Transp Porous Med (2011) 90:529–544
DOI 10.1007/s11242-011-9798-6

Ferromagnetic Convection in a Heterogeneous Darcy Porous Medium Using a Local Thermal Non-equilibrium (LTNE) Model

I. S. Shivakumara · Chiu-On Ng · M. Ravisha

Received: 2 April 2011 / Accepted: 21 June 2011 / Published online: 7 July 2011

Abstract The combined effects of vertical heterogeneity of permeability and local thermal non-equilibrium (LTNE) on the onset of ferromagnetic convection in a ferrofluid saturated Darcy porous medium in the presence of a uniform vertical magnetic field are investigated. A two-field model for temperature representing the solid and fluid phases separately is used. The eigenvalue problem is solved numerically using the Galerkin method for different forms of permeability heterogeneity function $\Gamma(z)$ and their effect on the stability characteristics of the system has been analyzed in detail. It is observed that the general quadratic variation of $\Gamma(z)$ with depth has more destabilizing effect on the system when compared to the homogeneous porous medium case. Besides, the influence of LTNE and magnetic parameters on the criterion for the onset of ferromagnetic convection is also assessed.

Keywords Heterogeneous porous medium · Local thermal non-equilibrium · Ferromagnetic convection

List of Symbols

$a = \sqrt{\ell^2 + m^2}$	Overall horizontal wave number
$\vec{B}$	Magnetic induction

I. S. Shivakumara · C.-O. Ng (✉)
Department of Mechanical Engineering, The University of Hong Kong,
Pokfulam Road, Hong Kong, Hong Kong
e-mail: cong@hku.hk

Present Address:
I. S. Shivakumara
UGC-CAS in Fluid Mechanics, Department of Mathematics, Bangalore University,
Bangalore 560001, India
e-mail: shivakumarais@gmail.com

M. Ravisha
Department of Mathematics, Smt. Rukmini Shedthi Memorial National Government
First Grade College, Barkur 576210, India

c	Specific heat
c_a	Acceleration coefficient
d	Thickness of the porous layer
$\mathrm{D} = \mathrm{d}/\mathrm{d}z$	Differential operator
$\vec{g}$	Acceleration due to gravity
h_t	heat transfer coefficient
$\vec{H}$	Magnetic field intensity
H_0	Imposed uniform vertical magnetic field
$H_t = hd^2/\varepsilon k_{tf}$	Scaled inter-phase heat transfer coefficient
$\hat{k}$	Unit vector in z-direction
K_0	The mean value of $K(z)$
$K(z)$	Permeability of the porous medium
k_f	Thermal conductivity of the fluid
k_s	Thermal conductivity of the solid
$K_p = -(\partial M/\partial T_f)_{H_0, T_a}$	Pyromagnetic co-efficient
ℓ, m	Wave numbers in the x and y directions
$\vec{M}$	Magnetization
$M_0 = M(H_0, T_a)$	Constant mean value of magnetization
$M_1 = \mu_0 K^2 \beta/(1+\chi)\alpha_t \rho_0 g$	Magnetic number
$M_3 = (1 + M_0/H_0)/(1+\chi)$	Non-linearity of magnetization parameter
p	Pressure
$\vec{q} = (u, v, w)$	Velocity vector
$R = \rho_0 \alpha_t g \beta k d^2/\varepsilon \mu_f \kappa_f$	Darcy–Rayleigh number
t	Time
T	Temperature
T_L	Temperature of the lower boundary
T_u	Temperature of the upper boundary
$T_a = (T_l + T_u)/2$	Reference temperature
W	Amplitude of vertical component of perturbed velocity
(x, y, z)	Cartesian co-ordinates

Greek Symbols

α_t	Thermal expansion coefficient
$\beta = \Delta T/d$	Temperature gradient
$\chi = (\partial M/\partial H)_{H_0}, T_0$	Magnetic susceptibility
$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$	Laplacian operator
$\nabla_h^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$	Horizontal Laplacian operator
ε	Porosity of the porous medium
$\Gamma(z)$	Non-dimensional permeability heterogeneity function
$\kappa_f = k_{tf}/(\rho_0 c)_f$	Thermal diffusivity of the fluid
μ_f	Dynamic viscosity
μ_0	Free space magnetic permeability of vacuum
φ	Magnetic potential
Φ	Amplitude of perturbed magnetic potential
$\gamma = \varepsilon k_{tf}/(1-\varepsilon)\, k_{ts}$	Porosity modified conductivity ratio
ρ_f	Fluid density

ρ_0 Reference density at T_a
Θ Amplitude of temperature

Subscripts

b Basic state
f Fluid
s Solid

1 Introduction

Buoyancy-driven convection in a layer of Newtonian viscous fluid saturated porous medium heated from below has received considerable attention during the last few decades because of its relevance in various applications such as biomedical engineering, drying processes, thermal insulation, radioactive waste management, transpiration cooling, geophysical systems, contaminant transport in groundwater, ceramic processing, solid-matrix compact heat exchangers, and many others. Both local thermal equilibrium (LTE) and local thermal non-equilibrium (LTNE) models have been utilized in investigating the problems. The growing volume of work is well documented by Ingham and Pop (1998), Vafai (2000, 2005), Nield and Bejan (2006), and Vadasz (2008).

Ferrofluids or magnetic nanofluids are not occurring in nature but they are synthesized in the laboratory because of their increasing importance in heat transfer applications in electronics, engines, micro and nanoelectromechanical systems, and other engineering applications. When a ferrofluid is submitted to a gradient of temperature, the momentum balance experiences a profound modification, through the Kelvin body force reflecting the magnetization of the ferrofluid. Such a study in a clear ferrofluid layer is well known since the classical works of Finlayson (1970), Neuringer and Rosensweig (1964), and Bashtovoi and Berkovski (1973). The analogous buoyancy-driven convection in a layer of ferrofluid saturating a porous medium heated from below in the presence of a uniform magnetic field has also attracted considerable attention in the literature owing to its importance in controlled emplacement of liquids or treatment of chemicals, and emplacement of geophysically imageable liquids into particular zones for subsequent imaging, etc. (for details see Borglin et al. 2000 and references therein). Sunil and Mahajan (2009) have used generalized energy method to study nonlinear convection in a magnetized ferrofluid saturated porous layer heated uniformly from below for the stress-free boundaries case. Shivakumara et al. (2008, 2009) have investigated theoretically the onset of convection in a layer of ferrofluid saturated porous medium for various types of velocity and temperature boundary conditions. Nanjundappa et al. (2010) have investigated the onset of buoyancy-driven convection in a ferromagnetic fluid saturated sparsely packed porous medium with fixed heat flux condition at the lower rigid boundary and a general thermal boundary condition at the upper free boundary. Recently, Shivakumara et al. (2011a) have analyzed the effect of Coriolis force due to rotation on the onset of ferromagnetic convection in a ferrofluid saturated porous layer.

As propounded by many researchers, the assumption of LTE is inadequate for proper understanding of the heat transfer problems in many practical applications involving hyperporous materials and also media in which there is a significant temperature difference between the fluid and solid phases (for details see Nield and Bejan 2006 and references therein). In such circumstances, the LTNE effects are to be taken into consideration.

Therefore, the recent trend in the study of thermal convective instability problems in porous media is to account for LTNE effects by considering a two-field model for energy equation each representing the fluid and solid phases separately. Realizing the importance, studies have been undertaken recently to know the effect of LTNE on ferromagnetic convection in a layer of porous medium heated from below but it is still in infancy. Lee et al. (2010) have investigated the effect of LTNE on the criterion for the onset of ferromagnetic convection in a horizontal layer of Darcy porous medium in the presence of a uniform vertical magnetic field, while Sunil et al. (2010) have discussed nonlinear aspects of the problem. Recently, Shivakumara et al. (2011b,c) have analyzed the problem of ferromagnetic convection in a layer of Brinkman porous medium using LTNE model.

