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AUTOMORPHISM GROUPS OF SPACES OF MINIMAL RATIONAL CURVES ON FANO MANIFOLDS OF PICARD NUMBER 1

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Abstract

Let X be a Fano manifold of Picard number 1 and M an irreducible component of the space of minimal rational curves on X. It is a natural problem to understand the extent to which the geometry of X is captured by the geometry of M. In this vein we raise the question as to whether the canonical map $\mathrm{Aut}_o(X) \to \mathrm{Aut}_o(M)$ is an isomorphism. After providing a number of examples showing that this may fail in general, we show that the map is indeed an isomorphism under the additional assumption that the subvariety of M consisting of members passing through a general point $x \in X$ is irreducible and of dimension ≥ 2.

1. Introduction

Let X be a Fano manifold of Picard number 1 embedded in a projective space $\mathbf{P}_N$. Suppose that X is covered by lines and let M be an irreducible component of the Hilbert scheme of lines covering X. For $x \in X$, let us denote by $M^x \subset M$ the subscheme consisting of members of M passing through x. By the condition that X has Picard number 1, the cardinality $\sharp(M^x)$ of the underlying variety M^x is strictly bigger than 1 for general $x \in X$ (e.g. by [HM1, Proposition 13]). $\mathrm{Aut}_o(X)$, the identity component of the automorphism group of X, acts naturally on M. This gives a natural homomorphism $\mathrm{Aut}_o(X) \to \mathrm{Aut}_o(M)$. From $\sharp(M^x) \geq 2$, this homomorphism is injective. It is very natural to ask when these two groups are isomorphic. Let us examine a few examples.

Example 1. When $X = \mathbf{P}_n = \mathbf{P}_N$, M is just the Grassmannian $Gr(2, n{+}1)$ of 2-planes in $\mathbf{C}^{n+1}$ and M^x is isomorphic to $\mathbf{P}_{n-1}$ for each $x \in X$. It is well known that $\mathrm{Aut}_o(\mathbf{P}_n) = \mathrm{Aut}_o(Gr(2, n+1))$.

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As a matter of fact, the same holds for the family of lines lying on a Grassmannian under the Plücker embedding, and also for many rational homogeneous spaces. However, this is not always true:

Example 2. For the three-dimensional smooth hyperquadric X, the space M of lines on X is isomorphic to $\mathbf{P}_3$ and $M^x \cong \mathbf{P}_1$. Thus $\mathrm{Aut}_o(X) = PSO(5, \mathbf{C})$, while $\mathrm{Aut}_o(M) = PSL(4, \mathbf{C})$. Another homogeneous example is the following. The exceptional simple Lie group G_2 has two simple roots, a long root σ_l and a short root σ_s. There are three parabolic subgroups up to conjugacy: P_l associated to σ_l, P_s associated to σ_s, and the Borel subgroup B. Let $X = G_2/P_l$, $Q = G_2/P_s$, and $F = G_2/B$. There are natural projections $\mu : F \to X$ and $\rho : F \to Q$ induced by the inclusions $B \subset P_l, B \subset P_s$. Both μ and ρ are $\mathbf{P}_1$-bundles. It turns out that fibers of ρ are sent to curves on X which are lines under the minimal projective embedding of X and in fact all lines on X arise this way. Thus Q is naturally isomorphic to the Hilbert scheme M of lines on X. It is well known that Q is isomorphic to the five-dimensional smooth hyperquadric. Since μ is a $\mathbf{P}_1$-bundle, $M^x \cong \mathbf{P}_1$ for each $x \in X$. In this case, $\mathrm{Aut}_o(X) = G_2$, while $\mathrm{Aut}_o(M) = PSO(7, \mathbf{C})$.

Example 3. Let $\lambda_1, \ldots, \lambda_{2g+2} \in \mathbf{C}$ be $2g + 2$ distinct points and B the hyperelliptic curve of genus g branched at these points. Consider two hyperquadrics

$$\begin{aligned} \mathbf{Q}_1 &= (\sum_{i=1}^{2g+2} X_i^2 = 0), \\ \mathbf{Q}_2 &= (\sum_{i=1}^{2g+2} \lambda_i X_i^2 = 0) \end{aligned}$$

