

TAME AUTOMORPHISMS FIXING A VARIABLE OF FREE ASSOCIATIVE ALGEBRAS OF RANK THREE

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ABSTRACT. We study automorphisms of the free associative algebra $K\langle x, y, z \rangle$ over a field K which fix the variable z . We describe the structure of the group of z -tame automorphisms and derive algorithms which recognize z -tame automorphisms and z -tame coordinates.

INTRODUCTION

Let K be an arbitrary field of any characteristic and let $K[x_1, \dots, x_n]$ and $K\langle x_1, \dots, x_n \rangle$ be, respectively, the polynomial algebra in n variables and of the free associative algebra of rank n , freely generated by x_1, \dots, x_n . We may think of $K\langle x_1, \dots, x_n \rangle$ as the algebra of polynomials in n noncommuting variables. The automorphism groups $\text{Aut } K[x_1, \dots, x_n]$ and $\text{Aut } K\langle x_1, \dots, x_n \rangle$ are well understood for $n \leq 2$ only. The description is trivial for $n = 1$, when the automorphisms φ are defined by $\varphi(x_1) = \alpha x_1 + \beta$, where $\alpha \in K^* = K \setminus \{0\}$ and $\beta \in K$. The classical results of Jung–van der Kulk [J, K] for $K[x_1, x_2]$ and of Czerniakiewicz–Makar-Limanov [Cz, ML1, ML2] give that all automorphisms of $K[x_1, x_2]$ and $K\langle x_1, x_2 \rangle$ are tame. Writing the automorphisms of $K[x_1, \dots, x_n]$ and $\text{Aut } K\langle x_1, \dots, x_n \rangle$ as n -tuples of the images of the variables, and using x, y instead of x_1, x_2 , this means that $\text{Aut } K[x, y]$ and $\text{Aut } K\langle x, y \rangle$ are generated by the affine automorphisms

$$\psi = (\alpha_{11}x + \alpha_{21}y + \beta_1, \alpha_{12}x + \alpha_{22}y + \beta_2), \quad \alpha_{ij}, \beta_j \in K,$$

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(and $\psi_1 = (\alpha_{11}x + \alpha_{21}y, \alpha_{12}x + \alpha_{22}y)$, the linear part of ψ , is invertible) and the triangular automorphisms

$$\rho = (\alpha_1x + p_1(y), \alpha_2y + \beta_2), \quad \alpha_1, \alpha_2 \in K^*, p_1(y) \in K[y], \beta_2 \in K.$$

It turns out that the groups $\text{Aut } K\langle x, y \rangle$ and $\text{Aut } K[x, y]$ are naturally isomorphic. As abstract groups they are described as the free product $A *_C B$ of the group A of the affine automorphisms and the group B of triangular automorphisms amalgamating their intersection $C = A \cap B$. Every automorphism φ of $K[x, y]$ and $K\langle x, y \rangle$ can be presented as a product

$$(1) \quad \varphi = \psi_m^{\varepsilon_m} \rho_m \psi_{m-1} \cdots \rho_2 \psi_1 \rho_1^{\varepsilon_1},$$

where $\psi_i \in A$, $\rho_i \in B$ (ε_1 and ε_m are equal to 0 or 1), and, if φ does not belong to the union of A and B , we may assume that $\psi_i \in A \setminus B$, $\rho_i \in B \setminus A$. The freedom of the product means that if φ has a nontrivial presentation of this form, then it is different from the identity automorphism.

In the case of arbitrary n , the tame automorphisms are defined similarly, as compositions of affine and triangular automorphisms. One studies not only the automorphisms but also the coordinates, i.e., the automorphic images of x_1 .

We shall mention few facts related with the topic of the present paper, for z -automorphisms of $K[x, y, z]$ and $K\langle x, y, z \rangle$, i.e., automorphisms fixing the variable z . For more details we refer to the books by van den Essen [E], Mikhalev, Shpilrain, and Yu [MSY], and our survey article [DY1].

Nagata [N] constructed the automorphism of $K[x, y, z]$

$$\nu = (x - 2(y^2 + xz)y - (y^2 + xz)^2z, y + (y^2 + xz)z, z)$$

which fixes z . He showed that ν is nontame, or wild, considered as an automorphism of $K[z][x, y]$, and conjectured that it is wild also as an element of $\text{Aut } K[x, y, z]$. This was the beginning of the study of z -automorphisms.

It is relatively easy to see (and to decide algorithmically) whether an endomorphism of $K[z][x, y]$ is an automorphism and whether this automorphism is z -tame, or tame as an automorphism of $K[z][x, y]$. When $\text{char } K = 0$, Drensky and Yu [DY2] presented a simple algorithm which decides whether a polynomial $f(x, y, z) \in K[x, y, z]$ is a z -coordinate and whether this coordinate is z -tame. This provided many new wild automorphisms and wild coordinates of $K[z][x, y]$. These results in

[DY2] are based on a similar algorithm of Shpilrain and Yu [SY1] which recognizes the coordinates of $K[x, y]$. Shestakov and Umirbaev [SU1, SU2, SU3] established that the Nagata automorphism is wild. They also showed that every wild automorphism of $K[z][x, y]$ is wild as an automorphism of $K[x, y, z]$. Umirbaev and Yu [UY] proved that the z -wild coordinates in $K[z][x, y]$ are wild also in $K[x, y, z]$. In this way, all z -wild examples in [DY2] give automatically wild examples in $K[x, y, z]$.

