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$f(R, T)$ gravity

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We consider $f(R, T)$ modified theories of gravity, where the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar R and of the trace of the stress-energy tensor T. We obtain the gravitational field equations in the metric formalism, as well as the equations of motion for test particles, which follow from the covariant divergence of the stress-energy tensor. Generally, the gravitational field equations depend on the nature of the matter source. The field equations of several particular models, corresponding to some explicit forms of the function $f(R, T)$, are also presented. An important case, which is analyzed in detail, is represented by scalar field models. We write down the action and briefly consider the cosmological implications of the $f(R, T^{\phi})$ models, where T^{ϕ} is the trace of the stress-energy tensor of a self-interacting scalar field. The equations of motion of the test particles are also obtained from a variational principle. The motion of massive test particles is nongeodesic, and takes place in the presence of an extra-force orthogonal to the four velocity. The Newtonian limit of the equation of motion is further analyzed. Finally, we provide a constraint on the magnitude of the extra acceleration by analyzing the perihelion precession of the planet Mercury in the framework of the present model.

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I. INTRODUCTION

The recent observational data [1,2] on the late-time acceleration of the Universe and the existence of dark matter have posed a fundamental theoretical challenge to gravitational theories. One possibility in explaining the observations is by assuming that at large scales the Einstein gravity model of general relativity breaks down, and a more general action describes the gravitational field. Theoretical models, in which the standard Einstein-Hilbert action is replaced by an arbitrary function of the Ricci scalar R [3], have been extensively investigated lately. The presence of a late-time cosmic acceleration of the Universe can indeed be explained by $f(R)$ gravity [4]. The conditions of the existence of viable cosmological models have been found in [5], and severe weak field constraints obtained from the classical tests of general relativity for the Solar System regime seem to rule out most of the models proposed so far [6,7]. However, viable models, passing Solar System tests, can be constructed [8–11]. $f(R)$ models that satisfy local tests and unify inflation with dark energy were considered in [12]. In the framework of $f(R)$ gravity models, the possibility that the galactic dynamic of massive test particles can be understood without the need for dark matter was considered in [13–17]. For reviews of $f(R)$ generalized gravity models see [3,18].

A generalization of $f(R)$ modified theories of gravity was proposed in [19], by including in the theory an explicit coupling of an arbitrary function of the Ricci scalar R with the matter Lagrangian density L_m. As a result of the coupling the motion of the massive particles is nongeodesic, and an extra-force orthogonal to the four velocity, arises. The connections with modified Newtonian dynamics and the Pioneer anomaly were also explored. This model was extended to the case of the arbitrary couplings in both geometry and matter in [20]. The astrophysical and cosmological implications of the nonminimal coupling matter-geometry coupling were extensively investigated in [21,22], and the Palatini formulation of the nonminimal geometry-coupling models was considered in [23]. In this

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context, a maximal extension of the Hilbert-Einstein action was proposed in [24], by assuming that the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar R and of the matter Lagrangian L_m. The gravitational field equations have been obtained in the metric formalism, as well as the equations of motion for test particles, which follow from the covariant divergence of the stress-energy tensor.

A specific application of the latter $f(R, L_m)$ gravity was proposed in [25], which may be considered a relativistically covariant model of interacting dark energy, based on the principle of least action. The cosmological constant in the gravitational Lagrangian is a function of the trace of the stress-energy tensor, and consequently the model was denoted "$\Lambda(T)$ gravity". It was argued that recent cosmological data favor a variable cosmological constant, which is consistent with $\Lambda(T)$ gravity, without the need to specify an exact form of the function $\Lambda(T)$ [25]. $\Lambda(T)$ gravity is more general than the Palatini $f(R)$ gravity, and reduces to the latter when we neglect the pressure of the matter.

It is the purpose of the present paper to consider another extension of standard general relativity, the $f(R, T)$ modified theories of gravity, where the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar R and of the trace of the stress-energy tensor T. Note that the dependence from T may be induced by exotic imperfect fluids or quantum effects (conformal anomaly). As a first step in our study we derive the field equations of the model from a variational, Hilbert-Einstein type, principle. The covariant divergence of the stress-energy tensor is also obtained. The $f(R, T)$ gravity model depends on a source term, representing the variation of the matter stress-energy tensor with respect to the metric. A general expression for this source term is obtained as a function of the matter Lagrangian L_m. Therefore each choice of L_m would generate a specific set of field equations. Some particular models, corresponding to specific choices of the function $f(R, T)$ are also presented, and their properties are briefly discussed. In fact, we also demonstrate the possibility of reconstruction of arbitrary Friedmann-Robertson-Walker cosmologies by an appropriate choice of a function $f(T)$. Scalar fields play a fundamental role in cosmology, i.e., as possible explanations for inflation, late-time acceleration, or dark matter, respectively. Therefore, we introduce and briefly discuss the $f(R, T^\phi)$ gravitational models, where T^ϕ is the trace of the stress energy of the scalar field. Some cosmological applications of this model are also presented.

Since in the present model, the covariant divergence of the stress-energy tensor is nonzero, the motion of massive test particles is nongeodesic, and an extra acceleration, due to the coupling between matter and geometry, is always present. The equations of motion of test particles are obtained from a variational principle. The same variational principle can be used to investigate the Newtonian limit of the model, and the expression of the extra acceleration is also obtained. We use the precession of the perihelion of the planet Mercury to obtain a general constraint on the magnitude of the extra acceleration.

The present paper is organized as follows. The field equations of $f(R, T)$ gravity are derived in Sec. II. Some particular cases of the model are considered in Sec. III. The case of the scalar fields is discussed in Sec. IV, and a Brans-Dicke type formulation of the model is obtained. The equations of motion of massive test particles are derived in Sec. V, where the Newtonian limit of the model is also obtained and analyzed. We discuss and conclude our results in Sec. VI. In the present paper we use the natural system of units with $G = c = 1$, so that the Einstein gravitational constant is defined as $\kappa^2 = 8\pi$.