in $\mathbf{P}_{2g+1}$. Desale and Ramanan ([DR]) have shown that the variety of $(g-2)$-dimensional planes in the intersection $\mathbf{Q}_1 \cap \mathbf{Q}_2$ is isomorphic to $\mathcal{S}U_B(2; 1)$, the moduli space of stable bundles of rank 2 with a fixed determinant of degree 1 on B, and the variety of $(g-1)$-dimensional planes in $\mathbf{Q}_1 \cap \mathbf{Q}_2$ is isomorphic to the Jacobian Jac_B. When $g = 3$, the 5-dimensional Fano variety $X = \mathbf{Q}_1 \cap \mathbf{Q}_2$ has $M \cong \mathcal{S}U_B(2; 1)$. For general $x \in X$, M^x is isomorphic to the complete intersection of two quadrics in $\mathbf{P}_4$. In this case, $\mathrm{Aut}_o(X) \cong \{1\} \cong \mathrm{Aut}_o(M)$ by [NR, Theorem 1 (a)]. On the other hand, when $g = 2$, $M \cong Jac_B$ for the 3-dimensional Fano variety $X = \mathbf{Q}_1 \cap \mathbf{Q}_2$. For a general point $x \in X$, M^x is isomorphic to the intersection of two conics in $\mathbf{P}_2$, namely, M^x consists of 4 points. In this case, $\mathrm{Aut}_o(X) = \{1\}$ while $\mathrm{Aut}_o(M) \cong Jac_B$.

Example 4. Let X be the smooth Fano threefold defined as the codimension-3 linear section of the Grassmannian $Gr(2, 5)$ of 2-planes in $\mathbf{C}^5$ under the Plücker embedding. By [MU], X is a smooth equivariant compactification of

$SL(2,\mathbf{C})$ modulo the octahedral group and $\mathrm{Aut}_o(X) \cong PSL(2,\mathbf{C})$. A line corresponds to the orbit of the Cartan subgroup of $SL(2,\mathbf{C})$. The space M of all lines lying on X is isomorphic to $\mathbf{P}_2$ ([Is, III.1.6]); thus $\mathrm{Aut}_o(M) \cong \mathrm{Aut}_o(\mathbf{P}_2) = PSL(3,\mathbf{C})$. M^x is finite.

We list the above examples with the dimension of M^x at general $x \in X$:

X	Ex. 1	Ex. 2	Ex. 3, g=3	Ex. 3, g=2	Ex. 4
$\dim(M^x)$	$n-1$	1	2	0	0
$\mathrm{Aut}_o(X) = \mathrm{Aut}_o(M)$?	Yes.	No.	Yes.	No.	No.

Note that $\mathrm{Aut}_o(X) = \mathrm{Aut}_o(M)$ when $\dim(M^x) \geq 2$ in the above table. Our main result says that this is true in general, with one additional assumption, the irreducibility of M^x. In fact, we will prove this for general Fano manifolds of Picard number 1 where minimal rational curves play the role of lines.

Theorem 1. *Let X be a Fano manifold of Picard number 1, M an irreducible component of the space of minimal rational curves on X as defined in Section 2, and let M^x be the subset consisting of members of M passing through a general point $x \in X$. If M^x is irreducible and* $\dim(M^x) \geq 2$*, then* $\mathrm{Aut}_o(X) = \mathrm{Aut}_o(M)$.

The definition of M given in Section 2 follows that of [Ko]. There is a natural morphism $M \to \mathrm{Chow}(X)$ which is the normalization of its image M^{Chow}. The natural inclusions

$$\mathrm{Aut}_o(X) \subset \mathrm{Aut}_o(M^{Chow}) \subset \mathrm{Aut}_o(M)$$

show that the statement of Theorem 1 holds when the space of minimal rational curves is understood as a subvariety of $\mathrm{Chow}(X)$. Let M^{Hilb} be the corresponding subvariety in the Hilbert scheme. Then the natural morphism $M^{Hilb} \to M^{Chow}$ is birational. Denoting the normalization of M^{Hilb} by $\widehat{M^{Hilb}}$, we have the following inclusions:

$$\mathrm{Aut}_o(X) \subset \mathrm{Aut}_o(M^{Hilb}) \subset \mathrm{Aut}_o(\widehat{M^{Hilb}}) \subset \mathrm{Aut}_o(M).$$

Thus the statement of Theorem 1 holds when the space of minimal rational curves is understood in the sense of Hilbert scheme.