Going to free algebras, the most popular candidate for a wild automorphism of $K\langle x, y, z \rangle$ is the example of Anick $(x + (y(xy - yz)), y, z + (zy - yz)y) \in \text{Aut } K\langle x, y, z \rangle$, see the book by Cohn [C], p. 343. It fixes one variable and its abelianization is a tame automorphism of $K[x, y, z]$. Exchanging the places of y and z , we obtain the automorphism $(x + z(xz - zy), y + (xz - zy)z, z)$ which fixes z (or a z -automorphism), and refer to it as the Anick automorphism. It is linear in x and y , considering z as a “noncommutative constant”. Drensky and Yu [DY3] showed that such z -automorphisms are z -wild if and only if a suitable invertible 2×2 matrix with entries from $K[z_1, z_2]$ is not a product of elementary matrices. In particular, this gives that the Anick automorphism is z -wild. When $\text{char } K = 0$, Umirbaev [U] described the defining relations of the group of tame automorphisms of $K[x, y, z]$. He showed that $\varphi = (f, g, h) \in \text{Aut } K\langle x, y, z \rangle$ is wild if the endomorphism $\varphi_0 = (f_0, g_0, z)$ of $K\langle x, y, z \rangle$ is a z -wild automorphism, where f_0, g_0 are the linear in x, y components of f, g , respectively. This implies that the Anick automorphism is wild. Recently Drensky and Yu [DY4, DY5] established the wildness of a big class of automorphisms and coordinates of $K\langle x, y, z \rangle$. Many of them cannot be handled with direct application of the methods of [DY3] and [U]. These results motivate the needs of systematic study of z -automorphisms of $K\langle x, y, z \rangle$. As in the case of z -automorphisms of $K[x, y, z]$, they are simpler than the arbitrary automorphisms of $K\langle x, y, z \rangle$ and provide important examples and conjectures for $\text{Aut } K\langle x, y, z \rangle$.

In the present paper we describe the structure of the group of z -tame automorphisms of $K\langle x, y, z \rangle$ as the free product of the groups of z -affine automorphisms and z -triangular automorphisms amalgamating the intersection. We also give algorithms which recognize z -tame automorphisms and coordinates of $K\langle x, y, z \rangle$. As an application, we show that all the z -automorphisms of the form $\sigma_h = (x + zh(xz - zy), y + h(xz - zy), z)$ are z -wild when the polynomials $h(xz - zy, z)$ are of positive

degree in x . This kind of automorphisms appear in [DY4, DY5] but the considerations there do not cover the case when $h(xz - zy, z)$ belongs to the square of the commutator ideal of $K\langle x, y, z \rangle$. Besides, the polynomial $x + zh(xz - zy, z)$ is a z -wild coordinate. Finally, we show that the z -endomorphisms of the form $\varphi = (x + u(x, y, z), y + v(x, y, z))$, where $(u, v) \neq (0, 0)$ and all monomials of u and v depend on both x and y , are not automorphisms. A partial case of this result was an essential step in the proof of the theorem of Czerniakiewicz and Makar-Limanov for the tameness of $\text{Aut } K\langle x, y \rangle$. The paper may be considered as a continuation of our paper [DY3].

1. THE GROUP OF z -TAME AUTOMORPHISMS

We fix the field K and consider the free associative algebra $K\langle x, y, z \rangle$ in three variables. We call the automorphism φ of $K\langle x, y, z \rangle$ a z -automorphism if $\varphi(z) = z$, and denote the automorphism group of the z -automorphisms by $\text{Aut}_z\langle x, y, z \rangle$. Since we want to emphasize that we work with z -automorphisms, we shall write $\varphi = (f, g)$, omitting the third coordinate z . The multiplication will be from right to left. If $\varphi, \psi \in \text{Aut}_z K\langle x, y, z \rangle$, then in $\varphi\psi$ we first apply ψ and then φ . Hence, if $\varphi = (f, g)$ and $\psi = (u, v)$, then

$$\varphi\psi = (u(f, g, z), v(f, g, z)).$$

The z -affine and z -triangular automorphisms of $K\langle x, y, z \rangle$ are, respectively, of the form

$$\psi = (\alpha_{11}x + \alpha_{21}y + \alpha_{31}z + \beta_1, \alpha_{12}x + \alpha_{22}y + \alpha_{32}z + \beta_2),$$

$\alpha_{ij}, \beta_j \in K$, the 2×2 matrix $(\alpha_{ij})_{i,j=1,2}$ being invertible,

$$\rho = (\alpha_1x + p_1(y, z), \alpha_2y + p_2(z)),$$

$\alpha_j \in K^*$, $p_1 \in K\langle y, z \rangle$, $p_2 \in K[z]$. The affine and the triangular z -automorphisms generate, respectively, the subgroups A_z and B_z of $\text{Aut}_z K\langle x, y, z \rangle$. We denote by $\text{TAut}_z K\langle x, y, z \rangle$ the group of z -tame automorphisms which is generated by the z -affine and z -triangular automorphisms. Of course, we may define the z -affine automorphisms as the z -automorphisms of the form $\psi = (f, g)$, where the polynomials $f, g \in K\langle x, y, z \rangle$ are linear in x and y . But, as we commented in [DY3], this definition is not convenient. For example, the Anick automorphism is affine in this sense but is wild.

