

AUTOMORPHISMS FIXING A VARIABLE OF  $K\langle x, y, z \rangle$ 

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ABSTRACT. We study automorphisms  $\varphi$  of the free associative algebra  $K\langle x, y, z \rangle$  over a field  $K$  such that  $\varphi(x), \varphi(y)$  are linear with respect to  $x, y$  and  $\varphi(z) = z$ . We prove that some of these automorphisms are wild in the class of all automorphisms fixing  $z$ , including the well known automorphism discovered by Anick, and show how to recognize the wild ones. This class of automorphisms induces tame automorphisms of the polynomial algebra  $K[x, y, z]$ . For  $n > 2$  the automorphisms of  $K\langle x_1, \dots, x_n, z \rangle$  which fix  $z$  and are linear in the  $x_i$ s are tame.

## INTRODUCTION

Let  $K$  be a field of any characteristic and let  $X = \{x_1, \dots, x_n\}$ ,  $n \geq 2$ , be a finite set. We denote by  $K[X]$  the polynomial algebra in the set of variables  $X$  and by  $K\langle X \rangle$  the free associative algebra (or the algebra of polynomials in the set  $X$  of noncommuting variables). We write the automorphisms of  $K[X]$  and  $K\langle X \rangle$  as  $n$ -tuples of the images of the coordinates, i.e.,  $\varphi = (f_1, \dots, f_n)$  means that  $\varphi(x_j) = f_j(X) = f_j(x_1, \dots, x_n)$ ,  $j = 1, \dots, n$ . We distinguish two kinds of  $K$ -algebra automorphisms of  $K[X]$  and  $K\langle X \rangle$ . The first kind are the affine automorphisms

$$\left( \sum_{i=1}^n \alpha_{i1} x_i + \beta_1, \dots, \sum_{i=1}^n \alpha_{in} x_i + \beta_n \right),$$

where  $\alpha_{ij}$  and  $\beta_j$  belong to  $K$  and the  $n \times n$  matrix  $(\alpha_{ij})$  is invertible. The second kind are the triangular automorphisms

$$(\alpha_1 x_1 + f_1(x_2, \dots, x_n), \dots, \alpha_{n-1} x_{n-1} + f_{n-1}(x_n), \alpha_n x_n + f_n),$$

where  $\alpha_j$  are invertible elements of  $K$  and the polynomials  $f_j(x_{j+1}, \dots, x_n)$  do not depend on the first  $j$  variables. The affine and the triangular automorphisms generate the group of the tame automorphisms. Instead of affine, one can consider linear automorphisms only, assuming that the polynomials  $f_1, \dots, f_n$  have no constant terms.

Many problems concerning automorphisms of free objects are stated in a similar way for free groups, polynomial algebras, free associative and free Lie algebras, for relatively free groups and algebras. Sometimes the solutions are obtained with similar methods but very often they require different techniques and the obtained

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results sound in different way, see the recent book [MSY] by Mikhalev, Shpilrain and Yu. The results and the open problems on automorphisms of polynomial algebras have served as the main motivation for the study of automorphisms of free associative algebras.

It is a classical result of Jung [J] (for  $K = \mathbb{C}$ ) and van der Kulk [K] (for any  $K$ ), that all automorphisms of  $K[x, y]$  are tame. Czerniakiewicz [Cz] and Makar-Limanov [ML1, ML2] proved that all automorphisms of  $K\langle x, y \rangle$  are also tame. This implies that  $\text{Aut } K\langle x, y \rangle \cong \text{Aut } K[x, y]$ .

It was an open problem whether there exist nontame (or wild) automorphisms of  $K[X]$  for  $n = |X| \geq 3$ . Nagata [N] constructed his famous example, the automorphism of  $K[x, y, z]$  defined by

