}{4}\left(u_R+u_L\right)-\log 2,\log\frac{3}{2}\right\}$.

Proof.

$$\begin{aligned}
\hat{a}\left(a\right) &= -\alpha\eta_0+\frac{\alpha}{2}\log\frac{e^{a-\frac{u_R}{2}}+e^{\theta}+\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{e^{-a-\frac{u_L}{2}}+e^{\theta}+\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}} \\
&\quad +\frac{1}{2}\log\left[\left(\frac{e^{a-\frac{u_R}{2}}+e^{\theta}+\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{e^{-a-\frac{u_L}{2}}+e^{\theta}+\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right)^2\frac{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}\right] \\
&= -\alpha\eta_0+\frac{\alpha}{2}\log\frac{e^{a-\frac{u_R}{2}}+e^{\theta}+\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{e^{-a-\frac{u_L}{2}}+e^{\theta}+\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}} \\
&\quad +\frac{1}{2}\log\frac{\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)^2}{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}+2\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)+\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{\frac{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)^2}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}+2\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)+\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}.
\end{aligned}$$

Because $\left(\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}-\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}\right)\left(e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}\right)>0$ and $\left(\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)^2}{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}-\frac{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)^2}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right)\left(e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}\right)>0$, it follows that

$$\left(e^{a-\frac{u_R}{2}}+e^{\theta}+\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}-\left(e^{-a-\frac{u_L}{2}}+e^{\theta}+\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}\right)\right)\left(e^{a-\frac{u_R}{2}}-e^{-a-\right.$$

and thus

$$\left(\hat{a}\left(a\right)+\alpha\eta_0\right)\left(e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}\right)>0.$$

Because $e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}=\left(2a-\frac{u_R-u_L}{2}\right)e^{\zeta}$ for some ζ between $a-\frac{u_r}{2}$ and $-a-\frac{u_l}{2}$. So $\left(\hat{a}\left(a\right)+\alpha\eta_0\right)\left(2a-\frac{u_R-u_L}{2}\right)>0$. It follows that $\left(\hat{a}\left(0\right)+\alpha\eta_0\right)\left(-u_R+u_L\right)>0$. ■

Observation If $\frac{3}{4}\left(u_R+u_L\right)-\log 2<\theta<-\frac{u_R+u_L}{4}$, then

$$\left|\hat{a}\left(a\right)+\alpha\eta_0\right|<\frac{1+\alpha}{2}\left|\log\frac{e^{\theta}+e^{a-\frac{1}{2}u_R}+\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{e^{\theta}+e^{-a-\frac{1}{2}u_L}+\sqrt{e^{2\theta}+e^{2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right|.$$

Proof. If $\theta > \frac{3}{4}(u_R + u_L) - \log 2$, then $\log \frac{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}$ has the same sign as $\log \frac{e^{\theta}+e^{a-\frac{1}{2}u_R}}{e^{\theta}+e^{-a-\frac{1}{2}u_L}}$ and hence $\log \frac{e^{\theta}+e^{a-\frac{1}{2}u_R}+\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{e^{\theta}+e^{-a-\frac{1}{2}u_L}+\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}$. If $\theta < -\frac{u_R+u_L}{4}$, then $\left|\log \frac{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right| > \log \frac{e^{\theta}+e^{a-\frac{1}{2}u_R}}{e^{\theta}+e^{-a-\frac{1}{2}u_L}}$, so $\left|\log \frac{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right| > \left|\log \frac{e^{\theta}+e^{a-\frac{1}{2}u_R}+\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{e^{\theta}+e^{-a-\frac{1}{2}u_L}+\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right|$. We have