In the commutative case, the z -automorphisms of $K[x, y, z]$ are simply the automorphisms of the $K[z]$ -algebra $K[z][x, y]$. A result of

Wright [Wr] states that over any field K the group $\text{TAut}_z K[x, y, z]$ has the amalgamated free product structure

$$\text{TAut}_z K[x, y, z] = A_z *_{C_z} B_z,$$

where A_z and B_z are defined as in the case of $K\langle x, y, z \rangle$ and $C_z = A_z \cap B_z$. (The original statement in [Wr] holds in a more general situation. In the case of $K[x, y, z]$ it involves affine and linear automorphisms with coefficients from $K[z]$ but this is not essential because every invertible matrix with entries in $K[z]$ is a product of elementary and diagonal matrices.)

Every z -tame automorphism φ of $K\langle x, y, z \rangle$ can be presented as a product in the form (1) where $\psi_i \in A_z$, $\rho_i \in B_z$ (ε_1 and ε_m are equal to 0 or 1), and, if φ does not belong to the union of A_z and B_z , we may assume that $\psi_i \in A_z \setminus B_z$, $\rho_i \in B_z \setminus A_z$. Fixing the linear nontriangular z -automorphism $\tau = (y, x)$, we can present φ in the canonical form

$$(2) \quad \varphi = \rho_n \tau \cdots \tau \rho_1 \tau \rho_0,$$

where $\rho_0, \rho_1, \dots, \rho_n \in B_z$ and only ρ_0 and ρ_n are allowed to belong to A_z , see for example p. 350 in [C]. Let

$$\rho_i = (\alpha_i x + p_i(y, z), \beta_i y + r_i(z)), \quad \alpha_i, \beta_i \in K^*, p_i \in K\langle y, z \rangle, r_i \in K[z].$$

Using the equalities for compositions of automorphisms

$$(\alpha x + p(y, z), \beta y + r(z)) = (x + \alpha^{-1}(p(y, z) - p(0, z)), y)(\alpha x + p(0, z), \beta y + r(z)),$$

$$(\alpha x + p(z), \beta y + r(z))\tau = (\beta y + r(z), \alpha x + p(z)) = \tau(\beta x + r(z), \alpha y + p(z)),$$

$p(z), r(z) \in K[z]$, we can do further simplifications in (2), assuming that $\rho_1, \dots, \rho_{n-1}$ are not affine and, together with ρ_n , are of the form $\rho_i = (x + p_i(y, z), y)$ with $p_i(0, z) = 0$ for all $i = 1, \dots, n$. We also assume that $\rho_0 = (\alpha_0 x + p_0(y, z), \beta_0 y + r_0(z))$. The condition that $\rho_1, \dots, \rho_{n-1}$ are not affine means that $\deg_y p_i(y, z) \geq 1$ and if $\deg_y p_i(y, z) = 1$, then $\deg_z p_i(y, z) \geq 1$, $i = 1, \dots, n-1$.

The following result shows that the structure of the group of z -tame automorphisms of $K\langle x, y, z \rangle$ is similar to the structure of the group of z -tame automorphisms of $K[x, y, z]$.

Theorem 1.1. *Over an arbitrary field K , the group $\text{TAut}_z K\langle x, y, z \rangle$ of z -tame automorphisms of $K\langle x, y, z \rangle$ is isomorphic to the free product $A_z *_{C_z} B_z$ of the group A_z of the z -affine automorphisms and the group B_z of z -triangular automorphisms amalgamating their intersection $C_z = A_z \cap B_z$.*

Proof. We define a bidegree of $K\langle x, y, z \rangle$ assuming that the monomial w is of bidegree $\text{bideg } w = (d, e)$ if $\deg_x w + \deg_y w = d$ and $\deg_z w = e$. We order the bidegrees (d, e) lexicographically, i.e., $(d_1, e_1) > (d_2, e_2)$ means that either $d_1 > d_2$ or $d_1 = d_2$ and $e_1 > e_2$. We denote by \bar{p} the leading bihomogeneous component of the nonzero polynomial $p(x, y, z)$. Let $\varphi = (f, g)$ be in the form (2), with all the restrictions fixed above, and let $q_i(y, z)$ be the leading component of $p_i(y, z)$. Direct computations give that, if ρ_n is not linear and $p_0(y, z) \neq \gamma_0 y + p'_0(z)$ in $\rho_0 = (\alpha_0 x + p_0(y, z), \beta_0 y + r_0(z))$, then

$$(3) \quad \begin{aligned} \bar{f} &= q_0(q_1(\dots q_{n-1}(q_n(y, z), z) \dots, z), z), \\ \bar{g} &= \beta_0 q_1(\dots q_{n-1}(q_n(y, z), z) \dots, z), \end{aligned}$$

and $\text{bideg } \bar{f} > (1, 0)$. Hence φ is not the identity automorphism. Similar considerations work when at least one of the automorphisms ρ_0 and ρ_n is affine. For example, if $\rho_0 = (\alpha_0 + \gamma_0 y + p'_0(z), \beta_0 y + r_0(z))$, $\gamma_0 \in K^*$, and $\text{bideg } p_n(y, z) > (1, 0)$, then

$$\begin{aligned} \bar{f} &= \gamma_0 q_1(\dots q_{n-1}(q_n(y, z), z) \dots, z), \\ \bar{g} &= \beta_0 q_1(\dots q_{n-1}(q_n(y, z), z) \dots, z). \end{aligned}$$