II. GRAVITATIONAL FIELD EQUATIONS OF $f(R, T)$ GRAVITY

We assume that the action for the modified theories of gravity takes the following form

$$S = \frac{1}{16\pi}\int f(R, T)\sqrt{-g}d^4x + \int L_m\sqrt{-g}d^4x, \tag{1}$$

where $f(R, T)$ is an arbitrary function of the Ricci scalar, R, and of the trace T of the stress-energy tensor of the matter, $T_{\mu\nu}$. L_m is the matter Lagrangian density, and we define the stress-energy tensor of matter as [26]

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}, \tag{2}$$

and its trace by $T = g^{\mu\nu}T_{\mu\nu}$, respectively. By assuming that the Lagrangian density L_m of matter depends only on the metric tensor components $g_{\mu\nu}$, and not on its derivatives, we obtain

$$T_{\mu\nu} = g_{\mu\nu}L_m - 2\frac{\partial L_m}{\partial g^{\mu\nu}}. \tag{3}$$

By varying the action S of the gravitational field with respect to the metric tensor components $g^{\mu\nu}$ provides the following relationship

$$\delta S = \frac{1}{16\pi}\int\Big[f_R(R,T)\delta R + f_T(R,T)\frac{\delta T}{\delta g^{\mu\nu}}\delta g^{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R,T)\delta g^{\mu\nu} + 16\pi\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}\Big]\sqrt{-g}d^4x, \tag{4}$$

where we have denoted $f_R(R, T) = \partial f(R, T)/\partial R$ and $f_T(R, T) = \partial f(R, T)/\partial T$, respectively. For the variation of the Ricci scalar, we obtain

$$\delta R = \delta(g^{\mu\nu}R_{\mu\nu}) = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}(\nabla_\lambda\delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu\delta\Gamma^\lambda_{\mu\lambda}), \tag{5}$$

where ∇_λ is the covariant derivative with respect to the symmetric connection Γ associated to the metric g. The variation of the Christoffel symbols yields

$$\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(\nabla_\mu \delta g_{\nu\alpha} + \nabla_\nu \delta g_{\alpha\mu} - \nabla_\alpha \delta g_{\mu\nu}), \quad (6)$$

and the variation of the Ricci scalar provides the expression

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}. \quad (7)$$

Therefore, for the variation of the action of the gravitational field we obtain

$$\delta S = \frac{1}{16\pi}\int\bigg[f_R(R,T)R_{\mu\nu}\delta g^{\mu\nu} + f_R(R,T)g_{\mu\nu}\Box\delta g^{\mu\nu} - f_R(R,T)\nabla_\mu\nabla_\nu\delta g^{\mu\nu} + f_T(R,T)\frac{\delta(g^{\alpha\beta}T_{\alpha\beta})}{\delta g^{\mu\nu}}\delta g^{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R,T)\delta g^{\mu\nu} + 16\pi\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}\bigg]\sqrt{-g}d^4x. \quad (8)$$

We define the variation of T with respect to the metric tensor as

$$\frac{\delta(g^{\alpha\beta}T_{\alpha\beta})}{\delta g^{\mu\nu}} = T_{\mu\nu} + \Theta_{\mu\nu}, \quad (9)$$

where

$$\Theta_{\mu\nu} \equiv g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}. \quad (10)$$

After partially integrating the second and third terms in Eq. (8), we obtain the field equations of the $f(R,T)$ gravity model as

$$f_R(R,T)R_{\mu\nu} - \frac{1}{2}f(R,T)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R(R,T) = 8\pi T_{\mu\nu} - f_T(R,T)T_{\mu\nu} - f_T(R,T)\Theta_{\mu\nu}. \quad (11)$$

Note that when $f(R,T) \equiv f(R)$, from Eqs. (11) we obtain the field equations of $f(R)$ gravity.

Contracting Eq. (11) gives the following relation between the Ricci scalar R and the trace T of the stress-energy tensor,

$$f_R(R,T)R + 3\Box f_R(R,T) - 2f(R,T) = 8\pi T - f_T(R,T)T - f_T(R,T)\Theta, \quad (12)$$

where we have denoted $\Theta = \Theta_\mu{}^\mu$.

By eliminating the term $\Box f_R(R,T)$ between Eqs. (11) and (12), the gravitational field equations can be written in the form

$$f_R(R,T)\left(R_{\mu\nu} - \frac{1}{3}Rg_{\mu\nu}\right) + \frac{1}{6}f(R,T)g_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{3}Tg_{\mu\nu}\right) - f_T(R,T)\left(T_{\mu\nu} - \frac{1}{3}Tg_{\mu\nu}\right) - f_T(R,T)\left(\Theta_{\mu\nu} - \frac{1}{3}\Theta g_{\mu\nu}\right) + \nabla_\mu\nabla_\nu f_R(R,T). \quad (13)$$

Taking into account the covariant divergence of Eq. (11), with the use of the following mathematical identity [27]

$$\nabla^\mu\bigg[f_R(R,T)R_{\mu\nu} - \frac{1}{2}f(R,T)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R(R,T)\bigg] \equiv 0, \quad (14)$$

where $f(R,T)$ is an arbitrary function of the Ricci scalar R and of the trace of the stress-energy tensor T, we obtain for the divergence of the stress-energy tensor $T_{\mu\nu}$ the equation

$$\nabla^\mu T_{\mu\nu} = \frac{f_T(R,T)}{8\pi - f_T(R,T)}\times[(T_{\mu\nu} + \Theta_{\mu\nu})\nabla^\mu \ln f_T(R,T) + \nabla^\mu\Theta_{\mu\nu}]. \quad (15)$$

Next we consider the calculation of the tensor $\Theta_{\mu\nu}$, once the matter Lagrangian is known. From Eq. (3) we obtain first

$$\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} = \frac{\delta g_{\alpha\beta}}{\delta g^{\mu\nu}}L_m + g_{\alpha\beta}\frac{\partial L_m}{\partial g^{\mu\nu}} - 2\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}} = \frac{\delta g_{\alpha\beta}}{\delta g^{\mu\nu}}L_m + \frac{1}{2}g_{\alpha\beta}g_{\mu\nu}L_m - \frac{1}{2}g_{\alpha\beta}T_{\mu\nu} - 2\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}. \quad (16)$$