$$\begin{aligned}
& \hat{a}(a) + \alpha\eta_0 \\
= \; & \frac{1+\alpha}{2} \log \frac{e^{\theta} + e^{a-\frac{1}{2}u_R} + \sqrt{e^{2\theta} + e^{2a-u_R} - e^{-\theta+a+\frac{1}{2}u_L}}}{e^{\theta} + e^{-a-\frac{1}{2}u_L} + \sqrt{e^{2\theta} + e^{-2a-u_L} - e^{-\theta-a+\frac{1}{2}u_R}}} \\
& + \frac{1}{2}\left(\log \frac{e^{\theta} + e^{a-\frac{1}{2}u_R} + \sqrt{e^{2\theta} + e^{2a-u_R} - e^{-\theta+a+\frac{1}{2}u_L}}}{e^{\theta} + e^{-a-\frac{1}{2}u_L} + \sqrt{e^{2\theta} + e^{-2a-u_L} - e^{-\theta-a+\frac{1}{2}u_R}}} - \log \frac{\sqrt{e^{2\theta} + e^{2a-u_R} - e^{-\theta+a+\frac{1}{2}u_L}}}{\sqrt{e^{2\theta} + e^{-2a-u_L} - e^{-\theta-a+\frac{1}{2}u_R}}}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
|\hat{a}(a)| \; = \; & \frac{1+\alpha}{2}\left|\log \frac{e^{\theta} + e^{a-\frac{1}{2}u_R} + \sqrt{e^{2\theta} + e^{2a-u_R} - e^{-\theta+a+\frac{1}{2}u_L}}}{e^{\theta} + e^{-a-\frac{1}{2}u_L} + \sqrt{e^{2\theta} + e^{-2a-u_L} - e^{-\theta-a+\frac{1}{2}u_R}}}\right| \\
& + \frac{1}{2}\left(\begin{array}{c} \left|\log \frac{e^{\theta}+e^{a-\frac{1}{2}u_R}+\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{e^{\theta}+e^{-a-\frac{1}{2}u_L}+\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right| \\ - \left|\log \frac{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right| \end{array}\right) \\
< \; & \frac{1+\alpha}{2}\left|\log \frac{e^{\theta} + e^{a-\frac{1}{2}u_R} + \sqrt{e^{2\theta} + e^{2a-u_R} - e^{-\theta+a+\frac{1}{2}u_L}}}{e^{\theta} + e^{-a-\frac{1}{2}u_L} + \sqrt{e^{2\theta} + e^{-2a-u_L} - e^{-\theta-a+\frac{1}{2}u_R}}}\right|.
\end{aligned}$$

■

Observation $a^{**} \notin \left(0, \frac{u_R-u_L}{4}\right)$ if $u_R - u_L > 0$ and $a^{**} \notin \left(\frac{u_R-u_L}{4}, 0\right)$ if $u_R - u_L < 0$.

Proof. Because $\hat{a}(a)\left(a - \frac{u_R-u_L}{4}\right) > 0$, If $0 < a < \frac{u_R-u_L}{4}$, then $\hat{a}(a) < 0 < a$, so a cannot be a fixed point. If $-\frac{u_R-u_L}{4} < a < 0$, but then $\hat{a}(a) > 0 > a$. ■

Lemma 6.7 *$\hat{a}'(a^{**}) \in \left(0, \frac{3}{4}(1+\alpha)\right)$ if $\theta > \max\left\{\frac{3}{4}(u_R+u_L) - \log 2, \log \frac{3}{2}\right\}$ and either*

1. *$\theta > -\frac{u_R+u_L}{4}$, or*
2. *$\theta > -\frac{\max\{u_R,u_L\}}{2} + \log 2$.*

Proof.

