The circumference of a graph with no $K_{3,t}$-minor

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Abstract

It was shown by Chen and Yu that every 3-connected planar graph G contains a cycle of length at least $|G|^{\log_3 2}$, where $|G|$ denotes the number of vertices of G. Thomas made a conjecture in a more general setting: there exists a function $\beta(t) > 0$ for $t \geq 3$, such that every 3-connected graph G with no $K_{3,t}$-minor, $t \geq 3$, contains a cycle of length at least $|G|^{\beta(t)}$. We prove that this conjecture is true with $\beta(t) = \log_{8t^{t+1}} 2$. We also show that every 2-connected graph with no $K_{2,t}$-minor, $t \geq 3$, contains a cycle of length at least $|G|/t^{t-1}$.

1 Introduction and notation

Over the past seven decades the Hamilton cycle problem has attracted tremendous research effort. Significant work has been done in characterizing those graphs that contain Hamilton cycles, and the earliest such results are concerned with planar graphs. In 1931, Whitney [22] proved that every 4-connected planar triangulation contains a Hamilton cycle. Tutte generalized this result to all 4-connected planar graphs [21], and Thomassen [20] strengthened this by showing that every 4-connected planar graph is in fact Hamilton connected.

With an attempt to generalize Tutte's theorem to other surfaces, Grünbaum [6] and Nash-Williams [12] independently conjectured that every 4-connected toroidal

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graph contains a Hamilton cycle. While this conjecture remains open, it has been shown that every 4-connected toroidal graph contains a Hamilton path [19], and that 5-connected toroidal graphs are Hamiltonian [18]. Moreover, Thomas and Yu [17] proved that 4-connected projective-planar graphs are also Hamiltonian. Generalizing to other surfaces, Yu [23] managed to show that every "locally planar" 5-connected triangulation of a surface contains a Hamilton cycle.

Notice that the above results are all concerning graphs with fairly high connectivity. If we relax the condition, then the situation changes dramatically: there are many 3-connected planar graphs that contain no Hamilton cycles, as exemplified in [8]. On the other hand, all cubic, bipartite, 3-connected, planar graphs are conjectured to be Hamiltonian by Barnette (see [10]).

When a graph G contains no Hamilton cycle, one may ask how long a cycle it contains. The length of a longest cycle in G, denoted by $c(G)$, is called the *circumference* of G. A good lower bound on $c(G)$ has also been the subject of extensive research. While studying paths in polytopes, Moon and Moser [11] implicitly conjectured that if G is a 3-connected planar graph then $c(G) \geq \alpha|G|^{\log_3 2}$, where α is a constant and $|G|$ denotes the number of vertices of G. (Grünbaum and Walther [7] made the same conjecture for the class of 3-connected cubic planar graphs.) Jackson and Wormald [9] gave the first polynomial lower bound on $c(G)$ for 3-connected planar graphs. This bound was improved by Gao and Yu [5] and further refined by Chung [4]. In [3], Chen and Yu fully established the Moon-Moser conjecture and showed that the same is true (within a constant factor) for 3-connected graphs embeddable in the torus or the Klein bottle. Based on these results, Böehme, Mohar, and Thomassen [2] proved that if G is a 3-connected graph of orientable genus g then $c(G) \geq \epsilon(g)|G|^{\log_3 2}$, where $\epsilon(g)$ is a constant dependent on g. Furthermore, $\epsilon(g)$ can be replaced by an absolute constant if G is also "locally planar" [15].

It is well known that a planar graph contains no $K_{3,3}$-minors. As a different generalization of the Chen-Yu result [3] on planar graphs, one may ask whether there is a similar result for 3-connected graphs with no $K_{3,t}$-minors. It is worthwhile pointing out that graphs containing no $K_{3,t}$-minors form an important class in the theory of graph minors. As discovered by Robertson and Seymour [14], in order to embed a graph in a given surface one must exclude large $K_{3,t}$-minors. In [13], Oporowski, Oxley, and Thomas showed that if G is a 3-connected graph with no $K_{3,t}$-minor, then it contains a large wheel. Inspired by this, Thomas and Seymour [16] made the following two conjectures.

(1.1) Conjecture (by Thomas). There exists a function $\beta(t) > 0$ for $t \geq 3$ such that, for any integer $t \geq 3$ and any 3-connected graph G with no $K_{3,t}$-minor, $c(G) \geq |G|^{\beta(t)}$.

(1.2) Conjecture (by Seymour and Thomas). There exist a constant $\beta > 0$ and a function $\alpha(t) > 0$ for $t \geq 3$ such that, for any integer $t \geq 3$ and any 3-connected graph G with no $K_{3,t}$-minor, $c(G) \geq \alpha(t)|G|^{\beta}$.

To prove the above conjectures, one reasonable approach is to find a structural description of 3-connected graphs with no $K_{3,t}$-minors. In this direction, Böhme, Maharry and Mohar [1] have obtained structural information about 7-connected graphs that contain no $K_{3,t}$-minors. However, a complete characterization of all 3-connected graphs seems to be very difficult to obtain. Alternatively, one might try to show the existence of a function $g(t) > 0$ such that any 3-connected graph containing no $K_{3,t}$-minor is embeddable in a surface of genus $g(t)$, in order to apply the result in [2]. Unfortunately this is not true, as there exist 3-connected graphs with no large $K_{3,t}$-minors but with arbitrarily large genus. (For example, let $C = v_0v_1 \dots v_{k-1}v_0$ be a cycle of length k. Let G be obtained from C be replacing each v_i by a complete graph with three vertices $x_{i,1}$, $x_{i,2}$, and $x_{i,3}$ such that $x_{i,j}$ and $x_{\ell,m}$ are adjacent if and only if $\ell = i - 1$ or $\ell = i + 1$, where the subscripts are taken modulo k. It was verified [1] that G contains no $K_{3,7}$-minor and the orientable genus of G is at least k.) One might therefore hope for some collection of simple reductions on 3-connected graphs with no $K_{3,t}$-minors which can be used to produce graphs embeddable in a surface of genus $g(t)$.

In this paper we approach the conjectures by direct construction of long cycles. Our main result is the following, which establishes Conjecture (1.1). (We shall actually prove a slightly stronger technical result, as stated in Section 2.)

(1.3) Theorem. *For any integer $t \geq 3$ and for any 3-connected graph G with no $K_{3,t}$-minor, $c(G) \geq |G|^{r(t)}$, where $r(t) = \log_{8t^{t+1}} 2$.*

From the graphs constructed by Moon and Moser [11] (also see [3]), we see that the exponent $\beta(t)$ cannot exceed $\log_3 2$. We feel that $\beta(t)$ can be improved to $\log_t 2$. Yet, it is still unknown what the best bound is and whether the method used in this paper can be further extended to establish Conjecture (1.2).

The remainder of this paper is organized as follows. In section 2, we state the main theorem, introduce some terminology, and exhibit some useful properties of the function $f(x) = x^{\log_b 2}$. In Section 3, we study graphs with weighted edges, and establish a result about paths in weighted graphs and a result about the circumference of 2-connected graphs with no $K_{2,t}$-minors. (We shall use weighted graphs to store information when performing certain reduction operations in the proof.) In Sections 4 - 6, we complete the proof of the technical result stated in Section 2. There are three statements in the technical result: (a), (b), and (c). The proof of the technical result is by induction on the number of vertices. The induction step for (a) is done in Section 4, and the induction step for (b) is done in Section 5. The induction step for (c) is done in Section 6, and the inductive proof will be completed in Section 6.

2 The technical theorem

The main goals of this section are to state a technical theorem which implies (1.3) and to prove some properties of the function $f(x) = x^{\log_b 2}$ which will be frequently used. First, we introduce notation and terminology necessary for stating and proving our results.

We only consider simple graphs. We use $A := B$ to rename B as A. For a graph G, $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively, and we let $|G| := |V(G)|$. For graphs H and G, $H \subseteq G$ means that H is a subgraph of G.

Let G be a graph and let $U \subseteq V(G)$. Then $G[U]$ denotes the subgraph of G induced by U. The set U is said to be a *connected set* of G if $G[U]$ is connected. Let $G - U := G[V(G) - U]$, and if $U = \{u\}$ then let $G - u := G - U$. We say that U is a *k-cut* of G if $G - U$ is not connected and $|U| = k$. If $\{u\}$ is a 1-cut of G, then u is called a *cutvertex* of G. Let $N_G(U) := \{x \in V(G) - U : x$ is adjacent to some vertex in $U\}$, and let $N_G(u) := N_G(\{u\})$. For convenience, let $N_G(H) := N_G(V(H))$ for any subgraph H of G. If there is no danger of confusion, we will simply drop the subscript G.

Let G be a graph. For two distinct vertices x, y of G, an *x-y path* in G is a path between x and y in G. If P is a path, we use $\ell(P)$ to denote the length of P, which is the number of edges of P. For any distinct vertices x, y of a path P, we use $P[x, y]$ to denote the subpath of P between x and y (inclusive), and define $P[x, y) := P[x, y] - y$, $P(x, y] := P[x, y] - x$, and $P(x, y) := P[x, y] - \{x, y\}$. An edge of G with ends u and v is often denoted by uv, or vu, or $\{u, v\}$. Let S be a set of 2-element subsets of $V(G)$. Then we use $G + S$ to denote the graph with vertex set $V(G)$ and edge set $E(G) \cup S$. (Note that each edge of G is a 2-element subset of $V(G)$.) If $S = \{\{u_i, v_i\} : i = 1, \ldots, k\}$, then we also write $G + \{u_i v_i : i = 1, \ldots, k\}$ instead of $G + S$. If $S = \{\{u, v\}\}$, then we let $G + uv := G + S$.

A graph H is a *minor* of a graph G if there exist disjoint connected sets V_x of G, indexed by $x \in V(H)$, such that, for any distinct $x, y \in V(H)$, $xy \in E(H)$ if and only if G has an edge with one end in V_x and the other in V_y. These sets form a *representation* of H in G. If H is a minor of G, then we say that G contains an *H-minor*. When there is no danger of confusion, we will not make an effort to distinguish between the edges of H and the edges of G. That is, we may view the edges of H as edges of G. Let G be a graph and let U be a connected subgraph of G; then we use G/U to denote the graph obtained from G by contracting U (and deleting resulting multiple edges and loops).

The graph $K_{3,t}$ is the complete bipartite graph with one part of size 3 and the other of size t. Let G be a graph and $\{x, y, z\} \subseteq V(G)$. We say that a $K_{3,t}$-minor H of G is *rooted at* $\{x, y, z\}$ if H has a representation in G such that $x \in V_1$, $y \in V_2$, $z \in V_3$, where V_1, V_2, V_3 are connected sets of G representing vertices of H in the partition set of size three. We define $\mu(G; x, y, z)$ to be the largest integer t such that H has a $K_{3,t}$-minor rooted at $\{x, y, z\}$.

We can now state the aforementioned technical theorem.

(2.1) Theorem. *Let $t \geq 3$ be an integer, let $r(t) := \log_{8t^{t+1}} 2$, and let G be a 3-connected graph with no $K_{3,t}$-minor. Then the following statements hold.*

(a) *For any distinct vertices x, y, z of G such that $xz, yz \in E(G)$, $G - z$ contains an x-y path of length at least $(\frac{|G|-1}{t^{\mu}})^{r(t)}$, where $\mu := \mu(G; x, y, z)$.*

(b) *For any $xy \in E(G)$, G contains an x-y path of length at least $|G|^{r(t)}$.*

(c) *For any two distinct edges xy, f of G, G contains an x-y path through f which has length at least $(\frac{|G|}{t^t})^{r(t)} + 1$.*

Note that (1.3) is an immediate consequence of (b) of Theorem (2.1). In order to prove Theorem (2.1), we need the following property of the function $f(x) = x^{\log_b 2}$.