It has been recognized that the effect of heterogeneity in either permeability or thermal conductivity or both on thermal convective instability in a layer of porous medium is of importance since there can be dramatic effects in the case of heterogeneity (Braester and Vadasz 1993; Simmons et al. 2001; Prasad and Simmons 2003). The effects of hydrodynamic and thermal heterogeneity, for the case of variation in both the horizontal and vertical directions as well as various other aspects, on the onset of convection in a horizontal layer of Newtonian fluid saturated porous medium, have been studied analytically and enough progress has been made in this direction in the recent past (Nield and Kuznetsov 2007a,b, 2010; Nield and Simmons 2007; Nield and Kuznetsov 2008a,b; Kuznetsov and Nield 2008; Nield et al. 2009, 2010; Kuznetsov et al. 2010, 2011; Simmons et al. 2010).

Both theoretical and experimental works are available on ferroconvection in homogeneous porous media. In fact, flow of ferrofluids through porous media was motivated by the potential use of ferrofluids to stabilize fingering in oil recovery processes. In such situations the presence of heterogeneities is common and it may affect the flow of ferrofluids through porous media. Since ferrofluids are stable colloidal suspensions of magnetic nanoparticles, these particles are buffeted by the fluid molecules. As a consequence, the fluid temperature may fluctuate rapidly with position which may lead to a thermal lagging between fluid and solid phases. Moreover, ferrofluids are considered to be very good conductors of heat and they are being used in many heat transfer related applications to augment heat transfer (Ganguly et al. 2004) and in some cases the temperatures involved may be high. For example, in a rotating shaft seal involving ferrofluids the temperature may rise above 100°C at high shaft surface speeds and a similar situation also arise in the use of ferrofluid in loud speaker coils (Popplewell et al. 1982). The use of porous media in many of these devices may be viable in the cooling of the systems still more effectively. Under the above circumstances, it is a prerequisite to know the LTNE effect on thermal convection in a ferrofluid saturated heterogeneous porous medium for a better understanding of heat transfer in such systems. To the best of our knowledge, the study has not received any attention in the literature. Therefore, it has motivated us to undertake this investigation. For simplicity the permeability heterogeneity is confined to just the vertical variation of that quantity in the present study and a two-field temperature model is used in analyzing the problem. The resulting eigenvalue problem is solved numerically using the Galerkin method for various forms of permeability heterogeneity function.

2 Mathematical Formulation

We consider an initially quiescent incompressible ferrofluid saturated horizontal heterogeneous porous layer of characteristic thickness d. The lower surface is held at constant temperature T_{L}, while the upper surface is at T_{U} ($<T_{\mathrm{L}}$). A Cartesian co-ordinate system (x, y, z) is

used with the origin at the bottom of the porous layer and the z-axis directed vertically upward in the presence of gravitational field. Based on the Oberbeck–Boussinesq approximation for the density, an LTNE model with a two-field model for temperature is used.

The governing basic equations are (Nield and Bejan 2006; Finlayson 1970):

$$\nabla \cdot \vec{q} = 0 \tag{1}$$

$$\rho_0 c_{\mathrm{a}} \frac{\partial \vec{q}}{\partial t} = -\nabla p + \rho_{\mathrm{f}} \vec{g} - \frac{\mu_{\mathrm{f}}}{K(z)} \vec{q} + \mu_0 (\vec{M} \cdot \nabla) \vec{H} \tag{2}$$

$$\varepsilon \left(\rho_0 c\right)_{\mathrm{f}} \frac{\partial T_{\mathrm{f}}}{\partial t} + \left(\rho_0 c\right)_{\mathrm{f}} \left(\vec{q} \cdot \nabla\right) T_{\mathrm{f}} = \varepsilon k_{\mathrm{f}} \nabla^2 T_{\mathrm{f}} + h_{\mathrm{t}} \left(T_{\mathrm{s}} - T_{\mathrm{f}}\right) \tag{3}$$

$$(1-\varepsilon) \left(\rho_0 c\right)_{\mathrm{s}} \frac{\partial T_{\mathrm{s}}}{\partial t} = (1-\varepsilon) k_{\mathrm{s}} \nabla^2 T_{\mathrm{s}} - h_{\mathrm{t}} \left(T_{\mathrm{s}} - T_{\mathrm{f}}\right) \tag{4}$$

$$\rho_{\mathrm{f}} = \rho_0 \left[1 - \alpha_{\mathrm{t}} \left(T_{\mathrm{f}} - T_{\mathrm{L}}\right)\right] \tag{5}$$

where $\vec{q}$ the velocity vector, p the excessive pressure over the reference hydrostatic value, ρ_{f} the fluid density, $K(z)$ the permeability of the porous medium, c_{a} the acceleration coefficient, ε the porosity of the porous medium, $\vec{M}$ the magnetization, $\vec{H}$ the magnetic field intensity, μ_{f} the fluid viscosity, μ_0 the magnetic permeability of vacuum, T_{f} the temperature of the fluid phase, T_{s} the temperature of the solid phase, c the specific heat, k_{f} the thermal conductivity of the fluid, k_{s} the thermal conductivity of the solid, α_{t} the thermal expansion coefficient of the fluid, and h_{t} is the inter-phase heat transfer coefficient which depends on the nature of the porous matrix and the saturating fluid and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian operator. The term $\mu_0(\vec{M} \cdot \nabla)\vec{H}$ in the above equation is the magnetic body force which appears as a result of polarization of the ferrofluid in the presence of magnetic field. It may be noted that large values of η correspond to a rapid transfer of heat between the phases which represents the LTE case, while moderate values of h_{t} corresponds to relatively strong LTNE effects. In other words, it measures the ease with which heat is transferred between the phases.

The Maxwell equations in the magnetostatic limit are:

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{H} = 0 \tag{6a}$$

or

$$\vec{H} = \nabla \varphi \tag{6b}$$

where $\vec{B}$ is the magnetic induction and φ is the magnetic potential.

Further, $\vec{B}$, $\vec{M}$, and $\vec{H}$ are related by

$$\vec{B} = \mu_0 (\vec{M} + \vec{H}). \tag{7}$$

It is assumed that the magnetization is aligned with the magnetic field, but may depend on the magnitude of the magnetic field as well as temperature (Finlayson 1970) and thus

$$\vec{M} = M(H, T_{\mathrm{f}}) \frac{\vec{H}}{H}. \tag{8}$$

where $M = |\vec{M}|$ and $H = |\vec{H}|$. The magnetic equation of state, following (Finlayson 1970), is taken as

$$M = M_0 + \chi (H - H_0) - K_{\mathrm{p}} (T_{\mathrm{f}} - T_{\mathrm{a}}) \tag{9}$$

where, $\chi = (\partial M/\partial H)_{H_0,\, T_{\mathrm{a}}}$ is the magnetic susceptibility, $K_{\mathrm{p}} = -(\partial M/\partial T_{\mathrm{f}})_{H_0, T_{\mathrm{a}}}$ is the pyromagnetic co-efficient and $M_0 = M(H_0, T_{\mathrm{a}})$.