From the condition $g_{\alpha\sigma}g^{\sigma\beta} = \delta^\beta_\alpha$, we have

$$\frac{\delta g_{\alpha\beta}}{\delta g^{\mu\nu}} = -g_{\alpha\sigma}g_{\beta\gamma}\delta^{\sigma\gamma}_{\mu\nu}, \quad (17)$$

where $\delta^{\sigma\gamma}_{\mu\nu} = \delta g^{\sigma\gamma}/\delta g^{\mu\nu}$ is the generalized Kronecker symbol. Therefore, for $\Theta_{\mu\nu}$ we find

$$\Theta_{\mu\nu} = -2T_{\mu\nu} + g_{\mu\nu}L_m - 2g^{\alpha\beta}\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}. \quad (18)$$

In the case of the electromagnetic field the matter Lagrangian is given by

$$L_m = -\frac{1}{16\pi}F_{\alpha\beta}F_{\gamma\sigma}g^{\alpha\gamma}g^{\beta\sigma}, \quad (19)$$

where $F_{\alpha\beta}$ is the electromagnetic field tensor. In this case we obtain $\Theta_{\mu\nu} = -T_{\mu\nu}$. In the case of a massless scalar field ϕ with Lagrangian $L_m = g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi$, we obtain $\Theta_{\mu\nu} = -T_{\mu\nu} + (1/2)Tg_{\mu\nu}$. The problem of the perfect fluids, described by an energy density ρ, pressure p and four velocity u^μ is more complicated, since there is no unique definition of the matter Lagrangian. However, in the present study we *assume* that the stress-energy tensor of the matter is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \quad (20)$$

and the matter Lagrangian can be taken as $L_m = -p$. The four velocity u_μ satisfies the conditions $u_\mu u^\mu = 1$ and $u^\mu\nabla_\nu u_\mu = 0$, respectively. Then, with the use of Eq. (18), we obtain for the variation of the stress-energy of a perfect fluid the expression

$$\Theta_{\mu\nu} = -2T_{\mu\nu} - pg_{\mu\nu}. \quad (21)$$

III. PARTICULAR CASES OF GRAVITATIONAL FIELD EQUATIONS IN THE $f(R,T)$ MODEL

In the present section we consider some particular classes of $f(R,T)$ modified gravity models, obtained by explicitly specifying the functional form of f. Generally, the field equations also depend, through the tensor $\Theta_{\mu\nu}$, on the physical nature of the matter field. Hence in the case of $f(R,T)$ gravity, depending on the nature of the matter source, for each choice of f we can obtain several theoretical models, corresponding to different matter models.

A. $f(R,T) = R + 2f(T)$

As a first case of a $f(R,T)$ modified gravity model we assume that the function $f(R,T)$ is given by $f(R,T) = R + 2f(T)$, where $f(T)$ is an arbitrary function of the trace of the stress-energy tensor of matter. The gravitational field equations immediately follow from Eq. (11), and are given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} - 2f'(T)T_{\mu\nu} - 2f'(T)\Theta_{\mu\nu} + f(T)g_{\mu\nu}, \tag{22}$$

where the prime denotes a derivative with respect to the argument.

If the matter source is a perfect fluid, $\Theta_{\mu\nu} = -2T_{\mu\nu} - pg_{\mu\nu}$, then the field equations become

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} + 2f'(T)T_{\mu\nu} + [2pf'(T) + f(T)]g_{\mu\nu}. \tag{23}$$

In the case of dust with $p = 0$ the gravitational field equations are given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} + 2f'(T)T_{\mu\nu} + f(T)g_{\mu\nu}. \tag{24}$$

These field equations were proposed in [25] to solve the cosmological constant problem. The simplest cosmological model can be obtained by assuming a dust universe ($p = 0$, $T = \rho$), and by choosing the function $f(T)$ so that $f(T) = \lambda T$, where λ is a constant. By assuming that the metric of the universe is given by the flat Robertson-Walker metric,

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \tag{25}$$

the gravitational field equations are given by

$$3\frac{\dot{a}^2}{a^2} = (8\pi + 3\lambda)\rho, \tag{26}$$

and

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = \lambda\rho, \tag{27}$$

respectively. Thus, this $f(R,T)$ gravity model is equivalent to a cosmological model with an effective cosmological constant $\Lambda_{\text{eff}} \propto H^2$, where $H = \dot{a}/a$ is the Hubble function [25]. It is also interesting to note that generally for this choice of $f(R,T)$ the gravitational coupling becomes an effective and time dependent coupling, of the form $G_{\text{eff}} = G \pm 2f'(T)$. Thus the term $2f(T)$ in the gravitational action modifies the gravitational interaction between matter and curvature, replacing G by a running gravitational coupling parameter.

The field equations reduce to a single equation for H,

$$2\dot{H} + 3\frac{8\pi + 2\lambda}{8\pi + 3\lambda}H^2 = 0, \tag{28}$$

with the general solution given by

$$H(t) = \frac{2(8\pi + 3\lambda)}{3(8\pi + 2\lambda)}\frac{1}{t}. \tag{29}$$

The scale factor evolves according to $a(t) = t^{\alpha}$, with $\alpha = 2(8\pi + 3\lambda)/3(8\pi + 2\lambda)$.