Note that $\dim(M^x) + 2$ is the anti-canonical degree of the minimal rational curves. So the condition $\dim(M^x) \geq 2$ is satisfied if the index of X is ≥ 4. When $X \subset \mathbf{P}_N$ and members of M are lines of $\mathbf{P}_N$, the tangent morphism $\tau_x : M^x \to \mathbf{P}T_x(X)$ assigning a line through x to its tangent vector at x, is an embedding because a line is determined by its tangent vector. Since M^x is smooth for general $x \in X$ (cf. Section 2), we see that M^x is irreducible if

$\dim(M^x) > \frac{1}{2}\dim(\mathbf{P}T_x(X))$. Thus

Corollary 1. *Let $X \subset \mathbf{P}_N$ be a Fano manifold of Picard number 1 covered by lines and M an irreducible component of the Hilbert scheme of lines covering X. If $K_X^{-1} = \mathcal{O}(k)$ with $k \geq \frac{\dim(X)}{2} + 2$, then $\mathrm{Aut}_o(X) = \mathrm{Aut}_o(M)$.*

For a smooth complete intersection X of multi-degree $(d_1, \ldots, d_k)$ in $\mathbf{P}_N$, X is covered by lines if $N - \sum_{i=1}^k d_i \geq 1$ (e.g. [Ko, V. 4.10]). In this case, the image of the tangent morphism $\tau_x(M^x) \subset \mathbf{P}T_x(X)$ is a complete intersection of dimension $N - 1 - \sum_{i=1}^k d_i$ for a general point $x \in X$. In fact, the equations defining $\mathcal{C}_x$ arise from [SR, I.3.1, equation (3)] when X is a hypersurface. The equations defining $\mathcal{C}_x$ for a smooth complete intersection are obtained by repeating the calculation for the hypersurface case. Thus

Corollary 2. *Let X be a smooth complete intersection of multi-degree $(d_1, \ldots, d_k)$ in $\mathbf{P}_N$. If $N - \sum_{i=1}^k d_i \geq 3$, then the identity component of the automorphism group of the Hilbert scheme of lines on X is isomorphic to that of X.*

Theorem 1 is a direct consequence of the following local Torelli type result. The statement below is somewhat sketchy. See Section 4 for the precise statement.

Theorem 2. *Let $\pi : \mathcal{X} \to \Delta$ be a regular family of Fano manifolds of Picard number 1 over the unit disc. Assume that the collection of a component M_t of the space of minimal rational curves on the fibers X_t of π for $t \in \Delta$ form a flat family $\psi : \mathcal{M} \to \Delta$ and for each $t \in \Delta$ and a general point $x_t \in X_t$, $M_t^{x_t}$ is irreducible and of dimension ≥ 2. If there exists a family of biregular morphisms $g_t : M_0 \to M_t$ with $g_0 = Id$, then $\{g_t\}$ is induced by a family of biregular morphisms $f_t : X_0 \to X_t$ with $f_0 = Id$.*

The proof of Theorem 2 is obtained by a study of special subvarieties in M_t corresponding to points of X_t. The condition $\dim(M_t^x) \geq 2$ is needed in the Kodaira vanishing of the cohomology of a natural line bundle. One technical difficulty arises from the possibility of singularity for minimal rational curves. To handle this difficulty, we need Kebekus' result ([Ke]) about singularity of minimal rational curves through a general point of X. For the case when X is covered by lines or conics under a projective embedding, this problem does not arise and our proof can be considerably shortened.

From the examples we examined, the condition $\dim(M^x) \geq 2$ is a necessary condition. But the irreducibility of M^x is satisfied in our examples as long as $\dim(M^x) \geq 1$. We are not aware of any example X and M for which $\dim(M^x) \geq 1$ for general $x \in X$ but M^x is not irreducible. In this sense, the irreducibility condition on M^x may eventually be removable in the statement of Theorem 1 and Theorem 2, although it is essential in the proof presented below.

2. Spaces of minimal rational curves

Let X be a Fano manifold of Picard number 1. Choose a component $\mathcal{H}$ of the space $\mathrm{Hom}_{bir}(\mathbf{P}_1, X)$ parametrizing morphisms from $\mathbf{P}_1$ to X which are birational over their images, such that

(i) the images of elements of $\mathcal{H}$ cover a dense open subset of X;

(ii) for an element $h \in \mathcal{H}$, the degree of $h^*K_X^{-1}$ is minimal among all possible choices of $\mathcal{H}$ satisfying (i).