The basic state is quiescent and there exists the following solution for the basic state:

$$
\begin{aligned}
&\vec{q}_\mathrm{b} = 0\\
&p_\mathrm{b}(z) = p_0 - \rho_0 g z - \frac{1}{2}\rho_0 \alpha_\mathrm{t} g \beta z\,(z-d) - \frac{\mu_0 M_0 K_\mathrm{p}\beta}{1+\chi} z - \frac{\mu_0 K_\mathrm{p}^2 \beta^2}{2(1+\chi)^2} z\,(z-d)\\
&T_\mathrm{fb}(z) = T_\mathrm{a} - \beta\,(z - d/2) = T_\mathrm{sb}(z)\\
&\vec{H}_\mathrm{b}\,(z) = \left[H_0 - \frac{K_\mathrm{p}\beta}{1+\chi}\left(z - \frac{d}{2}\right)\right]\hat{k}, \quad \vec{M}_\mathrm{b}\,(z) = \left[M_0 + \frac{K_\mathrm{p}\beta}{1+\chi}\left(z - \frac{d}{2}\right)\right]\hat{k}
\end{aligned}
\tag{10}
$$

where $\beta = \Delta T/d = (T_\mathrm{L} - T_\mathrm{U})/d$ is the temperature gradient, $\hat{k}$ is the unit vector in the z-direction, and the subscript b denotes the basic state.

3 Linear Stability Theory

To investigate the conditions under which the quiescent solution is stable against small disturbances, we consider a perturbed state in the form

$$
\begin{aligned}
&\vec{q} = \vec{q}\,', \quad p = p_\mathrm{b}(z) + p', \quad T_\mathrm{f} = T_\mathrm{fb}(z) + T_\mathrm{f}', \quad T_\mathrm{s} = T_\mathrm{sb}(z) + T_\mathrm{s}'\\
&\vec{H} = \vec{H}_\mathrm{b}(z) + \vec{H}', \quad \vec{M} = \vec{M}_\mathrm{b}(z) + \vec{M}'
\end{aligned}
\tag{11}
$$

where $\vec{q}\,' = (u', v', w')$, p', $T_\mathrm{f}', T_\mathrm{s}'$, $\vec{H}' = (H_x', H_\mathrm{y}', H_z')$, and $\vec{M}' = (M_x', M_y', M_z')$ are perturbed variables and are assumed to be small. Substituting Eq. 11 into Eqs. 7 and 8, and using Eq. 6, we obtain (after dropping the primes)

$$
\begin{aligned}
&H_x + M_x = (1 + M_0/H_0)\,H_x, \quad H_y + M_y = (1 + M_0/H_0)\,H_y\\
&H_z + M_z = (1+\chi)\,H_z - K T_\mathrm{f}.
\end{aligned}
\tag{12}
$$

Again substituting Eq. 11 into momentum Eq. 2, linearizing, eliminating the pressure term by taking curl twice and using Eq. 12 the z-component of the resulting equation can be obtained as (after dropping the primes):

$$
\begin{aligned}
\left[\rho_0 c_\mathrm{a}\frac{\partial}{\partial t} + \frac{\mu_\mathrm{f}}{K(z)}\right]\nabla^2 w + \frac{\partial}{\partial z}\left\{\frac{\mu_\mathrm{f}}{K(z)}\right\}\frac{\partial w}{\partial z} &= -\mu_0 K_\mathrm{p}\beta\frac{\partial}{\partial z}\left(\nabla_\mathrm{h}^2\varphi\right) + \frac{\mu_0 K_\mathrm{p}^2\beta}{1+\chi}\nabla_\mathrm{h}^2 T_\mathrm{f}\\
&\quad + \rho_0\alpha_\mathrm{t} g \nabla_\mathrm{h}^2 T_\mathrm{f}
\end{aligned}
\tag{13}
$$

where $\nabla_\mathrm{h}^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian operator.

Equations 3 and 4, after using Eq. 11 and linearizing, take the following form (after dropping the primes):

$$
\varepsilon\,(\rho_0 c)_\mathrm{f}\,\frac{\partial T_\mathrm{f}}{\partial t} + (\rho_0 c)_\mathrm{f}\,w\frac{\mathrm{d}T_\mathrm{fb}}{\mathrm{d}z} = \varepsilon k_\mathrm{f}\nabla^2 T_\mathrm{f} + h\,(T_\mathrm{s} - T_\mathrm{f}) \tag{14}
$$

$$
(1-\varepsilon)\,(\rho_0 c)_\mathrm{s}\,\frac{\partial T_\mathrm{s}}{\partial t} = (1-\varepsilon)\,k_\mathrm{s}\nabla^2 T_\mathrm{s} - h\,(T_\mathrm{s} - T_\mathrm{f})\,. \tag{15}
$$

Equations (6a, 6b), after substituting Eq. 11 and using Eq. 12, may be written as (after dropping the primes)

$$
\left(1 + \frac{M_0}{H_0}\right)\,\nabla_\mathrm{h}^2\varphi + (1+\chi)\,\frac{\partial^2\varphi}{\partial z^2} - K_\mathrm{p}\frac{\partial T_\mathrm{f}}{\partial z} = 0. \tag{16}
$$

Since there are no physical mechanisms to set up oscillatory convection, it is obvious to reason out that the exchange of stability holds for the problem considered. Accordingly, the normal mode expansion of the dependent variables is assumed in the form

$$\{w, T_\mathrm{f}, T_\mathrm{s}, \varphi\} = \{W(z), \Theta_\mathrm{f}(z), \Theta_\mathrm{s}(z), \Phi(z)\} \exp\left[i(\ell x + my)\right] \tag{17}$$

where ℓ and m are wave numbers in the x and y directions, respectively.

Substituting Eq. 17 into Eqs. 13–16, and non-dimensionalizing the variables by setting

$$\begin{gathered}\left(x^*, y^*, z^*\right) = \left(\frac{x^*}{d}, \frac{y^*}{d}, \frac{z^*}{d}\right), \quad t^* = \frac{\kappa_\mathrm{f}}{d^2} t, \quad W^* = \frac{d}{\varepsilon \kappa_\mathrm{f}} W, \quad \Theta_\mathrm{f}^* = \frac{1}{\beta d}\,\Theta_\mathrm{f} \\ \Theta_\mathrm{s}^* = \frac{1}{\beta d}\,\Theta_\mathrm{s}, \quad \Phi^* = \frac{(1+\chi)}{K_\mathrm{p} \beta d^2} \Phi, \quad \Gamma(z) = \frac{K_0}{K(z)}\end{gathered} \tag{18}$$

where $\kappa_\mathrm{f} = k_\mathrm{f}/(\rho_0 c)_\mathrm{f}$ is the effective thermal diffusivity of the fluid, $\Gamma(z)$ is the non-dimensional permeability heterogeneity function, and K_0 is the mean value of $K(z)$, we obtain (after dropping the asterisks for simplicity)

$$\Gamma(z)\left(\mathrm{D}^2 - a^2\right) W + \mathrm{D}\Gamma(z)\mathrm{D}W = a^2 R_\mathrm{D}\left[M_1 \mathrm{D}\Phi - (1+M_1)\,\Theta_\mathrm{f}\right] \tag{19}$$

$$\left(\mathrm{D}^2 - a^2\right)\Theta_\mathrm{f} + H_\mathrm{t}\left(\Theta_\mathrm{s} - \Theta_\mathrm{f}\right) = -W \tag{20}$$

$$\left(\mathrm{D}^2 - a^2\right)\Theta_\mathrm{s} + \gamma H_\mathrm{t}\left(\Theta_\mathrm{f} - \Theta_\mathrm{s}\right) = 0 \tag{21}$$

$$\left(\mathrm{D}^2 - a^2 M_3\right)\Phi - \mathrm{D}\Theta_\mathrm{f} = 0. \tag{22}$$

In the equations above, $R_\mathrm{D} = \rho_0 \alpha_\mathrm{t} g\,(T_\mathrm{L} - T_\mathrm{U})\,K_0 d/\varepsilon\mu_\mathrm{f}\kappa_\mathrm{f}$ is the Darcy–Rayleigh number, $H_\mathrm{t} = h_\mathrm{t} d^2/\varepsilon k_\mathrm{f}$ is the scaled inter-phase heat transfer coefficient, $\mathrm{D} = \mathrm{d}/\mathrm{d}z$ is the differential operator, $\gamma = \varepsilon k_\mathrm{f}/(1-\varepsilon)\,k_\mathrm{s}$ is the porosity modified conductivity ratio, $M_1 = \mu_0 K_\mathrm{p}^2 \beta/(1+\chi)\,\alpha_\mathrm{t}\rho_0 g$ is the magnetic number and it is a ratio of magnetic to gravitational forces and $M_3 = (1 + M_0/H_0)/(1+\chi)$ is a measure of nonlinearity of magnetization. The function $\Gamma(z)$ is chosen in the following form:

$$\Gamma(z) = 1 + \delta_1\left(z - \frac{1}{2}\right) + \delta_2\left(z^2 - \frac{1}{3}\right) \tag{23}$$

where $\delta_1$and δ_2 are constants and it may be noted that the quadratic function above has a unit mean. For the homogeneous porous medium case, $\delta_1 = 0 = \delta_2$.