(2.2) Lemma. *For any integer $b \geq 4$ and for any $m \geq n > 0$,*

$$m^{\log_b 2} + n^{\log_b 2} \geq (m + (b-1)n)^{\log_b 2}.$$

Proof. By dividing both sides of the above inequality by $m^{\log_b 2}$, it suffices to show that, for any s with $0 \leq s \leq 1$,

$$1 + s^{\log_b 2} \geq (1 + (b-1)s)^{\log_b 2}.$$

Let $f(s) = 1 + s^{\log_b 2} - (1 + (b-1)s)^{\log_b 2}$. Clearly, $f(0) = f(1) = 0$. Differentiating with respect to s, we have

$$f'(s) = \log_b 2 \cdot (s^{(\log_b 2)-1} - (b-1)(1 + (b-1)s)^{(\log_b 2)-1}).$$

A simple calculation shows that $f'(s) = 0$ has a unique solution. Therefore, since $f(0) = f(1) = 0$, either 0 is the absolute maximum of $f(s)$ over $[0, 1]$ or 0 is the absolute minimum of $f(s)$ over $[0, 1]$. That is, either $f(s) \geq 0$ for all $s \in [0, 1]$ or $f(s) \leq 0$ for all $s \in [0, 1]$. Note that $0 < \frac{1}{b} < 1$ (since $b \geq 4$) and

$$\begin{aligned} f(\frac{1}{b}) &= (1 + \frac{1}{2}) - (1 + \frac{b-1}{b})^{\log_b 2} \\ &= \frac{3}{2} - \frac{(2b-1)^{\log_b 2}}{2} \\ &> \frac{3}{2} - \frac{(2b)^{\log_b 2}}{2} \\ &= \frac{3}{2} - 2^{\log_b 2}. \end{aligned}$$

Since $b \geq 4$, $2^{\log_b 2} \leq 2^{\log_4 2} = \sqrt{2} < 3/2$. Thus, $f(\frac{1}{b}) > 0$ for $b \geq 4$. Therefore, we have $f(s) \geq 0$ for all $s \in [0, 1]$. $\square$

(2.3) Corollary. *Let $a \geq 1$ and $b \geq 4$ be integers, and let $m > 0$ and $n > 0$. If $m \geq \frac{n}{a}$, then*

$$m^{\log_b 2} + n^{\log_b 2} \geq (m + \frac{b-1}{a}n)^{\log_b 2}.$$

Proof. Since $m \geq \frac{n}{a} > 0$ and by Lemma (2.2), we have $m^{\log_b 2} + (\frac{n}{a})^{\log_b 2} \geq (m + (b-1)\frac{n}{a})^{\log_b 2}$. Since $a \geq 1$, $m^{\log_b 2} + n^{\log_b 2} \geq m^{\log_b 2} + (\frac{n}{a})^{\log_b 2}$. So (2.3) holds. $\square$

By repeatedly applying (2.3), we obtain the following

(2.4) Corollary. *Suppose $m, n_1, \dots, n_k$ are positive numbers such that $m \geq \frac{n_i}{a}$ for all $1 \leq i \leq k$. Then, for any integer $b \geq 4$,*

$$m^{\log_b 2} + \sum_{i=1}^{k} n_i^{\log_b 2} \geq (m + \frac{b-1}{a} \sum_{i=1}^{k} n_i)^{\log_b 2}.$$

3 Circumferences of 2-connected graphs

In this section, we prove a result about long paths in weighted graphs, which will be useful for proving Theorem (2.1). We will also see that a similar argument can be used to prove an interesting result about the circumference of a 2-connected graph with no $K_{2,t}$-minor.

For convenience, we introduce the concept of bridge. Let G be a graph and H a subgraph of G. An *H-bridge* of G is a subgraph of G which is induced by either (i) an edge in $E(G) - E(H)$ with both ends in $V(H)$ or (ii) the edges in a component D of $G - V(H)$ and edges of G from D to H. The H-bridges satisfying (ii) are said to be *non-trivial.* If $U \subseteq V(G)$, we may view U as a subgraph of G with vertex set U and no edges. Hence, we will also speak of U-bridges or bridges of G *associated* with U.

In the proof of (2.1), we need to replace certain bridges of a graph associated with 2-cuts by edges, and each such edge will be assigned a weight which records the number of vertices in the corresponding bridge. The following concepts will allow us to relate subgraphs in the weighted graph to $K_{3,t}$-minors and rooted $K_{3,t}$-minors in the original graph.

Let G be a graph and let $S \subseteq E(G)$. An *S-link of size m* in G consists of two disjoint connected subgraphs A, B of G and a subset S' of S such that $|S'| = m$ and each edge in S' has one end in $V(A)$ and the other in $V(B)$, and we denote it by $(A, B; S')$. If, in addition, $x \in V(A)$ and $y \in V(B)$ or $x \in V(B)$ and $y \in V(A)$, then $(A, B; S')$ is said to be *between x and y*.

Again, let G be a graph and let $S \subseteq E(G)$. Let P be a path in G. For any $e \in E(P)$, an *$(S; P)$-ladder with top e and m rungs* is an S-link $(A, B; S')$ of size m such that $e \notin S'$, one component of $P - e$ is contained in A, and the other component of $P - e$ is contained in B. The edges in S' are called the *rungs* of the ladder. We use $\alpha(P; e)$ to denote the maximum number t such that G has an $(S; P)$-ladder with top e and t rungs.

Let $\mathbb{R}^+$ denote the set of non-negative real numbers. For any function $\omega : E(G) \mapsto \mathbb{R}^+$ and a subgraph H of G, we define $\omega(H) := \sum_{e \in E(H)} \omega(e)$. We can now state and prove the main result of this section.

(3.1) Theorem. *Let $t \geq 2$ be an integer, G be 2-connected graph, $\omega : E(G) \mapsto \mathbb{R}^+$, $S = \{e \in E(G) : \omega(e) > 0\}$, and $x, y \in V(G)$ be distinct. If G does not contain any*

S-link of size t between x and y, then there is an x-y path P in G such that

$$\sum_{e \in E(P)} t^{\alpha(P;e)} \omega(e) \geq \omega(G).$$

Proof. Note that $\omega(G) = \omega(S)$. We will apply induction on $|G| + |S|$. If $|S| = 0$ then $\omega(G) = 0$, and hence, any x-y path P in G gives the desired path. If $|S| = 1$ then since G is 2-connected, G has an x-y path P containing the edge in S, and clearly $\sum_{e \in E(P)} t^{\alpha(P;e)} \omega(e) \geq \omega(G)$. So we may assume $|S| \geq 2$.

Suppose $|G| = 3$. Then G is a triangle. Let P, Q denote the x-y paths in G, and assume without loss of generality that $\omega(P) \geq \omega(Q)$. Therefore, since $t \geq 2$, $t\omega(P) \geq \omega(G)$. If $S \subseteq E(P)$, then $\sum_{e \in E(P)} t^{\alpha(P;e)} \omega(e) \geq \omega(G)$. So assume that $S \cap E(Q) \neq \emptyset$. Then for any $e \in E(P)$, $\alpha(P;e) \geq 1$. Hence, $\sum_{e \in E(P)} t^{\alpha(P;e)} \omega(e) \geq t\omega(P) \geq \omega(G)$. Therefore, we may assume that $|G| \geq 4$. We distinguish between two cases.

Case 1. $\{x, y\}$ is a 2-cut of G or some edge in S is incident with both x and y.

In this case, there exist subgraphs G_1 and G_2 of G such that $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = \{x, y\}$, and either $|G_1| \geq 3 \leq |G_2|$ or, for some $i \in \{1, 2\}$, G_i is induced by an edge in S. (See Figure 1(a) for an illustration.) Without loss of generality, we may assume that $\omega(G_1) \geq \omega(G_2)$. Then, since $t \geq 2$, $t \cdot \omega(G_1) \geq \omega(G)$.

First, let us assume that G_1 is induced by an edge $f \in S$. Then f is incident with both x and y. Let $P = G_1$. Since G is 2-connected and $|S| \geq 2$, there exists an edge $g \in S - \{f\}$ and an x-y path R in G_2 containing g. Let A, B denote the components of $R - f$. Then $(A, B; \{g\})$ is an $(S; P)$-ladder with top f and one rung. Thus, $\alpha(P; f) \geq 1$. So $\sum_{e \in E(P)} t^{\alpha(P;e)} \omega(e) \geq t \cdot \omega(f) = t \cdot \omega(G_1) \geq \omega(G)$.

Now assume that $|G_1| \geq 3$. Let $G^* := G_1 + xy$ and let $S^* = S \cap E(G_1)$. Define $\omega^* : E(G^*) \to \mathbb{R}^+$ as follows: for any $e \in E(G_1)$, $\omega^*(e) = \omega(e)$; and if $xy \notin E(G_1)$ then $\omega^*(xy) = 0$. Note that G^* is 2-connected and $|G^*| + |S^*| < |G| + |S|$. So by the induction hypothesis, there is an x-y path P in G^* such that

$$\sum_{e \in E(P)} t^{\alpha^*(P;e)} \omega^*(e) \geq \omega^*(G^*) = \omega(S^*) = \omega(G_1),$$

where $\alpha^*(P; e)$ denotes the greatest integer m such that G^* has an $(S^*; P)$-ladder with top e and m rungs.

If $S \cap E(G_2) = \emptyset$, then $\omega(G_1) = \omega(G)$ and $\alpha(P; e) = \alpha^*(P; e)$ for all $e \in E(P)$. Hence,

$$\sum_{e \in E(P)} t^{\alpha(P;e)} \omega(e) = \sum_{e \in E(P)} t^{\alpha^*(P;e)} \omega^*(e) \geq \omega(G_1) = \omega(G).$$

So we may assume that $S \cap E(G_2) \neq \emptyset$. Then, since G is 2-connected, G_2 has an x-y path R containing an edge $f \in S$. For any $(S^*; P)$-ladder $(A, B; S^*)$ in G^* between x and y with top e and m rungs, we can form an $(S; P)$-ladder with top e and $m+1$ rungs by adding to A the component of $R-f$ containing x, adding to B the component of $R-f$ containing y, and adding f to S^*. Hence, $\alpha(P;e) \geq \alpha^*(P;e)+1$ for all $e \in E(P)$. So

$$\sum_{e\in E(P)} t^{\alpha(P;e)}\omega(e) \geq \sum_{e\in E(P)} t^{\alpha^*(P;e)+1}\omega^*(e) \geq t\cdot\omega(G_1) \geq \omega(G).$$

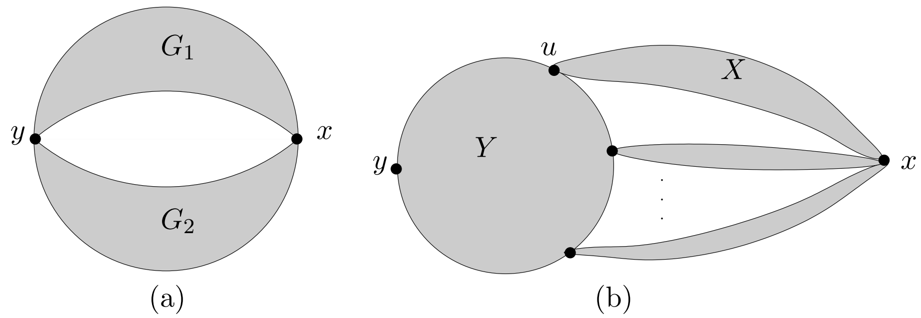

Figure 1: Two cases in the proof of (3.1).

Case 2. $\{x,y\}$ is not a 2-cut of G, and no edge in S is incident with both x and y.

Then y is contained in a unique block of $G-x$, say Y. Let X be a $(Y\cup\{x\})$-bridge of G with $\omega(X)$ maximum, and let u be the unique vertex in $V(X)\cap V(Y)$. (See Figure 1(b).) Since we are in Case 2, $u \neq y$. Because G has no S-link of size t, there are at most $t-1$ $(Y\cup\{x\})$-bridges of G that contain edges in S. So $t\omega(X) \geq \omega(G)-\omega(Y)$. Let $S_X = S\cap E(X)$ and $S_Y = S\cap E(Y)$. Clearly $|X|+|S_X| < |G|+|S| > |Y|+|S_Y|$. Define $\omega_X : E(X)\to\mathbb{R}^+$ such that, for any $e\in E(X)$, $\omega_X(e)=\omega(e)$, and define $\omega_Y : E(X)\to\mathbb{R}^+$ such that, for any $e\in E(Y)$, $\omega_Y(e)=\omega(e)$. So $\omega_X(X)=\omega(X)$ and $\omega_Y(Y)=\omega(Y)$. In the next two paragraphs, we will find an x-u path P_x in X and a u-y path P_y in Y.

If $|X|=2$, then let $P_x := X$, which is an x-u path. If $|X|\geq 3$, then by the induction hypothesis, X has an x-u path P_x such that $\sum_{e\in E(P_x)} t^{\alpha_X(P_x;e)}\omega_X(e) \geq \omega_X(X) = \omega(X)$, where $\alpha_X(P_x;e)$ is the greatest integer m such that X has an $(S_X;P_x)$-ladder with top e and m rungs.

If $|Y|=2$, then let $P_y := Y$, which is a u-y path. If $|Y|\geq 3$, then by the induction hypothesis, Y has a u-y path P_y such that $\sum_{e\in E(P_y)} t^{\alpha_Y(P_y;e)}\omega_Y(e) \geq \omega_Y(Y) = \omega(Y)$, where $\alpha_Y(P_y;e)$ is the greatest integer m such that Y has an $(S_Y;P_y)$-ladder with top e and m rungs.