B. $f(R,T) = f_1(R) + f_2(T)$

As a second example we consider the case in which the function f is given by $f(R,T) = f_1(R) + f_2(T)$, where $f_1(R)$ and $f_2(T)$ are arbitrary functions of R and T, respectively. In this case for an arbitrary matter source the gravitational field equations are given by

$$f_1'(R)R_{\mu\nu} - \frac{1}{2}f_1(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_1(R) = 8\pi T_{\mu\nu} - f_2'(T)T_{\mu\nu} - f_2'(T)\Theta_{\mu\nu} + \frac{1}{2}f_2(T)g_{\mu\nu}. \tag{30}$$

Assuming that the matter content consists of a perfect fluid, the gravitational field equations become

$$f_1'(R)R_{\mu\nu} - \frac{1}{2}f_1(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_1'(R) = 8\pi T_{\mu\nu} + f_2'(T)T_{\mu\nu} + \left[f_2'(T)p + \frac{1}{2}f_2(T)\right]g_{\mu\nu}. \tag{31}$$

In the case of dust with $p = 0$, the gravitational field equations reduce to

$$f_1'(R)R_{\mu\nu} - \frac{1}{2}f_1(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_1'(R) = 8\pi T_{\mu\nu} + f_2'(T)T_{\mu\nu} + \frac{1}{2}f_2(T)g_{\mu\nu}. \tag{32}$$

In the case $f_2(T) \equiv 0$, we reobtain the field equations of standard $f(R)$ gravity. Equation (31) can be reformulated as an effective Einstein field equation of the form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G_{\text{eff}}T_{\mu\nu} + T^{\text{eff}}_{\mu\nu}, \tag{33}$$

where we have denoted

$$G_{\text{eff}} = \frac{1}{f_1'(R)}\left[1 + \frac{f_2'(T)}{8\pi}\right], \tag{34}$$

and

$$T^{\rm eff}_{\mu\nu} = \frac{1}{f_1'(R)}\left\{\frac{1}{2}[f_1(R) - Rf_1'(R) + 2f_2'(T)p + f_2(T)]g_{\mu\nu} - (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_1'(R)\right\}. \quad (35)$$

The gravitational coupling is again given by an effective, matter (and time) dependent coupling, which is proportional to the derivative of the function f_2 with respect to T. The gravitational field equations can be recast in such a form that the higher order corrections, coming both from the geometry, and from the matter-geometry coupling, provide a stress-energy tensor of geometrical and matter origin, describing an "effective" source term on the right hand side of the standard Einstein field equations. In the $f(R,T)$ scenario, the cosmic acceleration may result not only from a geometrical contribution to the total cosmic energy density, but it is also dependent on the matter content of the universe, which provides new corrections to the Hilbert-Einstein Lagrangian via the matter-geometry coupling.

The (t,t) component of Eq. (32) has the following form:

$$3H^2 = \frac{8\pi}{f_1'(R)}\left[1 + \frac{f_2'(T)}{8\pi}\right]\rho + \frac{1}{f_1'(R)}\left[-\frac{1}{2}(f_1(R) - 6(\dot{H} + 2H^2)f_1'(R)) + 2f_2'(T) - 9(\ddot{H} + 4H\dot{H})f_1''(R)\right]. \quad (36)$$

Here $R = 6(\dot{H} + 2H^2)$. For simplicity, $T_{\mu\nu}$ corresponds to the matter with a constant equation of state parameter w. If we now define the e-folding N by $a = a_0e^N$, ρ and T are given by

$$\rho = \rho_0 e^{-3(1+w)N}, \qquad T = -(1-3w)\rho_0 e^{-3(1+w)N}. \quad (37)$$

We now consider an arbitrary development of the expansion in the Universe given by

$$H = h(N), \quad (38)$$

where $h(N)$ is an arbitrary function of N. Then Eq. (36) can be written as

$$\begin{aligned} f_2'(T) = F_2(N) \equiv \frac{3}{1+\rho_0 e^{-3(1+w)N}}\Big\{&-\frac{8\pi}{3}\rho_0 e^{-3(1+w)N} + \frac{1}{6}f_1[6(h(N)h'(N) + 2h(N)^2)] \\ &- [h(N)h'(N) + h(N)^2]f_1'[6(h(N)h'(N) + 2h(N)^2)] \\ &+ 3[h(N)^2h''(N) + h(N)h'(N)^2 + 4h(N)^2h'(N)] \\ &\times f_1''[6(h(N)h'(N) + 2h(N)^2)]\Big\}. \end{aligned} \quad (39)$$

Equation (39) dictates that for an arbitrary $f_1(R)$, and for the following specific choice

$$f_2'(T) = F_2\left(-\frac{\ln(-\frac{T}{(1-3w)\rho_0})}{3(1+w)}\right), \quad (40)$$

an arbitrary development of the expansion in the Universe given by (38) can be realized. Hence, for viable $f(R)$ gravitational models, using the above reconstruction method, the possibility arises to modify the universe evolution by adding the corresponding function depending on the trace of the stress-energy tensor.

C. $f(R,T) = f_1(R) + f_2(R)f_3(T)$

As a third case of generalized $f(R,T)$ gravity models, we consider that the action is given by $f(R,T) = f_1(R) + f_2(R)f_3(T)$, where f_i, $i = 1, 2, 3$ are arbitrary functions of the argument. For an arbitrary matter source the gravitational field equations are given by

$$\begin{aligned} &[f_1'(R) + f_2'(R)f_3(T)]R_{\mu\nu} - \frac{1}{2}f_1(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)[f_1'(R) + f_2'(R)f_3(T)] \\ &= 8\pi T_{\mu\nu} - f_2(R)f_3'(T)T_{\mu\nu} - f_2(R)f_3'(T)\Theta_{\mu\nu} + \frac{1}{2}f_2(R)f_3(T)g_{\mu\nu}. \end{aligned} \quad (41)$$

In the case of a perfect fluid we find the field equations

$$\begin{aligned} &[f_1'(R) + f_2'(R)f_3(T)]R_{\mu\nu} - \frac{1}{2}f_1(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)[f_1'(R) + f_2'(R)f_3(T)] \\ &= 8\pi T_{\mu\nu} + f_2(R)f_3'(T)T_{\mu\nu} + f_2(R)\left[f_3'(T)p + \frac{1}{2}f_3(T)\right]g_{\mu\nu}. \end{aligned} \quad (42)$$

For the case of dust matter we obtain

$$\begin{aligned} &[f_1'(R) + f_2'(R)f_3(T)]R_{\mu\nu} - \frac{1}{2}f_1(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)[f_1'(R) + f_2'(R)f_3(T)] \\ &= 8\pi T_{\mu\nu} + f_2(R)f_3'(T)T_{\mu\nu} + \frac{1}{2}f_2(R)f_3(T)g_{\mu\nu}. \end{aligned} \quad (43)$$

In this class of models both, the effective cosmological constant $\Lambda_{\rm eff}$ and the running gravitational coupling $G_{\rm eff}$ are functions of both matter and geometry.