The boundaries are impermeable, paramagnetic with fixed temperatures and hence we have

$$W = \Theta_\mathrm{f} = \Theta_\mathrm{s} = 0 \quad \text{at } z = 0, 1. \tag{24a}$$

$$(1+\chi)\mathrm{D}\Phi - a\Phi = 0 \quad \text{at } z = 0 \tag{24b}$$

$$(1+\chi)\mathrm{D}\Phi + a\Phi = 0 \quad \text{at } z = 1. \tag{24c}$$

4 Numerical Solution

Equations 19–22 together with the boundary conditions given by Eq. 24 constitute an eigenvalue problem with R_D as the eigenvalue. The resulting eigenvalue problem is solved numerically using the Galerkin technique. In this method, the test (weighted) functions are the same as the base (trial) functions. Thus, $W(z)$, $\Theta_\mathrm{f}(z)$, $\Theta_\mathrm{s}(z)$, and $\Phi(z)$ are expanded in the series form

$$W = \sum_{i=1}^{n} A_i\, W_i(z),\, \Theta_\mathrm{f}(z) = \sum_{i=1}^{n} B_i\, \Theta_{fi}(z),\, \Theta_\mathrm{s}(z) = \sum_{i=1}^{n} C_i\, \Theta_{si}(z),\, \Phi(z) = \sum_{i=1}^{n} E_i\, \Phi_i(z) \tag{25}$$

where A_i, B_i, and C_i are unknown coefficients. Multiplying Eq. 19 by $W_j(z)$, Eq. 20 by $\Theta_{fj}(z)$, Eq. 21 by $\Theta_{sj}(z)$, and Eq. 22 by $\Phi_j(z)$; performing the integration by parts with respect to z between $z = 0$ and 1, and using the boundary conditions, we obtain the following system of linear homogeneous algebraic equations:

$$C_{ji}A_i + D_{ji}B_i + E_{ji}E_i = 0 \tag{26}$$

$$F_{ji}A_i + G_{ji}B_i + H_{ji}C_i = 0 \tag{27}$$

$$I_{ji}B_i + J_{ji}C_i = 0 \tag{28}$$

$$K_{ji}B_i + L_{ji}E_i = 0. \tag{29}$$

The coefficients C_{ji}–I_{ji} involve the inner products of the base functions and are given by

$$\begin{aligned}
C_{ji} &= \left\langle\left(1 + \delta_1(z - 1/2) + \delta_2(z^2 - 1/3)\right) \mathrm{D}W_j\mathrm{D}W_i\right\rangle - \left\langle(\delta_1 + 2\delta_2 z)\, W_j\mathrm{D}W_i\right\rangle \\
&\quad + a^2\left\langle\left(1 + \delta_1(z - 1/2) + \delta_2(z^2 - 1/3)\right) W_jW_i\right\rangle \\
D_{ji} &= a^2 R_\mathrm{D}(1 + M_1)\left\langle W_j\Theta_{fi}\right\rangle, \quad E_{ji} = -a^2 R_\mathrm{D} M_1\left\langle W_j\mathrm{D}\Phi_i\right\rangle \\
F_{ji} &= -\langle\Theta_{fj}W_i\rangle, \quad G_{ji} = \langle\mathrm{D}\Theta_{fj}\mathrm{D}\Theta_{fi}\rangle + (a^2 + H_\mathrm{t})\langle\Theta_{fj}\Theta_{fi}\rangle \\
H_{ji} &= -H_\mathrm{t}\langle\Theta_{fj}\Theta_{si}\rangle, \quad I_{ji} = -\gamma H_\mathrm{t}\langle\Theta_{sj}\Theta_{fi}\rangle \\
J_{ji} &= \langle\mathrm{D}\Theta_{sj}\mathrm{D}\Theta_{si}\rangle + (a^2 + \gamma H_\mathrm{t})\langle\Theta_{sj}\Theta_{si}\rangle, \quad K_{ji} = -\langle\mathrm{D}\Phi_j\Theta_{fi}\rangle \\
L_{ji} &= \frac{a}{1+\chi}\left[\Phi_j(1)\Phi_i(1) + \Phi_j(0)\Phi_i(0)\right] + \langle\mathrm{D}\Phi_j\mathrm{D}\Phi_i\rangle + a^2 M_3\langle\Phi_j\Phi_i\rangle
\end{aligned} \tag{30}$$

where the inner product is defined as $\langle\cdots\rangle = \int_0^1(\cdots)\mathrm{d}z$.

The base functions $W_i(z)$, $\Theta_{fi}(z)$, $\Theta_{si}(z)$, and $\Phi_i(z)$ are assumed in the following form:

$$W_i = z(1 - z)T_{i-1}^*, \quad \Theta_{fi} = z(z - 1)T_{i-1}^* = \Theta_{si}, \quad \Phi_i = (z - 1/2)T_{i-1}^* \tag{31}$$

where T_i^*s ($i \in N$) are the modified Chebyshev polynomials, such that $W_i(z)$, $\Theta_{fi}(z)$, and $\Theta_{si}(z)$ satisfy the corresponding boundary conditions. The magnetic potential Φ_i does not satisfy the respective boundary conditions but the boundary residuals technique is used for the function Φ_i (see Finlayson 1970) and the first term in L_{ji} represents this residual term. The characteristic equation formed from Eqs. 25–28 for the existence of non-trivial solution is solved numerically for different values of physical parameters as well as for different forms of $\Gamma(z)$. The Newton–Raphson method is used to obtain the Darcy–Rayleigh number R_D as a function of wave number a when all the parameters and functions are fixed and the bisection method is built-in to locate the critical stability parameters (R_Dc, a_c) to the desired degree of accuracy. It is observed that the results are converged by taking six terms in the Galerkin expansion.