Let $P := P_x\cup P_y$. For any $e\in E(P_y)$, $\alpha(P;e) \geq \alpha_Y(P_y;e)$, and so, $\sum_{e\in E(P_y)} t^{\alpha(P;e)}\omega(e) \geq \sum_{e\in E(P_y)} t^{\alpha_Y(P_y;e)}\omega_Y(e) \geq \omega(Y)$. If $S_X\cup S_Y = S$ then $\sum_{e\in E(P_x)} t^{\alpha(P;e)}\omega(e) \geq \sum_{e\in E(P_x)} t^{\alpha_X(P_x;e)}\omega_X(e) \geq \omega(X) = \omega(G)-\omega(Y)$. If $S_X\cup S_Y \neq S$ then it is easy to see that $\alpha(P;e)\geq\alpha_X(P_x;e)+1$ for all $e\in E(P_x)$, and so, $\sum_{e\in E(P_x)} t^{\alpha(P;e)}\omega(e) \geq \sum_{e\in E(P_x)} t^{\alpha_X(P_x;e)+1}\omega_X(e) \geq t\cdot\omega(X) \geq \omega(G)-\omega(Y)$.

Therefore,

$$\begin{aligned}\sum_{e\in E(P)} t^{\alpha(P;e)}\omega(e) &\geq \sum_{e\in E(P_x)} t^{\alpha_X(P_x;e)}\omega_X(e) + \sum_{e\in E(P_y)} t^{\alpha_Y(P_y;e)}\omega_Y(e)\\ &\geq \omega(G)-\omega(Y)+\omega(Y)\\ &= \omega(G).\end{aligned}$$

□

In Theorem (3.1), if G does not contain any S-link of size t between x and y, then $\alpha(P;e)\leq t-1$ for all $e\in E(P)$. Hence, we have following corollary.

(3.2) Corollary. *Let G be a 2-connected graph, let $\omega : E(G)\mapsto \mathbb{R}^+$, and let $S=\{e\ :\omega(e)>0\}$. Let $x,y\in V(G)$ be distinct, and assume that G does not contain any S-link of size t between x and y. Then there is an x-y path P in G such that*

$$\sum_{e\in E(P)} \omega(e) \geq \frac{\omega(G)}{t^{t-1}}.$$

Next we use an argument similar to the proof of (3.1) to derive a result on the circumference of 2-connected graphs.

(3.3) Proposition. *Let $t\geq 2$ be an integer, and let G be a 2-connected graph with no $K_{2,t}$-minors. Then, for any distinct vertices x,y of G, there is an x-y path in G of length at least $|G|/t^{t-1}$. In particular, $c(G)\geq |G|/t^{t-1}$.*

Proof. We will prove the following stronger result from which (3.3) follows.

(*) *Let G be a 2-connected graph containing no $K_{2,t}$-minors, let $x,y\in V(G)$ be distinct, and let $\mu:=\mu(G;x,y)$ denote the largest integer m such that G has a $K_{2,m}$-minor rooted at $\{x,y\}$. Then G contains an x-y path of length at least $|G|/t^{\mu}$.*

Since $t^{\mu}\geq 2$, (*) holds when $|G|\leq 3$. So assume that $|G|\geq 4$ and (*) holds for all graphs with less than $|G|$ vertices. We consider two cases (see Figure 1 for an illustration).

Case 1. $\{x,y\}$ is a 2-cut of G.

In this case, there exist subgraphs G_1,G_2 of G such that $V(G_1)\cap V(G_2)=\{x,y\}$, $E(G_1)\cap E(G_2)=\emptyset$, and $|G_1|\geq 3\leq |G_2|$. Without loss of generality, we may assume that $|G_1|\geq |G_2|$. Since G contains no $K_{2,t}$-minor, $t|G_1|\geq |G|$. Since G is 2-connected and $G_2-\{x,y\}\neq\emptyset$, $|G_1+xy|<|G|$ and $\mu(G;x,y)\geq \mu(G_1+xy;x,y)+1$. Clearly, G_1+xy contains no $K_{2,t}$-minors. Hence, by applying induction to G_1+xy, we conclude that G_1+xy contains an x-y path P of length at least $\frac{|G_1+xy|}{t^{\mu(G_1+xy;x,y)}}\geq \frac{t|G_1|}{t^{\mu}}\geq |G|/t^{\mu}$. Note that we can always choose P to be a path in G.

Case 2. $\{x,y\}$ is not a 2-cut of G.

Then y is contained in a unique block of $G-x$, say Y. Let X be a $(Y\cup\{x\})$-bridge of G with $|X|$ maximum, and let u be the unique vertex in $V(X)\cap V(Y)$. Since we are in Case 2, $u\neq y$. Since G contains no $K_{2,t}$-minor, there are at most $t-1$ $(Y\cup\{x\})$-bridges in G. So $t|X|\geq |G|-|Y|$. Note that $|X|<|G|>|Y|$. Next, we find an x-u path P_x in X and a u-y path P_y in Y.

If $|X|=2$, then let $P_x:=X$. In this case, it follows from the choice of X that all $(Y\cup\{x\})$-bridges of G are trivial. So $\ell(P_x)=1$ and $|G|=|Y|+1$. Now assume that $|X|\geq 3$. Then $X+xu$ is a 2-connected graph containing no $K_{2,t}$-minor. By applying induction to $X+xu$, we find an x-u path P_x in $X+xu$ of length at least $\frac{|X+xu|}{t^{\mu(X+xu;x,u)}}$. We can always choose P_x to be a path in G.

If $|Y|=2$, then let $P_y:=Y$. In this case, $\ell(P_y)\geq\frac{|Y|}{t^{\mu}}$. Now assume that $|Y|\geq 3$. By applying induction to Y,u,y, we find a u-y path P_y of length at least $\frac{|Y|}{t^{\mu(Y;u,y)}}\geq |Y|/t^{\mu}$. We can always choose P_y to be a path in G.

Let $P:=P_x\cup P_y$; then P is an x-y path in G and $\ell(P)=\ell(P_x)+\ell(P_y)$.

If $|X|=2$, then $|Y|\geq 3$ and $\ell(P)\geq 1+|Y|/t^{\mu}\geq |G|/t^{\mu}$. So assume that $|X|\geq 3$. Note that $\mu(X+xu;x,u)\leq\mu(G;x,y)$, and if G has at least two non-trivial $(Y\cup\{x\})$-bridges then $\mu(X+xu;x,u)+1\leq\mu(G;x,y)$. So $\ell(P_x)\geq\frac{|X+xu|}{t^{\mu(X+xu;x,u)}}\geq\frac{|G|-(|Y|-1)}{t^{\mu}}$. Therefore, $\ell(P)\geq\frac{|G|-|Y|+1}{t^{\mu}}+|Y|/t^{\mu}\geq |G|/t^{\mu}$. □

To prove (2.1), we also need to consider 2-connected graphs which are obtained from 3-connected graphs by contracting connected subgraphs.

(3.4) Lemma. *Let G be a 3-connected graph, let H be an induced subgraph of G such that $U:=G-V(H)$ is connected, and let $H^*:=G/U$. Then (1) H^* is a minor of G, and (2) if H is 2-connected then H^* is 3-connected.*

Proof. Since U is connected, H^* is a minor of G. Now assume that H is 2-connected. Then $|H|\geq 3$, and so, H^* is 2-connected (since G is 3-connected). Suppose for a contradiction that H^* is not 3-connected. Let T be a 2-cut of H^*, and let u denote the vertex of H^* resulting from the contraction of U. If $u\in T$, then $T-\{u\}$ is a 1-cut of H, contradicting the assumption that H is 2-connected. Thus, $u\notin T$. Hence H^*-T has a component, say D, not containing u. Then D is also a component of $G-T$, contradicting the assumption that G is 3-connected. □

4 Paths avoiding a vertex

Here we prove the following lemma which will serve as the induction step for proving (a) of Theorem (2.1).

(4.1) Lemma. *Suppose $n\geq 5$ and Theorem (2.1) holds for graphs with at most $n-1$ vertices. Then (a) of Theorem (2.1) holds for graphs with n vertices.*

Proof. Let $t \geq 3$ be an integer, let G be a 3-connected graph with no $K_{3,t}$-minor, and let $|G| = n$. Let $\{x, y, z\} \subseteq V(G)$, and assume that $\{zx, zy\} \subseteq E(G)$. For convenience, we let $b := 8t^{t+1}$, $r := \log_b 2$, and $H := G - z$.

Claim 1. We may assume that H is not 3-connected.

Suppose H is 3-connected. Since $|H| < n$, (2.1) holds for H. In particular, (b) of (2.1) holds for H. Therefore, H has an x-y path of length at least $|H|^r = (|G| - 1)^r > (\frac{|G|-1}{t^\mu})^r$ (because $\mu \geq 1$ and $t \geq 3$). Hence (a) of (2.1) holds for G.

Claim 2. We may assume that $\{x, y\}$ is not a 2-cut of H.

Suppose on the contrary that $\{x, y\}$ is a 2-cut of H. Let H_1, H_2, …, H_s be the non-trivial $\{x, y\}$-bridges of H. Note that $s \geq 2$. (See Figure 2.) Without loss of generality, we may assume that $|H_1| \geq |H_i|$ for all $i = 1, \ldots, s$. Since G is 3-connected, z has a neighbor in $H_i - \{x, y\}$ for each $1 \leq i \leq s$. Since G has no $K_{3,t}$-minor, $s \leq t - 1$. By the choice of H_1, $|H_1| \geq |H|/s \geq |H|/t$.

First, let us assume that H_1 is 2-connected. See Figure 2(a). Since G is 3-connected, $U := G - V(H_1)$ is connected. Let $H_1^* := G/U$ and let u denote the vertex of H_1^* resulting from the contraction of U. Note that $ux, uy \in E(H_1^*)$. Since H_1 is 2-connected, it follows from (3.4) that H_1^* is 3-connected and contains no $K_{3,t}$-minor. Let $\mu_1 := \mu(H_1^*; x, y, u)$. Recall that $\mu = \mu(G; x, y, z)$. Since z has a neighbor in $V(H_i) - \{x, y\}$ for each $2 \leq i \leq s$, $\mu \geq \mu_1 + (s - 1) \geq \mu_1 + 1$. Since $|H_1^*| < n$, (2.1) holds for H_1^*. In particular, (a) of (2.1) holds for H_1^*. So $H_1 = H_1^* - u$ contains an x-y path P such that

$$\ell(P) \geq (\frac{|H_1^*| - 1}{t^{\mu_1}})^r \geq (\frac{|H_1|}{t^{\mu_1}})^r \geq (\frac{|H|}{t^\mu})^r = (\frac{|G| - 1}{t^\mu})^r.$$

Hence (a) of (2.1) holds for G.

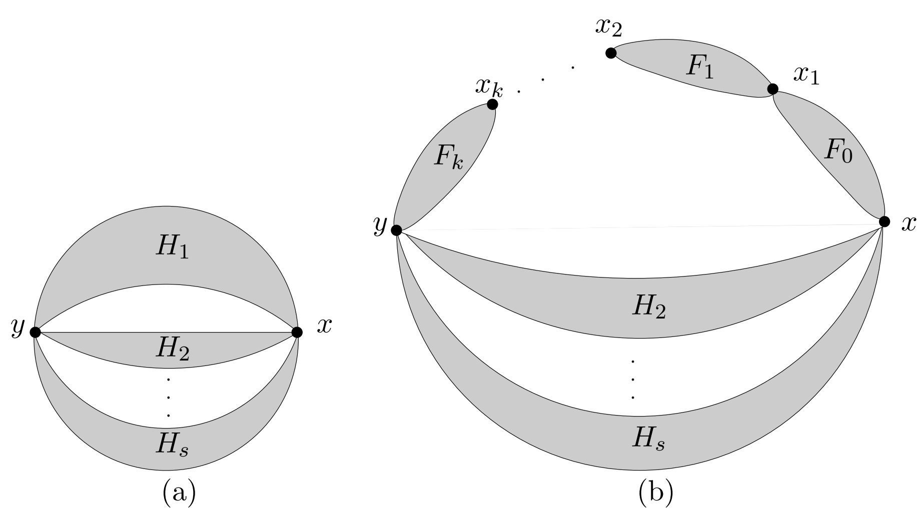

Figure 2: Two cases in the proof of Claim 2.