From (i), a general element of $\mathcal{H}$ is free, namely, the locally free sheaf $h^*T(X)$ on $\mathbf{P}_1$ is semi-positive. Let $\mathcal{H}^{free}$ be the open subset of $\mathcal{H}$ corresponding to free rational curves. The scheme $\mathcal{H}$ is smooth at each point $[h] \in \mathcal{H}^{free}$ because $H^1(\mathbf{P}_1, h^*T(X)) = 0$ and the tangent space of $\mathcal{H}$ at $[h]$ is naturally isomorphic to $H^0(\mathbf{P}_1, h^*T(X))$. The group $\mathrm{Aut}(\mathbf{P}_1)$ acts on $\mathcal{H}$. Under this action, $\mathcal{H}$ has a structure of $\mathrm{Aut}(\mathbf{P}_1)$-principal bundle over an irreducible quasi-projective variety M which is a component of $\mathrm{RatCurves}^n(X)$, the normalization of the subvarieties in $\mathrm{Chow}(X)$ corresponding to rational curves of X ([Ko, II.2.15]). We call M a **minimal dominating component** of $\mathrm{RatCurves}^n(X)$ and members of M **minimal rational curves**. The subset $M^{free} \subset M$ corresponding to $\mathcal{H}^{free}$ is Zariski open and smooth. The tangent space of M at a point corresponding to $[h] \in \mathcal{H}^{free}$ is naturally isomorphic to the quotient

$$H^0(\mathbf{P}_1, h^*T(X))/H^0(\mathbf{P}_1, T(\mathbf{P}_1))$$

where $T(\mathbf{P}_1)$ is regarded as a subsheaf of $h^*T(X)$ by the differential of h.

Let $o \in \mathbf{P}_1$ be a marked base point and $\mathrm{Aut}(\mathbf{P}_1, o)$ the subgroup of $\mathrm{Aut}(\mathbf{P}_1)$ consisting of automorphisms fixing o. For a general point x of X let $\mathcal{H}^x$ be the subscheme of $\mathcal{H}$ corresponding to elements of $\mathcal{H}$ sending o to x. Then $\mathcal{H}^x \subset \mathcal{H}^{free}$. Since $h^*T(X)$ is a semi-positive bundle on $\mathbf{P}_1$, $H^1(\mathbf{P}_1, h^*T(X) \otimes \mathbf{m}_o) = 0$ for any $[h] \in \mathcal{H}^x$, where $\mathbf{m}_o$ is the maximal ideal at o. Thus $\mathcal{H}^x$ is smooth and the tangent space at a point $[h] \in \mathcal{H}^x$ is naturally isomorphic to $H^0(\mathbf{P}_1, h^*T(X) \otimes \mathbf{m}_o)$.

By the natural action of $\mathrm{Aut}(\mathbf{P}_1, o)$ on $\mathcal{H}^x$, $\mathcal{H}^x$ has a structure of $\mathrm{Aut}(\mathbf{P}_1, o)$-principal bundle over a smooth quasi-projective variety M^x with finitely many components ([Ko, II.2.16, II.3.11.5]). The minimality of the degree of $h^*K_X^{-1}$ in the choice of $\mathcal{H}$ implies that M^x is a projective variety. The tangent space of M^x at a point corresponding to $[h] \in \mathcal{H}^x$ is naturally isomorphic to the quotient

$$H^0(\mathbf{P}_1, h^*T(X) \otimes \mathbf{m}_o)/H^0(\mathbf{P}_1, T(\mathbf{P}_1) \otimes \mathbf{m}_o).$$

Although $\mathcal{H}^x$ is a submanifold of $\mathcal{H}$, the induced morphism $\eta : M^x \to M^{free} \subset M$ is not necessarily an embedding. Let us see at what point η is not injective. Two elements $h_1, h_2 \in \mathcal{H}^x$ are sent to the same point in M if and only if they have the same image in X. If they are different as elements of M^x, $h_1(o) = h_2(o) = x$ must correspond to a multiple point of the image $C := h_1(\mathbf{P}_1) = h_2(\mathbf{P}_2)$ and an analytic neighborhood of $o \in \mathbf{P}_1$ is sent to different branches of C at x by h_1 and h_2.

Proposition 1. *The morphism $\eta : M^x \to M^{free}$ is an immersion for a general point $x \in X$.*

Proof. Since $\mathcal{H}^x \subset \mathcal{H}^{free}$ is an embedding, η is an immersion at $[h] \in M^x$ if the $\mathrm{Aut}(\mathbf{P}_1)$-orbit of $[h]$ slices $\mathcal{H}^x$ along the $\mathrm{Aut}(\mathbf{P}_1, o)$-orbit of $[h]$ in a transversal way. This is equivalent to

$$H^0(\mathbf{P}_1, T(\mathbf{P}_1)) \cap H^0(\mathbf{P}_1, h^*T(X) \otimes \mathbf{m}_o) \quad = \quad H^0(\mathbf{P}_1, T(\mathbf{P}_1) \otimes \mathbf{m}_o).$$