$$(x - 2(y^2 + xz)y - (y^2 + xz)^2z, y + (y^2 + xz)z, z).$$

It fixes  $z$  and, as Nagata showed, is wild considered as an automorphism of  $K[z][x, y]$ , i.e., cannot be presented as a product of tame automorphisms of  $K[x, y, z]$  which fix  $z$ . It was conjectured that the Nagata automorphism is wild also as an element of  $\text{Aut}(K[x, y, z])$  and it was one of the main open problems in the theory of polynomial automorphisms for more than 30 years. Recently, Shestakov and Umirbaev [SU1, SU2, SU3] have developed a special technique based on noncommutative algebra (Poisson algebras) and have established that the Nagata automorphism is wild. They have also proved that every automorphism of  $K[x, y, z]$ , which fixes  $z$  and is wild as an automorphism of  $K[z][x, y]$ , is wild as an automorphism of  $K[x, y, z]$ . Umirbaev and Yu [UY] have established even a stronger version of this result: If  $\varphi \in \text{Aut } K[z][x, y]$  is wild, then there exists no tame automorphism of  $K[x, y, z]$  which sends  $x$  to  $\varphi(x)$ . In this way, if  $f(x, y, z)$  is a  $K[z]$ -wild coordinate in  $K[z][x, y]$ , then it is immediately wild also in  $K[x, y, z]$ . (A coordinate means an automorphism image of a variable.) In [DY1] the authors of the present paper commenced the systematic study of the wild automorphisms and wild coordinates of  $K[z][x, y]$ , see also their survey [DY2]. In particular, they provided many new wild automorphisms and wild coordinates of  $K[z][x, y]$  which are automatically wild automorphisms and wild coordinates of  $K[x, y, z]$ . For  $n > 3$  the problem for existing of wild automorphisms of  $K[X]$  is still open.

Up till now no wild automorphisms of the free algebras with more than two generators are known. There are some candidates to be wild, see the book by Cohn [C2]. One of them is the example of Anick  $(x + y(xy - yz), y, z + (zy - yz)y) \in \text{Aut } K\langle x, y, z \rangle$ , see [C2], p. 343. Although it fixes one variable, its abelianization is a tame automorphism of  $K[x, y, z]$  and we cannot apply the results of Shestakov, Umirbaev and Yu. Of course, if we were able to lift the Nagata automorphism (or any wild automorphism of  $K[x, y, z]$ ) to any automorphism of  $K\langle x, y, z \rangle$ , this would give an example of a wild automorphism of  $K\langle x, y, z \rangle$ .

The present paper is motivated by the idea that the results on the automorphisms of  $K[z][x, y]$  give important information on the automorphisms of  $K[x, y, z]$ . We study automorphisms of the free algebra  $K\langle x, y, z \rangle$  which fix the variable  $z$ . We restrict our considerations to the automorphisms  $\varphi$  such that  $\varphi(x), \varphi(y)$  are linear with respect to  $x, y$ . We call such automorphisms linear  $K[z]$ -automorphisms. We prove that some of these automorphisms are wild in the class of all automorphisms fixing  $z$ , including the automorphism discovered by Anick, and show how to recognize the wild ones. This class of automorphisms induces tame automorphisms of

the polynomial algebra  $K[x, y, z]$ . For  $n > 2$  the automorphisms of  $K\langle x_1, \dots, x_n, z \rangle$  which fix  $z$  and are linear in the  $x_i$ s are tame.

A forthcoming paper will be devoted to the description of the structure of the group of all  $K[z]$ -automorphisms of  $K\langle x, y, z \rangle$ .

## 1. PRELIMINARIES

We fix a field  $K$  of any characteristic, a set of variables  $X = \{x_1, \dots, x_n\}$ ,  $n \geq 2$ , and one more variable  $z$ . The main object of our paper is the free algebra  $K\langle X, z \rangle$  in the set of free generators  $X \cup \{z\}$ . The algebra  $K\langle X, z \rangle$  is isomorphic to the free product  $K[z] *_{K} K\langle X \rangle$ . We call an endomorphism  $\varphi$  of  $K\langle X, z \rangle$  a  $K[z]$ -endomorphism if it fixes  $z$  (and hence  $K[z] \subset K\langle X, z \rangle$ ) and write  $\varphi = (f_1, \dots, f_n)$ , where  $f_j = \varphi(x_j)$ ,  $j = 1, \dots, n$ . We denote the group of  $K[z]$ -automorphisms by  $\text{Aut}_{K[z]}K\langle X, z \rangle$ .

Defining the tame  $K[z]$ -automorphisms of  $K\langle X, z \rangle$ , the group of the triangular automorphisms consists obviously of the automorphisms

$$(\alpha_1 x_1 + f_1(x_2, \dots, x_n, z), \dots, \alpha_{n-1} x_{n-1} + f_{n-1}(x_n, z), \alpha_n x_n + f_n(z)),$$

$\alpha_j \in K^*$ ,  $f_j \in K\langle X, z \rangle$ , but we have to decide which are the linear automorphisms.