If $\text{bideg } p_0(y, z) > (1, 0)$ and $\rho_n = (x + \gamma_n y, y)$, $\gamma_n \in K^*$, then

$$\begin{aligned} \bar{f} &= q_0(q_1(\dots q_{n-1}(q_n(x + \gamma_n y, z), z) \dots, z), z), \\ \bar{g} &= \beta_0 q_1(\dots q_{n-1}(q_n(x + \gamma_n y, z), z) \dots, z). \end{aligned}$$

In all the cases, φ is not the identity automorphism. Hence, if φ has a nontrivial presentation in the form (2), then it is different from the identity automorphism, and we conclude that $\text{TAut}_z K\langle x, y, z \rangle$ is a free product with amalgamation of the groups A_z and B_z . \square

Following our paper [DY3] we identify the group of z -automorphisms which are linear in x and y with the group $GL_2(K[z_1, z_2])$. Let $f \in K\langle x, y, z \rangle$ be linear in x, y . Then f has the form

$$f = \sum \alpha_{ij} z^i x z^j + \sum \beta_{ij} z^i y z^j, \quad \alpha_{ij}, \beta_{ij} \in K.$$

The z -derivatives f_x and f_y are defined by

$$f_x = \sum \alpha_{ij} z_1^i z_2^j, \quad f_y = \sum \beta_{ij} z_1^i z_2^j.$$

Here f_x and f_y are in $K[z_1, z_2]$ and are polynomials in two commuting variables. The z -Jacobian matrix of the linear z -endomorphism $\varphi =$

(f, g) of $K\langle x, y, z \rangle$ is defined as

$$J_z(\varphi) = \begin{pmatrix} f_x & g_x \\ f_y & g_y \end{pmatrix}.$$

By [DY3] the mapping $\varphi \rightarrow J_z(\varphi)$ is an isomorphism of the group of the z -automorphisms which are linear in x, y and $GL_2(K[z_1, z_2])$. Also, such an automorphism is z -tame if and only if its z -Jacobian matrix belongs to $GE_2(K[z_1, z_2])$. (By the further development of this result by Umirbaev [U], the z -wild automorphisms of the considered type are wild also as automorphisms of $K\langle x, y, z \rangle$.)

Corollary 1.2. *The group $\text{TAut}_z K\langle x, y, z \rangle$ is isomorphic to the free product with amalgamation $GE_2(K[z_1, z_2]) *_{C_1} B_z$, where $GE_2(K[z_1, z_2])$ is identified as above with the group of z -tame automorphisms which are linear in x and y , and $C_1 = GE_2(K[z_1, z_2]) \cap B_z$.*

Proof. Everything follows from the observations that: (i) in the form (2), $\rho_j \tau \cdots \tau \rho_i \in GE_2(K[z_1, z_2])$ if and only if all ρ_j, \dots, ρ_i belong to $GE_2(K[z_1, z_2])$; (ii) $\rho_j \tau \cdots \tau \rho_i \in C_1$ if and only if $i = j$ and $\rho_i \in GE_2[z_1, z_2]$; (iii) $\tau \in GE_2(K[z_1, z_2])$. \square

2. RECOGNIZING z -TAME AUTOMORPHISMS AND COORDINATES

Now we use Theorem 1.1 to present algorithms which recognize z -tame automorphisms and coordinates of $K\langle x, y, z \rangle$. Of course, in all algorithms we assume that the field K is constructive. We start with an algorithm which determines whether a z -endomorphism of $K\langle x, y, z \rangle$ is a z -tame automorphism. The main idea is similar to that of the well known algorithm which decides whether an endomorphism of $K[x, y]$ is an automorphism, see Theorem 6.8.5 in [C], but the realization is more sophisticated. In order to simplify the considerations, we shall use the trick introduced by Formanek [F] in his construction of central polynomials of matrices.

Let H_n be the subspace of $K\langle x, y, z \rangle$ consisting of all polynomials which are homogeneous of degree n with respect to x and y . We define an action of $K[t_0, t_1, \dots, t_n]$ on H_n in the following way. If

$$w = z^{a_0} u_1 z^{a_1} u_2 \cdots z^{a_{n-1}} u_n z^{a_n},$$

where $u_i = x$ or $u_i = y$, $i = 1, \dots, n$, then

$$t_0^{b_0} t_1^{b_1} \cdots t_n^{b_n} * w = z^{a_0+b_0} u_1 z^{a_1+b_1} u_2 \cdots z^{a_{n-1}+b_{n-1}} u_n z^{a_n+b_n},$$

and then extend this action by linearity. Clearly, H_n is a free $K[t_0, t_1, \dots, t_n]$ -module with basis consisting of the 2^n monomials $u_1 \cdots u_n$, where $u_i = x$ or $u_i = y$. The proof of the following lemma is obtained by easy direct computation.

Lemma 2.1. *Let $\beta \in K^*$,*

$$(4) \quad v(x, y, z) = \sum \theta_i(t_0, t_1, \dots, t_k) * u_{i_1} \cdots u_{i_k} \in H_k,$$

$$(5) \quad q(y, z) = \omega(t_0, t_1, \dots, t_d) * y^d \in H_d,$$

where $\theta_i \in K[t_0, t_1, \dots, t_k]$, $\omega \in K[t_0, t_1, \dots, t_d]$, $u_{i_j} = x$ or $u_{i_j} = y$.
Then

$$(6) \quad \begin{aligned} u(x, y, z) &= q(v(x, y, z)/\beta, z) = \omega(t_0, t_d, t_{2d}, \dots, t_{kd})/\beta^d \\ &\quad (\sum \theta_i(t_0, t_1, \dots, t_k) * u_{i_1} \cdots u_{i_k}) \\ &\quad (\sum \theta_i(t_k, t_{k+1}, \dots, t_{2k}) * u_{i_1} \cdots u_{i_k}) \cdots \\ &\quad (\sum \theta_i(t_{k(d-1)}, t_{k(d-1)+1}, \dots, t_{kd}) * u_{i_1} \cdots u_{i_k}). \end{aligned}$$

Algorithm 2.2. Let $\varphi = (f, g)$ be a z -endomorphism of $K\langle x, y, z \rangle$. We make use of the bidegree defined in the proof of Theorem 1.1.