5 Results and Discussion

The effect of LTNE and different forms of permeability heterogeneity function $\Gamma(z)$ on the criterion for the onset of ferromagnetic convection in a layer of ferrofluid saturated Darcy

Table 1 Various forms of vertical heterogeneity of permeability $\Gamma(z)$

Models	δ_1	δ_2	Nature of $\Gamma(z)$
$F1$	0	0	Homogeneous
$F2$	1	0	Linear variation in z
$F3$	0	1	Only quadratic variation in z
$F4$	1	1	General quadratic variation in z

Table 2 Comparison of critical Darcy–Rayleigh and the corresponding wave numbers for different values of H_t with $M_1 = 1 = M_3$ and $\gamma = 1$ for a homogeneous porous medium case

Present analysis									Lee et al. (2010)	
H_t	$i = j = 1$		$i = j = 2$		$i = j = 5$		$i = j = 6$		Exact solution	
	R_{Dc}	a_c	R_{Dc}	a_c	R_{Dc}	a_c	R_{Dc}	a_c	R_{Dc}	a_c
10^{-2}	20.010	3.163	25.877	3.686	25.653	3.673	25.653	3.673	25.653	3.673
10^{-1}	20.099	3.170	25.975	3.693	25.751	3.680	25.751	3.680	25.751	3.680
10^{0}	20.943	3.232	26.906	3.760	26.683	3.742	26.683	3.742	26.683	3.742
10^{1}	26.469	3.458	33.284	4.037	33.047	4.024	33.047	4.024	33.047	4.024
10^{2}	36.610	3.294	46.680	3.874	46.317	3.859	46.317	3.859	46.317	3.859
10^{3}	39.607	3.178	51.133	3.708	50.697	3.695	50.697	3.695	50.697	3.695
10^{4}	39.960	3.164	51.670	3.687	51.224	3.674	51.224	3.674	51.224	3.674
10^{5}	39.996	3.162	51.725	3.685	51.278	3.672	51.278	3.672	51.278	3.672

porous medium heated from below in the presence of a uniform vertical magnetic field has been investigated numerically using the Galerkin method. As shown in Table 1, four different forms of $\Gamma(z)$ denoted by $F1$, $F2$, $F3$, and $F4$ have been considered on the stability characteristics of the system. To validate the numerical procedure employed and also to know the process of convergence, the critical stability parameters (R_{Dc}, a_c) computed under different limiting conditions and at various levels of the Galerkin expansion are exhibited and compared with the earlier published ones in Tables 2 and 3. The results tabulated in Tables 2 and 3 are for a homogeneous porous medium case in the limit as $\chi \to \infty$ for different values of H_t. It is seen that our numerical results are in excellent agreement with those obtained exactly by Lee et al. (2010) for $M_1 = 1 = M_3 = \gamma$, and Banu and Rees (2002) for $M_1 = 0$, $\gamma = 0.01$, 1, respectively. From the exhibited values, it is also clear that the results converge for six terms in the Galerkin expansion and thus verify the accuracy of the numerical procedure employed. Table 4 shows the numerically computed values of R_{Dc} and the corresponding a_c for the full problem considered when $H_t = 100$, $M_1 = M_3 = 1$ and $\chi = 6$ at various levels of the Galerkin approximation for various forms of $\Gamma(z)$. From the tabulated values, it is clear that the results converge for six terms in the Galerkin expansion. Further inspection of the above table reveals that R_{Dc} turns out to be the same for permeability heterogeneity functions of type $F1$ and $F2$ as well as $F3$ and $F4$ if a single term is considered in the Galerkin expansion. Similar type of result is observed by Nield and Kuznetsov (2010) in the case of an ordinary viscous fluid ($M_1 = 0$) saturating a heterogeneous porous medium for the LTE ($H_t = 0$) case, where they have considered $\sin(\pi z)$ as a trial function (see Table 5). Whereas, the values of R_{Dc} differ slightly for all types of permeability heterogeneity functions at higher order Galerkin method.

Table 3 Comparison of critical Darcy–Rayleigh and the corresponding wave numbers for various values of $\log_{10} H_t$ and for two values of γ when $M_1 = 0$ for a homogeneous porous medium case

γ	$\log_{10} H_t$	Banu and Rees (2002) Exact solution		Present analysis	
		R_c	a_c	R_c	a_c
0.01	−2	39.498	3.142	39.498	3.142
	−1	39.678	3.149	39.678	3.149
	0	41.453	3.218	41.453	3.218
	1	57.675	3.737	57.675	3.737
	2	182.564	5.607	182.564	5.607
	3	1097.63	7.028	1097.63	7.028
	4	3310.29	3.463	3310.29	3.463
	5	3910.54	3.172	3910.54	3.172
1.0	−2	39.498	3.142	39.498	3.142
	−1	39.677	3.149	39.677	3.149
	0	41.362	3.211	41.362	3.211
	1	52.359	3.436	52.359	3.436
	2	72.339	3.270	72.339	3.270
	3	78.190	3.156	78.190	3.156
	4	78.879	3.143	78.879	3.143
	5	78.949	3.141	78.949	3.141

Table 4 Comparison of critical Darcy–Rayleigh and the corresponding wave numbers for different orders of approximations in the Galerkin expansion for $H_t = 100$, $\chi = 6$, and $M_1 = M_3 = 1$

Approximations									
γ	Model	$i = j = 1$		$i = j = 2$		$i = j = 5$		$i = j = 6$	
		R_{Dc}	a_c	R_{Dc}	a_c	R_{Dc}	a_c	R_{Dc}	a_c
0.5	$F1$	58.175	3.979	58.175	3.979	58.818	4.058	58.818	4.059
	$F2$	58.175	3.979	55.041	4.008	56.123	4.104	56.124	4.104
	$F3$	60.061	4.215	57.041	4.263	57.664	4.334	57.664	4.334
	$F4$	60.061	4.215	48.503	4.389	48.325	4.654	48.326	4.653
1	$F1$	44.835	3.718	44.835	3.718	45.418	3.794	45.418	3.794
	$F2$	44.835	3.718	42.509	3.734	43.441	3.823	43.442	3.823
	$F3$	46.610	3.920	44.378	3.950	44.928	4.019	44.928	4.019
	$F4$	46.610	3.920	37.991	4.030	38.178	4.220	38.179	4.219

Figure 1a–d exhibits neutral curves (R_D against a) for two values of γ $(= 1, 5)$ (with $H_t = 100$, $M_1 = 1 = M_3$, $\chi = 6$), H_t $(= 10, 100)$ (with $\gamma = 0.5$, $M_1 = 1 = M_3$ $\chi = 6$), $M_1 (= 0.0001, 0.1)$ (with $H_t = 100$, $\gamma = 0.5$, $M_3 = 1$ and $\chi = 6$) and $M_3 (= 1, 10)$ (with $\chi = 6$, $H_t = 100$, $\gamma = 0.5$ and $M_1 = 1$), respectively, for different forms of $\Gamma(z)$. The neutral curves exhibit single but different minimum with respect to the wave number for various forms of $\Gamma(z)$ but their shape is identical in form to that of the Darcy–Benard problem. For

Table 5 Comparison of critical Darcy–Rayleigh and the corresponding wave numbers for an ordinary viscous fluid ($M_1 = 0$) and LTE ($H_t = 0$) case

Nield and Kuznetsov (2010)			Present analysis			
Model	R_{Dc}	a_c	$i = j = 1$		$i = j = 6$	
			R_{Dc}	a_c	R_{Dc}	a_c
$F1$	39.478	3.142	40.000	3.162	39.478	3.142
$F2$	39.478	3.142	40.000	3.162	37.861	3.157
$F3$	41.585	3.142	42.272	3.327	39.878	3.319
$F4$	41.585	3.142	42.272	3.327	34.439	3.427

