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Periodic solutions of a derivative nonlinear Schrödinger equation: Elliptic integrals of the third kind

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ABSTRACT

The nonlinear Schrödinger equation (NLSE) is an important model for wave packet dynamics in hydrodynamics, optics, plasma physics and many other physical disciplines. The 'derivative' NLSE family usually arises when further nonlinear effects must be incorporated. The periodic solutions of one such member, the Chen – Lee – Liu equation, are studied. More precisely, the complex envelope is separated into the absolute value and the phase. The absolute value is solved in terms of a polynomial in elliptic functions while the phase is expressed in terms of elliptic integrals of the third kind. The exact periodicity condition will imply that only a countable set of elliptic function moduli is allowed. This feature contrasts sharply with other periodic solutions of envelope equations, where a continuous range of elliptic function moduli is permitted.

1. Introduction

The nonlinear Schrödinger equation (NLSE) is an important model in many branches of physics, e.g. hydrodynamics [1], optics [2 – 4] and nonlinear science [5, 6]. NLSE incorporates a quadratic dispersion law and a cubic nonlinearity, and is written here in canonical form (* = complex conjugate):

$$i\frac{\partial A}{\partial t}+\frac{\partial^2 A}{\partial x^2}-\lambda A^2 A^* = 0\,. \tag{1}$$

If λ is positive (negative), plane waves are stable (unstable) [1 – 6]. Solitons can propagate as a permanent entity by a balance of dispersion and nonlinearity. In the context of water waves, packets of sufficiently large steepness, i.e., fourth order in the (small) amplitude parameter, call for the introduction of higher order nonlinear terms. From the perspective of optical physics, short waves demand the inclusion of 'self steepening' terms too. Frequently such higher order NLSE models are not amenable to analytical treatment. The goal here is study one such family of models, the 'derivative' NLSE, where exact solutions can be obtained, and thus insight and qualitative properties can be deduced.

More precisely, we study the Chen – Lee – Liu equation (CLL) [7 – 13]:

$$iA_t + A_{xx} - \lambda A^2 A^* + i\sigma A A^* A_x = 0\,, \tag{2}$$

where the last term represents 'self steepening' effects in hydrodynamics and optics. In this paper, we shall focus on the regime λ > 0, σ > 0, and a comprehensive description on the other possible combinations will be left for future study. The objective is to investigate a class of periodic solutions of these 'derivative' NLSE models analytically, with CLL being one typical example.

Periodic solutions of NLS

It will be instructive to review the periodic solutions of NLSE. Several classes of such solutions are known earlier in the literature. A brief, but certainly incomplete, overview can be provided:

(1) Rational solutions of hyperbolic and trigonometric functions with nonzero asymptotic values in the far field – physically, they correspond to waves of elevation moving on a continuous wave background [14 – 17].

(2) Real Jacobi elliptic functions in x and a complex exponential in t – Physically, the wave profile is periodic in space (x) and harmonic / oscillatory in time (t), and the wave intensity ($|A|^2$) is independent of time [18 – 20].

(3) Jacobi elliptic functions in *both* x and t – The wave intensity pattern is now doubly periodic, i.e. periodic in both space and time [3, 21, 22].

(4) Both the absolute value and argument of the complex envelope A depend on the spatial coordinate x, but the time dependence is just simple harmonic. Such solutions have usually been known in the literature as the '*nontrivial phase*' type [23 – 27].

(5) Computer algebra software has recently been employed in the search for traveling wave solution [28].

Periodic solutions of CLL

Although brief discussions on the periodic solutions of derivative nonlinear Schrödinger equations have been found scattered in the literature, most, if not all, of these works either have a trivial phase or have neglected the treatment of the phase factor [29, 30].

The main goal here is to extend the class of 'nontrivial phase' solutions of NLSE [23 – 27] to other integrable envelope equations. The family of derivative NLSE, with CLL as illustrative example, will constitute the focus of the present work.