$$\begin{aligned}
& \hat{a}'(a; u_R, u_L) \\
= \;& \frac{1}{2} + \frac{1}{2}\frac{e^{a-\frac{u_R}{2}} - e^{\theta}}{\sqrt{e^{2\theta} + e^{2a-u_R} - e^{-\theta+a+\frac{1}{2}u_L}}}\left(1 - \frac{\frac{e^{\theta}+e^{a-\frac{u_R}{2}}}{2}}{\sqrt{e^{2\theta} + e^{2a-u_R} - e^{-\theta+a+\frac{1}{2}u_L}}}\right) \\
& + \frac{1}{2}\frac{e^{-a-\frac{u_L}{2}} - e^{\theta}}{\sqrt{e^{2\theta} + e^{-2a-u_L} - e^{-\theta-a+\frac{1}{2}u_R}}}\left(1 - \frac{\frac{e^{\theta}+e^{-a-\frac{u_L}{2}}}{2}}{\sqrt{e^{2\theta} + e^{-2a-u_L} - e^{-\theta-a+\frac{1}{2}u_R}}}\right) \\
& + \frac{\alpha}{2}\left(1 + \frac{1}{2}\frac{e^{a-\frac{u_R}{2}} - e^{\theta}}{\sqrt{e^{2\theta} + e^{2a-u_R} - e^{-\theta+a+\frac{1}{2}u_L}}} + \frac{1}{2}\frac{e^{-a-\frac{u_L}{2}} - e^{\theta}}{\sqrt{e^{2\theta} + e^{-2a-u_L} - e^{-\theta-a+\frac{1}{2}u_R}}}\right) \\
> \;& 0
\end{aligned}$$

because $1 - \frac{\frac{e^{\theta}+e^{a-\frac{u_R}{2}}}{2}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}} \in \left(0, \frac{1}{2}\right)$ and $\frac{e^{a-\frac{u_R}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}} \in (-1,1)$ and $\left(1 - \frac{\frac{e^{\theta}+e^{-a-\frac{u_L}{2}}}{2}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right) \in \left(0, \frac{1}{2}\right)$ and $\frac{e^{-a-\frac{u_L}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}} \in (-1,1)$.

If either $a-\frac{u_R}{2} < \theta$ or $-a-\frac{u_L}{2} < \theta s$, then $\hat{a}'(a; u_R, u_L) < \frac{1}{2}+\frac{1}{4}+\frac{\alpha}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4}(1+\alpha)$.

Case 3 $\theta > -\frac{u_R+u_L}{4}$

Proof. Then $a - \frac{u_R}{2} - \theta + \left(-a - \frac{u_L}{2} - \theta\right) < 0$, so either $a - \frac{u_R}{2} < \theta$ or $-a - \frac{u_L}{2} < \theta$ and $\hat{a}'(a) < \frac{3}{4}(1+\alpha)$. ∎

Case 4 $\theta < -\frac{u_R+u_L}{4}$.

Proof. If $a \notin \left(\theta + \frac{u_R}{2}, -\theta - \frac{u_L}{2}\right)$, then either $a - \frac{u_R}{2} < \theta$ or $-a - \frac{u_L}{2} < \theta$. Therefore, the statement does not hold only if $a^{**} \in \left(\theta + \frac{u_R}{2}, -\theta - \frac{u_L}{2}\right)$. Because $\theta \in \left(u_R + u_L, -\frac{u_R+u_L}{4}\right)$,

$$\begin{aligned}
|\hat{a}(a)| &< \frac{1+\alpha}{2}\left|\log \frac{e^{\theta} + e^{a-\frac{1}{2}u_R} + \sqrt{e^{2\theta} + e^{2a-u_R} - e^{-\theta+a+\frac{1}{2}u_L}}}{e^{\theta} + e^{-a-\frac{1}{2}u_L} + \sqrt{e^{2\theta} + e^{-2a-u_L} - e^{-\theta-a+\frac{1}{2}u_R}}}\right| \\
&< \frac{1+\alpha}{2}\left(\log 2 + \left|\log e^{2a-\frac{u_R-u_L}{2}}\right|\right)
\end{aligned}$$