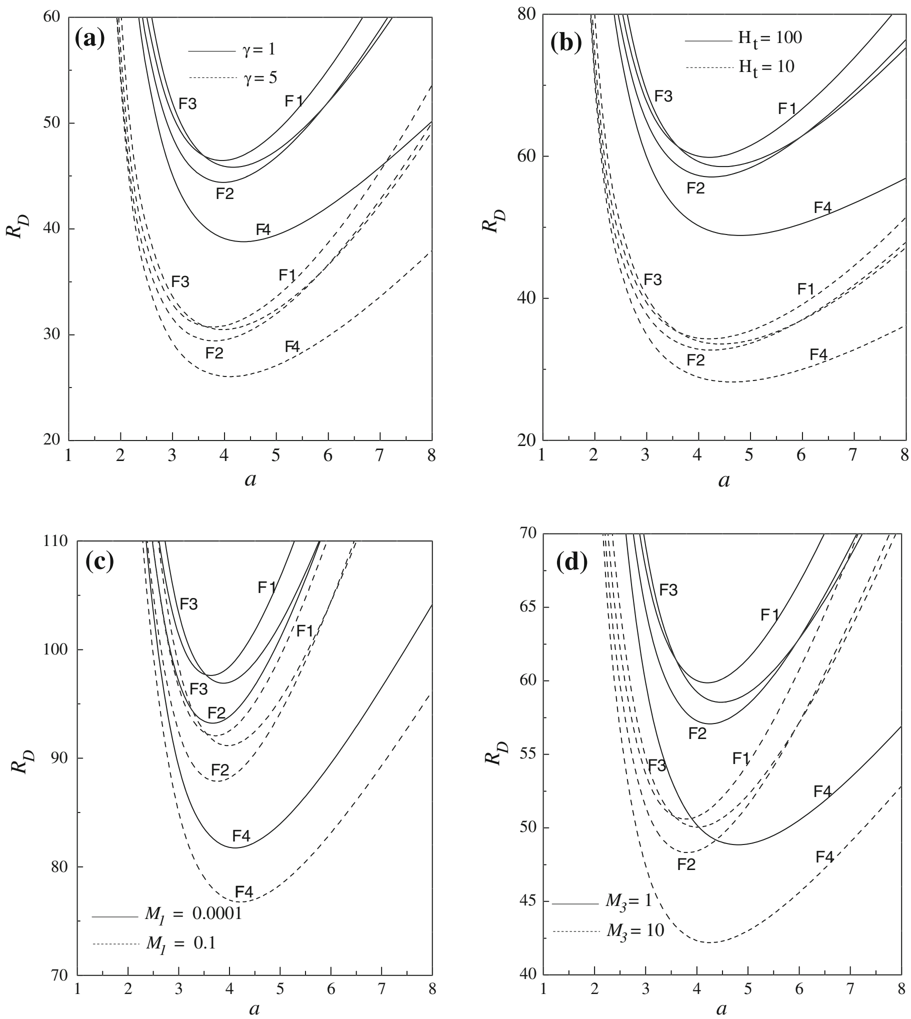

Fig. 1 Neutral curves for different values of **a** γ when $H_t = 100$, $M_1 = 1$, $M_3 = 1$, and $\chi = 6$; **b** H_t when $\gamma = 0.5$, $M_1 = 1$, $M_3 = 1$, and $\chi = 6$; **c** M_1 when $\gamma = 0.5$, $H_t = 100$ $M_3 = 1$, and $\chi = 6$; and **d** M_3 when, $\gamma = 0.5$, $H_t = 100$, $M_1 = 1$, and $\chi = 6$ for different forms of $\Gamma(z)$

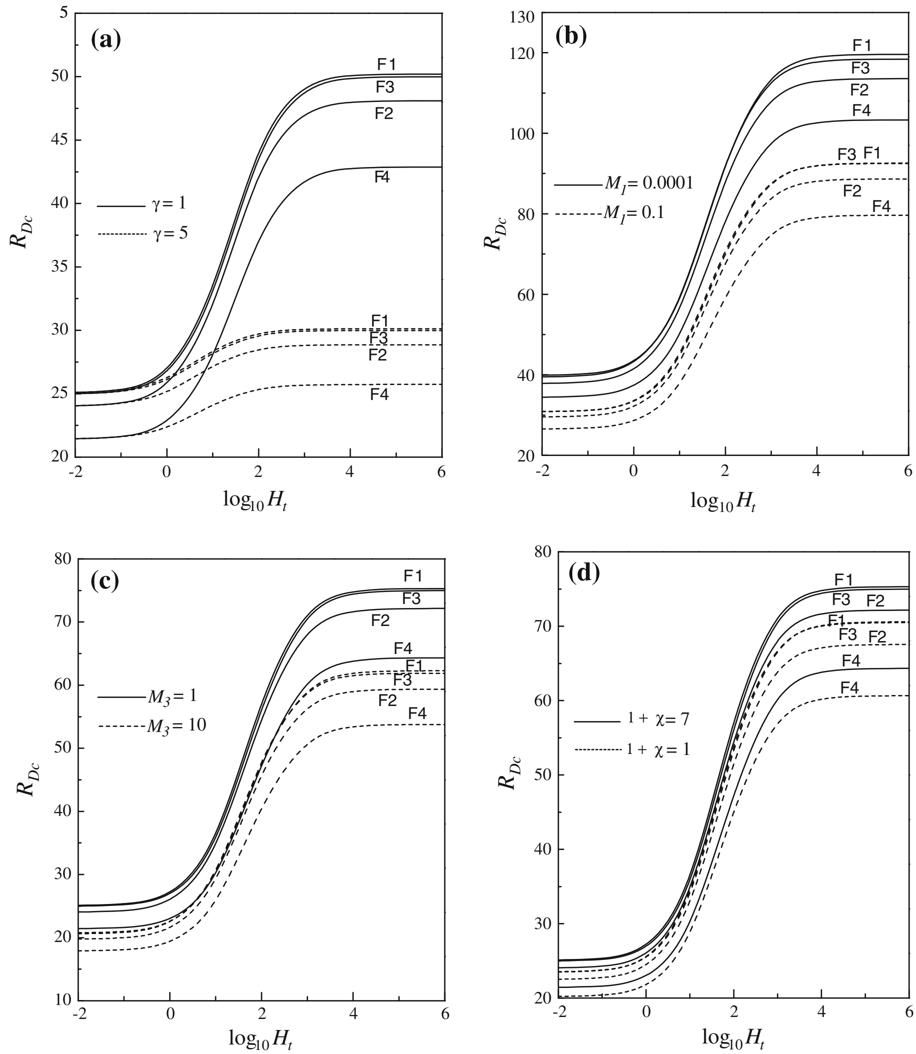

Fig. 2 Variation of R_{Dc} with $\log_{10} H_{\mathrm{t}}$ for different values of **a** γ when $M_1 = 1$, $M_3 = 1$, and $\chi = 6$; **b** M_1 when $\gamma = 0.5$, $M_3 = 1$, and $\chi = 6$; **c** M_3 when $\gamma = 0.5$, $M_1 = 1$, and $\chi = 6$; and **d** χ when $\gamma = 0.5$, $M_1 = 1$, and $M_3 = 1$ for different forms of $\Gamma(z)$

each of the forms of $\Gamma(z)$, the effect of increasing γ (see Fig. 1a), M_1 (see Fig. 1c) and M_3 (see Fig. 1d) is to reduce the Darcy–Rayleigh number and to decrease the region of stability, while opposite is the trend with increasing H_{t} (see Fig. 1b).

To determine the criterion for the onset of convection, the critical Darcy–Rayleigh number R_{Dc} is the important value and hence its variation is summarized in Fig. 2a–d as a function of $\log_{10} H_{\mathrm{t}}$ for various values of $\gamma(= 1, 5$ with $M_1 = 1 = M_3$ and $\chi = 6)$, $M_1(= 0.0001, 0.5$ with $\gamma = 0.5$, $M_3 = 1$ and $\chi = 6)$, $M_3(= 1, 10$ with $\gamma = 0.5$, $M_1 = 1$ and $\chi = 6)$ and $\chi(= 0, 6$ with $\gamma = 0.5$, $M_1 = 1$, and $M_3 = 1)$, respectively. The variation of the corresponding critical wave numbers a_{c} is shown in Fig. 3a–d. For various forms of $\Gamma(z)$, it is noted