The theoretical treatment is first presented in Section 2. These nontrivial phase solutions have usually been studied in two contexts, either the Gross – Pitaevskii equation with a periodic potential or in the language of algebraic geometry and curves. In the former, the restrictions in the modulus and the phase are usually masked by the underlying periodic potential. In the latter case, the techniques of algebraic geometry, while elegant mathematically, are usually beyond the everyday language of most practitioners of hydrodynamics and optics. Furthermore, it is difficult to impossible to implement the physically significant quantities, e.g. frequency and wave speed, from the predictions of algebraic curves theory to common computer algebra software.

On employing polar representation, the absolute value and phase of the complex envelope *A* are expressed in terms of a polynomial in elliptic functions and elliptic integrals of the third kind respectively. For the wave profile to be periodic, these two entities must have commensurate periods. This imposes a 'quantized' condition for the permissible modulus of the associated elliptic functions / integrals (Sections 2, 3).

Although elliptic functions have been employed extensively in the literature [31], elliptic integral of the third kind has been used much less frequently in science and engineering [32 – 34]. Hence a brief introduction is provided in Section 3. A consistency check, and the computational results which demonstrate the existence of solutions explicitly, are presented in Section 4. For any given CLL, the precise periodicity condition will select a 'discrete' set of allowable elliptic function moduli. This contrasts sharply with other families of periodic solutions of envelope equations, where a continuous range of moduli is permitted.

2. Polar representations

The polar or Madelung representation calls for

$$A = R\exp(i\Theta - i\omega t), \quad R = R(x), \quad \Theta = \Theta(x). \tag{3}$$

Solutions with Θ being different from a constant or a linear function in x will be termed 'nontrivial phase solutions'. To minimize algebraic complexity, we first study the case where the angular frequency, ω, is assumed to be zero for the rest of this paper. Substituting (3) into (2) and simple manipulations now lead to

$$\Theta_x = \frac{C_0}{R^2} - \frac{\sigma R^2}{4}, \quad \text{and} \tag{4}$$

$$R_x{}^2 + \frac{C_0{}^2}{R^2} - 2C_1 - \frac{\sigma C_0\, R^2}{2} - \frac{\lambda R^4}{2} + \frac{\sigma^2 R^6}{16} = 0\,, \tag{5}$$

where C_0, C_1 are constants. This is a canonical equation for an elliptic function in R^2 [18 – 27]. In fact, if we let $Y = R^2$, then equation (5) reduces to $(dY/dx)^2 = P(Y)$, where P is a polynomial of degree four. It is well known that Y is a rational function, rational in the exponential function or elliptic function if P has a triple zero and a simple zero, two double zeros and four distinct zeros respectively. For the generic case where P has four distinct zeros, by applying a well established procedure, the differential equation $(dY/dx)^2 = P(Y)$ can be further transformed into the Jacobi normal form, and the solution is the Jacobi *sine amplitude* sn(x). Hence $Y = R^2$ can always be expressed as certain algebraic functions in sn(x). A comprehensive listing of all possible elliptic functions solutions as well as their degenerations will be left for future study.

At present we shall be content with one particular solution as:

$$R^2 = \frac{2\lambda}{\sigma^2}\left[1 + \frac{\sigma\, r\, \mathrm{dn}\,(rx)}{\lambda}\right], \tag{6}$$

with dn being the Jacobi elliptic function and k is the associated modulus. The function dn has a period $2K$, where K is the complete elliptic integral of the first kind,

$$K = K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}\quad . \tag{7}$$

The other parameters are given by

$$C_0 = \frac{(2-k^2)\, r^2 \sigma^2 - 6\lambda^2}{2\sigma^3}\,, \qquad C_1 = \frac{\lambda[(2-k^2)\, r^2 \sigma^2 - 2\lambda^2]}{2\sigma^4}\ . \tag{8}$$