We may call a  $K[z]$ -automorphism  $(f_1, \dots, f_n)$  linear if the polynomials  $f_j$  are linear with respect to  $X$ , with coefficients depending on  $z$ . But for  $n = 2$  it is not known whether a big class of automorphisms similar to this of Anick are tame, although they all are linear with respect to  $X$ . So, we prefer to introduce the group of elementary linear automorphisms generated by the automorphisms

$$(\alpha_1 x_1, \dots, \alpha_n x_n), \quad (x_1, \dots, x_{j-1}, x_j + a(z)x_i b(z), x_{j+1}, \dots, x_n), \quad i \neq j,$$

$\alpha_j \in K^*$ ,  $a, b \in K[z]$ , and to generate the group of the tame  $K[z]$ -automorphisms by the elementary linear automorphisms and the triangular automorphisms. As we shall see, the celebrated Suslin theorem [Su] gives that for  $n = |X| \geq 3$  the two possible definitions are equivalent. Our paper is concentrated around  $K[z]$ -endomorphisms of  $K\langle X, z \rangle$  which are linear with respect to  $X$ . We call such endomorphisms linear  $K[z]$ -endomorphisms.

One of the main tools in the study of automorphisms is the Jacobian matrix. If  $\varphi = (f_1, \dots, f_n)$  is an automorphism of  $K[X]$ , then the Jacobian matrix  $J(\varphi) = (\partial f_j / \partial x_i)$  is invertible. The famous Jacobian conjecture states that, if  $\text{char } K = 0$  and  $J(\varphi)$  is invertible for an endomorphism  $\varphi$ , then  $\varphi$  is an automorphism, see the book by van den Essen [E] for the history and the state-of-the-art of the problem. There are various analogues of the Jacobian matrix in the case of (relatively) free groups and (relatively) free associative and Lie algebras, see [MSY]. In the case of free associative algebras, the exact analogue of the Jacobian matrix was introduced by Dicks and Lewin [DL] who proved the Jacobian conjecture for the free associative algebra  $K\langle x, y \rangle$  with two generators. The complete solution, also into affirmative, was given by Schofield in his book [Sc]. Implicitly, in terms of endomorphisms, the Jacobian matrix appeared in the paper [Y] by Yagzhev who used it to construct an algorithm which recognizes whether an endomorphism of  $K\langle X \rangle$  is an automorphism.

We recall the construction of the partial derivatives of Dicks and Lewin. Let  $y_0 = z, y_1 = x_1, \dots, y_n = x_n, Y = \{y_0, y_1, \dots, y_n\}$  and  $F = K\langle Y \rangle$ . The algebra  $\mathcal{M}(F)$  of the multiplications of  $F$  is a subalgebra of the algebra of  $K$ -linear operators

acting on  $F$  and is generated by the operators  $\lambda(u) : F \rightarrow F$  and  $\rho(u) : F \rightarrow F$ ,  $u \in F$ , of left and right multiplications defined, respectively, by

$$w^{\lambda(u)} = uw, \quad w^{\rho(u)} = wu, \quad w \in F.$$

The operators  $\lambda(u)$  and  $\rho(v)$  commute and  $\mathcal{M}(F) \cong F^{\text{op}} \otimes_K F$ , where  $F^{\text{op}}$  is the opposite algebra of  $F$ . The isomorphism is given by

$$\sum \lambda(u_p) \rho(v_p) \rightarrow \sum u_p \otimes v_p$$

and the opposite algebra appears because  $w^{\lambda(u_1)\lambda(u_2)} = u_2 u_1 w = w^{\lambda(u_2 u_1)}$ . We define the partial derivatives on the monomials  $w = y_{j_1} \cdots y_{j_k} \in F$  by

$$\frac{\partial w}{\partial y_i} = \sum_{p=1}^k (y_{j_1} \cdots y_{j_{p-1}}) \otimes (y_{j_{p+1}} \cdots y_{j_k}) \delta_{pi} \in F^{\text{op}} \otimes_K F,$$

where  $\delta_{pi} = 0, 1$  is the Kronecker symbol, and then extend them on  $F$  by linearity. The Jacobian matrix of an endomorphism  $\varphi = (f_0, f_1, \dots, f_n)$  of  $F$  is defined by

$$J(\varphi) = \begin{pmatrix} \frac{\partial f_0}{\partial y_0} & \frac{\partial f_1}{\partial y_0} & \cdots & \frac{\partial f_n}{\partial y_0} \\ \frac{\partial f_0}{\partial y_1} & \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_0}{\partial y_n} & \frac{\partial f_1}{\partial y_n} & \cdots & \frac{\partial f_n}{\partial y_n} \end{pmatrix}.$$