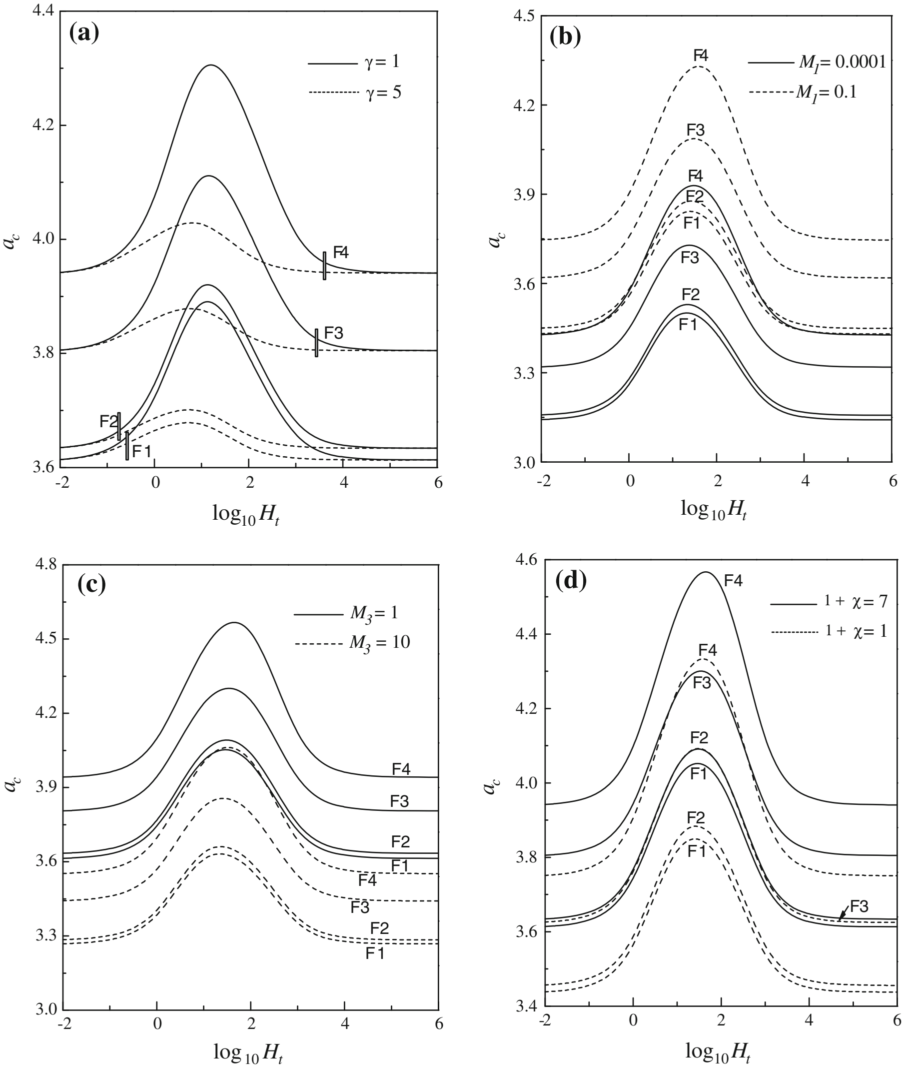

Fig. 3 Variation of a_c with $\log_{10} H_t$ for different values of **a** γ when $M_1 = 1$, $M_3 = 1$, and $\chi = 6$; **b** M_1 when $\gamma = 0.5$, $M_3 = 1$, and $\chi = 6$; **c** M_3 when $\gamma = 0.5$, $M_1 = 1$, and $\chi = 6$; and **d** χ when $\gamma = 0.5$, $M_1 = 1$, and $M_3 = 1$ for different forms of $\Gamma(z)$

that the curves of R_{Dc} for different γ coalesce and asymptote to a single value when H_t is small (see Fig. 2a). However, R_{Dc} decreases with increasing γ as the value of H_t increases and remains independent of H_t at higher values of the same. This is because, for very small values of H_t there is no significant transfer of heat between the fluid and solid phases and hence the condition for the onset of convection is not affected by the properties of the solid phase. Besides, increasing γ is to hasten the onset of ferromagnetic convection because heat is transported to the system through both solid and fluid phases. Further inspection of the figure reveals that the system is more stable if the form of $\Gamma(z)$ is of the type $F1$ than of the types $F3$ and $F2$, and the least stable is for $F4$. Thus, the onset of ferromagnetic convection can be either hastened or delayed depending on the type of heterogeneity in the permeability of the porous medium.

Although a similar trend as noted above could be seen with increasing M_1 (see Fig. 2b), M_3 (see Fig. 2c), and χ (see Fig. 2d), the curves of R_{Dc} for different M_1, M_3, and χ do not coalesce for small values of H_t. The size of M_1 is related to the importance of magnetic forces as compared to gravitational forces. The case $M_1 = 0$ corresponds to convective instability in an ordinary viscous fluid saturating a heterogeneous porous medium. It is seen that increase in the value of M_1 is to hasten the onset of ferromagnetic convection (i.e., to decrease the critical Darcy–Rayleigh number) suggesting that the ferrofluids carry heat more efficiently than the ordinary viscous fluids. This is due to an increase in the destabilizing magnetic force with increasing M_1, which favors the ferrofluid to flow more easily. Similar is the case with increasing M_3. This is because, a higher value of M_3 would arise from either a larger pyromagnetic coefficient or larger temperature gradient. Both these factors are conducive to generating a larger gradient in the Kelvin body force field, possibly promoting the instability. The effect of increasing χ has a stabilizing effect on the system because of its dampening effect due to an increase in magnetic induction.

The variation of the critical wave number a_c is shown in Fig. 3a–d. Irrespective of the forms of $\Gamma(z)$ considered, it is seen that the critical wave number remains the same for different values of γ in both the small and large H_t limits and this is evident from Fig. 3a. However, at moderate values of H_t, the critical wave number reaches its peak value and increasing γ decreases the value of a_c. In other words, increase in the value of γ is to enlarge the size of convection cells only at moderate values of H_t and the size of convection cells remains independent of γ when $H_t \ll 1$ and $H_t \gg 1$. Although the critical wave number remains invariant when $H_t \ll 1$ and $H_t \gg 1$ for a fixed value of M_1 (Fig. 3b), M_3 (Fig. 3c), and χ (Fig. 3d), the curves of a_c for different M_1, M_3, and χ do not join together under these two limiting cases of H_t. We note that increasing M_1 and χ is to increase the critical wave number, while opposite is the case with increasing M_3. That is, increase in the value of M_1 and χ as well as decrease in M_3 is to diminish the size of convection cells for all values of H_t. A closer inspection of the figures also reveals that the critical wave number is higher for model $F4$ followed by $F3$, then $F2$ and the least for $F1$.

6 Conclusions

The principal results of the foregoing linear stability analysis of ferromagnetic convection in a layer of ferrofluid saturated heterogeneous Darcy porous medium heated from below in the presence of a uniform vertical magnetic field using an LTNE model may be summarized as follows.

(i) Irrespective of different forms of permeability heterogeneity function $\Gamma(z)$, the onset of ferromagnetic convection retains its unimodal shape with one distinct minimum which defines the critical Darcy–Rayleigh number and the critical wave number for various values of physical parameters.

(ii) The system is more stable when $\Gamma(z) = 1$ and the least stable if $\Gamma(z)$ is of general quadratic variation with depth z. Thus, the onset of ferromagnetic convection can be either hastened or delayed depending on the type of heterogeneity of the porous medium. The porosity modified conductivity ratio γ has no effect on the onset of ferromagnetic convection in the small-H_t limit, while for other values of H_t increasing γ is to hasten the onset of ferromagnetic convection. Also, increasing M_1, M_3 and decreasing χ is to advance the onset of ferromagnetic convection.

(iii) The critical wave number for different values of γ in the small-H_t and large-H_t limits (LTE case) remain invariant and coincide, but attain a maximum value at the intermediate values of H_t (LTNE case) and in that case increasing γ decreases the critical wave number. Although the critical wave number assumes the same value when $H_t \ll 1$ and $H_t \gg 1$ for different values of M_1, M_3, and χ, the curves of a_c for these values do not coalesce. Increasing M_1 and χ is to increase the critical wave number but opposite is the case with increasing M_3.