The coefficients of the cubic nonlinearity (λ) and 'self steepening' (σ) cannot be independent, and must be related through

$$y = \frac{r\sigma}{\lambda} \tag{9}$$

where y satisfies the equation

$$k^4 y^4 - 8(2-k^2)y^2 + 32 = 0\ . \tag{10}$$

The discriminant of this quadratic in y^2 is given by

$$64(4 - 4k^2 - k^4)$$

which is positive as long as

$$k^2 < 2(\sqrt{2}-1)\,. \tag{11}$$

The sum and product of roots of (10), as a quadratic in y^2, are both positive. Hence (10) has two positive roots for y^2, and thus four real roots for y. The phase function $\Theta(x)$ must be extracted from (4). The period of the absolute value R is $2MK$, where M is an integer. For the complex variable A to be truly periodic, the periods of the functions R and Θ cannot be independent. As a concrete example, consider the Madelung description in hydrodynamics, where R would represent density and Θ denotes a velocity potential. Consequently, for the flow configuration to be periodic, both the modulus and the phase of the wave function should be periodic with that same period. Similar interpretations apply to other applications in science and engineering.

More precisely, the phase Θ defined by (3) must change by $2N\pi$ for an integral period in R^2, where N is integer. The quantitative expression for this constraint can now be written as

$$\frac{\lambda}{4\sigma}\left[\frac{\sigma^2 r^2(2-k^2)}{\lambda^2}-6\right]\int_0^{\frac{2MK}{r}}\frac{dv}{1+\sigma r\,\mathrm{dn}\,(rv)/\lambda}-\frac{\lambda}{2\sigma}\int_0^{\frac{2MK}{r}}\left(1+\frac{\sigma r\,\mathrm{dn}\,(rv)}{\lambda}\right)dv=2N\pi\ . \tag{12}$$

Due to the periodicity of dn, it is actually sufficient to consider the integrals for the interval (0, $2K$) and multiply the result by M. This first integral will be converted to an elliptic integral of the third kind documented in the next section.

3. Elliptic integrals of the third kind

For the present purpose, the elliptic integral of the third kind is taken as

$$\Lambda(n,\phi,k^2)=\int_0^{\phi}\frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}, \tag{13}$$

which depends on the parameter n, in addition to the modulus k and the upper limit of integration. A purely algebraic form can be obtained by

$$t=\sin\theta,\qquad \xi=\sin\phi, \tag{14}$$

$$\int_0^{\xi}\frac{dt}{(1-nt^2)\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{15}$$

where obviously the elliptic integral of the first kind is recovered for n = 0. To relate to the formulation in Section 2, the substitution

$$t=\mathrm{sn}\ v,\qquad \xi=\mathrm{sn}\ u, \tag{16}$$

will lead to

$$\Lambda\,(n, u, k^2) = \int_0^u \frac{dv}{(1 - n\,\mathrm{sn}^2\,v)}, \tag{17}$$

which will now be employed to calculate the polar angle Θ.

One now reduces Θ and R defined by (4) and (6) to a form related to (17), which will be taken as the definition of the elliptic integral of the third kind. To accomplish this objective, an indefinite integral from the literature [35], which can also be verified by direct calculations, is

$$\int_0^u \frac{dv}{1 + \alpha\,\mathrm{dn}\,(v)}$$

$$= \left(\frac{1}{1-\alpha^2}\right) \Lambda \left(-\frac{k^2\,\alpha^2}{1-\alpha^2}, u, k^2\right)$$

$$- \frac{\alpha}{\left[(1-\alpha^2)\,(1-(1-k^2)\,\alpha^2)\right]^{1/2}} \tan^{-1}\left(\sqrt{\frac{1-(1-k^2)\,\alpha^2}{1-\alpha^2}}\,\frac{\mathrm{sn}(u)}{\mathrm{cn}(u)}\right), \tag{18}$$

where sn and cn are the standard Jacobi elliptic functions. Since sn(0) and sn(2K) both vanish, only the first term on the right hand side contributes to a definite integral of range (0, 2K) as given in (12).