Now assume that H_1 is not 2-connected. Since H is 2-connected, the blocks of H_1 can be labeled as $F_0, \ldots, F_k$ and the cutvertices of H_1 can be labeled as $x_1, \ldots, x_k$ such that (i) for each $0 \le i \le k-1$, $V(F_i) \cap V(F_{i+1}) = \{x_{i+1}\}$, (ii) for any $1 \le i, j \le k-1$ with $|i-j| \ge 2$, $V(F_i) \cap V(F_j) = \emptyset$, and (iii) $x_0 := x \in V(F_0) - \{x_1\}$ and $x_{k+1} := y \in V(F_k) - \{x_k\}$. Since G is 3-connected, $W_i := G - V(F_i)$ is connected for each $0 \le i \le k$. Let $F_i^* := G/W_i$ and let w_i denote the vertex of F_i^* resulting from the contraction of W_i. Then $w_i x_i, w_i x_{i+1} \in E(F_i^*)$. Let $\mu_i := \mu(F_i^*, x_i, x_{i+1}, w_i)$ if $|F_i| \ge 3$, and let $\mu_i = 1$ if $|F_i| = 2$. Then $\mu \ge \mu_i + (s-1) \ge \mu_1 + 1$. If $|F_i| = 2$ then let $P_i := F_i$, and it is easy to see that $\ell(P_i) = 1 \ge (\frac{|F_i|}{t^{\mu_i}})^r$ (because $t \ge 3$ and $\mu_i = 1$ in this case). If $|F_i| \ge 3$, then it follows from (3.4) that F_i^* is 3-connected and has no $K_{3,t}$-minor. Since $|F_i^*| < n$, (2.1) holds for F_i^*. In particular, (a) of (2.1) holds for F_i^*. So $F_i = F_i^* - w_i$ has an x_i-x_{i+1} path P_i such that $\ell(P_i) \ge (\frac{|F_i^*|-1}{t^{\mu_i}})^r = (\frac{|F_i|}{t^{\mu_i}})^r$. Let $P := \bigcup_{i=0}^k P_i$; then P is an x-y path in H_1. Since $\sum_{0 \le i \le k}(|F_i| - 1) = |H_1| - 1$ and by (2.4),

$$\ell(P) \ge \sum_{i=0}^{k} (\frac{|F_i|}{t^{\mu_i}})^r \ge \sum_{i=0}^{k} (\frac{|F_i|}{t^{\mu-1}})^r \ge (\frac{|H_1|}{t^{\mu-1}})^r \ge (\frac{|H|}{t^{\mu}})^r = (\frac{|G|-1}{t^{\mu}})^r.$$

So (a) of (2.1) holds for G. This completes the proof of Claim 2.

Let $\mathcal{U} := \{\{u_1, v_1\}, \ldots, \{u_k, v_k\}\}$ denote a maximal collection of 2-cuts of H satisfying the following three properties:

(C1) for each $1 \le i \le k$, $\{x, y\}$ is contained in a $\{u_i, v_i\}$-bridge B_i of H;

(C2) for each $1 \le i \le k$, for any 2-cut T of H with $T \ne \{u_i, v_i\}$, and for any T-bridge B of H containing $\{x, y\}$, $B \not\subseteq B_i$; and

(C3) $(H - V(B_i)) \cap (H - V(B_j)) = \emptyset$ for $1 \le i \ne j \le k$.

Let $X := (\bigcap_{i=1}^k B_i) + \{xy, u_i v_i : i = 1, \ldots, k\}$ and let $G_i := (G - (V(B_i) - \{u_i, v_i\})) + \{zu_i, zv_i, u_i v_i\}$.

Claim 3. (1) X is a minor of G, (2) either X is a triangle or X is 3-connected, and (3) for each $1 \le i \le k$, G_i is a minor of G and G_i is 3-connected.

We may view X as obtained from G by contracting connected subgraphs $G_i - \{z, u_i\}$ to v_i and by contracting z to x (because $xz, yz \in E(G)$). So X is a minor of G.

Clearly, X is 2-connected. If $|X| = 3$ then X is a triangle. Now assume $|X| \ge 4$. Suppose that T is a 2-cut of X, and let B_T denote a non-trivial T-bridge of X not containing $\{x, y\}$. Then T may be viewed as a 2-cut of H, and by (C2), $E(B_T) \cap \{u_i v_i : i = 1, \ldots, k\} = \emptyset$. Hence, $\mathcal{U} \cup \{T\}$ contradicts the maximality of $\mathcal{U}$.

Since H is 2-connected, $H - (V(G_i) - \{u_i, v_i\})$ has disjoint paths P_u, P_v from $\{x, y\}$ to $\{u_i, v_i\}$, with $u_i \in V(P_u)$ and $v_i \in V(P_v)$. So $H - (V(G_i) - \{u_i, v_i\})$ is the disjoint union of connected graphs P'_u, P'_v such that $P_u \subseteq P'_u$ and $P_v \subseteq P'_v$. Since $H - (V(G_i) - \{u_i, v_i\})$ is connected, there is an edge of G between P'_u and P'_v. Therefore, we may view G_i as obtained from G by contracting P'_u and P'_v to u_i and

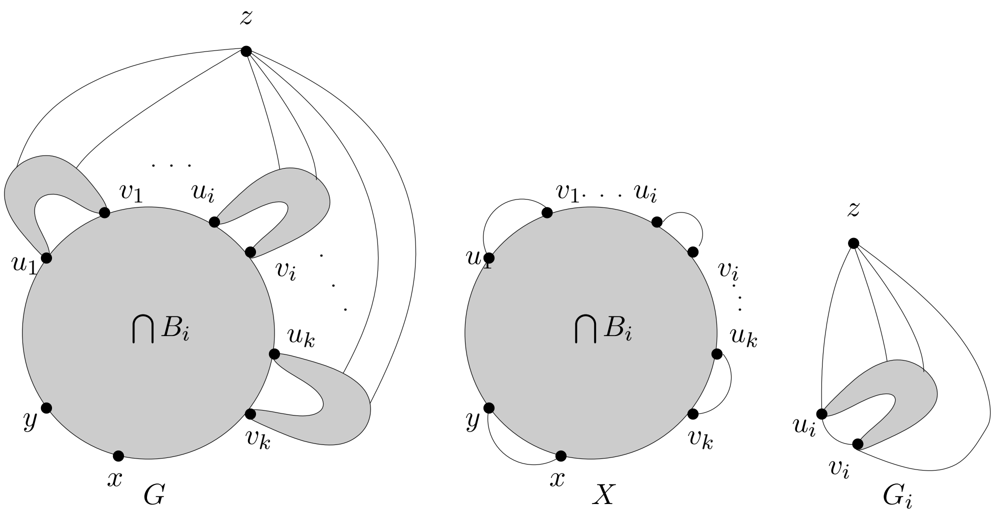

Figure 3: The description of $\mathcal{U} = \{\{u_1, v_1\}, \dots, \{u_k, v_k\}\}$, X and G_i.

v_i, respectively. So G_i is a minor of G. Clearly G_i is 3-connected. This completes the proof of Claim 3.

Claim 4. We may assume that $|X| < \frac{|H|}{t}$.

Suppose $|X| \geq \frac{|H|}{t}$. By Claim 3, either X is a triangle or X is 3-connected. If X is a triangle, then let P_X be the x-y path in X of length 2. Then $\ell(P_X) = 2 > 3^r = |X|^r$. Now assume that X is 3-connected. By (1) of Claim 3, X is a minor of G. So X contains no $K_{3,t}$-minor. Since $|X| < n$, (2.1) holds for X. In particular, (b) of (2.1) holds for X. Recall that $xy \in E(X)$. Hence, X contains an x-y path P_X such that $\ell(P_X) \geq |X|^r$.

In any case, X contains an x-y path P_X such that

$$\ell(P_X) \geq |X|^r \geq (\frac{|H|}{t})^r \geq (\frac{|H|}{t^\mu})^r \geq (\frac{|G|-1}{t^\mu})^r.$$

Clearly, P_X can be extended to the desired x-y path P in H by replacing each edge u_iv_i in $E(P_X)$ with a u_i-v_i path in $G_i - z$ of length at least 2. So we have Claim 4.

Next, we define $\omega : E(X) \to \mathbb{R}^+$ as follows: $\omega(e) = 0$ if $e \in E(X) - \{u_iv_i : i = 1, \dots, k\}$, and $\omega(u_iv_i) = |G_i| - 3$ for $i = 1, \dots, k$. Let $S := \{u_iv_i : i = 1, \dots, k\} = \{e \in E(X) : \omega(e) > 0\}$. Since G contains no $K_{3,t}$-minor, X contains no S-link of size t between x and y. By Theorem (3.1), we have the following.

Claim 5. X contains an x-y path P_X such that

$$\sum_{e \in E(P_X)} t^{\alpha(P_X;e)}\omega(e) \geq \omega(S) = |H| - |X|,$$

where $\alpha(P_X; e)$ is the greatest integer m such that X has an $(S; P_X)$-ladder with top e and m rungs.

Let $\mu_i := \mu(G_i; u_i, v_i, z)$ for each $1 \leq i \leq k$. From an $(S; P_X)$-ladder, if we replace each rung u_iv_i by G_i, we see that G has a $K_{3,p}$-minor rooted at $\{x, y, z\}$, where $p = \mu_i + \alpha(P_X; u_iv_i)$. So we have

Claim 6. $t - 1 \geq \mu \geq \mu_i + \alpha(P_X; u_iv_i)$ for each $1 \leq i \leq k$.

By Claims 4 and 5, $\omega(S) > 0$. Hence $E(P_X) \cap S \neq \emptyset$. Without loss of generality, we may assume that $u_1v_1 \in E(P_X)$ and $\frac{\omega(u_1v_1)}{t^{\mu_1}} \geq \frac{\omega(u_iv_i)}{t^{\mu_i}}$ for all $u_iv_i \in E(P_X)$. We distinguish two cases.

Case 1. $|X| \geq \sum_{e \in E(P_X)-\{u_1v_1\}} t^{\alpha(P_X;e)}\omega(e)$.

By Claim 5, we have

$$\begin{aligned}
\omega(u_1v_1) &\geq \frac{1}{t^{\alpha(P_X;u_1v_1)}}(|H| - |X| - \sum_{e \in E(P_X)-\{u_1v_1\}} t^{\alpha(P_X;e)}\omega(e)) \\
&\geq \frac{1}{t^{\alpha(P_X;u_1v_1)}}(|H| - 2|X|) \quad \text{(by Case 1)} \\
&> \frac{1}{t^{\alpha(P_X;u_1v_1)}}(|H| - \frac{2|H|}{t}) \quad \text{(by Claim 4).}
\end{aligned}$$

Therefore, since $\alpha(P_X; u_1v_1) \leq t - 1$, we have

(1) $\omega(u_1v_1) \geq \frac{t-2}{t^{\alpha(P_X;u_1v_1)+1}}|H| \geq \frac{t-2}{t^t}|H|$.

By Claim 3, G_1 is 3-connected and has no $K_{3,t}$-minor. Since $|G_1| < n$, (2.1) holds for G_1. In particular, (a) of (2.1) holds. Recall that $\{zu_1, zv_1\} \subseteq E(G_1)$. Hence, $G_1 - z$ contains a u_1-v_1 path P_1 (other than u_1v_1 so that $P_1 \subseteq G$) such that

(2) $\ell(P_1) \geq (\frac{|G_1|-1}{t^{\mu_1}})^r \geq (\frac{\omega(u_1v_1)}{t^{\mu_1}})^r$.

By Claim 3, either X is a triangle or X is a 3-connected minor of G. If X is a triangle, then $u_1v_1 \neq xy$ (since $\{x, y\}$ is not a 2-cut of H), and so, X has an x-y path Q_X of length 2 and through u_1v_1. Because $t \geq 3$, $\ell(Q_X) = 2 > 3^r + 1 > (\frac{|X|}{t^t})^r + 1$. If X is a 3-connected minor of G, then (2.1) holds for X. In particular, (c) of (2.1) holds for X. Recall that $\{xy, u_1v_1\} \subseteq E(X)$. Hence X has an x-y path Q_X through u_1v_1 such that $\ell(Q_X) \geq (\frac{|X|}{t^t})^r + 1$.

In any case, X has an x-y path Q_X through u_1v_1 such that

(3) $\ell(Q_X) \geq (\frac{|X|}{t^t})^r + 1$.

Let $P := (Q_X - u_1v_1) \cup P_1$. Clearly, P is an x-y path in H.

