

Semi-Global Stabilization of Linear Time-Delay Systems with Input Energy Constraint

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Abstract: This paper is concerned with semi-global stabilization of linear systems with actuator delay and energy constraints. Under the condition of null controllability by vanishing energy, the parametric Lyapunov equation based L_2 low gain feedback is adopted to solve the problem. If the delay in the system is exactly known, a delay-dependent controller is designed and if the delay in the system is either time-varying or not exactly known, a delay-independent controller is established. The proposed approach is used in the linearized model of the relative motion in the orbit plane of a spacecraft with respect to another in a circular orbit around the Earth to validate its effectiveness.

1. INTRODUCTION

Linear systems with actuator magnitude saturation have wide engineering background and are difficult to control. This class of systems have been extensively studied in the past several decades and many control problems have been studied. Among these problems are global stabilization (Kaliora and Astolfi (2004), Sussmann et al. (1994), Teel (1992)), semi-global stabilization (Lin (1998), Lin et al. (1996)), finite gain stabilization (Liu et al. (1996)), and local stabilization and estimation of domain of attraction (Hu and Lin (2001)).

On the other hand, control of linear systems in the presence of time delay, especially input delay, has also been attracting significant attention for several decades. The delays in the control signals arise from a variety of sources such as signal transmission and computation. In fact, the analysis and design of control systems that takes into account delays in the control input is a classical problem, and many related problems have been studied in the literature (see Chen et al. (1995), Gu and Liu (2009), Hale (1977), Lam et al. (2007), Zhang et al. (2004) and the references given there). Control systems with both actuator delay and magnitude saturation have also received much attention in recent years (see, for example,

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Lin and Fang (2007), Mazenc et al. (2003), Tarbouriech and da Silva Jr. (2000), Yakoubi and Chitour (2007) and the references therein).

Similar to magnitude constraints, energy constraints are also encountered naturally in practical systems because any physical system can only be powered with finite energy. However, the problem of controlling energy constrained system has not received as much attention that for magnitude constrained system. Only very recently, has the null controllability with vanishing energy problem been considered in Ichikawa (2008) and Priola and Zabczyk (2003). Under the assumption of null controllability with vanishing energy, we recently solved the semi-global stabilization problem for this class of constrained systems by using L_2 low gain feedback (Zhou et al. (2010)).

In the present paper, we go a further step over Zhou et al. (2010) by showing that semi-global stabilization of an input delayed linear system subject to energy constraint can also be achieved by a special kind of L_2 low gain feedback, namely, parametric Lyapunov equation based low gain feedback, provided that the open-loop system is null controllable with vanishing energy in the absence of actuator delay. By semi-global stabilization we mean that a controller is designed such that the closed-loop system is locally asymptotically stable with its domain of attraction containing an arbitrarily large bounded set of the state space. For the case that the delay is constant and exactly known and for the case that the delay is either time-varying or unknown, a delay dependent controller and a delay independent controller are respectively established. It is shown that the delay in the actuator can be any arbitrarily large finite value. These results complement the relating results in Zhou et al. (2010). The effectiveness of

the proposed approach is validated by the linearized model of the relative motion in the orbit plane of a spacecraft with respect to another in a circular orbit around the Earth subject to actuator delay.

The remainder of this paper is organized as follows. Some preliminaries and the problem formulation are given in Section 2. Section 3 contains the main results of this paper. A numerical example is worked out in Section 4 to show the effectiveness of the proposed methodology. Section 5 concludes the paper.

2. PROBLEM FORMULATION AND PRELIMINARIES

2.1 Problem Formulation

Consider a linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (1)$$

where $x(t) \in \mathbf{R}^n$ and $u(t) \in \mathbf{R}^m$, are, respectively, the state and input vectors. Let $x(t, x_0, u)$ denote the solutions of (1) with initial condition x_0 and input u . Denote

$$L_2(0, T, \mathbf{R}^m) \triangleq \left\{ f(t) : [0, T] \rightarrow \mathbf{R}^m \mid \int_0^T \|f(t)\|^2 dt < \infty \right\}.$$

We recall the following definition of null controllability with vanishing energy for system (1).