If  $\varphi$  fixes  $z = y_0$ , the first column of this matrix consists of one 1 and  $n$  zeros and the matrix is invertible if and only if the matrix

$$J_{K[z]}(\varphi) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial y_n} & \cdots & \frac{\partial f_n}{\partial y_n} \end{pmatrix}$$

is invertible and we call the latter matrix  $J_{K[z]}(\varphi)$  the Jacobian matrix of the  $K[z]$ -endomorphism  $\varphi$ .

It is well known (and easy to check) that the Jacobian matrices satisfy the chain rule

$$J(\varphi\psi) = J(\varphi)\varphi(J(\psi)),$$

where  $\varphi(J(\psi))$  means that  $\varphi$  acts on the entries of  $J(\psi)$ . In particular, if  $\varphi$  is an automorphism, then  $J_\varphi$  is invertible, i.e. belongs to  $GL_{n+1}(\mathcal{M}(F))$ . The theorem of Dicks-Lewin-Schofield gives that the invertibility of  $J(\varphi)$  implies that  $\varphi$  is an automorphism. It is obvious, that the same holds for the matrix  $J_{K[z]}(\varphi)$  when  $\varphi$  is a  $K[z]$ -endomorphism of  $F = K\langle X, z \rangle$ . If  $\varphi = (f_1, \dots, f_n)$  is a linear  $K[z]$ -endomorphism of  $K\langle X, z \rangle$ , then

$$f_j = \sum_{i=1}^n \sum_{p=1}^{k_{ij}} b_{ijp}(z) x_i c_{ijp}(z)$$

and the entries  $a_{ij}$  of  $J_{K[z]}(\varphi) = (a_{ij})$  are of the form

$$a_{ij} = \sum_{p=1}^{k_{ij}} b_{ijp}(z) \otimes c_{ijp}(z) \in K[z]^{\text{op}} \otimes_K K[z] \subset F^{\text{op}} \otimes_K F.$$

Since  $K[z]^{\text{op}} \otimes_K K[z] \cong K[z_1, z_2]$ , identifying  $z \otimes 1$  with  $z_1$  and  $1 \otimes z$  with  $z_2$ , we can consider  $J_{K[z]}(\varphi)$  as a matrix with entries from  $K[z_1, z_2]$ .

**Lemma 1.** *Let  $\varphi$  be a linear  $K[z]$ -endomorphism of  $F = K\langle X, z \rangle$ . Then  $\varphi$  is an automorphism of  $F$  if and only if its Jacobian matrix  $J_{K[z]}(\varphi)$  belongs to the group  $GL_n(K[z_1, z_2])$  of the invertible matrices with entries from  $K[z_1, z_2]$ . The group of the linear  $K[z]$ -automorphisms of  $F$  is isomorphic to  $GL_n(K[z_1, z_2])$ .*

*Proof.* The first part of the lemma follows from the fact that an  $n \times n$  matrix with entries from  $K[z]^{\text{op}} \otimes_K K[z] \subset F^{\text{op}} \otimes_K F$  is invertible over  $F^{\text{op}} \otimes_K F$  if and only if it is invertible over  $K[z]^{\text{op}} \otimes_K K[z]$ . For the second part, if  $\varphi, \psi$  are linear  $K[z]$ -endomorphisms, then the matrices  $J_{K[z]}(\varphi), J_{K[z]}(\psi)$  depend on  $z_1, z_2$  only and the chain rule gives that  $J_{K[z]}(\varphi\psi) = J_{K[z]}(\varphi)J_{K[z]}(\psi)$ , i.e., the Jacobian matrix of the product of two linear  $K[z]$ -endomorphisms is equal to the product, over  $K[z_1, z_2]$ , of the Jacobian matrices of the factors.  $\square$