because $\sqrt{e^{2\theta} + e^{2a-u_R} - e^{-\theta+a+\frac{1}{2}u_L}} \in \left(0, e^{\theta} + e^{a-\frac{1}{2}u_R}\right)$ and $\sqrt{e^{2\theta} + e^{-2a-u_L} - e^{-\theta-a+\frac{1}{2}u_R}} \in \left(0, e^{\theta} + e^{-a-\frac{1}{2}u_L}\right)$. Consider $u_R > u_L$. By assumption, $\theta + \frac{u_R}{2} > 0$. Because $a^{**} \notin \left(0, \frac{u_R-u_L}{4}\right)$, if $a^{**} \in \left(\theta + \frac{u_R}{2}, -\theta - \frac{u_L}{2}\right)$ then $a^{**} > \frac{u_R-u_L}{4}$. Because

$\hat{a}(a)\left(a - \frac{u_R - u_L}{4}\right) > 0$,

$$\begin{aligned} & \hat{a}(a) \\ < \; & \frac{1+\alpha}{2}\left(\log 2 + 2a - \frac{u_R - u_L}{2}\right) \\ = \; & \frac{1+\alpha}{2}\left(\log 2 - \frac{u_R - u_L}{2}\right) + (1+\alpha)\,a \end{aligned}$$

if $a \in \left(\frac{u_R - u_L}{4}, -\theta - \frac{u_L}{2}\right)$. If a fixed point a^{**} exists in $\left(\frac{u_R - u_L}{4}, -\theta - \frac{u_L}{2}\right)$, then

$$\begin{aligned} a^{**} &= \hat{a}(a^{**}) \\ &< \frac{1+\alpha}{2}\left(\log 2 - \frac{u_R - u_L}{2}\right) + (1+\alpha)\,a^{**}, \end{aligned}$$

so

$$\begin{aligned} a^{**} &> \left(1 + \frac{1}{\alpha}\right)\frac{1}{2}\left(\frac{u_R - u_L}{2} - \log 2\right) \\ &\geq \frac{u_R - u_L}{2} - \log 2 \text{ (because } \alpha \leq 1) \\ &= \frac{u_R}{2} - \log 2 - \frac{u_L}{2} \\ &> -\theta - \frac{u_L}{2} \text{ (because we assume that } \theta > -\frac{\max\{u_R, u_L\}}{2} + \log 2), \end{aligned}$$

contradiction to the hypothesis that $a^{**} \in \left(\frac{u_R - u_L}{4}, -\theta - \frac{u_L}{2}\right)$. The case where $u_R < u_L$ is analogous. ■ ■

Lemma 6.8 *$a^{**}(u_R - u_L) < 0$ if $\theta < u_R + u_L$, $\alpha < \frac{1}{4}$ and either*

1. *$\theta > -\frac{u_R + u_L}{4}$, or*
2. *$\theta > -\frac{\max\{u_R, u_L\}}{2} + \log 2$.*

Proof. This follows because $\hat{a}'(a^{**}) < \frac{3}{4}(1+\alpha) < 1$ and $\hat{a}(0)\left(-\frac{u_R - u_L}{2}\right) > 0$. ■

6.3 Proofs for Proposition 2

Lemma 6.9 *$\frac{\partial \eta_R(a; u_r, u_l)}{\partial a} \in (0, 1)$ if $\theta > \frac{u_R + u_L}{4}$*

Proof. Again, if $\eta_R > \left\{\theta, a - \frac{1}{2}\tilde{u}_R, a + \frac{\tilde{u}_L}{2}\right\}$, then

$$\begin{aligned} \frac{\partial p_R(\eta_R; a)}{\partial a} &= -F(\eta_R - \theta) + p_R(\eta_R) \\ \frac{\partial p_L(\eta_R; a)}{\partial a} &= p_L(\eta_R). \end{aligned}$$