IV. $f(R, T^\phi)$ GRAVITY

Scalar fields are supposed to play a fundamental role in physics and cosmology [28]. In particular, cosmological inflation, the late-time acceleration of the universe, or dark matter and its properties can be explained in the framework of specific scalar field models. However, obtaining more general gravitational models with scalar fields as a source may give a better insight in the general properties of the gravitational field, and could also provide some possibilities for observationally testing the generalizations of gravity models. In the present section, we consider the $f(R, T)$ gravity model in the case of self-interacting scalar fields.

A. The action of the $f(R, T^\phi)$ gravity

We start with the following action for matter,

$$S_{\mathrm{matter}}(g_{\mu\nu}, \psi_i) = \int d^4x\sqrt{-g}\mathcal{L}(g_{\rho\sigma}, \psi_i). \quad (44)$$

In Eq. (44) the ψ_i's, $i = 1, 2, \ldots$ represent the matter fields. By using the matter action (44), we now introduce the action for the gravitational field with matter sources as

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}F(R, \phi) + S_{\mathrm{matter}}(e^{\phi}g_{\mu\nu}, \psi_i), \quad (45)$$

where $F(R, \phi)$ is an algebraic function of R and of the scalar field ϕ. Then by the variation of the action with respect to ϕ, we obtain first

$$\frac{1}{2\kappa^2}F_\phi(R, \phi) + \frac{1}{2}T_\phi = 0, \quad (46)$$

where we denoted $F_\phi(R, \phi) \equiv \partial F(, \phi)/\partial\phi$ and

$$T_\phi \equiv e^{-\phi}g^{\mu\nu}T_{\phi\mu\nu}, T_{\phi\mu\nu} \equiv T_{\mu\nu}|_{g_{\mu\nu}\to e^{\phi}g_{\mu\nu}}, \quad (47)$$

respectively. The stress-energy tensor of matter is defined, as usual, by

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{matter}}(g_{\rho\sigma}, \psi_i)}{\delta g_{\mu\nu}}. \quad (48)$$

By the assumption that $F(R, \phi)$ is an algebraic function of R and ϕ, Eq. (46) can be algebraically solved with respect to ϕ. Thus we can obtain ϕ as a function of R and T_ϕ, i.e., $\phi = \phi(R, T_\phi)$. Then by substituting the expression of ϕ into the action (45), we obtain an example of $F(R, T_\phi)$ gravity, with the following action

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\tilde{F}(R, T_\phi) + S_{\mathrm{matter}}(e^{\phi}g_{\mu\nu}, \psi_i), \quad (49)$$

where we have denoted

$$\tilde{F}(R, T_\phi) \equiv F[R, \phi(R, T_\phi)]. \quad (50)$$

With the use of the conformal Weyl transformation $g_{\mu\nu} \to e^{-\phi}g_{\mu\nu}$, the action (45) or (49) is transformed as

$$\begin{aligned} S &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}e^{-2\phi}F\left[\left(R + 3\Box\phi - \frac{3}{2}\partial_\sigma\phi\partial^\sigma\phi\right)e^{\phi}, \phi\right] \\ &\quad + S_{\mathrm{matter}}(g_{\mu\nu}, \psi_i) \\ &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\tilde{F}\left[\left(R + 3\Box\phi - \frac{3}{2}\partial_\sigma\phi\partial^\sigma\phi\right)e^{\phi}, T\right] \\ &\quad + S_{\mathrm{matter}}(g_{\mu\nu}, \psi_i), \end{aligned} \quad (51)$$

with $T \equiv g^{\mu\nu}T_{\mu\nu}$. In the action $S_{\mathrm{matter}}(g_{\mu\nu}, \psi_i)$ in Eq. (51), the matter fields have only a minimal coupling with gravity, and they do not couple with ϕ. Then the frame in the action (51) might be regarded as a physical frame.

B. Example of $f(R, T^\phi)$ scalar field gravity, and reconstruction

As an example of $f(R, T^\phi)$ gravity of the form $R + f(T^\phi)$, we consider the case of a scalar field with a self-interaction potential $V(\phi)$. The action is given by

$$S^\phi = \int d^4x\sqrt{-g}\left[-\frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi)\right], \quad (52)$$

where we have included $\omega(\phi)$ for later convenience. For the scalar field model described by Eq. (52), the trace of the stress-energy tensor is given by

$$T^\phi = -\omega(\phi)\partial_\mu\phi\partial^\mu\phi - 4V(\phi). \quad (53)$$

Consequently we may define the $f(R, T^\phi) = R + f(T^\phi)$ gravity model in the following form:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R + f(T^\phi) - \frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi)\right]. \quad (54)$$

For the model (54), in a flat Friedmann-Robertson-Walker geometry, the Friedmann equations have the following form:

$$\begin{aligned} \frac{3}{\kappa^2}H^2 &= \frac{1}{2}\omega(\phi)\dot{\phi}^2 + V(\phi) - f[\omega(\phi)\dot{\phi}^2 - 4V(\phi)] \\ &\quad + 2f'[\omega(\phi)\dot{\phi}^2 - 4V(\phi)]\omega(\phi)\dot{\phi}^2, \end{aligned} \quad (55)$$

$$\begin{aligned} -\frac{1}{\kappa^2}(3H^2 + 2\dot{H}) &= \frac{1}{2}\omega(\phi)\dot{\phi}^2 - V(\phi) \\ &\quad + f[\omega(\phi)\dot{\phi}^2 - 4V(\phi)], \end{aligned} \quad (56)$$

where $H = \dot{a}/a$.