Step 0. If some of the polynomials f, g depends on z only, then φ is not an automorphism.

Step 1. Let u, v be the homogeneous components of highest bidegree of f, g , respectively. If both u, v are of bidegree $(1, 0)$, i.e., linear, then we check whether they are linearly independent. If yes, then φ is a product of a linear automorphism (from $GL_2(K)$) and a translation $(x + p(z), y + r(z))$. If u, v are linearly dependent, then φ is not an automorphism.

Step 2. Let $\text{bideg } u > (1, 0)$ and $\text{bideg } u \geq \text{bideg } v$. Hence $u \in H_l$, $v \in H_k$ for some k and l . Taking into account (3), we have to check whether $l = kd$ for a positive integer d and to decide whether $u = q(v/\beta, z)$ for some $\beta \in K^*$ and some $q(y, z) \in H_d$. In the notation of Lemma 2.1, we know u in (6) and v in (4) up to the multiplicative constant β . Hence, up to β , we know the polynomials $\theta_i(t_0, t_1, \dots, t_n)$ in the presentation of v . We compare some of the nonzero polynomial coefficients of $u = \sum \lambda_j(t_0, \dots, t_{kd})u_{j_1} \cdots u_{i_{kd}}$ with the corresponding coefficient of $q(v/\beta, z)$. Lemma 2.1 allows to find explicitly, up to the value of β^d , the polynomial $\omega(t_0, t_1, \dots, \omega_d)$ in (5) using the usual

division of polynomials. If $l = kd$ and $u = q(v/\beta, z)$, then we replace $\varphi = (f, g)$ with $\varphi_1 = (f - q(g/\beta, z), g)$. Then we apply Step 0 to φ_1 . If u cannot be presented in the desired form, then φ is not an automorphism.

Step 3. If $\text{bideg } v > (1, 0)$ and $\text{bideg } u < \text{bideg } v$, we have similar considerations, as in Step 2, replacing $\varphi = (f, g)$ with $\varphi_1 = (f, g - q(f/\alpha, z))$ for suitable $q(y, z)$. Then we apply Step 0 to φ_1 . If v cannot be presented in this form, then φ is not an automorphism.

Corollary 2.3. *Let $h(t, z) \in K\langle t, z \rangle$ and let $\deg_u h(u, z) > 0$. Then*

$$\sigma_h = (x + zh(xz - zy, z), y + h(xz - zy, z)z, z)$$

is a z -wild automorphism of $K\langle x, y, z \rangle$.

Proof. It is easy to see that σ_h is a z -automorphism of $K\langle x, y, z \rangle$ with inverse σ_{-h} . We apply Algorithm 2.2. Let w be the homogeneous component of highest bidegree of $h(xz - zy, z)$. Clearly, w has the form $w = \bar{h}(xz - zy, z) = q(xz - zy, z)$ for some bihomogeneous polynomial $q(t, z) \in K\langle t, z \rangle$. The leading components of the coordinates of σ_h are $zq(xz - zy, z)$ and $q(xz - zy, z)z$, and are of the same bidegree. If σ_h is a z -tame automorphism, then we can reduce the bidegree using a linear transformation, which is impossible because $zq(xz - zy, z)$ and $q(xz - zy, z)z$ are linearly independent. \square

The algorithm in Theorem 6.8.5 in [C] which recognizes the automorphisms of $K[x, y]$ can be easily modified to recognize the coordinates of $K[x, y]$. Such an algorithm is explicitly stated in [SY3], where Shpilrain and Yu established an algorithm which gives a canonical form, up to automorphic equivalence, of a class of polynomials in $K[x, y]$. (The automorphic equivalence problem for $K[x, y]$ asks how to decide whether, for two given polynomials $p, q \in K[x, y]$, there exists an automorphism φ such that $q = \varphi(p)$. It was solved over \mathbb{C} by Wightwick [Wi] and, over an arbitrary algebraically closed constructive field K , by Makar-Limanov, Shpilrain, and Yu [MLSY].) When $\text{char } K = 0$, Shpilrain and Yu [SY1] gave a very simple algorithm which decides whether a polynomial $f(x, y) \in K[x, y]$ is a coordinate. Their approach is based on an idea of Wright [Wr] and the Euclidean division algorithm applied for the partial derivatives of a polynomial in $K[x, y]$. Using the isomorphism of $\text{Aut } K[x, y]$ and $\text{Aut } K\langle x, y \rangle$ and reducing the considerations to the case of $K[x, y]$, Shpilrain and Yu [SY2] found the first algorithm which recognizes the coordinates of $K\langle x, y \rangle$. Now we want to modify

Algorithm 2.2 to decide whether a polynomial $f(x, y, z)$ is a z -tame coordinate of $K\langle x, y, z \rangle$.