So

$$\begin{aligned}
\frac{\partial \eta_R(a)}{\partial a} &= -\frac{2\frac{\partial p_R(\eta_R;a)}{\partial a}+\frac{\partial p_L(\eta_R;a)}{\partial a}}{2p'_R(\eta_R)+p'_L(\eta_R)} = \frac{2F(\eta_R-\theta)-1}{1-e^{-\eta_R+a-\frac{\tilde{u}_R}{2}}+1-e^{-(\eta_R-\theta)}} \\
&= \frac{1-e^{-(\eta_R-\theta)}}{1-e^{-\eta_R+a-\frac{\tilde{u}_R}{2}}+1-e^{-(\eta_R-\theta)}} \in (0,1) \\
&= \frac{1}{2}\frac{e^{a-\frac{u_R}{2}}-e^{\theta}+\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}} \\
&= \frac{1}{2}\left(1+\frac{e^{a-\frac{u_R}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\right)
\end{aligned}$$

because $\eta_R(a) > \max\left\{a-\frac{u_R}{2},\theta\right\}$. ■

Lemma 6.10 $\frac{\partial \eta_R(a;\tilde{u}_R,\tilde{u}_L)}{\partial \tilde{u}_R} < 0$ *if* $\theta > \frac{u_R+u_L}{4}$

Proof.

$$\begin{aligned}
\frac{\partial \eta_R}{\partial u_R} &= -\frac{2\frac{\partial p_R(\eta_R;a)}{\partial u_R}+\frac{\partial p_L(\eta_R;a)}{\partial u_R}}{2p'_R(\eta_R)+p'_L(\eta_R)} \\
&= -\frac{2\left(F(\eta_R-\theta)\left(1-F\left(\eta_R-a+\frac{1}{2}\tilde{u}_R\right)\right)\right)\frac{1}{2}}{1-e^{-\eta_R+a-\frac{\tilde{u}_R}{2}}+1-e^{-(\eta_R-\theta)}} \\
&\quad \text{(this shows that it is negative)} \\
&= -\frac{\left(1-\frac{1}{2}e^{-\eta_R+\theta}\right)\frac{1}{2}e^{-\eta_R+a-\frac{u_R}{2}}}{1-e^{-\eta_R+a-\frac{\tilde{u}_R}{2}}+1-e^{-(\eta_R-\theta)}} \\
&= -\frac{\frac{1}{2}e^{a-\frac{u_R}{2}}\left(e^{a-\frac{u_R}{2}}+\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}\right)}{e^{\eta_R}\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}} \\
&= -\frac{e^{a-\frac{u_R}{2}}}{e^{\theta}+e^{a-\frac{u_R}{2}}+\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\left(1+\frac{e^{a-\frac{u_R}{2}}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\right)
\end{aligned}$$

■

Lemma 6.11 $\frac{\partial \hat{a}(a;u_R,u_L)}{\partial \tilde{u}_R} < 0$ *if* $\theta > \frac{u_R+u_L}{4}$

Proof.

$$\begin{aligned}
\frac{\partial\alpha\left(a;u_R,u_L,\theta\right)}{\partial u_R} &= -\frac{1}{2}\left(1+\alpha\right)\frac{e^{a-\frac{u_R}{2}}}{e^{\eta_R}} \\
&\quad +\frac{1}{4}\frac{e^{2a-u_R}}{\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}}\left(\frac{1}{\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}}-\frac{1+\alpha}{e^{\eta_R}}\right) \\
&\quad -\frac{1}{4}\frac{\frac{1}{2}e^{-\theta-a+\frac{1}{2}u_R}}{\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}}\left(\frac{1}{\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}}-\frac{1+\alpha}{e^{-\eta_L}}\right) \\
&= -\frac{1}{2}\left(1+\alpha\right)\frac{e^{a-\frac{u_R}{2}}}{e^{\eta_R}} \\
&\quad +\frac{1}{4}\frac{e^{a-\frac{u_R}{2}}}{e^{\eta_R}}\frac{e^{a-\frac{u_R}{2}}\left(\begin{array}{c}\frac{1}{2}e^{\theta}+\frac{1}{2}e^{a-\frac{u_R}{2}} \\ -\frac{1}{2}\left(1+2\alpha\right)\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}\end{array}\right)}{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}} \\
&\quad -\frac{1}{4}\frac{\frac{1}{2}e^{-\theta-a+\frac{1}{2}u_R}}{\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}}\left(\frac{1}{\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}}-\frac{1+\alpha}{e^{-\eta_L}}\right).
\end{aligned}$$