Definition 1. (Ichikawa (2008)) The system (1) (or the matrix pair (A, B)) is said to be null controllable with vanishing energy (NCVE) if for each initial $x(0) = x_0$, there exists a sequence of pairs (T_N, u_N) , $0 \leq T_N < \infty$, $u_N \in L_2(0, T_N, \mathbf{R}^m)$ such that $x(T_N, x_0, u_N) = 0$ and

$$\lim_{N \rightarrow \infty} \int_0^{T_N} \|u_N(t)\|^2 dt = 0.$$

Roughly speaking, a system is NCVE if, for any initial condition, there exists a control sequence with arbitrary small energy such that it can steer the state of the system to the origin. This class of systems and the relating control problems have many applications in practice. For example, the relative motion of a spacecraft with respect to another in a circular orbit around the Earth is captured by a non-linear system whose linearized version is NCVE (Ichikawa (2008)). Certainly, it is important in accomplishing a control objective with arbitrary energy expended.

Regarding the criterion for null controllability with vanishing energy, we recall the following condition from Priola and Zabczyk (2003) in which the results are developed for infinite dimensional linear systems.

Lemma 2. Linear system (1) is NCVE if and only if (A, B) is controllable in the ordinary sense and all the eigenvalues of A are located in the closed-left half s -plane.

It follows that the conditions for null controllability with vanishing energy happens to be the conditions for asymptotical null controllability with bounded controls (Sussmann et al. (1994) and Zhou et al. (2010)).

In this paper, we consider the following linear system with input delay

$$\dot{x}(t) = Ax(t) + Bu(t - \tau), \quad (2)$$

where $x(t) \in \mathbf{R}^n$ and $u(t) \in \mathbf{R}^m$ are, respectively, the state and input vectors, and $\tau > 0$ is a known constant scalar representing the delay in the control input. Throughout this paper, we use $\mathcal{C}_{n,\tau} = \mathcal{C}([-\tau, 0], \mathbf{R}^n)$ to denote the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into \mathbf{R}^n with the topology of uniform convergence, and $x_t \in \mathcal{C}_{n,\tau}$ to denote the restriction of $x(t)$ to the interval $[t - \tau, t]$ translated to $[-\tau, 0]$, that is, $x_t(\theta) = x(t + \theta)$, $\forall \theta \in [-\tau, 0]$. For any $\psi \in \mathcal{C}_{n,\tau}$, we define $\|\psi\|_c = \sup_{\theta \in [-\tau, 0]} \|\psi(\theta)\|$.

The problem we are interested in is as follows:

Problem 3. (L_2 -Semi-global Stabilization of Input-delayed Linear System) Consider the linear time delay system (2). Let $\Omega \subset \mathcal{C}_{n,\tau}$ be a bounded compact set. Find a control $u(t) \in \mathcal{U}_E$ with

$$\mathcal{U}_E = \left\{ u(t) : [-\tau, \infty) \rightarrow \mathbf{R}^m \mid \int_{-\tau}^{\infty} \|u(t)\|^2 dt \leq 1 \right\}, \quad (3)$$

such that the closed-loop system is asymptotically stable for arbitrary initial condition $\psi \in \Omega \subset \mathcal{C}_{n,\tau}$.

Toward solving the above problem, we first recall the L_2 low gain feedback approach studied in Zhou et al. (2010). Assume that the matrix $A(\varepsilon) : [0, 1] \rightarrow \mathbf{R}^{n \times n}$ is a continuous matrix function of ε and such that

$$\lambda(A(\varepsilon)) \subset \mathbf{C}^-, \forall \varepsilon \in (0, 1]$$

$$\lambda(A(0)) \subset \mathbf{C}^{-0} \triangleq \{s : \operatorname{Re}\{s\} \leq 0\}.$$

Basically, it means that the parametric matrix $A(\varepsilon)$ should be stable for $\forall \varepsilon \in (0, 1]$ and $A(0)$ should be marginally unstable. The following definition of L_2 -vanishment is initially given in Zhou et al. (2010).