(iv) The critical wave number is higher if $\Gamma(z)$ is of general quadratic variation with depth z and the least if $\Gamma(z) = 1$.

Acknowledgments One of the authors (ISS) wishes to thank the University of Hong Kong for inviting him as a visiting Professor and also the Bangalore University for sanctioning sabbatical leave. The work was financially supported by the Research Grants Council of the Hong Kong Special Administrative Region, China, through Project Nos. HKU 715609E and HKU 715510E. We thank the reviewers for useful suggestions.

References

Banu, N., Rees, D.A.S.: Onset of Darcy–Benard convection using a thermal non- equilibrium model. Int. J. Heat Mass Transf. **45**, 2221–2228 (2002)

Bashtovoi, V.G., Berkovski, B.M.: Thermomechanics of ferromagnetic fluids. Magn. Gidrodyn. **3**, 3–14 (1973)

Borglin, S.E., Mordis, J., Oldenburg, C.M.: Experimental studies of the flow of ferrofluid in porous media. Transp. Porous Med. **41**, 61–80 (2000)

Braester, C., Vadasz, P.: The effect of a weak heterogeneity of a porous medium on natural convection. J. Fluid Mech. **254**, 345–362 (1993)

Finlayson, B.A.: Convective instability of ferromagnetic fluids. J. Fluid Mech. **40**, 753–767 (1970)

Ganguly, R., Sen, S., Puri, I.K.: Heat transfer augmentation using a magnetic fluid under the influence of a line dipole. J. Magn. Magn. Mater. **271**, 63–73 (2004)

Ingham, D.B., Pop, I. (eds.): Transport Phenomenon in Porous Media. Pergamon, Oxford (1998)

Kuznetsov, A.V., Nield, D.A.: The effects of combined horizontal and vertical heterogeneity on the onset of convection in a porous medium: double diffusive case. Transp. Porous Med. **72**, 157–170 (2008)

Kuznetsov, A.V., Nield, D.A., Simmons, C.T.: The effect of strong heterogeneity on the onset of convection in a porous medium: periodic and localized variation. Transp. Porous Med. **81**, 123–139 (2010)

Kuznetsov, A.V., Nield, D.A., Simmons, C.T.: The onset of convection in a strongly heterogeneous porous medium with transient temperature profile. Transp. Porous Med. **86**, 851–865 (2011)

Lee, J., Shivakumara, I.S., Ravisha, M.: Effect of thermal non-equilibrium on convective instability in a ferromagnetic fluid saturated porous medium. Transp. Porous Med. **86**, 103–124 (2010)

Nanjundappa, C.E., Shivakumara, I.S., Ravisha, M.: The onset of buoyancy-driven convection in a ferromagnetic fluid saturated porous medium. Meccanica **45**, 213–226 (2010)

Neuringer, J.L., Rosensweig, R.E.: Ferrohydrodynamics. Phys. Fluids **7**(12), 1927–1937 (1964)

Nield, D.A., Bejan, A.: Convection in Porous Media. 3rd edn. Springer-Verlag, New York (2006)

Nield, D.A., Kuznetsov, A.V.: The effects of combined horizontal and vertical heterogeneity on the onset of convection in a porous medium. Int. J. Heat Mass Transf. **50**, 3329–3339 (2007a)

Nield, D.A., Kuznetsov, A.V.: The onset of convection in a shallow box occupied by a heterogeneous porous medium with constant flux boundaries. Transp. Porous Med. **67**, 441–451 (2007b)

Nield, D.A., Simmons, C.T.: A discussion on the effect of heterogeneity on the onset of convection in a porous medium. Transp. Porous Med. **68**, 413–421 (2007)

Nield, D.A., Kuznetsov, A.V.: Effects of combined horizontal and vertical heterogeneity on the onset of transient convection in a porous medium. J. Porous Med. **11**, 377–387 (2008a)

Nield, D.A., Kuznetsov, A.V.: The effects of combined horizontal and vertical heterogeneity on the onset of convection in a porous medium: moderate heterogeneity. Int. J. Heat Mass Transf. **51**, 2361–2367 (2008b)
Nield, D.A., Kuznetsov, A.V.: The onset of convection in a heterogeneous porous medium with transient temperature profile. Transp. Porous Med. **85**, 691–702 (2010)
Nield, D.A., Kuznetsov, A.V., Simmons, C.T.: The effect of strong heterogeneity on the onset of convection in a porous medium. Transp. Porous Med. **77**, 169–186 (2009)
Nield, D.A., Kuznetsov, A.V., Simmons, C.T.: The effect of strong heterogeneity on the onset of convection in a porous medium: 2D/3D localization and spatially correlated random permeability fields. Transp. Porous Med. **83**, 465–477 (2010)
Popplewell, J., Al-Qenaie, A., Charles, W., Moskowitz, R., Raj, K.: Thermal conductivity measurements on ferrofluids. Colloid Polym. Sci. **260**, 333–338 (1982)
Prasad, A., Simmons, C.T.: Unstable density-driven flow in heterogeneous porous media: a stochastic study of the Elder [1967b] 'short heater' problem. Water Resour. Res. **39**, 1007 (2003)
Shivakumara, I.S., Nanjundappa, C.E., Ravisha, M.: Thermomagnetic convection in a magnetic nanofluids fluid saturated porous medium. Int. J. Appl. Math. Eng. Sci. **2**(2), 157–170 (2008)
Shivakumara, I.S., Nanjundappa, C.E., Ravisha, M.: Effect of boundary conditions on the onset of thermomagnetic convection in a ferrofluid saturated porous medium. ASME J. Heat Transf. **131**, 101003 (2009)
Shivakumara, I.S., Lee, J., Nanjundappa, C.E., Ravisha, M.: Ferromagnetic convection in a rotating ferrofluid saturated porous layer. Transp. Porous Med. **87**, 251–273 (2011a)
Shivakumara, I.S., Lee, J., Ravisha, M., Gangadhara Reddy, R.: The onset of Brinkman ferroconvection using a thermal non-equilibrium model. Int. J. Heat Mass Transf. **54**, 2116–2125 (2011b)
Shivakumara, I.S., Lee, J., Ravisha, M., Gangadhara Reddy, R.: Effects of MFD viscosity and LTNE on the onset of ferromagnetic convection in a porous medium. Int. J. Heat Mass Transf. **54**, 2630–2641 (2011c)
Simmons, C.T., Fenstemaker, T.R., Sharp, J.M.: Variable-density flow and solute transport in heterogeneous porous media: approaches, resolutions and future challenges. J. Contam. Hydrol. **52**, 245–275 (2001)
Simmons, C.T., Kuznetsov, A.V., Nield, D.A.: Effect of strong heterogeneity on the onset of convection in a porous medium: importance of spatial dimensionality and geologic controls. Water Resour. Res. **46**, W09539 (2010)
Sunil, M.A.: A nonlinear stability analysis for thermoconvective magnetized ferrofluid saturating a porous medium. Transp. Porous Med. **76**, 327–343 (2009)
Sunil, S.P., Mahajan, A.: Nonlinear ferroconvection in a porous layer using a thermal nonequilibrium model. Spec. Top. Rev. Porous Med. **1**(2), 105–121 (2010)
Vadasz, P. (ed.): Emerging Topics in Heat and Mass Transfer in Porous Media. Springer, New York (2008)
Vafai, K. (ed.): Handbook of Porous Media. Marcel Dekker, New York (2000)
Vafai, K. (ed.): Handbook of Porous Media. 2nd edn. Taylor and Francis (CRC), Boca Raton (2005)