In the following, we consider, for simplicity, the case $V(\phi) = 0$. Then the action (54) has the following form:

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R + F[-\omega(\phi)\partial_\mu\phi\partial^\mu\phi]\right\}, \quad (57)$$

and the Friedmann equations (55) and (56) take the form,

$$\frac{3}{\kappa^2}H^2 = -F[\omega(\phi)\dot{\phi}^2] + 2F'[\omega(\phi)\dot{\phi}^2]\omega(\phi)\dot{\phi}^2, \quad (58)$$

$$-\frac{1}{\kappa^2}(3H^2 + 2\dot{H}) = F[\omega(\phi)\dot{\phi}^2]. \quad (59)$$

In the action (57), $F[-\omega(\phi)\partial_\mu\phi\partial^\mu\phi]$ is defined by

$$\begin{aligned} F[-\omega(\phi)\partial_\mu\phi\partial^\mu\phi] &\equiv f[-\omega(\phi)\partial_\mu\phi\partial^\mu\phi] \\ &\quad - \frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi. \end{aligned} \quad (60)$$

The action (57) gives a model of k-essence [29–31]. In [32], it has been shown that the Friedmann equations (58) and (59) do not admit the de Sitter solution, except in the trivial case where ϕ is a constant, and $F(0) > 0$. In [32], the formalism of the general reconstruction has also been explicitly given. An explicit model of modified gravity in which a crossing of the phantom divide can be realized was reconstructed in [33].

As a simple example, we consider the model

$$F[-\omega(\phi)\partial_\mu\phi\partial^\mu\phi] = -F_0 e^{-2\ln((\phi)/(\phi_0))\partial_\mu\phi\partial^\mu\phi}, \quad (61)$$

where F_0 and ϕ_0 are constants. The Friedmann equations have a solution where the universe expands by a power law,

$$H = \frac{h_0}{t}, \qquad \phi = t. \quad (62)$$

The constant h_0 can be obtained by solving the following algebraic equation

$$3h_0^2 - 2h_0 = \kappa^2\phi_0^2 F_0. \quad (63)$$

V. THE EQUATION OF MOTION OF TEST PARTICLES AND THE NEWTONIAN LIMIT IN $f(R,T)$ GRAVITY

Since in the general $f(R,T)$ type gravity models the stress-energy tensor of matter is not covariantly conserved, it follows that the test particles, moving in a gravitational field, do not follow geodesic lines. This situation is similar to the case of the $f(R,L_m)$ models [24], where the coupling between matter and geometry induces a supplementary acceleration acting on the particle. In the present section, we derive the equations of motion of test particles in $f(R,T)$ gravity models, and obtain the Newtonian limit of the theory. We also investigate the constraints on the magnitude of the extra acceleration that can be obtained from the available observational data on the perihelion precession of the planet Mercury.

A. The equations of motion of test particles

In the case of a perfect fluid, with the stress-energy tensor given by Eq. (20), the divergence of the stress-energy tensor is given by

$$\nabla^\mu T_{\mu\nu} = -\frac{1}{8\pi + f_T(R,T)}\{T_{\mu\nu}\nabla^\mu f_T(R,T) + g_{\mu\nu}\nabla^\mu[f_T(R,T)p]\}. \quad (64)$$

We also introduce the projection operator $h_{\mu\lambda} = g_{\mu\lambda} - u_\mu u_\lambda$ for which we have $h_{\mu\lambda}u^\mu = 0$ and $h_{\mu\lambda}T^{\mu\nu} = -h_\lambda^\nu p$, respectively.

Explicitly, Eq. (64) can be written in the form

$$\nabla_\nu(\rho+p)u^\mu u^\nu + (\rho+p)[u^\nu\nabla_\nu u^\mu + u^\mu\nabla_\nu u^\nu] - g^{\mu\nu}\nabla_\nu p = -\frac{1}{8\pi + f_T(R,T)}\{T^{\mu\nu}\nabla_\nu f_T(R,T) + g^{\mu\nu}\nabla_\nu[f_T(R,T)p]\}. \quad (65)$$

By contracting Eq. (65) with $h_{\mu\lambda}$ we obtain

$$g_{\mu\lambda}u^\nu\nabla_\nu u^\mu = 8\pi\frac{\nabla_\nu p}{(\rho+p)[8\pi + f_T(R,T)]}h_\lambda^\nu. \quad (66)$$

After multiplying with $g^{\alpha\lambda}$ and by taking into account the identity

$$u^\nu\nabla_\nu u^\mu = \frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda, \quad (67)$$

we obtain the equation of motion of a test fluid in $f(R,T)$ gravity as

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda = f^\mu, \quad (68)$$

where

$$f^\mu = 8\pi\frac{\nabla_\nu p}{(\rho+p)[8\pi + f_T(R,T)]}(g^{\mu\nu} - u^\mu u^\nu). \quad (69)$$

The extra force f^μ is perpendicular to the four velocity, $f^\mu u_\mu = 0$. When $f_T(R,T) = 0$, we reobtain the equation of motion of perfect fluids with pressure in standard general relativity, which follows from the conservation of the energy-momentum tensor, $\nabla_\mu T^\mu_\nu = 0$ [34]. In the limit $p \to 0$, corresponding to a pressureless fluid (dust), in standard general relativity the motion of the test particles becomes geodesic. The same result holds true in the $f(R,T)$ gravity model. Even if $f_T(R,T) \neq 0$, the motion of the dust particles always follows the geodesic lines of the geometry. By assuming that the term $8\pi\nabla_\nu p/(\rho+p)\times[8\pi + f_T(R,T)]$ can be formally represented as $\nabla_\nu \ln\sqrt{Q}$,

$$8\pi\frac{\nabla_\nu p}{(\rho+p)[8\pi + f_T(R,T)]} = \nabla_\nu \ln\sqrt{Q}, \quad (70)$$

the equation of motion Eq. (68) can be obtained from the variational principle

$$\delta S_p = \delta\int L_p ds = \delta\int\sqrt{Q}\sqrt{g_{\mu\nu}u^\mu u^\nu}ds = 0, \quad (71)$$

where S_p and $L_p = \sqrt{Q}\sqrt{g_{\mu\nu}u^\mu u^\nu}$ are the action and the Lagrangian density for the test particles, respectively.