If $\frac{\omega(u_1v_1)}{t^{\mu_1}} \leq \frac{|X|}{t^t}$, then

$$\begin{aligned}
\ell(P) &\geq \ell(P_1) + (\ell(Q_X) - 1) \\
&\geq (\frac{\omega(u_1v_1)}{t^{\mu_1}})^r + (\frac{|X|}{t^t})^r \quad \text{(by (2) and (3))} \\
&\geq ((b-1)\frac{\omega(u_1v_1)}{t^{\mu_1}} + \frac{|X|}{t^t})^r \quad \text{(by (2.3) and since } \tfrac{\omega(u_1v_1)}{t^{\mu_1}} \leq \tfrac{|X|}{t^t}\text{)} \\
&\geq (\frac{3\omega(u_1v_1)}{t^{\mu_1}})^r \quad \text{(because } b \geq 4 \text{ and } \tfrac{\omega(u_1v_1)}{t^{\mu_1}} \leq \tfrac{|X|}{t^t}\text{)}
\end{aligned}$$

$$
\begin{aligned}
&\geq (\frac{3(t-2)|H|}{t^{\alpha(P_X;u_1v_1)}tt^{\mu_1}})^r \quad \text{(by (1))}\\
&\geq (\frac{|H|}{t^{\alpha(P_X;u_1v_1)+\mu_1}})^r \quad \text{(since } t \geq 3\text{)}\\
&\geq (\frac{|H|}{t^{\mu}})^r \quad \text{(by Claim 6)}\\
&= (\frac{|G|-1}{t^{\mu}})^r.
\end{aligned}
$$

Hence (a) of (2.1) holds for G.

So we may assume $\frac{\omega(u_1v_1)}{t^{\mu_1}} \geq \frac{|X|}{t^t}$. Since we are in Case 1, $t^{\alpha(P_X;u_1v_1)}\omega(u_1v_1) + |X| \geq \sum_{e\in E(P_X)} t^{\alpha(P_X;e)}\omega(e)$. By Claim 5, $t^{\alpha(P_X;u_1v_1)}\omega(u_1v_1) + |X| \geq |H| - |X|$. Thus, we have

(4) $|X| \geq \frac{1}{2}(|H| - t^{\alpha(P_X;u_1v_1)}\omega(u_1v_1))$.

Then

$$
\begin{aligned}
\ell(P) &\geq \ell(P_1) + (\ell(Q_X) - 1)\\
&\geq (\frac{\omega(u_1v_1)}{t^{\mu_1}})^r + (\frac{|X|}{t^t})^r \quad \text{(by (2) and (3))}\\
&\geq (\frac{\omega(u_1v_1)}{t^{\mu_1}} + (b-1)(\frac{|X|}{t^t}))^r \quad \text{(Since } \tfrac{\omega(u_1v_1)}{t^{\mu_1}} \geq \tfrac{|X|}{t^t} \text{ and by (2.3))}\\
&\geq (\frac{\omega(u_1v_1)}{t^{\mu_1}} + \frac{b-1}{2t^t}(|H| - t^{\alpha(P_X;u_1v_1)}\omega(u_1v_1)))^r \quad \text{(by (4))}\\
&\geq (\frac{\omega(u_1v_1)}{t^{\mu_1}} + \frac{1}{t^{\mu_1+\alpha(P_X;u_1v_1)}}(|H| - t^{\alpha(P_X;u_1v_1)}\omega(u_1v_1)))^r \quad \text{(by Claim 6 and } b = 8t^{t+1}\text{)}\\
&\geq (\frac{|H|}{t^{\mu_1+\alpha(P_X;u_1v_1)}})^r\\
&\geq (\frac{|H|}{t^{\mu}})^r \quad \text{(by Claim 6)}\\
&= (\frac{|G|-1}{t^{\mu}})^r.
\end{aligned}
$$

Hence (a) of (2.1) holds for G.

Case 2. $|X| \leq \sum_{e\in E(P_X)-\{u_1v_1\}} t^{\alpha(P_X;e)}\omega(e)$.

For each $1 \leq i \leq k$, $|G_i| < n$. By Claim 3, G_i is 3-connected and contains no $K_{3,t}$-minor. So (2.1) holds for G_i. In particular, (a) of (2.1) holds for G_i. Recall that $\{zu_i, zv_i\} \subset E(G_i)$. Hence $G_i - z$ contains a u_i-v_i path P_i such that $\ell(P_i) \geq (\frac{|G_i|-1}{t^{\mu_i}})^r \geq (\frac{\omega(u_iv_i)}{t^{\mu_i}})^r$. Let $P := (P_X - S) \cup (\bigcup_{u_iv_i\in E(P_X)} P_i)$. Clearly, P is an x-y path in H and

$$
\begin{aligned}
\ell(P) &\geq \sum_{u_iv_i\in E(P_X)} \ell(P_i)\\
&\geq \sum_{u_iv_i\in E(P_X)} (\frac{\omega(u_iv_i)}{t^{\mu_i}})^r
\end{aligned}
$$

$$
\begin{aligned}
&= (\frac{\omega(u_1v_1)}{t^{\mu_1}})^r + \sum_{u_iv_i \in E(P_X)-\{u_1v_1\}} (\frac{\omega(u_iv_i)}{t^{\mu_i}})^r \\
&\geq (\frac{\omega(u_1v_1)}{t^{\mu_1}} + (b-1) \sum_{u_iv_i \in E(P_X)-\{u_1v_1\}} \frac{\omega(u_iv_i)}{t^{\mu_i}})^r.
\end{aligned}
$$

The final inequality above follows from the assumption that $\frac{\omega(u_1v_1)}{t^{\mu_1}} \geq \frac{\omega(u_iv_i)}{t^{\mu_i}}$ for all $u_iv_i \in E(P_X)$ (see before Case 1) and by applying Lemma (2.4). By Claim 6, $t^{\mu_i+\alpha(P_X;u_iv_i)} \leq t^{\mu}$. Hence,

$$
\begin{aligned}
\ell(P) &\geq (\frac{\omega(u_1v_1)}{t^{\mu_1}} + \frac{b-1}{t^{\mu}} \sum_{u_iv_i \in E(P_X)-\{u_1v_1\}} t^{\alpha(P_X;u_iv_i)}\omega(u_iv_i))^r \\
&\geq (\frac{\omega(u_1v_1)}{t^{\mu_1}} + \frac{b-1}{2t^{\mu}}(|X| + \sum_{u_iv_i \in E(P_X)-\{u_1v_1\}} t^{\alpha(P_X;u_iv_i)}\omega(u_iv_i)))^r \quad \text{(by Case 2)} \\
&\geq (\frac{\omega(u_1v_1)}{t^{\mu_1}} + \frac{b-1}{2t^{\mu}}(|H| - t^{\alpha(P_X;u_1v_1)}\omega(u_1v_1))^r \quad \text{(by Claim 5)} \\
&\geq (\frac{\omega(u_1v_1)}{t^{\mu_1}} + \frac{1}{t^{\mu_1+\alpha(P_X;u_1v_1)}}(|H| - t^{\alpha(P_X;u_1v_1)}\omega(u_1v_1))^r \ (\text{since } b-1 \geq \tfrac{2t^{\mu}}{t^{\mu_1+\alpha(P_X;u_1v_1)}}) \\
&= (\frac{|H|}{t^{\mu_1+\alpha(P_X;u_1v_1)}})^r \\
&\geq (\frac{|H|}{t^{\mu}})^r \quad \text{(by Claim 6)}.
\end{aligned}
$$

Hence (a) of (2.1) holds for G. □

5 Paths in 3-connected graphs

We now prove the following result which will serve as the induction step for part (b) in the proof of Theorem (2.1).

(5.1) Lemma. *Suppose $n \geq 5$ and Theorem (2.1) holds for graphs with at most $n-1$ vertices. Then (b) of Theorem (2.1) holds for graphs with n vertices.*

Proof. Let $t \geq 3$ be an integer, let G be a 3-connected graph with no $K_{3,t}$-minor, and let $|G| = n$. Let $xy \in E(G)$. For convenience, we let $b := 8t^{t+1}$ and $r := \log_b 2$. It is easy to see that (b) of (2.1) holds when $n \leq 8t^{t+1}$. So we may assume that $n \geq 8t^{t+1}$. Therefore, $\frac{n}{4(t-1)t^{t-1}} > 1$.

To find the desired x-y path in (b) of (2.1), we start from x and "extend" our path to y. At a certain point, the remaining graph is no longer 3-connected, and we are forced to choose one out of several parts of the graph. While our choice may be "good" at certain stage, it may become undesirable at some later stage. In that case, we need to come back and modify our choice. This is a very complicated process, and the following concept of "magic minor" will help us explain things in a precise and concise way.

Let H_0 be an induced subgraph of G and let $x_0, y_0 \in V(H_0)$ be distinct such that $H_0 + x_0y_0$ is 2-connected. We say that (H_0, x_0, y_0) is a *magic minor* of (G, x, y) if the following conditions hold:

(M1) $G - (V(H_0) - \{x_0, y_0\})$ contains vertex disjoint paths X_0, Y_0 from x, y to x_0, y_0, respectively;

(M2) $U_0 := G - V(H_0)$ is connected and H_0^* is 3-connected, where $H_0^* := G/U_0$ if H_0 is 2-connected and $H_0^* := (G/U_0) + x_0y_0$ otherwise;

(M3) U_0 is the disjoint union of Λ_0 and Ω_0 such that $V(X_0) \subseteq \Lambda_0 \cup \{x_0\}$, $V(Y_0) \subseteq \Omega_0 \cup \{y_0\}$, both $G[\Lambda_0]$ and $G[\Omega_0]$ are connected, and $N(V(H_0) - y_0) \subseteq \Lambda_0 \cup \{y_0\}$; and

(M4) $|H_0| \geq n/2$ and, for any $a \geq \frac{n}{2t^t}$ with $a \geq 1$,

$$a^r + \ell(X_0) + \ell(Y_0) \geq (a + 4(n - |H_0|))^r.$$

We say that (H_0, x_0, y_0) is a *minor* of (G, x, y) if (M1), (M2), and (M3) hold.

Let $\mathcal{M}$ denote the set of all magic minors of (G, x, y). Then

(1) $\mathcal{M} \neq \emptyset$ and we may choose $(H_0, x_0, y_0) \in \mathcal{M}$ such that $|H_0|$ is minimum.

Let $H_0 := G - x$, let $y_0 := y$, and let x_0 be a neighbor of x other than y. Then $G - (V(H_0) - \{x_0, y_0\})$ consists of vertex disjoint paths X_0 and Y_0 with $V(X_0) = \{x, x_0\}$ and $V(Y_0) = \{y_0\}$. So (M1) holds. Clearly, $U_0 := G - V(H_0)$ consists of only one vertex (namely x), and so, is connected. Since G is 3-connected, H_0 is 2-connected. So $H_0^* := G/U_0 = G$ is 3-connected, and (M2) holds. Set $\Lambda_0 = \{x\}$ and $\Omega_0 = \emptyset$. It is easy to see that (M3) holds. Obviously, $|H_0| = n - 1 \geq n/2$. Also, for any $a \geq \frac{n}{2t^t}$ with $a \geq 1$, $a^r + \ell(X_0) + \ell(Y_0) = a^r + 1 \geq (a+b)^r \geq (a+4)^r$ by Lemma (2.2) (because $a \geq 1$ and $b \geq 4$). Since $|H_0| = n - 1$, $(a + 4(n - |H_0|))^r = (a+4)^r$. Hence (M4) also holds. Therefore $(H_0, x_0, y_0) \in \mathcal{M}$, and so, we have (1).

Next we recursively define minors of (G, x, y) starting from (H_0, x_0, y_0). Suppose we have already defined a minor (H_i, x_i, y_i) (for some $i \geq 0$) of (G, x, y, z). That is, (m0) H_i is an induced subgraph of G and $H_i + x_iy_i$ is 2-connected, (m1) $G - (V(H_i) - \{x_i, y_i\})$ contains vertex disjoint paths from x, y to x_i, y_i, respectively, (m2) $U_i := G - V(H_i)$ is connected and H_i^* is 3-connected, where $H_i^* := G/U_i$ if H_i is 2-connected and $H_i^* := (G/U_i) + x_iy_i$ otherwise, (m3) U_i is the disjoint union of Λ_i and Ω_i such that both $G[\Lambda_i]$ and $G[\Omega_i]$ are connected, and $N(V(H_i) - y_i) \subseteq \Lambda_i \cup \{y_i\}$, and (m4) $|H_i| \geq n/2$.

According to the rules (R1), (R2) and (R3) below, we define the following: $(H_{i+1}, x_{i+1}, y_{i+1})$, U_{i+1}, u_{i+1}, and H_{i+1}^*; $(F_{i+1}, x'_{i+1}, y'_{i+1})$, W_{i+1}, w_{i+1}, and F_{i+1}^*; $(H_{i+1,j}, x_{i+1}, y_{i+1,j})$, $U_{i+1,j}, u_{i+1,j}$, and $H_{i+1,j}^*$; $(F_{i+1,j}, x'_{i+1}, y'_{i+1,j})$; and Λ_{i+1} and Ω_{i+1}. See Figure 4 for an illustration.