Definition 4. Let $S(\varepsilon) : [0, 1] \rightarrow \mathbf{R}^{m \times n}$ and $A(\varepsilon) : [0, 1] \rightarrow \mathbf{R}^{n \times n}$ be stated above. Then $(S(\varepsilon), A(\varepsilon))$ is called L_2 -vanishing if

$$\lim_{\varepsilon \rightarrow 0^+} \left\| S(\varepsilon) e^{A(\varepsilon)t} \right\|_{L_2} \triangleq \lim_{\varepsilon \rightarrow 0^+} \left(\int_0^{\infty} \left\| S(\varepsilon) e^{A(\varepsilon)t} \right\|^2 dt \right)^{\frac{1}{2}} = 0. \quad (4)$$

A couple of characterizations for the L_2 -vanishment were presented in Zhou et al. (2010), based on which the following new design method named as L_2 low gain feedback is introduced.

Definition 5. (L_2 Low Gain Feedback) Assume that $(A, B) \in (\mathbf{R}^{n \times n}, \mathbf{R}^{n \times m})$ is NCVE. A stabilizing feedback gain $K(\varepsilon) : [0, 1] \in \mathbf{R}^{m \times n}$ is said to be an L_2 low gain feedback if $(K(\varepsilon), A - BK(\varepsilon))$ is L_2 -vanishing, namely,

$$\lim_{\varepsilon \rightarrow 0^+} \left\| K(\varepsilon) e^{(A - BK(\varepsilon))t} \right\|_{L_2} = 0.$$

By using this L_2 low gain feedback, it is shown in Zhou et al. (2010) that Problem 3 can be solved in the case $\tau = 0$ under the condition of null controllability with vanishing energy. In this paper, we will further show that Problem 3 is also solvable if $\tau \neq 0$ under the same condition.

Without loss of generality, we assume that (A, B) is given in the following form

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & A_- \end{bmatrix}, \quad B = \begin{bmatrix} B_0 \\ B_- \end{bmatrix},$$

where A_- contains all eigenvalues of A that have negative real parts and A_0 contains all eigenvalues of A that are on

the imaginary axis. Since (A, B) is NCVE, we know that (A_0, B_0) is controllable. Clearly, the subsystem (A_-, B_-) does not affect the solvability of Problem 3. In what follows, we will further impose, without loss of generality, the following assumption on the system:

Assumption 6. $(A, B) \in (\mathbf{R}^{n \times n}, \mathbf{R}^{n \times m})$ is controllable and all the eigenvalues of A are on the imaginary axis.

2.2 Extension of L_2 -Vanishment to Nonlinear Systems

In this subsection, we will take a new look at Definition 4 and then introduce some basic ideas for proving the desired result regarding solutions to Problem 3.

Consider the following family of linear systems:

$$\begin{cases} \dot{x}(t) = A(\varepsilon)x(t), & x(0) = x_0 \in \mathbf{R}^n, \\ y(t) = S(\varepsilon)x(t), \end{cases} \quad (5)$$

where $(S(\varepsilon), A(\varepsilon))$ is defined in Definition 4 and $\varepsilon \in [0, 1]$. Notice that

$$\|y\|_{L_2} = \left(\int_0^\infty \|S(\varepsilon)e^{A(\varepsilon)t}x_0\|^2 dt \right)^{\frac{1}{2}}.$$

Then it follows from Definition 4 that $(S(\varepsilon), A(\varepsilon))$ is L_2 -vanishing if and only if the L_2 norm of the output of system (5) with arbitrary bounded initial condition $x_0 \in \mathbf{R}^n$ approaches to zero as ε does. This observation implies the possibility of extending the definition of L_2 -vanishment for matrix pair $(S(\varepsilon), A(\varepsilon))$ to nonlinear systems.