So $\frac{\partial\alpha(a;u_R,u_L,\theta)}{\partial u_R}<0$ if

$$\begin{aligned}
& e^{a-\frac{u_R}{2}}\left(e^{\theta}+e^{a-\frac{u_R}{2}}-\left(1+2\alpha\right)\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}\right) \\
< \;& 4\left(e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}\right).
\end{aligned}$$

If $2\theta>\frac{u_R+u_L}{2}$, then

$$\begin{aligned}
& e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}} \\
=\; & \left(e^{\theta}-e^{a-\frac{u_R}{2}}\right)^2+2e^{\theta+a-\frac{u_R}{2}}-e^{a-\theta+\frac{u_L}{2}} \\
\geq\; & \left(e^{\theta}-e^{a-\frac{u_R}{2}}\right)^2+e^{\theta+a-\frac{u_R}{2}}.
\end{aligned}$$

Then

$$\begin{aligned}
& e^{a-\frac{u_R}{2}}\left(e^{\theta}+e^{a-\frac{u_R}{2}}-\left(1+2\alpha\right)\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}\right) \\
\leq\; & e^{a-\frac{u_R}{2}}\left(e^{\theta}+e^{a-\frac{u_R}{2}}-\left|e^{\theta}-e^{a-\frac{u_R}{2}}\right|\right) \\
=\; & e^{a-\frac{u_R}{2}}2*\min\left\{e^{\theta},e^{a-\frac{u_R}{2}}\right\} \\
\leq\; & 2e^{\theta+a-\frac{u_R}{2}} \\
\leq\; & 2\left(\left(e^{\theta}-e^{a-\frac{u_R}{2}}\right)^2+e^{\theta+a-\frac{u_R}{2}}\right) \\
\leq\; & 4\left(e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}\right).
\end{aligned}$$

So $2\theta > \frac{u_R+u_L}{2}$ is sufficient for $\alpha'(a) \in (0,1)$ for all a and $\frac{\partial\alpha(a;u_R,u_L,\theta)}{\partial u_R} < 0$.

In fact, because

$$\begin{aligned} & e^{a-\frac{u_R}{2}}\left(e^{\theta}+e^{a-\frac{u_R}{2}}-(1+2\alpha)\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}\right) \\ < \quad & 2\left(\left(e^{\theta}-e^{a-\frac{u_R}{2}}\right)^2+e^{\theta+a-\frac{u_R}{2}}\right) \end{aligned}$$

we get

$$\begin{aligned} \frac{\partial\alpha(a;u_R,u_L,\theta)}{\partial u_R} \quad < \quad & -\frac{1}{2}\frac{e^{a-\frac{u_R}{2}}}{e^{\eta_R}}\left(1+\alpha-\frac{e^{a-\frac{u_R}{2}}\left(e^{\theta}+e^{a-\frac{u_R}{2}}-(1+2\alpha)\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}\right)}{4\left(e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}\right)}\right) \\ < \quad & -\frac{1}{2}\frac{e^{a-\frac{u_R}{2}}}{e^{\eta_R}}\frac{\begin{array}{c}2\left(e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}\right)+2\left(e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}\right)\\ -2\left(\left(e^{\theta}-e^{a-\frac{u_R}{2}}\right)^2+e^{\theta+a-\frac{u_R}{2}}\right)\end{array}}{4\left(e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}\right)} \\ = \quad & -\frac{1}{2}\frac{e^{a-\frac{u_R}{2}}}{e^{\eta_R}}\left(\frac{1}{2}+\frac{2\left(e^{\theta+a-\frac{u_R}{2}}-e^{a-\theta+\frac{\tilde{u}_L}{2}}\right)}{4\left(e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}\right)}\right) \\ < \quad & -\frac{1}{4}\frac{e^{a-\frac{u_R}{2}}}{e^{\eta_R}} \end{aligned}$$