(R1) Suppose $\{x_i, y_i\}$ is a 2-cut of H_i. See Figure 4(a). Let B_i denote an $\{x_i, y_i\}$-bridge of H_i with the maximum number of vertices, and let $H_{i+1} := G[V(B_i)]$.

Let $H_{i+1,j}$, $j = 1, \dots, s_{i+1}$, denote the non-trivial $\{x_i, y_i\}$-bridges of H_i different from B_i. Let $x_{i+1} = x_i$, $y_{i+1} = y_i$, and $y_{i+1,j} = y_i$ for $1 \le j \le s_{i+1}$. Set $\Lambda_{i+1} := \Lambda_i \cup (V(H_i) - V(H_{i+1}))$ and $\Omega_{i+1} := \Omega_i$. In this case, F_{i+1} and $F_{i+1,j} = \emptyset$ are not defined.

(R2) Suppose $\{x_i, y_i\}$ is not a 2-cut of H_i. See Figure 4(b) and Figure 4(c). Let B_i denote the unique block of $H_i - x_i$ containing y_i. Let $B_{i,x}$ be a $(B_i \cup \{x_i\})$-bridge of H_i with the maximum number of vertices, and let $z_i \in V(B_i) \cap V(B_{i,x})$. Let $B_{i,y}$ be a maximum $\{y_i, z_i\}$-bridge of B_i. (Possibly $B_{i,y} = B_i$.) If $|B_{i,x}| \ge |B_{i,y}|$, then let $H_{i+1} := G[V(B_{i,x})]$, $x_{i+1} := x_i$, and $y_{i+1} := z_i$; let $H_{i+1,j}$, $j = 1, \dots, s_{i+1}$, denote the non-trivial $(B_i \cup \{x_i\})$-bridges of H_i different from $B_{i,x}$, let $y_{i+1,j}$ denote the vertex in $V(H_{i+1,j}) \cap V(B_i)$; let $F_{i+1} := G[V(B_{i,y})]$, $x'_{i+1} := z_i$, and $y'_{i+1} := y_i$; let $F_{i+1,j}$, $j = 1, \dots, t_{i+1}$, denote the non-trivial $\{y_i, z_i\}$-bridges of B_i different from $B_{i,y}$, and let $y'_{i+1,j} = y_i$. Set $\Lambda_{i+1} := \Lambda_i \cup (V(H_i) - (V(H_{i+1}) \cup V(F_{i+1})))$ and $\Omega_{i+1} := \Omega_i \cup V(F_{i+1}) - \{y_{i+1}\}$. See Figure 4(b). If $|B_{i,x}| < |B_{i,y}|$, then let $H_{i+1} := G[V(B_{i,y})]$, $x_{i+1} := z_i$, and $y_{i+1} := y_i$; let $H_{i+1,j}$, $j = 1, \dots, s_{i+1}$, denote the non-trivial $\{z_i, y_i\}$-bridges of B_i different from $B_{i,y}$, and let $y_{i+1,j} = y_i$; let $F_{i+1} := G[V(B_{i,x})]$, $x'_{i+1} := x_i$, and $y'_{i+1} := z_i$; let $F_{i+1,j}$, $j = 1, \dots, t_{i+1}$, denote the non-trivial $(B_i \cup \{x_i\})$-bridges of H_i different from $B_{i,x}$, and let $y'_{i+1,j}$ be the vertex in $V(F_{i+1,j}) \cap V(B_i)$. Set $\Lambda_{i+1} := \Lambda_i \cup (V(H_i) - V(H_{i+1}))$ and $\Omega_{i+1} := \Omega_i$. See Figure 4(c).

(R3) Let $U_{i+1} := G - V(H_{i+1})$, let $H^*_{i+1} := G/U_{i+1}$ if H_{i+1} is 2-connected and let $H^*_{i+1} := (G/U_{i+1}) + x_{i+1}y_{i+1}$ otherwise, and let u_{i+1} denote the vertex of H^*_{i+1} resulting from the contraction of U_{i+1}. Let $U_{i+1,j} := G - V(H_{i+1,j})$, let $H^*_{i+1,j} := G/U_{i+1,j}$ if $H_{i+1,j}$ is 2-connected and let $H^*_{i+1,j} := (G/U_{i+1,j}) + x_{i+1}y_{i+1,j}$ otherwise, and let $u_{i+1,j}$ denote the vertex of $H^*_{i+1,j}$ resulting from the contraction of $U_{i+1,j}$. Let $W_{i+1} := G - V(F_{i+1})$, let $F^*_{i+1} := G/W_{i+1}$ if F_{i+1} is 2-connected and let $F^*_{i+1} := (G/W_{i+1}) + x'_{i+1}y'_{i+1}$ otherwise, and let w_{i+1} denote the vertex of F^*_{i+1} resulting from the contraction of W_{i+1}.

Next we derive some useful properties (assuming the graphs involved are defined).

(2) U_{i+1}, $U_{i+1,j}$ and W_{i+1} are connected subgraphs of G, H_{i+1} and F_{i+1} are induced subgraphs of G, $H_{i+1}+x_{i+1}y_{i+1}$ is 2-connected, H^*_{i+1}, $H^*_{i+1,j}$ and F^*_{i+1} are 3-connected minors of G, $\{u_{i+1}x_{i+1}, u_{i+1}y_{i+1}\} \subseteq E(H^*_{i+1})$, $\{y_{i+1,j}x_{i+1}, y_{i+1,j}u_{i+1,j}\} \subseteq E(H^*_{i+1,j})$, and $\{w_{i+1}x'_{i+1}, w_{i+1}y'_{i+1}\} \subseteq E(F^*_{i+1})$. Moreover, U_{i+1} is the disjoint union of Λ_{i+1} and Ω_{i+1}, both $G[\Lambda_{i+1}]$ and $G[\Omega_{i+1}]$ are connected, and $N(V(H_{i+1}) - y_{i+1}) \subseteq \Lambda_{i+1} \cup \{y_{i+1}\}$.

Since G is 3-connected and U_i is connected (see (m2)), it follows from (R1), (R2), and (R3) that U_{i+1}, $U_{i+1,j}$, and W_{i+1} are connected. Since H_i is an induced subgraph of G (by (m0)), it follows from (R1) and (R2) that H_{i+1} and F_{i+1} are induced subgraphs of G. Since $H_i + x_iy_i$ is 2-connected (by (m0)) and $|H_i| \ge n/2$ (by (m4)), we see that $|H_{i+1}| \ge 3$ and $H_{i+1} + x_{i+1}y_{i+1}$ is 2-connected. If H_{i+1} is 2-connected then H^*_i is 3-connected by (3.4). If H_{i+1} is not 2-connected then, since

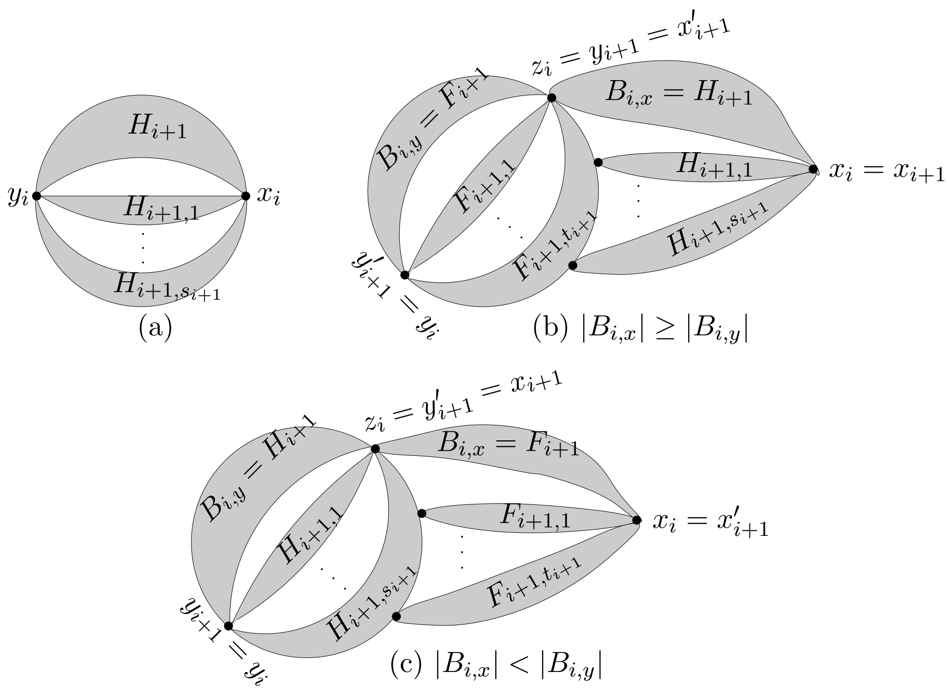

Figure 4: The descriptions of (R1) and (R2).

$H_{i+1} + x_{i+1}y_{i+1}$ is 2-connected, $H^*_{i+1} = (G/U_i) + x_{i+1}y_{i+1} = (G + x_{i+1}y_{i+1})/U_i$ is 3-connected. Similarly, we can show that F^*_{i+1} (if $F_{i+1} \neq \emptyset$) and $H^*_{i+1,j}$ are 3-connected. The properties enjoyed by Λ_{i+1} and Ω_{i+1} follow instantly from (m3) and the construction of Λ_{i+1} and Ω_{i+1}. The rest of (2) follows from (R3).

From (R1) and (R2), we have (3) and (4) below.

(3) $H_i - (V(H_{i+1}) - \{x_{i+1}, y_{i+1}\})$ contains vertex disjoint paths from x_{i+1}, y_{i+1} to x_i, y_i, respectively, and $H_i - (V(H_{i+1,j}) - \{x_{i+1}, y_{i+1,j}\})$ contains vertex disjoint paths from $x_{i+1}, y_{i+1,j}$ to x_i, y_i, respectively. Also if F_{i+1} is defined, then $H_i - (V(F_{i+1}) - \{x'_{i+1}, y'_{i+1}\})$ contains vertex disjoint paths from x'_{i+1}, y'_{i+1} to x_i, y_i, respectively.

(4) H_{i+1} and F_{i+1} intersect at $z_i \in \{x_{i+1}, y_{i+1}\}$, $|V(H_{i+1,j}) \cap V(F_{i+1})| \leq 1$ and $V(H_{i+1,j}) \cap V(F_{i+1}) \subseteq \{x_{i+1}, y_{i+1,j}\}$, and $H_{i+1} - \{x_{i+1}, y_{i+1}\}$ and $H_{i+1,j} - \{x_{i+1}, y_{i+1,j}\}$, $j = 1, \ldots, s_{i+1}$, are disjoint.

By (m0), H_i is an induced subgraph of G. Since G is 3-connected and has no $K_{3,t}$-minor, $s_{i+1} \leq t - 2$ and $t_{i+1} \leq t - 2$. Because $|H_{i+1}| \geq |H_{i+1,j}|$ for $j = 1, \ldots, s_{i+1}$ and $|F_{i+1}| \geq |F_{i+1,j}|$ for $j = 1, \ldots, t_{i+1}$, it follows from (R1) and (R2) that

(5) $(t-1)|H_{i+1}| + (t-1)|F_{i+1}| \geq |H_i|$.

Now suppose $\{(H_i, x_i, y_i) : i = 0, \ldots, k\}$ is a maximal sequence constructed recursively starting from (H_0, x_0, y_0) by rules (R1) and (R2), subject to the following two conditions:

(S1) $|H_k| \geq \frac{n}{2}$, and

(S2) for each $1 \leq s \leq k$,

$$\sum_{i=1}^{s} \sum_{j=1}^{s_i} |H_{i,j}| \leq \frac{1}{2}(n - |H_s|).$$

By (R1), (R2) and (R3), we can construct from (H_k, x_k, y_k) the following: $(H_{k+1}, x_{k+1}, y_{k+1})$, $(H_{k+1,j}, x_{k+1}, y_{k+1,j})$ for $j = 1, \ldots, s_{k+1}$, $(F_{k+1}, x'_{k+1}, y'_{k+1})$, $U_{k+1}, W_{k+1}, H^*_{k+1}$, F^*_{k+1}, u_{k+1}, w_{k+1}, Λ_{k+1}, and Ω_{k+1}.

By (2) and (3) and since (H_0, x_0, y_0) is a minor of (G, x, y), (H_i, x_i, y_i) is a minor of (G, x, y) for all $1 \leq i \leq k+1$. Also (2)-(5) hold for $i = 1, \ldots, k$.