Definition 7. Consider the following family of nonlinear systems

$$\begin{cases} \dot{x}(t) = A(\varepsilon, x(t)), & x(0) = x_0 \in \mathbf{R}^n, \\ y(t) = S(\varepsilon, x(t)), \end{cases} \quad (6)$$

where $A(\varepsilon, x) : [0, 1] \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ is continuous with respect to ε and globally Lipschitz with respect to x , $S(\varepsilon, x) : [0, 1] \times \mathbf{R}^n \rightarrow \mathbf{R}^m$ is continuous with respect to ε and $\varepsilon \in [0, 1]$. Assume that for arbitrary $\varepsilon \in [0, 1]$, the system in (6) is globally asymptotically stable. Then the system in (6) is called L_2 -vanishing if

$$\|x_0\| \leq D < \infty \Rightarrow \lim_{\varepsilon \rightarrow 0^+} \|y\|_{L_2} = 0,$$

where D is any positive scalar.

The following simple results for L_2 -vanishment can be derived easily by definition. The idea found in the proof of this result will be adopted to prove our main results in the next section.

Proposition 8. The system in (6) is L_2 -vanishing if there exists a scalar $\varepsilon^* > 0$ and a function $V(\varepsilon, x) : [0, \varepsilon^*] \times \mathbf{R}^n \rightarrow \mathbf{R}^+$, $V(\varepsilon, 0) = 0$ such that

$$\|x\| \leq D < \infty \Rightarrow \lim_{\varepsilon \rightarrow 0^+} V(\varepsilon, x) = 0, \quad (7)$$

and

$$\dot{V}(\varepsilon, x(t)) \leq -\kappa(\varepsilon)\|y(t)\|^2, \quad (8)$$

where $\kappa(\varepsilon) : [0, \varepsilon^*] \rightarrow \mathbf{R}^+$ is bounded for all $\varepsilon \in [0, \varepsilon^*]$.

Proof. Since system (6) is globally asymptotically stable, we have

$$\|x_0\| \leq D < \infty \Rightarrow \lim_{t \rightarrow \infty} \|x(t)\| = 0. \quad (9)$$

Taking integral on both sides of (8) from 0 to ∞ gives

$$\lim_{t \rightarrow \infty} V(\varepsilon, x(t)) - V(\varepsilon, x_0) \leq -\kappa(\varepsilon) \int_0^\infty \|y(t)\|^2 dt.$$

Then it follows from (9) and $V(\varepsilon, 0) = 0$ that

$$\|x_0\| \leq D < \infty \Rightarrow \int_0^\infty \|y(t)\|^2 dt \leq \frac{1}{\kappa(\varepsilon)} V(\varepsilon, x_0).$$

Therefore, by invoking (7) and the boundness of $\kappa(\varepsilon)$, we get

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^\infty \|y(t)\|^2 dt \leq \lim_{\varepsilon \rightarrow 0^+} \left(\frac{1}{\kappa(\varepsilon)} V(\varepsilon, x_0) \right) = 0.$$

The result then follows from Definition 7. ■

Remark 9. We can assume without loss of generality that $\kappa(\varepsilon) = 1$ since we can replace $V(\varepsilon, x)$ in Proposition 8 by $\kappa^{-1}(\varepsilon)V(\varepsilon, x)$.

Remark 10. If the Lyapunov function $V(\varepsilon, x)$ in Proposition 8 is chosen as $x^T P(\varepsilon)x$ and the nonlinear system (6) is replaced by the linear plant (5), the conditions in Proposition 8 can be written as the conditions in Theorem 4 proven in Zhou et al. (2010) where it is further shown that this condition is both necessary and sufficient for the L_2 vanishment of linear system (5).