■

$$
\begin{aligned}
\frac{\partial \eta_R^{**}}{\partial u_L} &= \frac{\partial \eta_R(a;u_R,u_L)}{\partial a}\frac{\partial a^{**}}{\partial u_L}+\frac{\partial \eta_R(a;u_R,u_L)}{\partial u_L} \\
&= \frac{\partial \eta_R(a;u_R,u_L)}{\partial a}\frac{\frac{\partial \hat{a}(a;u_R,u_L)}{\partial u_L}}{1-\hat{a}'(a^{**})}+\frac{\partial \eta_R(a;u_R,u_L)}{\partial u_L} \\
&> \frac{1}{4}\frac{1}{e^{\eta_R^{**}}}\frac{1}{2\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}}\frac{1}{e^{-\eta_L}(1-\alpha'(a^{**}))} \\
&\quad\times\left[\begin{array}{c}\left(\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}e^{a-\frac{u_R}{2}}+e^{2a-u_R}-\frac{1}{2}e^{-\theta+a+\frac{u_L}{2}}\right)e^{-a-\frac{u_L}{2}} \\ -(1-\alpha'(a^{**}))e^{-\eta_L}e^{-\theta+a+\frac{1}{2}u_L}\end{array}\right] \\
&\propto \left[\begin{array}{c}\left(\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}e^{a-\frac{u_R}{2}}+e^{2a-u_R}-\frac{1}{2}e^{-\theta+a+\frac{u_L}{2}}\right)e^{-a-\frac{u_L}{2}} \\ -(1-\alpha'(a^{**}))e^{-\eta_L}e^{-\theta+a+\frac{1}{2}u_L}\end{array}\right] \\
&> \frac{1}{2}e^{\theta-\frac{u_R+u_L}{2}}+\frac{3}{2}e^{a-u_R-\frac{u_L}{2}}-\frac{1}{2}e^{-\theta} \\
&\quad -\left(e^{\theta}+e^{-a-\frac{1}{2}u_L}\right)e^{-\theta+a+\frac{1}{2}u_L} \\
&> \frac{1}{2}e^{\theta-\frac{u_R+u_L}{2}}-\left(e^{-\theta}+e^{a+\frac{u_L}{2}}\right)+\frac{3}{2}e^{a-u_R-\frac{u_L}{2}}-\frac{1}{2}e^{-\theta} \\
&= \frac{1}{2}e^{\theta-\frac{u_R+u_L}{2}}-\frac{3}{2}e^{-\theta}+e^{a^{**}+\frac{u_L}{2}}\left(\frac{3}{2}e^{-(u_R+u_L)}-1\right).
\end{aligned}
$$

Therefore, $\frac{\partial \eta_R^{**}}{\partial u_L}>0$ if

1. $u_R+u_L<\log\frac{3}{2}$ and $2\theta-\frac{u_R+u_L}{2}>\log 3$, or
2. $\frac{1}{2}e^{\theta-\frac{u_R+u_L}{2}}-\frac{3}{2}e^{-\theta}+e^{\frac{u_L}{2}}\left(\frac{3}{2}e^{-(u_R+u_L)}-1\right)>0$ and $u_R>u_L$ because in that case, $a^{**}<0$.