This is precisely the case if $h : \mathbf{P}_1 \to X$ is an immersion at o. Thus Proposition 1 follows from the following result of S. Kebekus. □

Proposition 2 ([Ke, Theorem 3.3 and 3.4]). *If $x \in X$ is general, any $[h] \in \mathcal{H}^x$ is an immersion at o, i.e., $h_* T_o(\mathbf{P}_1) \neq 0$. Furthermore, the tangent morphism $\tau_x : M^x \to \mathbf{P}T_x(X)$ defined by $\tau_x([h]) = h_* T_o(\mathbf{P}_1)$ is a finite morphism over its image.*

The restriction of the universal $\mathbf{P}_1$-bundle $\mathrm{Univ}^{rc}(X) \to \mathrm{RatCurves}^n(X)$ to M gives the universal $\mathbf{P}_1$-bundle $\rho : \mathcal{U} \to M$ with a natural morphism $\mu : \mathcal{U} \to X$. Furthermore, there exists a natural morphism $U : \mathbf{P}_1 \times \mathcal{H} \to \mathcal{U}$ so that $\rho \circ U = u \circ p_2$ where $p_2 : \mathbf{P}_1 \times \mathcal{H} \to \mathcal{H}$ is the projection and $u : \mathcal{H} \to M$ is the quotient by $\mathrm{Aut}(\mathbf{P}_1)$ ([Ko, II.2.15]):

$$\begin{array}{ccc} \mathbf{P}_1 \times \mathcal{H} & \stackrel{U}{\longrightarrow} & \mathcal{U} \\ \downarrow p_2 & & \downarrow \rho \\ \mathcal{H} & \stackrel{u}{\longrightarrow} & M. \end{array}$$

Let $j : \mathcal{H}^x \to \mathcal{U}$ be the restriction of $U(o, *) : \mathcal{H} \to \mathcal{U}$. This descends to $\iota : M^x \to \mathcal{U}$ which is injective. The composition $\rho \circ \iota : M^x \to M$ is exactly the immersion η in Proposition 1. It follows that ι is an embedding and we can identify M^x with the submanifold $\iota(M^x)$ in $\mathcal{U}$. Then the restriction of $\rho : \mathcal{U} \to M$ to $\iota(M^x)$ coincides with η in Proposition 1. Under this identification, $M^x = \mu^{-1}(x)$. In the following, by $T(\mathcal{U})$ and $T(M)$, we denote the tangent bundles of the smooth parts of $\mathcal{U}$ and M, respectively.

Proposition 3. *Let L be the line subbundle of $T(\mathcal{U})$ defined by the relative tangent vectors of the $\mathbf{P}_1$-bundle $\rho : \mathcal{U} \to M$. Then L^{-1} restricted to M^x is ample.*

Proof. By the definition of the tangent morphism $\tau_x : M^x \to \mathbf{P}T_x(X)$ in Proposition 2, L restricted to M^x is equivalent to the pull-back of the tautological line bundle $\mathcal{O}(-1)$ on $\mathbf{P}T_x(X)$ by τ_x. Since τ_x is finite by Proposition 2, L^{-1} is ample on M^x. □

Proposition 4. *On M^x, we have the following short exact sequence*

$$0 \longrightarrow L \longrightarrow \mathcal{O}^n \longrightarrow \eta^*T(M)/T(M^x) \longrightarrow 0$$

where $T(M^x)$ is regarded as a subbundle of $\rho^*T(M)$ by the immersion $\eta : M^x \to M$.

Proof. By the definition of L, we have the following exact sequence on $\mathcal{U}$:

$$0 \longrightarrow L \longrightarrow T(\mathcal{U}) \longrightarrow \rho^*T(M) \longrightarrow 0.$$

Let $N_{M^x \subset \mathcal{U}}$ be the normal bundle of M^x in $\mathcal{U}$. Restricting the above sequence to M^x and quotienting out by $T(M^x)$, we get

$$0 \longrightarrow L \longrightarrow N_{M^x \subset \mathcal{U}} \longrightarrow \eta^*T(M)/T(M^x) \longrightarrow 0.$$

Since $N_{M^x \subset \mathcal{U}}$ is a trivial bundle, we get the desired sequence. $\square$

3. Spaces of immersed subvarieties

Let Y be an irreducible quasi-projective variety. An irreducible projective subvariety $Z \subset Y$ is called an **immersed subvariety** if Z lies in the smooth locus of Y, its normalization $\hat{Z}$ is smooth and the normalization morphism $\nu : \hat{Z} \to Z$ is unramified. The set of immersed subvarieties of Y form a subscheme Imm_Y of the Hilbert scheme of Y. Let $\theta : \mathcal{W} \to \mathrm{Imm}_Y$ be the universal family. Given a subscheme $V \subset \mathrm{Imm}_Y$, the restriction of θ to V will be denoted by $\theta_V : \mathcal{W}_V \to V$.