Note that, for each $1 \leq s \leq k+1$, the vertices of G outside H_s are either outside H_0, or in $H_{i,j}$ for some $1 \leq i \leq s$ and $1 \leq j \leq s_i$, or in F_i for some $1 \leq i \leq s$, or in $F_{i,j}$ for some $1 \leq i \leq s$ and $1 \leq j \leq t_i$. Also note that $n - |H_0|$ is the number of vertices of G outside H_0, $\sum_{i=1}^{s} \sum_{j=1}^{s_i} |H_{i,j}|$ is the number of vertices of G in $H_{i,j}$ for $1 \leq i \leq s$ and $1 \leq j \leq s_i$, $(t-1)\sum_{1 \leq i \leq s} |F_i|$ is at least the number of vertices of G in F_i or $F_{i,j}$ for $1 \leq i \leq s$ and $1 \leq j \leq t_i$, and $n - |H_s|$ is the number of vertices of G outside H_s. Hence, we have

(6) For each $1 \leq s \leq k+1$,

$$\sum_{i=1}^{s} \sum_{j=1}^{s_i} |H_{i,j}| + (t-1) \sum_{i=1}^{s} |F_i| + (n - |H_0|) \geq n - |H_s|.$$

Since $|H_{k+1}| < |H_0|$ and by (1), $(H_{k+1}, x_{k+1}, y_{k+1})$ is not a magic minor of (G, x, y). By (2) and (3), $(H_{k+1}, x_{k+1}, y_{k+1})$ is a minor of (G, x, y). Thus the maximality of k implies that either (S1) or (S2) fails with respect to $(H_{k+1}, x_{k+1}, y_{k+1})$; that is,

(7) $|H_{k+1}| < n/2$, or $|H_{k+1}| \geq n/2$ and $\sum_{i=1}^{k+1} \sum_{j=1}^{s_i} |H_{i,j}| > \frac{1}{2}(n - |H_{k+1}|)$.

Since $|H^*_{k+1}| < n > |F^*_i|$ for $1 \leq i \leq k+1$, it follows from (2) that (2.1) holds for H^*_{k+1} and F^*_i. In particular, (a) of Theorem (2.1) holds for H^*_{k+1} and F^*_i. Recall from (2) that $\{u_{k+1}x_{k+1}, u_{k+1}y_{k+1}\} \subseteq E(H^*_{k+1})$, and $\{w_i x'_i, w_i y'_i\} \subseteq E(F^*_i)$. Hence, since $\mu \geq t-1$, we have the following.

(8) $H_{k+1} = H^*_{k+1} - u_{k+1}$ contains an x_{k+1}-y_{k+1} path Q_{k+1} such that $\ell(Q_{k+1}) \geq (\frac{|H_{k+1}|}{t^{t-1}})^r$, and for each $1 \leq i \leq k+1$, $F_i = F^*_i - w_i$ contains an x'_i-y'_i path R_i such that $\ell(R_i) \geq (\frac{|F_i|}{t^{t-1}})^r$.

Recall that $|H_i| \geq \frac{n}{2}$ for $1 \leq i \leq k$. It follows from (5) (with $i = k$) and (S1) that

(9) $|H_{k+1}| \geq \frac{|H_k|}{2(t-1)} \geq \frac{n}{4(t-1)}$, and hence, $\frac{|H_{k+1}|}{t^{t-1}} \geq \frac{n}{4(t-1)t^{t-1}} \geq \frac{|F_i|}{2(t-1)t^{t-1}}$ for $1 \leq i \leq k$, and $\frac{|H_{k+1}|}{t^{t-1}} \geq \frac{n}{4(t-1)t^{t-1}} \geq \frac{|H_{i,j}|}{2(t-1)t^{t-1}}$ for $1 \leq i \leq k$ and $1 \leq j \leq s_i$.

So by (5) (with $i = k$) and (S1), we have

(10) $\frac{|H_{k+1}|}{t^{t-1}} + 4(t-1)\sum_{i=1}^{k+1}|F_i| \geq \frac{1}{t^t}(t|H_{k+1}| + (t-1)|F_{k+1}|) \geq \frac{1}{t^t}|H_k| \geq \frac{n}{2t^t}$.

Let $Q_k := Q_{k+1} \cup R_{k+1}$. Then, by (4) and (8), Q_k is an x_k-y_k path in H_k and

$$\begin{aligned}
\ell(Q_k) &= \ell(Q_{k+1}) + \ell(R_{k+1}) \\
&\geq (\frac{|H_{k+1}|}{t^{t-1}})^r + (\frac{|F_{k+1}|}{t^{t-1}})^r \quad \text{(by (8))} \\
&\geq (\frac{|H_{k+1}|}{t^{t-1}} + \frac{b-1}{t^{t-1}}|F_{k+1}|)^r \quad \text{(since } |H_{k+1}| \geq |F_{k+1}| \text{ and by (2.2))} \\
&\geq (\frac{|H_{k+1}|}{t^{t-1}} + 4(t-1)|F_{k+1}|)^r \quad \text{(because } b-1 \geq 4(t-1)t^{t-1}\text{)}.
\end{aligned}$$

Similarly, let $Q_{k-1} := Q_{k+1} \cup R_{k+1} \cup R_k = Q_k \cup R_k$. Then, by (4) and (8), Q_{k-1} is an x_{k-1}-y_{k-1} path in H_{k-1}. By the above inequality, we have

$$\begin{aligned}
\ell(Q_{k-1}) &= \ell(Q_k) + \ell(R_k) \\
&\geq (\frac{|H_{k+1}|}{t^{t-1}} + 4(t-1)|F_{k+1}|)^r + (\frac{|F_k|}{t^{t-1}})^r \quad \text{(by (8))} \\
&\geq (\frac{|H_{k+1}|}{t^{t-1}} + 4(t-1)|F_{k+1}| + \frac{b-1}{2(t-1)}\frac{|F_k|}{t^{t-1}})^r \quad \text{(by (9) and by (2.3))} \\
&\geq (\frac{|H_{k+1}|}{t^{t-1}} + 4(t-1)|F_{k+1}| + 4(t-1)|F_k|)^r \quad \text{(because } b-1 \geq 8(t-1)^2 t^{t-1}\text{)}.
\end{aligned}$$

Continuing in this fashion, let $Q_0 := Q_{k+1} \cup (\bigcup_{i=1}^{k+1} R_i)$. Then by (4) and (8), Q_0 is an x_0-y_0 path in H_0 and

$$\ell(Q_0) \geq (\frac{|H_{k+1}|}{t^{t-1}} + 4(t-1)\sum_{i=1}^{k+1}|F_i|)^r.$$

Let $P := X_0 \cup Q_0 \cup Y_0$. Recall X_0 and Y_0 from (M1). Then P is an x-y path in G, and

$$\begin{aligned}
\ell(P) &= \ell(Q_0) + \ell(X_0) + \ell(Y_0) \\
&\geq (\frac{|H_{k+1}|}{t^{t-1}} + 4(t-1)\sum_{i=1}^{k+1}|F_i|)^r + \ell(X_0) + \ell(Y_0) \\
&\geq (\frac{|H_{k+1}|}{t^{t-1}} + 4(t-1)\sum_{i=1}^{k+1}|F_i| + 4(n-|H_0|))^r \quad \text{(by (M4) and (10))}.
\end{aligned}$$

(11) We may assume that $\sum_{i=1}^{k+1}\sum_{j=1}^{s_i}|H_{i,j}| > \frac{1}{2}(n - |H_{k+1}|)$.

For, suppose $\sum_{i=1}^{k+1}\sum_{j=1}^{s_i}|H_{i,j}| \leq \frac{1}{2}(n - |H_{k+1}|)$. Then by (7), $|H_{k+1}| < n/2$. By (6),

$$(t-1)\sum_{i=1}^{k+1}|F_i| + (n - |H_0|) \geq n - |H_{k+1}| - \sum_{i=1}^{k+1}\sum_{j=1}^{s_i}|H_{i,j}| \geq \frac{1}{2}(n - |H_{k+1}|).$$

Hence

$$\ell(P) \geq (\frac{|H_{k+1}|}{t^{t-1}} + 4(t-1)\sum_{i=1}^{k+1} |F_i| + 4(n - |H_0|))^r \geq (2(n - |H_{k+1}|))^r \geq n^r.$$

So (b) of (2.1) holds for G. Hence, we have (11).

Statement (11) suggests that we route the desired path through $H_{i,j}$, $1 \leq i \leq k+1$ and $1 \leq j \leq s_i$. Since $|H^*_{i,j}| < n$ and by (2), (2.1) holds for $H^*_{i,j}$. In particular, (a) of (2.1) holds for $H^*_{i,j}$. Recall that $\{y_{i,j}u_{i,j}, y_{i,j}x_i\} \subseteq E(H^*_{i,j})$. Hence, $H^*_{i,j} - y_{i,j}$ has a $u_{i,j}$-x_i path $Q^*_{i,j}$ of length at least $(\frac{|H^*_{i,j}|-1}{t^\mu})^r \geq (\frac{|H_{i,j}|}{t^{t-1}})^r$ (because $\mu \geq t-1$).

Since $G[\Lambda_i]$ is connected, $V(P[x, x_i)) \subseteq \Lambda_i$, and $N(V(H_i) - y_i) \subseteq \Lambda_i \cup \{y_i\}$, we can extend $Q^*_{i,j}$ in $G[\Lambda_i \cup \{x_i\}]$ to obtain a path $Q_{i,j}$ in G such that (i) $Q_{i,j}$ is an $x_{i,j}$-x_i path in G for some $x_{i,j} \in V(P[x, x_i))$, (ii) $V(Q_{i,j}) \cap V(P) = \{x_{i,j}, x_i\}$, and (iii) subject to (i) and (ii), $|P[x_i, x_{i,j}]|$ is maximum. (Note that (ii) holds since $V(P[y, y_i)) \subseteq \Omega_i$ and $\Lambda_i \cap \Omega_i = \emptyset$.) Then we have (12) and (13) below.

(12) for any $(i_1, j_1) \neq (i_2, j_2)$, $E(P[x_{i_1}, x_{i_1,j_1}]) \cap E(P[x_{i_2}, x_{i_2,j_2}]) = \emptyset$ implies that $V(Q_{i_1,j_1}) \cap V(Q_{i_2,j_2}) \subseteq V(P)$.

(13) $\ell(Q_{i,j}) \geq \ell(Q^*_{i,j}) \geq (\frac{|H_{i,j}|}{t^{t-1}})^r$.

Next, we show that some $Q_{i,j}$'s can be used to construct our desired path. For convenience, we define an auxiliary graph $\mathcal{A}$ with vertex set $V(\mathcal{A}) := \{Q_{i,j} : 1 \leq i \leq k+1 \text{ and } 1 \leq j \leq s_i\}$ such that Q_{i_1,j_1} and Q_{i_2,j_2} are adjacent in $\mathcal{A}$ if and only if $E(P[x_{i_1}, x_{i_1,j_1}]) \cap E(P[x_{i_2}, x_{i_2,j_2}]) \neq \emptyset$. By definition,

(14) $\mathcal{A}$ is an interval graph, and therefore is perfect.

Let θ be the cardinality of a maximum clique of $\mathcal{A}$.

(15) We claim that $\theta \leq t-1$.

For, let $\mathcal{C}$ be a clique of $\mathcal{A}$ with $V(\mathcal{C}) = \{Q_{i_1,j_1}, Q_{i_2,j_2}, \ldots, Q_{i_t,j_t}\}$. Without loss of generality, we may assume that $i_1 \leq i_2 \leq \ldots \leq i_t$. Then, $x_{i_s,j_s} \in V(P[x, x_{i_1}))$ for all $s = 2, \ldots, t$. Note that (from (3)) all $y_{i,j}$'s are contained in a connected subgraph Y of $G - V(P[x, x_{k+1}])$ containing $P[y_{k+1}, y_0]$. Since $x_{i_s} \in V(P[x_{i_1}, x_{k+1}])$ for all $2 \leq s \leq t$, we can produce a $K_{3,t}$-minor in G by contracting Y, $P[x, x_{i_1})$, and $P[x_{i_1}, x_{k+1}]$. But this is a contradiction. So $\theta \leq t-1$.

It follows from (14) and (15) that the chromatic number $\chi(\mathcal{A}) \leq \theta \leq t-1$. Therefore, there is an independent set $\mathcal{I}$ of $\mathcal{A}$ such that

(16) $\sum_{Q_{i,j} \in \mathcal{I}} |H_{i,j}| \geq \frac{1}{t-1} \sum_{i=1}^{k+1} \sum_{j=1}^{s_i} |H_{i,j}|$.

Hence by (11), we have

(17) $\sum_{Q_{i,j} \in \mathcal{I}} |H_{i,j}| \geq \frac{n - |H_{k+1}|}{2(t-1)}$.