The second integral in (12) is much easier to handle:

$$\int_0^{2K} \mathrm{dn}\,(x)\,dx = 2\left[\sin^{-1}(\mathrm{sn}\,(x))\right]_0^K = \pi\,.$$

On using the shorthand notation,

$$\alpha = \frac{\sigma r}{\lambda}, \tag{19}$$

the periodicity condition (12) can now be written in a compact form:

$$M\left\{\left[\frac{(2-k^2)\,\alpha^2-6}{4\alpha\,(1-\alpha^2)}\right]\Lambda\left(-\frac{k^2\,\alpha^2}{1-\alpha^2},\,2K,\,k^2\right)-\left(\frac{K}{\alpha}+\frac{\pi}{2}\right)\right\}=2N\pi\,, \qquad (20)$$

with M, N integers.

4. Consistency checks and numerical results

Consistency checks

The objective of the consistency check is to show that, for special parameter values (k = 0 here), the elliptic function algorithm developed in the previous solution reduces to special, ‘intuitively obvious’ solutions of CLL.

For the special case k = 0, (9, 10) dictate that this class of solutions only exists for

$$\sigma=\frac{\lambda\sqrt{2}}{r}\ . \qquad (21)$$

The envelope A itself, as defined by (3), has constant radial part,

$$R^2=\frac{r^2}{\lambda}(1+\sqrt{2})\,, \qquad (22)$$

and a linear phase

$$\Theta=-\,rx. \qquad (23)$$

This solution for A as defined by (3) will satisfy the steady state CCL equation

$$A_{xx}-\lambda A^2A^*+i\sigma AA^*A_x=0, \qquad (24)$$

when (21) holds. The periodicity condition (20) degenerates to a simple form $M = -4N$, which dictates that there is only one choice for M for any given N.

On the other hand, one cannot study the long wave limit ($k \rightarrow 1$), due to the restriction (11).

Numerical results

To demonstrate explicitly the existence of solutions for nonzero values of k^2, typical numerical results for (20) will be presented.

Fixed k, σ and λ vary – For a solution of a fixed period (k and r), we demonstrate how (20) is satisfied by selecting a typical value of M, and calculating (20) for various α. Figure 1 plots the left hand side of (20) versus α for $M = 3$. As the graph ranges from a minimum to arbitrary large values (numerically), it will always attain a value $2N\pi$, for a sufficiently large integer N. Thus for any large integer N, there would be two possible values of $\alpha = \sigma r/\lambda$ which would satisfy this periodicity requirement. From the perspective of applications, one can always the tune the magnitude of 'self steepening' (the value of σ), to attain a solution of a given period.

As a numerical example, $M = 3$, $N = 8$, $k^2 = 0.5$, $r = 1$, one needs $\alpha = -0.512$.

Fixed σ and λ, k varies – It is of course more common to have (2) given (σ, λ fixed) and a periodic solution is sought. One then first chooses a value for M, and attempts to satisfy (20) with a particular choice of k, by simply plotting (20) versus k using standard computer algebra software. Typically only one or at most a few values of k can accomplish this goal. We demonstrate the dynamics by presenting several illustrative results for $\alpha = 0.5$:

$M = 1$, $N = -3$, $k^2 = 0.376$;

$M = 1$, $N = -4$, $k^2 = 0.750$;

$M = 2$, $N = -6$, $k^2 = 0.375$;

$M = 2$, $N = -7$, $k^2 = 0.619$;

$M = 2$, $N = -8$, $k^2 = 0.750$;

$M = 2$, $N = -9$, $k^2 = 0.821$;

$M = 3$, $N = -8$, $k^2 = 0.095$;

$M = 3$, $N = -9$, $k^2 = 0.375$;

$M = 3$, $N = -10$, $k^2 = 0.554$;

$M = 3$, $N = -11$, $k^2 = 0.672$;

$M = 3$, $N = -12$, $k^2 = 0.750$;

$M = 3$, $N = -13$, $k^2 = 0.802$.