Lemma 6.12 *When preference intensity on both sides are equal, the ex ante probability that over coordination happens decreases with θ if $u<0$ and $\theta>\log\frac{3}{2}$.*

Proof. This is because

$$
\begin{aligned}
\frac{\partial \eta_R^{*}(u,u,\theta,\alpha)}{\partial \theta} &= e^{-\eta_R^*}\frac{e^{\theta}+\frac{e^{2\theta}+\frac{1}{2}e^{-\theta+\frac{u}{2}}}{\sqrt{e^{2\theta}+e^{-\frac{u}{2}}-e^{-\theta+\frac{u}{2}}}}}{2} \\
&< e^{-\eta_R^*}\frac{e^{\theta}+e^{\frac{u_r}{2}}+\frac{e^{2\theta}+e^{-\frac{u}{2}}-e^{-\theta+\frac{u}{2}}}{\sqrt{e^{2\theta}+e^{-\frac{u}{2}}-e^{-\theta+\frac{u}{2}}}}}{2} \\
&< 1
\end{aligned}
$$

because $\frac{3}{2}e^{-\theta+\frac{u}{2}}-e^{\frac{-u}{2}}=e^{-\frac{u}{2}}\left(\frac{3}{2}e^{-\theta+u}-1\right)<e^{-\frac{u}{2}}\left(\frac{3}{2}e^{-\theta}-1\right)<0$ because $u<0$ and $\theta>\log\frac{3}{2}$. Thus the derivative of the ex ante probability of over coordination w.r.t. θ is

$$\begin{aligned}\frac{\partial\left(e^{-\alpha\theta}-e^{-\alpha\eta_R^*}\right)}{\partial\theta} &= -\alpha\left(e^{-\alpha\theta}-e^{-\alpha\eta_R^*}\frac{\partial\eta_R^*\left(u,u,\theta,\alpha\right)}{\partial\theta}\right)\\ &< \alpha\left(e^{-\alpha\eta_R^*}-e^{-\alpha\theta}\right)<0.\end{aligned}$$

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References

[1] Adams, W. (1987), As New Hampshire Goes. In: Orren, G., Polsby, N. (Eds.), Media and Momentum (Chatham House).

[2] Dekel, E., and M. Piccione (2000), "Sequential Voting Procedures in Symmetric Binary Elections," Journal of Political Economy, 108, 34-55.

[3] Duverger, M. (1954), Political Parties: Their Organization and Activity in the Modern State (New York: Wiley).

[4] Feddersen, T., and W. Pesendorfer (1997), "Voting Behavior and Information Aggregation in Elections with Private Information," Econometrica, 65, 1029-1058.

[5] Feddersen, T., and W. Pesendorfer (1998), "Convicting the Innocent: The Inferiority of Unanimous Jury Verdicts under Strategic Voting," The American Political Science Review, 92(1), 23-35.

[6] Klumpp, T. A., and M. K. Polborn (2006), "Primaries and the New Hampshire Effect," Journal of Public Economics, 31, 1073-1114.

[7] Lo (2007), "Language and Coordination Games," mimeo.

[8] Malbin, M., 1985. You get what you pay for, but is that what you want? In: Grassmuck, G. (Ed.), Before Nomination: Our Primary Problems (American Enterprise Institute for Public Policy Research).

[9] Morton, R. B., and K. C. Williams (1999), "Information Asymmetries and Simultaneous versus Sequential Voting," American Political Science Review, 93(1), 51-67.

[10] Myatt, D. P. (2007), "On the Theory of Strategic Voting," Review of Economic Studies, 74, 255-281.

[11] Myerson, R. (2000), "Large Poisson Games," Journal of Economic Theory, 94, 7-45.

[12] Myerson, R. (2002), "Comparison of Scoring Rules in Poisson Voting Games," 103,217-251.

[13] Myerson, R. B. and Weber, R. J. (1993), "A Theory of Voting Equilibria," American Political Science Review, 87 (1), 102-114.

[14] Riker, W. H. (1982), "The Two Party System and Duverger's Law: An Essay on the History of Political Science," American Political Science Review, 76 (4), 753-766.