To prove this result we start with the Lagrange equations corresponding to the action (71),

$$\frac{d}{ds}\left(\frac{\partial L_p}{\partial u^\lambda}\right) - \frac{\partial L_p}{\partial x^\lambda} = 0. \quad (72)$$

Since

$$\frac{\partial L_p}{\partial u^\lambda} = \sqrt{Q}u_\lambda \quad (73)$$

and

$$\frac{\partial L_p}{\partial x^\lambda} = \frac{1}{2}\sqrt{Q}g_{\mu\nu,\lambda}u^\mu u^\nu + \frac{1}{2}\frac{Q_{,\lambda}}{Q}, \quad (74)$$

a straightforward calculation gives the equations of motion of the particle as

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda + (u^\mu u^\nu - g^{\mu\nu})\nabla_\nu \ln\sqrt{Q} = 0. \quad (75)$$

When $\sqrt{Q} \to 1$ we reobtain the standard general relativistic equation for geodesic motion.

As an example of the application of the previous formalism we consider the case in which the pressure can be expressed as a function of the density by a linear barotropic equation of state of the form $p = w\rho$, where the constant w satisfies the condition $w \ll 1$. Therefore $\rho + p \approx \rho$ and $T = \rho - 3p \approx \rho$, respectively. Moreover, for simplicity, we also assume that the function f_T is a function of $T \approx \rho$ only. We can expand f_T near a fixed value ρ_0 of the density, so that $f_T(\rho) = f_T(\rho_0) + (\rho - \rho_0)[df_T/d\rho)/d\rho]|_{\rho=\rho_0} = 8\pi[a_0 + b_0(\rho - \rho_0)]$, where $a_0 = f_T(\rho_0)/8\pi$ and $b_0 = [df_T/d\rho)/d\rho]|_{\rho=\rho_0}/8\pi$, respectively. Equation (70) of the definition of $\sqrt{Q}$ becomes

$$\frac{w}{1 + a_0 - b_0\rho_0}\nabla_\nu \ln\frac{\rho}{1 + a_0 + b_0(\rho - \rho_0)} = \nabla_\nu \ln\sqrt{Q}, \quad (76)$$

giving

$$\sqrt{Q}(\rho) \approx \left[\frac{C\rho}{1 + a_0 + b_0(\rho - \rho_0)}\right]^{w/(1+a_0-b_0\rho_0)}, \quad (77)$$

where C is an arbitrary constant of integration. Equation (70) is also valid for a fluid satisfying a linear barotropic equation of state of the form $p = (\gamma - 1)\rho$, $\gamma =$ constant, and for a model with $f_T(R, T) =$ constant $= f_T$. In this case $\sqrt{Q} = C_T\rho^{8\pi(\gamma-1)/\gamma(8\pi+f_T)}$, where C_T is an arbitrary integration constant. Therefore, Eq. (70) is valid in both the nonrelativistic and the extreme relativistic limits of the model. On the other hand, we have to mention that the function $\sqrt{Q}$ can always be obtained by formally integrating the left-hand side of Eq. (70). However, generally this function cannot be expressed in an exact analytical form, and to find its functional form approximate methods have to be used.

B. The Newtonian limit

The variational principle (71) and the pressureless dust model, described by Eqs. (76) and (77), can be used to study the Newtonian limit of the model. In the limit of the weak gravitational fields,

$$ds \approx \sqrt{1 + 2\phi - \vec{v}^2}dt \approx (1 + \phi - \vec{v}^2/2)dt, \quad (78)$$

where ϕ is the Newtonian potential and $\vec{v}$ is the usual tridimensional velocity of the fluid. By using the relation $x^\alpha = \exp(\alpha \ln x) \approx 1 + \alpha \ln x$, we can approximate $\sqrt{Q}(\rho)$ given by Eq. (77) as

$$\sqrt{Q}(\rho) \approx 1 + \frac{w}{(1 + a_0 - b_0\rho_0)}\ln\left[\frac{C\rho}{1 + a_0 + b_0(\rho - \rho_0)}\right] = 1 + U(\rho), \quad (79)$$

where we have denoted

$$U(\rho) = \frac{w}{(1 + a_0 - b_0\rho_0)}\ln\left[\frac{C\rho}{1 + a_0 + b_0(\rho - \rho_0)}\right]. \quad (80)$$

In the first order of approximation the equations of motion of the fluid can be derived from the variational principle

$$\delta\int\left[1 + U(\rho) + \phi - \frac{\vec{v}^2}{2}\right]dt = 0, \quad (81)$$

and are given by

$$\vec{a} = -\nabla\phi - \nabla U(\rho) = \vec{a}_N + \vec{a}_p + \vec{a}_E, \quad (82)$$

where $\vec{a}$ is the total acceleration of the system, $\vec{a}_N = -\nabla\phi$ is the Newtonian gravitational acceleration and

$$\vec{a}_p = -\frac{C}{1 + a_0 - b_0\rho_0}\frac{1}{\rho}\nabla p = -\frac{1}{\rho}\nabla p, \quad (83)$$

is the hydrodynamical acceleration. Equation (83) also allows us to fix the value of the arbitrary integration constant C as $C = 1 + a_0 - b_0\rho_0$. Finally,

$$\vec{a}_E(\rho, p) = \frac{b_0}{1 + a_0 - b_0\rho_0}\frac{\nabla p}{1 + a_0 + b_0(\rho - \rho_0)}, \quad (84)$$

is a supplementary acceleration induced due to the modification of the action of the gravitational field.