It is reasonable to conjecture that for a given M, there will eventually be an increasing number of discrete, allowable values of k^2 which satisfy the periodicity requirement (20).

5. Conclusions

The nonlinear Schrödinger equation (NLSE), an important model in many branches of physics, possesses periodic solutions known frequently in the literature as the 'nontrivial phase' type. This mechanism of solving envelope equations and this type of wave profiles are conjectured here to be valid for *derivative* NLSE family of models as well. The distinctive characteristic is that the phase of the wave function, expressed in polar coordinates, must be a nonlinear function of the relevant variable. Typically an elliptic integral of the third kind is involved. The details of a typical example of derivative NLSE, the Chen – Lee – Liu equation, are worked out explicitly. The precise periodicity condition is that both the absolute value and the phase must be periodic. This condition is formulated in terms of an elliptic integral of third kind, and is computed numerically. Only a countable set of elliptic function moduli is permitted, in sharp contrast with other periodic solutions of NLSE and derivative NLSE. Other family of solutions for CLL, e.g. by employing other elliptic functions for the radial function R and/or invoking a nonzero angular frequency (ω) in the initial formulation (2, 3) remains to be explored. Further variants of derivative NLSE, e.g. the Kaup – Newell equation, will be studied in the future.

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Figure Captions

(1) Figure 1 – The periodicity condition, left hand side of Equation (20), versus α, $M = 3$, $N = 0$, $k^2 = 0.5$.

Periodicity condition (20) versus α

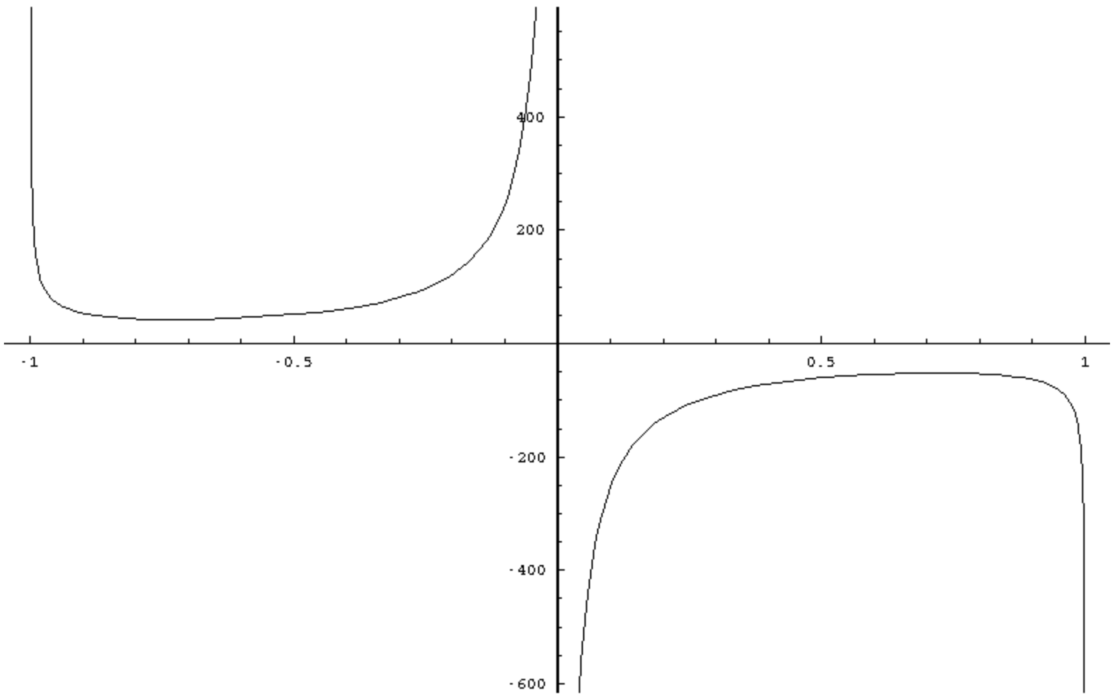

Figure 1