Note, that if $\varphi = (f, g)$ and $\varphi' = (f, g')$ are two z -automorphisms of $K\langle x, y, z \rangle$ with the same first coordinate, then $\varphi^{-1}\varphi'$ fixes x . Hence $\varphi^{-1}\varphi' = (x, g'')$ and, obligatorily, $g'' = \beta y + r(x, z)$. In this way, if we know one z -coordinate mate g of f , then we are able to find all other z -coordinate mates. These arguments and Corollary 2.3 give immediately:

Corollary 2.4. *Let $h(t, z) \in K\langle t, z \rangle$ and let $\deg_u h(u, z) > 0$. Then $f(x, y, z) = x + zh(xz - zy, z)$ is a z -wild coordinate of $K\langle x, y, z \rangle$.*

Theorem 2.5. *There is an algorithm which decides whether a polynomial $f(x, y, z) \in K\langle x, y, z \rangle$ is a z -tame coordinate.*

Proof. We start with the analysis of the behavior of the first coordinate f of φ in (2). Let h be the first coordinate of $\psi = \rho_{n-1}\tau \cdots \tau\rho_1\tau\rho_0$ and let, as in (2), $\rho_n = (x + p_n(y, z), y)$ and $p_n(0, z) = 0$. Then

$$(7) \quad f(x, y, z) = \rho_n\tau(h(x, y, z)) = h(y, x + p_n(y, z), z).$$

In order to make the inductive step, we have to recover the polynomials $h(x, y, z)$ and $p_n(y, z)$ or, at least their leading components with respect to a suitable grading.

For a pair of positive integers (a, b) , we define the (a, b) -bidegree of a monomial $w \in K\langle x, y, z \rangle$ by

$$\text{bideg}_{(a,b)} w = (a\deg_x w + b\deg_y w, \deg_z w)$$

and order the bidegrees in the lexicographic order, as in Algorithm 2.2. For a nonzero polynomial $f \in K\langle x, y, z \rangle$ we denote by $|f|_{(a,b)}$ the homogeneous component of maximal (a, b) -bidegree. We write $\varphi = (f, g) \in \text{TAut}_z K\langle x, y, z \rangle$ in the form (2). Let us assume again that $\text{bideg } p_i(y) > (1, 0)$ for all $i = 0, 1, \dots, n$, and let h be the first coordinate of $\psi = \rho_{n-1}\tau \cdots \tau\rho_1\tau\rho_0$. Then the highest bihomogeneous component of h is

$$\bar{h}(y, z) = q_0(q_1(\dots(q_{n-1}(y, z), z)\dots), z).$$

The homogeneous component of maximal $(d_n, 1)$ -bidegree of $x + p_n(y, z)$ is $|x + q_n(y, z)|_{(d_n, 1)} = x + \xi_n y^{d_n}$ if $\deg_z q_n(y, z) = 0$ and $|x + q_n(y, z)|_{(d_n, 1)} = q_n(y, z)$ if $\deg_z q_n(y, z) > 0$. Direct calculations give

$$|f|_{(d_n, 1)} = |\rho_n\tau(\bar{h})|_{(d_n, 1)} = |\bar{h}(x + q_n(y, z))|_{(d_n, 1)}.$$

If $f'(x, z)$ and $f''(y, z)$ are the components of $f(x, y, z)$ which do not depend on y and x , respectively, we can recover the degree d_n of $p_n(y, z)$ as the quotient $d_n = \deg_x f' / \deg_y f''$. Now the problem is to recover $q_n(y, z)$ and $\bar{h}(y, z)$. Since $\bar{h}(y, z)$ does not depend on x , we have that

$$\bar{h}(y, z) = \overline{h(x, y, z)} = \overline{h(0, y, z)}.$$

From the equality (7) and the condition $p_n(0, z) = 0$ we obtain that

$$f(x, 0, z) = h(0, x + p_n(0, z), z) = h(0, x, z).$$

Hence $h(0, y, z) = f(y, 0, z)$ and we are able to find $\bar{h}(y, z)$. We write \bar{h} and \bar{q}_n in the form

$$\bar{h}(y, z) = \theta(t_0, t_1, \dots, t_k) * y^k, \quad q_n(y, z) = \omega(t_0, t_1, \dots, t_d) * y^d,$$

where $\theta(t_0, t_1, \dots, t_k) \in K[t_0, t_1, \dots, t_k]$ is known explicitly and $\omega(t_0, t_1, \dots, t_d) \in K[t_0, t_1, \dots, t_d]$. Similarly, the part of the component of maximal bidegree of $f(x, y, z)$ which does not depend on x has the form

$$\overline{f''}(y, z) = \zeta(t_0, t_1, \dots, t_{kd}) * y^{kd}, \quad \zeta(t_0, t_1, \dots, t_{kd}) \in K[t_0, t_1, \dots, t_{kd}].$$

Since $\bar{h}(q_n(y, z), z) = \overline{f''}(y, z)$, by Lemma 2.1 we obtain

$$\begin{aligned} \zeta(t_0, t_1, \dots, t_{kd}) &= \theta(t_0, t_d, t_{2d}, \dots, t_{kd}) \omega(t_0, t_1, \dots, t_d) \\ &\quad \omega(t_d, t_{d+1}, \dots, t_{2d}) \cdots \omega(t_{(k-1)d}, t_{(k-1)d+1}, \dots, t_{kd}). \end{aligned}$$