## 2. THE MAIN RESULTS

By the theorem of Suslin [Su], for  $n \geq 3$ , every matrix in  $GL_n(K[z_1, \dots, z_p])$  can be presented as a product of a diagonal matrix and elementary matrices, i.e., belongs to the group  $GE_n(K[z_1, \dots, z_p])$ . For  $n = 2$ , this is not true. Cohn [C1] showed that the matrix

$$\begin{pmatrix} 1 + z_1 z_2 & z_2^2 \\ z_1^2 & 1 - z_1 z_2 \end{pmatrix} \in SL_2(K[z_1, z_2])$$

cannot be presented as a product of elementary  $2 \times 2$  matrices with entries from  $K[z_1, z_2]$ .

**Theorem 2.** (i) *A linear  $K[z]$ -automorphism of  $K\langle x, y, z \rangle$  is tame if and only if its Jacobian matrix belongs to the group  $GE_2(K[z_1, z_2])$ .*

(ii) *Every linear  $K[z]$ -automorphism of  $K\langle x, y, z \rangle$  induces a tame automorphism of  $K[x, y, z]$ .*

(iii) *For  $n > 2$ , any linear  $K[z]$ -automorphism of  $K\langle X, z \rangle$  is tame.*

*Proof.* (i) The algebra  $F = K\langle x, y, z \rangle$  has an augmentation assuming that the variables  $x, y$  are linear and  $z$  is a ‘‘constant’’, i.e. of zero degree. The corresponding augmentation ideal  $\omega_F$  consists of all polynomials without terms depending only on  $z$ . Every element  $f$  of  $F$  has the form

$$f = f_0(z) + f_1(x, y, z) + f_2(x, y, z),$$

where  $f_0(z) \in K[z]$ ,  $f_1(x, y, z)$  is linear with respect to  $x, y$  and  $f_2(x, y, z) \in \omega_F^2$ . It is easy to see (as in the case of endomorphisms of  $K[X]$  and  $K\langle X \rangle$ ) that the  $K[z]$ -endomorphism  $\varphi = (f(x, y, z), g(x, y, z))$ ,  $f = f_0 + f_1 + f_2$ ,  $g = g_0 + g_1 + g_2$ , is an automorphism if and only if the augmentation preserving endomorphism  $\varphi_0 = (f_1 + f_2, g_1 + g_2)$  is an automorphism. Also,  $\varphi$  is tame if and only if  $\varphi_0$  is tame. Then we can decompose  $\varphi_0$  as a product of elementary augmentation preserving automorphisms. So, we may restrict our considerations to augmentation preserving  $K[z]$ -automorphisms only. If  $\varphi_0$  is an automorphism, then the linear  $K[z]$ -endomorphism  $\varphi' = (f_1, g_1)$  is also an automorphism. If  $\varphi_0$  is tame, then  $\varphi'$  is a product of elementary linear  $K[z]$ -automorphisms. Hence, a linear  $K[z]$ -automorphism  $\varphi$  of  $K\langle x, y, z \rangle$  is tame if and only if it is a product of elementary

linear  $K[z]$ -automorphisms. By Lemma 1 this means that  $J_{K[z]}(\varphi)$  belongs to  $GE_2(K[z_1, z_2])$ .

(ii) The linear  $K[z]$ -automorphism  $\varphi$  of  $K\langle x, y, z \rangle$  induces a linear  $K[z]$ -automorphism  $\bar{\varphi}$  of  $K[x, y, z]$ . Its Jacobian matrix  $J_{K[z]}(\bar{\varphi})$  belongs to  $GL_2(K[z])$ . Since  $K[z]$  is a principal ideal domain, the groups  $GL_2(K[z])$  and  $GE_2(K[z])$  coincide. This gives that  $J_{K[z]}(\bar{\varphi})$  is a product of a diagonal matrix and elementary matrices with entries from  $K[z]$ . Since the diagonal and the elementary matrices correspond to elementary linear automorphisms, we derive that  $\bar{\varphi}$  is a tame  $K[z]$ -automorphism.