Since $\mathcal{I}$ is an independent set in $\mathcal{A}$ and by (12), two distinct members Q_{i_1,j_1} and Q_{i_2,j_2} of $\mathcal{I}$ have one vertex in common if and only if either $x_{i_1,j_1} = x_{i_2}$ or $x_{i_2,j_2} = x_{i_1}$. So no three members of $\mathcal{I}$ share a common vertex of P. Thus, $(\bigcup_{Q_{i,j} \in \mathcal{I}} Q_{i,j}) \cup$

$P[x, x_{k+1}]$ contains an x-x_{k+1} path X_{k+1} which contains $\bigcup_{Q_{i,j}\in\mathcal{I}} Q_{i,j}$. Let $Y_{k+1} = P[y_{k+1}, y_0] \cup Y_0$.

Note that X_{k+1} and Y_{k+1} are vertex disjoint paths in $G - (V(H_{k+1}) - \{x_{k+1}, y_{k+1}\})$ from x, y to x_{k+1}, y_{k+1}, respectively. So (M1) holds for $(H_{k+1}, x_{k+1}, y_{k+1})$. By (2), H_{k+1} is an induced subgraph of G, $U_{k+1} := G - V(H_{k+1})$ is connected, and H^*_{k+1} is a 3-connected minor of G. So (M2) holds for $(H_{k+1}, x_{k+1}, y_{k+1})$. Recall (2), U_{k+1} is the disjoint union of Λ_{k+1} and Ω_{k+1}, both $G[\Lambda_{k+1}]$ and $G[\Omega_{k+1}]$ are connected, and $N(V(H_{k+1}) - y_{k+1}) \subseteq \Lambda_{k+1} \cup \{y_{k+1}\}$. ¿From the construction of Λ_{k+1} and Ω_{k+1}, it can be seen that $V(X_{k+1}) \subseteq \Lambda_{k+1} \cup \{x_{k+1}\}$, $V(Y_{k+1}) \subseteq \Omega_{k+1} \cup \{y_{k+1}\}$. Hence (M3) also holds for $(H_{k+1}, x_{k+1}, y_{k+1})$. For any $a \geq \frac{n}{2t^t}$, we have $a \geq 1$ (since $n \geq 8t^{t+1}$) and

$$\begin{aligned}
& a^r + \ell(X_{k+1}) + \ell(Y_{k+1}) \\
\geq\ & a^r + \sum_{Q_{i,j}\in\mathcal{I}} \ell(Q_{i,j}) \\
\geq\ & a^r + \sum_{Q_{i,j}\in\mathcal{I}} (\frac{|H_{i,j}|}{t^{t-1}})^r \quad \text{(by (13))} \\
\geq\ & (a + \frac{b-1}{t}(\frac{\sum_{Q_{i,j}\in\mathcal{I}} |H_{i,j}|}{t^{t-1}}))^r \quad \text{(because } a \geq \tfrac{1}{t}(\tfrac{n}{2t^{t-1}}) \geq \tfrac{1}{t}(\tfrac{|H_{i,j}|}{t^{t-1}}) \text{ and by (2.4))} \\
\geq\ & (a + 8(t-1) \sum_{Q_{i,j}\in\mathcal{I}} |H_{i,j}|)^r \quad \text{(because } t \geq 3 \text{ and } b-1 \geq 8(t-1)t^t) \\
\geq\ & (a + 8 \sum_{i=1}^{k+1} \sum_{j=1}^{s_i} |H_{i,j}|)^r \quad \text{(by (16))} \\
\geq\ & (a + 4(n - |H_{k+1}|))^r \quad \text{(by (11))}.
\end{aligned}$$

Since $|H_{k+1}| < |H_0|$, it follows from (1) that $(H_{k+1}, x_{k+1}, y_{k+1})$ is not a magic minor of (G, x, y). Hence (M4) does not hold for $(H_{k+1}, x_{k+1}, y_{k+1})$. Therefore,

(18) $|H_{k+1}| < \frac{n}{2}$.

Let $Q := X_{k+1} \cup Q_{k+1} \cup Y_{k+1}$. Recall that $\bigcup_{Q_{i,j}\in\mathcal{I}} \subseteq X_{k+1}$ and X_{k+1} is an x-x_{k+1} path. Then Q is an x-y path in G and

$$\begin{aligned}
\ell(Q) &\geq \ell(Q_{k+1}) + \ell(X_{k+1}) \\
&\geq (\frac{|H_{k+1}|}{t^{t-1}})^r + \sum_{Q_{i,j}\in\mathcal{I}} (\frac{|H_{i,j}|}{t^{t-1}})^r \quad \text{(by (8) and (13))} \\
&\geq (\frac{|H_{k+1}|}{t^{t-1}} + \frac{b-1}{2(t-1)} \sum_{Q_{i,j}\in\mathcal{I}} \frac{|H_{i,j}|}{t^{t-1}})^r \quad \text{(by (9) and (2.4))} \\
&\geq (\frac{|H_{k+1}|}{t^{t-1}} + 4(t-1) \sum_{Q_{i,j}\in\mathcal{I}} |H_{i,j}|)^r \quad \text{(since } b-1 \geq 8(t-1)^2 t^{t-1}) \\
&\geq (2(n - |H_{k+1}|))^r \quad \text{(by (17))} \\
&\geq n^r \quad \text{by (18))}.
\end{aligned}$$

Therefore, (b) of (2.1) holds for G. $\square$

6 Paths through a given edge

In this section, we first prove a result which serves as the induction step for part (c) in the proof of Theorem (2.1). We then complete the proof of (2.1).

(6.1) Lemma. *Suppose $n \geq 5$ and (a) of Theorem (2.1) holds for graphs with at most n vertices. Then (c) of Theorem (2.1) holds for graphs with n vertices.*

Proof. Let $t \geq 3$ be an integer, let G be a 3-connected graph with no $K_{3,t}$-minor, and let $|G| = n$. Let $xy, f \in E(G)$. For convenience, we let $b := 8t^{t+1}$ and $r := \log_b 2$.

First, assume that f is incident with one of $\{x, y\}$. By symmetry, we may assume that f is incident with y. Let y' denote the other end of f. Since $f \neq xy$, $y' \neq x$. By applying (4.1) to G, x, y', y (as G, x, y, z, respectively), we see that (a) of (2.1) holds for G, x, y', y. That is, $G - y$ contains an x-y' path P' such that $\ell(P') \geq (\frac{|G|-1}{t^\mu})^r \geq (\frac{|G|-1}{t^{t-1}})^r$. We can extend P' to an x-y path P through f in G such that $\ell(P) \geq (\frac{|G|-1}{t^{t-1}})^r + 1 \geq (\frac{|G|}{t^t})^r + 1$ (since $t \geq 3$). Hence (c) of (2.1) holds for G.

Therefore, we may assume that f is incident with neither x nor y. Since G is 3-connected, G contains an x-y path Q through f. Let Q_x and Q_y be the components of $Q - f$ containing x and y, respectively.

Let X denote the minimal union of blocks of $G - V(Q_y)$ containing Q_x. Then the blocks of X can be labeled as $X_0, X_1, \ldots, X_p$ and the cutvertices of X can be labeled as $x_1, \ldots, x_p$ such that

(X1) $V(X_i) \cap V(X_{i+1}) = \{x_i\}$,

(X2) $V(X_i) \cap V(X_j) = \emptyset$ if $j \geq i + 2$, and

(X3) $x_0 := x \in V(X_0) - \{x_1\}$, $x_{p+1} \in V(X_p) - \{x_p\}$, and x_{p+1} is incident with f.

See Figure 5. Since G is 3-connected, $U_i := G - V(X_i)$ is connected for each $0 \leq i \leq p$. By (3.4), $X_i^* := G/U_i$ is either a triangle or a 3-connected minor of G. Let u_i denote the vertex of X_i^* resulting from the contraction of U_i. Since $xy, f \in E(G)$, $u_i x_i, u_i x_{i+1} \in E(X_i^*)$. Since $|Q_y| \geq 2$, $|X_i^*| < n$.

Since G is 3-connected, $Y := G - V(X)$ has all its cutvertices contained in $V(Q_y)$. So the blocks of Y can be labeled as $Y_0, Y_1, \ldots, Y_q$ and the cutvertices of Y can be labeled as $y_1, \ldots, y_q$ such that

(Y1) $V(Y_i) \cap V(Y_{i+1}) = \{y_i\}$,

(Y2) $V(Y_i) \cap V(Y_j) = \emptyset$ if $j \geq i + 2$, and

(Y3) $y_0 := y \in V(Y_0) - \{y_1\}$, $y_{q+1} \in V(Y_q) - \{y_q\}$, and y_{q+1} is incident with f.

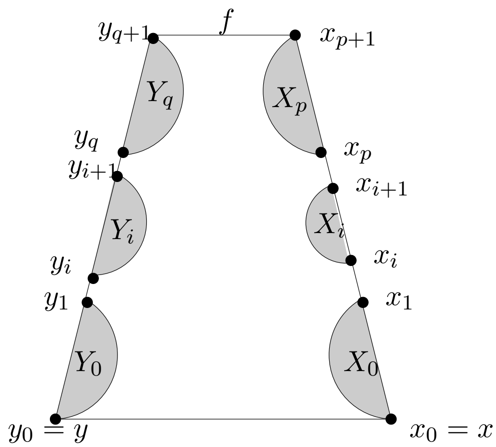

Figure 5: The blocks $X_0, \ldots, X_p$ and $Y_0, \ldots, Y_q$.

See Figure 5. Since $xy \in E(G)$, for each $0 \leq i \leq q$, $W_i := G - V(Y_i)$ is connected. By (3.4), $Y_i^* := G/W_i$ is either a triangle or a 3-connected minor of G. Let w_i denote the vertex of Y_i^* resulting from the contraction of W_i. Since $xy, f \in E(G)$, $w_i y_i, w_i y_{i+1} \in E(Y_i^*)$. Because $|X| \geq 2$, $|Y_i^*| < n$.

If $|X_i| = 2$ then let $P_i := X_i$, and so $\ell(P_i) = 1 \geq (\frac{|X_i|}{t^{t-1}})^r$ (since $t \geq 3$). If $|X_i| \geq 3$ then, since $|X_i^*| < n$, (2.1) holds for X_i^*. In particular, (a) of (2.1) holds for X_i^*. Thus $X_i := X_i^* - u_i$ has an x_i-x_{i+1} path P_i such that $\ell(P_i) \geq (\frac{|X_i^*|-1}{t^{t-1}})^r = (\frac{|X_i|}{t^{t-1}})^r$.

If $|Y_i| = 2$ then let $Q_i := Y_i$, and so $\ell(Q_i) = 1 \geq (\frac{|Y_i|}{t^{t-1}})^r$ (since $t \geq 3$). If $|Y_i| \geq 3$ then, since $|Y_i^*| < n$, (2.1) holds for Y_i^*. In particular, (a) of (2.1) holds for Y_i^*. Thus $Y_i := Y_i^* - w_i$ has an y_i-y_{i+1} path Q_i such that $\ell(Q_i) \geq (\frac{|Y_i^*|-1}{t^{t-1}})^r = (\frac{|Y_i|}{t^{t-1}})^r$.

Now let $P := ((\bigcup_{i=1}^p P_i) \cup (\bigcup_{i=1}^q Q_i)) + f$. Then P is an x-y path in G through f and

$$\begin{aligned}
\ell(P) &= \sum_{i=1}^{p} \ell(P_i) + \sum_{i=1}^{q} \ell(Q_i) + 1 \\
&\geq \sum_{i=1}^{p} (\frac{|X_i|}{t^{t-1}})^r + \sum_{i=1}^{q} (\frac{|Y_i|}{t^{t-1}})^r + 1 \\
&\geq (\frac{|G|}{t^{t-1}})^r + 1 \\
&> (\frac{|G|}{t^t})^r + 1.
\end{aligned}$$

Thus (c) of (2.1) holds for G. □

Proof of Theorem (2.1). For convenience, we let $b := 8t^{t+1}$ and $r := \log_b 2$. We apply induction on $n := |G|$. First assume that $n = 4$. Then G is isomorphic to the complete graph on 4 vertices. It is easy to see that $G - z$ contains an x-y path P such that $\ell(P) = 2$. Since $t \geq 3$, $\ell(P) = 2 \geq (\frac{3}{t^{t-1}})^r = (\frac{n-1}{t^{t-1}})^r$. So (a) holds.

Clearly, G contains an x-y path Q such that $\ell(Q) = 3$. Hence $\ell(Q) = 3 \geq 4^r = n^r$, and (b) holds. Finally, G contains an x-y path R through f such that $\ell(R) = 3$. Hence $\ell(R) = 3 \geq (\frac{4}{t^t})^r + 1 = (\frac{n}{t^t})^r + 1$.

So we may assume that $n \geq 5$ and (2.1) holds for graphs with at most $n - 1$ vertices. By Lemmas (4.1), (5.1) and (6.1), we see that (2.1) also holds for G. $\square$

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