C. The precession of the perihelion of Mercury

An estimation of the effect of the extra force, generated by the coupling between matter and geometry, on the orbital parameters of the motion of the planets around the Sun can be obtained in a simple way by using the properties of the Runge-Lenz vector, defined as $\vec{A} = \vec{v} \times \vec{L} - \alpha\vec{e}_r$, where $\vec{v}$ is the velocity relative to the Sun, with mass $M_\odot$, of a planet of mass m, $\vec{r} = r\vec{e}_r$ is the two-body position vector, $\vec{p} = \mu\vec{v}$ is the relative momentum, $\mu = mM_\odot/(m + M_\odot)$ is the reduced mass, $\vec{L} = \vec{r} \times \vec{p} = \mu r^2\dot{\theta}\vec{k}$ is the angular momentum, and $\alpha = GmM_\odot$ [35]. For an elliptical orbit of eccentricity e, major semiaxis a, and period T, the equation of the orbit is given by $(L^2/\mu\alpha)r^{-1} = 1 + e\cos\theta$. The Runge-Lenz vector can be expressed as

$$\vec{A} = \left(\frac{\vec{L}^2}{\mu r} - \alpha\right)\vec{e}_r - \dot{r}L\vec{e}_\theta, \quad (85)$$

and its derivative with respect to the polar angle θ is given by

$$\frac{d\vec{A}}{d\theta} = r^2\left[\frac{dV(r)}{dr} - \frac{\alpha}{r^2}\right]\vec{e}_\theta, \tag{86}$$

where $V(r)$ is the potential of the central force [35]. The potential term consists of the post-Newtonian potential, $V_{\rm PN}(r) = -\alpha/r - 3\alpha^2/mr^2$, plus the contribution from the general coupling between matter and geometry. Thus we have

$$\frac{d\vec{A}}{d\theta} = r^2\left[6\frac{\alpha^2}{mr^3} + m\vec{a}_E(r)\right]\vec{e}_\theta, \tag{87}$$

where we have also assumed that $\mu \approx m$. The change in direction $\Delta\phi$ of the perihelion with a change of θ of 2π is obtained as $\Delta\phi = (1/\alpha e)\int_0^{2\pi}|\dot{\vec{L}} \times d\vec{A}/d\theta|d\theta$, and it is given by

$$\Delta\phi = 24\pi^3\left(\frac{a}{T}\right)^2\frac{1}{1-e^2} + \frac{L}{8\pi^3 me}\frac{(1-e^2)^{3/2}}{(a/T)^3} \times \int_0^{2\pi}\frac{a_E[L^2(1+e\cos\theta)^{-1}/m\alpha]}{(1+e\cos\theta)^2}\cos\theta d\theta, \tag{88}$$

where we have used the relation $\alpha/L = 2\pi(a/T)/\sqrt{1-e^2}$. The first term of this equation corresponds to the standard general relativistic precession of the perihelion of the planets, while the second term gives the contribution to the perihelion precession due to the presence of the coupling between matter and geometry.

As an example of the application of Eq. (88) we consider the case for which the extra force may be considered as a constant, $a_E \approx$ constant, an approximation that could be valid for small regions of spacetime. In the Newtonian limit the extra acceleration generated by the coupling between matter and geometry can be expressed in a similar form [19]. With the use of Eq. (88) one finds for the perihelion precession the expression

$$\Delta\phi = \frac{6\pi GM_\odot}{a(1-e^2)} + \frac{2\pi a^2\sqrt{1-e^2}}{GM_\odot}a_E, \tag{89}$$

where we have also used Kepler's third law, $T^2 = 4\pi^2a^3/GM_\odot$. For the planet Mercury $a = 57.91 \times 10^{11}$ cm, and $e = 0.205\,615$, respectively, while $M_\odot = 1.989 \times 10^{33}$ g. With these numerical values the first term in Eq. (89) gives the standard general relativistic value for the precession angle, $(\Delta\phi)_{\rm GR} = 42.962$ arcsec per century, while the observed value of the precession is $(\Delta\phi)_{\rm obs} = 43.11 \pm 0.21$ arcsec per century [36]. Therefore the difference $(\Delta\phi)_E = (\Delta\phi)_{\rm obs} - (\Delta\phi)_{\rm GR} = 0.17$ arcsec per century can be attributed to other physical effects. Hence the observational constraints requires that the value of the constant a_E must satisfy the condition $a_E \leq 1.28 \times 10^{-9}$ cm/s^2.

VI. DISCUSSIONS AND FINAL REMARKS

In the present paper, we have considered a generalized gravity model with an arbitrary coupling between matter (described by the trace of the stress-energy tensor) and geometry, with the Lagrangian given by an arbitrary function of T and of the Ricci scalar. We have derived the gravitational field equations corresponding to this model, and considered several particular cases that may be relevant in explaining some of the open problems of cosmology and astrophysics. The new matter and time dependent terms in the gravitational field equations play the role of an effective cosmological constant. We have also demonstrated the possibility of reconstruction of arbitrary Friedmann-Robertson-Walker cosmologies by an appropriate choice of a function $f(T)$. The equations of motion corresponding to this model show the presence of an extra force acting on test particles, and the motion is generally nongeodesic. We have obtained, by using the perihelion precession of Mercury, an upper limit on the magnitude of the extra acceleration in the Solar System. This value of a_E, obtained from the solar system observations, is somewhat smaller than the value of the extra acceleration $a_E \approx 10^{-8}$ cm/s^2, necessary to explain the "dark matter" properties, as well as the Pioneer anomaly [19,37,38]. However, it does not rule out the possibility of the presence of some extra gravitational effects acting at both the solar system and galactic levels, since the assumption of a constant extra force may not be correct on larger astronomical scales.

Therefore, the predictions of the $f(R, T)$ gravity model could lead to some major differences, as compared to the predictions of standard general relativity, or other generalized gravity models, in several problems of current interest, such as cosmology, gravitational collapse or the generation of gravitational waves. The study of these phenomena may also provide some specific signatures and effects, which could distinguish and discriminate between the various gravitational models. In order to explore in more detail the connections between the $f(R, T)$ gravity model and the cosmological evolution, some explicit physical models are necessary to be built. This will be done in forthcoming work.

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