Take the underlying reduced varieties of all the schemes involved and regard $\mathcal{W}_V$ and V as varieties. Let $\varphi : \hat{\mathcal{W}}_V \to \mathcal{W}_V$ be the normalization of $\mathcal{W}_V$. Then a general fiber of $\theta_V \circ \varphi$ gives the normalization of the corresponding fiber of $\theta_V : \mathcal{W}_V \to V$ by Seidenberg's theorem (e.g. [BS, Theorem 1.7.1]). In particular, there exists a nonempty open subvariety V' of V so that we have a simultaneous normalization of fibers of $\theta_{V'} : \mathcal{W}_{V'} \to V'$. Thus there exists a natural stratification of Imm_Y into countably many subschemes so that over each stratum the universal family can be simultaneously normalized in the above sense. We will call this **SN-stratification** (SN standing for "simultaneous normalization").

For an immersed subvariety $\nu : \hat{Z} \to Z \subset Y$, the quotient $\nu^*T(Y)/T(\hat{Z})$ is a vector bundle on $\hat{Z}$, which will be denoted by $N_{\hat{Z}}$. Here $T(Y)$ means the tangent bundle of the smooth locus of Y.

Proposition 5. *For an immersed subvariety $Z \subset Y$ of a quasi-projective variety Y, let $\hat{Z}$ be its normalization and $\mathcal{S}$ the SN-stratum of Imm_Y containing the point $[Z]$. Then $\dim(\mathcal{S}) \leq \dim H^0(\hat{Z}, N_{\hat{Z}})$.*

Proof. Let $\hat{\mathcal{W}}_{\mathcal{S}} \to \mathcal{S}$ be the simultaneous normalizations of the universal family over $\mathcal{S}$. To prove the claimed inequality, we may assume that $[Z]$ is a general point of $\mathcal{S}$, by the upper-semi-continuity of $\dim H^0(\hat{Z}, N_{\hat{Z}})$ as Z varies.

By the natural morphism $\mu : \mathcal{W} \to Y$ associated to the universal family, $\hat{\mathcal{W}}_{\mathcal{S}} \to \mathcal{S}$ can be viewed as a deformation of the morphism $\nu : \hat{Z} \to Y$ in the sense of [Ho]. We have the Kodaira-Spencer map defined in section 1 of [Ho]

$$\kappa : T_{[Z]}(\mathcal{S}) \to H^0(\hat{Z}, N_{\hat{Z}}),$$

which must be injective because $[Z]$ is general in $\mathcal{S}$. □

Let us apply the above discussions to the case $Y = M$, a minimal dominating component of $\mathrm{RatCurves}^n(X)$ for a Fano manifold X of Picard number 1. Assume that M^x is irreducible for general $x \in X$. Let $\rho : \mathcal{U} \to M, \mu : \mathcal{U} \to X$ be the universal family morphisms. Restricting to a Zariski dense open subset $X' \subset X$, we may assume that $\mu' : \mathcal{U}|_{X'} \to X'$ is a smooth morphism with connected fibers whose fibers over $x \in X'$ correspond to M^x. By shrinking X' if necessary, we may assume that ρ gives an immersion of each fiber of μ' into M^{free} by Proposition 1. Thus we have, by a simultaneous normalization, a morphism $\sigma : X' \to \mathrm{Imm}_M$ defined by

$$\sigma(x) \quad := \quad [\text{the image of } M^x \text{ in } M].$$

By shrinking X' further, we can assume that $\sigma(X')$ lies in a single SN-stratum $\mathcal{S}$ of Imm_M and $\mu' : \mathcal{U}|_{X'} \to X'$ is the pull-back of the simultaneous normalization of the universal family over $\mathcal{S}$.