Here we know ζ and θ and want to determine ω . Let

$$\begin{aligned} \zeta'(t_0, t_1, \dots, t_{kd}) &= \zeta(t_0, t_1, \dots, t_{kd}) / \theta(t_0, t_d, t_{2d}, \dots, t_{kd}) \\ &= \omega(t_0, t_1, \dots, t_d) \omega(t_d, t_{d+1}, \dots, t_{2d}) \cdots \omega(t_{(k-1)d}, t_{(k-1)d+1}, \dots, t_{kd}). \end{aligned}$$

The greatest common divisor of the polynomials $\zeta'(t_0, t_1, \dots, t_{kd})$ and $\zeta'(t_{(k-1)d}, t_{(k-1)d+1}, \dots, t_{(2k-1)d})$ in $K[t_0, t_1, \dots, t_{(2k-1)d}]$ is equal, up to a multiplicative constant β , to $\omega(t_{(k-1)d}, t_{(k-1)d+1}, \dots, t_{kd})$. Hence the knowledge of ζ' allows to determine $\beta\omega(t_0, t_1, \dots, t_k)$ as well as the value of β^d . This means that we know also all the possible values of β and the polynomial $q_n(y, z)$. Now we apply on $f(x, y, z)$ the z -automorphism $\sigma = (x - q_n(y, z), y)$. Since $f(x, y, z) - \bar{h}(x + q_n(y, z), z)$ is lower in the $(d_n, 1)$ -biordering than $f(x, y, z)$ itself, we may replace f with $\sigma(f)$ and to make the next step. The considerations are almost the same when some of the automorphisms ρ_0 and ρ_n is affine. For example, if $f = \varphi(x)$ and $\rho_n = (x + \gamma y, y)$, $\gamma \in K$, in (2), then the leading bihomogeneous component of $h = \tau\rho_n^{-1}(f)$ does not depend on y , and we can do the next step. If f is a z -tame coordinate, then the above process will stop when we reduce f to a polynomial in the form $\alpha x + p(y, z)$. If f is not

a z -tame coordinate, then the process will also stop by different reason. In some step we shall reduce $f(x, y, z)$ to a polynomial $f_1(x, y, z)$. It may turn out that the degree $d = \deg_x f_1(x, 0, z) / \deg_y f_1(0, y, z)$ is not integer. Or, the commutative polynomials θ and ω corresponding to f_1 do not exist. \square

The following corollary is stronger than Corollary 2.4.

Corollary 2.6. *Let $h(t, z) \in K\langle t, z \rangle$ and let $\deg_u h(u, z) > 0$. Then $f(x, y, z) = x + h(xz - zy, z)$ is not a z -tame coordinate of $K\langle x, y, z \rangle$.*

Proof. We apply the algorithm in the proof of Theorem 2.5. Let $f(x, y, z)$ be a z -tame coordinate and let $h'(x, z) = h(xz, z)$ and $h''(y, z) = h(-zy, z)$ be the polynomials obtained from $h(xz - zy, z)$ replacing, respectively, y and x by 0. Clearly, $\text{bideg}_x h' = \text{bideg}_y h''$. Hence, as in the proof of Theorem 2.5 we can replace $f(x, y, z)$ with $\sigma(f)$, where $\sigma = (x - \alpha y, y)$, for a suitable $\alpha \in K^*$, and the leading bihomogeneous component of $\sigma(f)$ in the $(1, 1)$ -ordering does not depend on y . But this brings to a contradiction. If $h_1(t, z) \in K\langle t, z \rangle$ is homogeneous with respect to t , and

$$h_1((x - \gamma y)z - zy, z) = h_2(x, z)$$

for some $h_2(x, z)$, then, replacing x with 0, we obtain $h_1(-(\gamma yz + zy), z) = 0$, which is impossible. \square

Remark 2.7. In Corollary 2.6, we cannot guarantee that the polynomial $f(x, y, z) = x + h(xz - zy, z)$ is a z -coordinate at all. For example, let $f(x, y, z) = x + (xz - zy)$ be a z -coordinate with a coordinate mate $g(x, y, z)$. If $g_1(x, y, z)$ is the linear in x, y component of g , then $\varphi_1 = (f, g_1)$ is also a z -automorphism. Then, for suitable polynomials $c, d \in K[z_1, z_2]$, the matrix

$$J_z(\varphi_1) = \begin{pmatrix} 1 + z_2 & c(z_1, z_2) \\ -z_1 & d(z_1, z_2) \end{pmatrix}$$

is invertible. If we replace z_1 with 0 in its determinant $\det(J_z) = (1 + z_2)d(z_1, z_2) - z_1c(z_1, z_2)$ we obtain that $(1 + z_2)d_2(0, z_2) \in K^*$ which is impossible.

3. ENDOMORPHISMS WHICH ARE NOT AUTOMORPHISMS

In this section we shall establish a z -analogue of the following proposition which is the main step of the proof of the theorem of Czerniakiewicz [Cz] and Makar-Limanov [ML1, ML2] for the tameness of the automorphisms of $K\langle x, y \rangle$.

Proposition 3.1. *Let $\varphi = (x + u, y + v)$ be an endomorphism of $K\langle x, y \rangle$, where u, v are in the commutator ideal of $K\langle x, y \rangle$ and at least one of them is different from 0. Then φ is not an automorphism of $K\langle x, y \rangle$.*

An essential moment in its proof, see the book by Cohn [C], is the following lemma.

Lemma 3.2. *If $f, g \in K\langle x, y \rangle$ are two bihomogeneous polynomials, then they either generate a free subalgebra of $K\langle x, y \rangle$ or, up to multiplicative constants, both are powers of the same bihomogeneous element of $K\langle x, y \rangle$.*

We shall prove a weaker version of the lemma for $K\langle x, y, z \rangle$ which will be sufficient for our purposes.