(iii) The Jacobian matrix  $J_{K[z]}(\varphi)$  of the linear  $K[z]$ -automorphism  $\varphi$  of  $K\langle X, z \rangle$  belongs to  $GL_n(K[z_1, z_2])$ . The theorem of Suslin gives that  $J_{K[z]}(\varphi)$  is a product of a diagonal matrix and elementary matrices. Again, these matrices correspond to elementary linear automorphisms and we obtain the proof.  $\square$

Recall that an automorphism  $(f_1, \dots, f_n)$  of  $K\langle X \rangle$  is called stably tame if the automorphism  $(f_1, \dots, f_n, x_{n+1}, \dots, x_{n+m})$  of  $K\langle X, x_{n+1}, \dots, x_{n+m} \rangle$  is tame for some  $m \geq 1$ . Theorem 2 (iii) immediately gives:

**Corollary 3.** *The linear  $K[z]$ -automorphisms of  $K\langle x, y, z \rangle$  are stably tame.*

There is an algorithm which decides whether a matrix in  $GL_2(K[z_1, \dots, z_p])$  belongs to  $GE_2(K[z_1, \dots, z_p])$ . It was suggested by Tolhuizen, Hollmann and Kalker [THK] for the partial ordering by degree and then generalized by Park [P] for any monomial ordering on  $K[z_1, \dots, z_p]$ . One applies Gaussian elimination process on the matrix based on the Euclidean division algorithm for  $K[z_1, \dots, z_p]$ . The matrix belongs to  $GE_2(K[z_1, \dots, z_p])$  if and only if this procedure brings it to an elementary or diagonal matrix. The result of Park was already used by Shpilrain and Yu [SY] to give an algorithm which recognizes whether a polynomial in  $K[x, y]$  is a coordinate, and by the authors in [DY1] to decide whether a polynomial in  $K[z][x, y]$  is a coordinate and a tame coordinate.

Consider the automorphism  $(x + y(xy - yz), y, z + (zy - yz)y)$  of  $K\langle x, y, z \rangle$  constructed by Anick. Exchanging the places of  $y$  and  $z$ , we obtain the automorphism

$$\varphi = (x + z(xz - zy), y + (xz - zy)z) \in \text{Aut}_{K[z]}K\langle x, y, z \rangle.$$

Its abelianization  $\bar{\varphi} = (x + z^2(x - y), y + z^2(x - y))$  is a tame automorphism of  $K[z][x, y]$ : Apply Theorem 2 (ii) or change the coordinates  $(x, y)$  of  $K[z][x, y]$  to  $(u, y) = (x - y, y)$ , then

$$\bar{\varphi} = (x - y, y + z^2(x - y)) = (u, y + z^2u).$$

Clearly,  $\varphi$  is a linear  $K[z]$ -automorphism. Its Jacobian matrix is

$$J_{K[z]}(\varphi) = \begin{pmatrix} 1 + z_1z_2 & z^2 \\ z_1^2 & 1 - z_1z_2 \end{pmatrix}$$

and is the matrix constructed by Cohn. It cannot be presented as a product of elementary  $2 \times 2$  matrices with entries from  $K[z_1, z_2]$ . (Direct arguments: Applying the Gaussian elimination process, we cannot reduce the entries of  $J_{K[z]}(\varphi)$  using only the Euclidean division algorithm because the leading terms of the entries of the columns are not divisible by each other.) Hence, Theorem 2 (i) gives that this automorphism is wild considered as a  $K[z]$ -automorphism. On the other hand, by Corollary 3, it is stably tame. An explicit decomposition of  $J_{K[z]}(\varphi)$  can be obtained

from the proof of the lemma of Mennicke, see Lemma 2.3 of [PW]. The sequence of elementary operations in [PW], p. 281, formula (2.1), gives the decomposition

$$\begin{pmatrix} 1 + z_1 z_2 & z_2^2 & 0 \\ z_1^2 & 1 - z_1 z_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= E_{13}(-z_2)E_{23}(-z_1)E_{31}(z_1)E_{32}(-z_2)E_{13}(z_2)E_{23}(z_1)E_{31}(-z_1)E_{32}(z_2),$$

where the matrix  $E_{ij}(\alpha z_1^a z_2^b)$  corresponds to the  $K[z]$ -automorphism of  $K\langle x, y, t, z \rangle$  defined by  $x_j \rightarrow x_j + \alpha z^a x_i z^b$ ,  $x_k \rightarrow x_k$ , when  $k \neq j$ , and  $x_1 = x$ ,  $x_2 = y$ ,  $x_3 = t$ .

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