Proposition 6. *If* $\dim(M^x) \geq 1$*, the morphism* $\sigma : X' \to \mathcal{S}$ *defined above is generically injective.*

Proof. Suppose $\sigma(x_1) = \sigma(x_2)$ for general $x_1 \neq x_2$ in X'. This means that $M^{x_1} = M^{x_2}$. Thus we have a positive dimensional family of rational curves belonging to M in X passing through x_1 and x_2. Such a family must degenerate to a reducible curve passing through x_1 and x_2 by bend-and-break ([Ko, II.5]). This is a contradiction to the minimal degree condition (ii) in the choice of $\mathcal{H}$. □

Proposition 7. *Suppose* $\dim(M^x) \geq 2$*. Then* $\sigma : X' \to \mathcal{S}$ *is birational.*

Proof. By Proposition 6, it suffices to show that $\dim(\mathcal{S}) \leq n := \dim(X)$. By Proposition 5, it is enough to show that for a general fiber F of $\mu' : \mathcal{U}|_{X'} \to X'$, $\dim H^0(F, N_F) = n$. F is an irreducible component of M^x for a general $x \in X$. From Proposition 4,

$$0 \longrightarrow H^0(F, L) \longrightarrow H^0(F, \mathcal{O}^n) \longrightarrow H^0(F, N_F) \longrightarrow H^1(F, L) \longrightarrow \cdots .$$

But from the assumption that $\dim(F) \geq 2$, by Kodaira's Vanishing Theorem for negative line bundles we get $H^1(F, L) = 0 = H^0(F, L)$ because L^{-1}

is ample on M^x by Proposition 3. It follows that $h^0(F, N_F) = h^0(F, \mathcal{O}^n) = n$. □

For a general $h \in M$, let $\mathcal{S}^h \subset \mathcal{S}$ be the subscheme consisting of members of $\mathcal{S}$ passing through h. From Proposition 7, the following is evident.

Proposition 8. *For a general member $h : \mathbf{P}_1 \to X$ of M,*

$$\mathcal{S}^h = \textit{closure of} \bigcup_{\textit{general } x \in h(\mathbf{P}_1)} [M^x]$$

where $[M^x]$ denotes the point of $\mathcal{S}$ corresponding to the immersed image of M^x in M. Moreover, under the birational map $\sigma : X \to \mathcal{S}$ of Proposition 7, $\sigma(h(\mathbf{P}_1)) = \mathcal{S}^h$ for general $h \in M$.

4. Proof of Theorem 2

We can state Theorem 1 in a precise form as follows.

Theorem 1. *Let X be a Fano manifold of Picard number 1 and M a minimal dominating component of* $\mathrm{RatCurves}^n(X)$*. Suppose for a general $x \in X$, M^x is irreducible and of dimension ≥ 2. Then $\mathrm{Aut}_o(X) = \mathrm{Aut}_o(M)$.*

Theorem 1 is a direct consequence of the following Theorem 2, applied to the case $\mathcal{X} = X \times \Delta$ and $\mathcal{M} = M \times \Delta$.

Theorem 2. *Let $\pi : \mathcal{X} \to \Delta$ be a regular family of Fano manifolds of Picard number 1 over the unit disc. Assume that there exists a component of $\mathrm{Hom}_{bir}(\mathbf{P}_1, \mathcal{X})$ which gives a flat family of projective varieties $\psi : \mathcal{M} \to \Delta$ so that $M_t := \psi^{-1}(t)$ is a minimal dominating component for $X_t := \pi^{-1}(t)$ for each $t \in \Delta$. Suppose $M_t^{x_t}$ is irreducible of dimension ≥ 2 for a general $x_t \in X_t$ for all $t \in \Delta$. If there exists a family of biregular morphisms $g_t : M_0 \to M_t$ with $g_0 = \mathrm{Id}$. Then $\{g_t\}$ is induced by a family of biregular morphisms $f_t : X_0 \to X_t$ with $f_0 = \mathrm{Id}$.*

Proof. The biregular morphism g_t induces a biregular morphism $g_{t*} : \mathrm{Imm}_{M_0} \to \mathrm{Imm}_{M_t}$ which preserves the SN-stratifications. Let $\mathcal{S}_t$ be the corresponding stratum of Imm_{M_t} arising from X_t for each $t \in \Delta$. By Proposition 7, we have the following sequence of birational maps:

$$X_0 \xrightarrow{\sigma_0} \mathcal{S}_0 \xrightarrow{g_{t*}} \mathcal{S}_t \xrightarrow{\sigma_t^{-1}} X_t.$$