Lemma 3.3. *Let $(0, 0) \neq (a, b) \in \mathbb{Z}^2$ and let $f_1, f_2 \in K\langle x, y, z \rangle$ be bihomogeneous with respect to the (a, b) -degree of $K\langle x, y, z \rangle$, i.e., $\text{adeg}_x w + \text{bdeg}_y w$ is the same for all monomials of f_1 , and similarly for f_2 . If f_1 and f_2 are algebraically dependent, then both $\text{deg}_{(a,b)} f_1$ and $\text{deg}_{(a,b)} f_2$ are either nonnegative or nonpositive.*

Proof. Let $v(f_1, f_2, z) = 0$ for some nonzero polynomial $v(u_1, u_2, z) \in K\langle u_1, u_2, z \rangle$. We may assume that both f_1, f_2 depend not on z only. We fix a term-ordering on $K\langle x, y, z \rangle$. Let \tilde{f}_1 and \tilde{f}_2 be the leading monomials of f_1 and f_2 , respectively. For each monomial $z^{k_0} u_{i_1} z^{k_1} \dots z^{k_{s-1}} u_{i_s} z^{k_s} \in K\langle u_1, u_2, z \rangle$ the leading monomial of $z^{k_0} f_{i_1} z^{k_1} \dots z^{k_{s-1}} f_{i_s} z^{k_s} \in K\langle x, y, z \rangle$ is $z^{k_0} \tilde{f}_{i_1} z^{k_1} \dots z^{k_{s-1}} \tilde{f}_{i_s} z^{k_s}$. Hence, the algebraic dependence of f_1 and f_2 implies that two different monomials $z^{k_0} \tilde{f}_{i_1} z^{k_1} \dots z^{k_{s-1}} \tilde{f}_{i_s} z^{k_s}$ and $z^{l_0} \tilde{f}_{j_1} z^{l_1} \dots z^{l_{t-1}} \tilde{f}_{j_t} z^{l_t}$ are equal. We write $\tilde{f}_1 = z^{p_1} g_1 z^{q_1}$ and $\tilde{f}_2 = z^{p_2} g_2 z^{q_2}$, where g_1, g_2 do not start and do not end with z . After some cancelation in the equation

$$z^{k_0} \tilde{f}_{i_1} z^{k_1} \dots z^{k_{s-1}} \tilde{f}_{i_s} z^{k_s} = z^{l_0} \tilde{f}_{j_1} z^{l_1} \dots z^{l_{t-1}} \tilde{f}_{j_t} z^{l_t}$$

we obtain a relation of the form

$$(8) \quad g_{a_1} z^{m_1} \dots z^{m_{k-1}} g_{a_k} z^{m_k} = g_{b_1} z^{n_1} \dots z^{n_{l-1}} g_{b_l} z^{n_l},$$

with different g_{a_1} and g_{b_1} . Hence, if $\deg g_1 \geq \deg g_2$, then $g_1 = g_2 g_3$ for some monomial g_3 (and $g_2 = g_1 g_3$ if $\deg g_1 < \deg g_2$). Again, g_2 and g_3 satisfy a relation of the form (8). Since $\deg g_1 \geq \deg g_2 > 0$, we obtain $\deg g_1 + \deg g_2 > \deg g_1 = \deg g_2 + \deg g_3$. Applying inductive arguments, we derive that both $\deg_{(a,b)} g_2$ and $\deg_{(a,b)} g_3$ are either nonnegative or nonpositive, and the same holds for f_1 and f_2 because $g_1 = g_2 g_3$, $\deg_{(a,b)} g_1 = \deg_{(a,b)} g_2 + \deg_{(a,b)} g_3$, and $\deg_{(a,b)} f_i = \deg_{(a,b)} g_i$, $i = 1, 2$. \square

The condition that $u(x, y)$ and $v(x, y)$ belong to the commutator ideal of $K\langle x, y \rangle$, as in Proposition 3.1, immediately implies that all monomials of u and v depend on both x and y , as required in the following theorem.

Theorem 3.4. *The z -endomorphisms of the form*

$$\varphi = (x + u(x, y, z), y + v(x, y, z)),$$

where $(u, v) \neq (0, 0)$ and all monomials of u and v depend on both x and y , are not automorphisms of $K\langle x, y, z \rangle$.

Proof. The key moment in the proof of Proposition 3.1 is the following. If $\varphi = (x + u, y + v)$ is an endomorphism of $K\langle x, y \rangle$, where u, v are in the commutator ideal of $K\langle x, y \rangle$ and at least one of them is different from 0, then there exist two integers a and b such that $(a, b) \neq (0, 0)$ and $a \leq 0 \leq b$ with the property that $\deg_{(a,b)}(x + u) = \deg_{(a,b)} x = a$ and $\deg_{(a,b)}(y + v) = \deg_{(a,b)} y = b$. Ordering in a suitable way the (a, b) -bidegrees, one concludes that the (a, b) -degrees of the leading bi-homogeneous components of $x + u$ and $y + v$ are with different signs. Then Lemma 3.2 shows that these leading components are algebraically independent and bidegree arguments as in the proof of Proposition 3.1 give that φ cannot be an automorphism. We repeat verbatim these arguments, working with the same (a, b) -(bi)degree and bidegree ordering Proposition 3.1, without counting the degree of z . In the final step, we use Lemma 3.3 instead of Lemma 3.2. \square

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