Let $f_t : X_0 \to X_t$ be the birational map which is the composition of the above, namely, $f_t = \sigma_t^{-1} \circ g_{t*} \circ \sigma_0$. For a general member $h : \mathbf{P}_1 \to X$ of M_0,

$$\begin{aligned} f_t(h(\mathbf{P}_1)) &= \sigma_t^{-1}(g_{t*}(\sigma_0(h(\mathbf{P}_1)))) \\ &= \sigma_t^{-1}(g_{t*}(\mathcal{S}_0^h)) \text{ by Proposition 8} \\ &= \sigma_t^{-1}(\mathcal{S}_t^{g_t(h)}) \\ &= \sigma_t^{-1}(\sigma_t(g_t(h)(\mathbf{P}_1))) \text{ by Proposition 8} \\ &= g_t(h)(\mathbf{P}_1) \end{aligned}$$

where the image under a rational map means the strict image. Thus the birational map $f_t : X_0 \to X_t$ sends members of M_0 to members of M_t and vice versa. It follows that f_t can be extended to a biregular morphism from X_0 to X_t by the following proposition, which is equivalent to Proposition 4.4 in [HM2]. We reproduce the proof for the readers' convenience. □

Proposition 9. *Let X_1, X_2 be two Fano manifolds of Picard number 1. Let M_1 resp. M_2 be a minimal rational component on X_1 resp. X_2. Let $\Phi : X_1 \to X_2$ be a birational map sending general members of M_1 to general members of M_2 and vice versa. Then Φ can be extended to a biregular morphism.*

Proof. We denote by $B \subset X$ the subvariety on which Φ fails to be a local biholomorphism.

First assume that B is of codimension ≥ 2. Since X_1 and X_2 are Fano we may choose k large enough so that both $K_{X_1}^{-k}$ and $K_{X_2}^{-k}$ are very ample. Let s be a pluri-anticanonical section on X_2 in $\Gamma(X_2, K_{X_2}^{-k})$. Then $\Phi^* s$ is a well-defined pluri-anticanonical section on $X_1 - B$. It extends across B under the assumption that B is of codimension ≥ 2. It follows that Φ induces a linear monomorphism $\phi : \Gamma(X_2, K_{X_2}^{-k}) \to \Gamma(X_1, K_{X_1}^{-k})$ and hence a linear isomorphism $\phi^* : \Gamma(X_1, K_{X_1}^{-k})^* \to \Gamma(X_2, K_{X_2}^{-k})^*$ by taking adjoints. Identifying X_1 resp. X_2 as a complex submanifold of $\mathbf{P}\Gamma(X_1, K_{X_1}^{-k})^*$ resp. $\mathbf{P}\Gamma(X_2, K_{X_2}^{-k})^*$, Φ is nothing other than the restriction of the projectivization $[\phi^*] : \mathbf{P}(\Gamma(X_1, K_{X_1}^{-k})^*) \to \mathbf{P}(\Gamma(X_2, K_{X_2}^{-k})^*)$ to X_1, thus a biholomorphism.

Now suppose B has an irreducible component R of codimension 1 in X. The strict image $\Phi(R)$ has codimension ≥ 2 in X'. Since X_1 has Picard number 1, all M_1-curves intersect R. Thus their images under Φ will intersect $\Phi(R)$. But these images are general M_2-curves by assumption, a contradiction to [Ko, II.3.7]. □

References

[BS] Beltrametti, M. and Sommese, A.: *The adjunction theory of complex projective varieties.* Walter de Gruyter, 1995.

[DR] Desale, U.V. and Ramanan, S.: Classification of vector bundles of rank 2 on hyperelliptic curves. Invent. Math. **38** (1976) 161-185.

[Ho] Horikawa, E.: On deformations of holomorphic maps I. J. Math. Soc. Japan **25** (1973) 372-396.

[HM1] Hwang, J.-M. and Mok, N.: Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation. Invent. Math. **131** (1998) 393-418.

[HM2] Hwang, J.-M. and Mok, N.: Cartan-Fubini type extension of holomorphic maps for Fano manifolds of Picard number 1. Journal Math. Pures Appl. **80** (2001) 563-575.

[Is] Iskovskikh, V.A.: Anticanonical models of 3-dimensional algebraic varieties. J. Soviet Math. **13** (1980) 745-814.

[Ke] Kebekus, S.: Families of singular rational curves. J. Alg. Geom. **11** (2002) 245-256.

[Ko] Kollár, J.: *Rational curves on algebraic varieties.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3 Folge, Band 32, Springer-Verlag, 1996.

[MU] Mukai, S. and Umemura, H.: Minimal rational threefolds. in *Algebraic Geometry, Tokyo/Kyoto 1982*. Lecture Notes in Math. **1016** (1983) 490-518.

[NR] Narasimhan, M.S. and Ramanan, S.: Deformations of the moduli space of vector bundles over an algebraic curve. Ann. of Math. **101** (1975) 391-417.

[SR] Semple, J. G. and Roth, L.: *Introduction to algebraic geometry.* Oxford University Press, 1985.

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