At the end of this subsection, we give some introduction to the parametric Lyapunov equation based L_2 low gain feedback (see Zhou et al. (2010)) which will be used in this paper to solve Problem 3. This kind of L_2 low gain design is based on solution to the following parametric ARE:

$$A^T P + PA - PBB^T P = -\varepsilon P. \quad (10)$$

Some relevant properties of this ARE are summarized in the following lemma whose proof can be found in Zhou et al. (2010).

Lemma 11. Assume that (A, B) satisfies Assumption 6. Then for arbitrary $\varepsilon > 0$, ARE (10) has a unique positive definite solution $P(\varepsilon) = W^{-1}(\varepsilon)$ which satisfies the following Lyapunov equation:

$$W \left(A + \frac{\varepsilon}{2} I \right)^T + \left(A + \frac{\varepsilon}{2} I \right) W = -BB^T. \quad (11)$$

Moreover,

$$\lim_{\varepsilon \rightarrow 0^+} P(\varepsilon) = 0$$

$$\frac{d}{d\varepsilon} P(\varepsilon) > 0, \forall \varepsilon > 0$$

$$P(\varepsilon) BB^T P(\varepsilon) \leq n\varepsilon P(\varepsilon)$$

$$e^{A^T t} P(\varepsilon) e^{At} \leq e^{\omega \varepsilon t} P(\varepsilon), \forall t \geq 0, \quad \forall \omega \geq n - 1.$$

3. MAIN RESULTS

In this section, we use the idea in proving Proposition 8 for nonlinear systems to study the semi-global stabilization problem of time-delayed linear system with energy constraints by using L_2 low gain feedback.

3.1 Delay-Dependent Feedback

Under the condition that the time delay τ is exactly known, we can propose the following delay dependent solution to Problem 3 by state feedback.

Theorem 12. Consider linear system (2) with an arbitrarily large but finite delay τ . Assume that (A, B) satisfies Assumption 6. Then there exists an $\varepsilon^* \in (0, 1]$ such that the family of linear state feedback

$$u(t) = -B^T P(\varepsilon) e^{A\tau} x(t), \quad \forall \varepsilon \in (0, \varepsilon^*], \quad (12)$$

where $P(\varepsilon)$ is the unique positive definite solution to the parametric ARE (10), solves Problem 3.

Proof. The closed-loop system consisting of (2) and (12) is given by

$$\dot{x}(t) = Ax(t) - BB^T P(\varepsilon) e^{A\tau} x(t - \tau), \quad (13)$$

whose initial condition is $x(\theta) = \psi(\theta), \forall \theta \in [-\tau, 0], \psi \in \Omega \subset \mathcal{C}_{n,\tau}, t \geq 0$. It follows that

$$\begin{cases} x(t) = e^{At}\psi(0) - \int_0^t e^{A(t-s)} BB^T \\ \quad \cdot P(\varepsilon) e^{A\tau} \psi(s - \tau) ds, t \in [0, \tau], \\ x(t) = e^{A\tau} x(t - \tau) - \int_{t-\tau}^t e^{A(t-s)} B \\ \quad \cdot B^T P(\varepsilon) e^{A\tau} x(s - \tau) ds, t \geq \tau. \end{cases} \quad (14)$$

We get easily from the first equation in (14) that (inequality (22) in Zhou et al. (2010)), $\forall t \in [0, \tau]$,

$$\|x(t)\| \leq \max_{s \in [0, \tau]} \{ \|e^{As}\| \} (1 + \tau \|BB^T P e^{A\tau}\|) \|\psi\|_c \quad (15)$$

which is bounded as $\psi \in \Omega$. Therefore, we need only to consider the closed-loop system (13) for $t \geq \tau$. In this case, inserting the second relation in (14) into the closed-loop system (13) gives, for all $t \geq \tau$,

$$\dot{x}(t) = (A - BB^T P(\varepsilon)) x(t) - BB^T P(\varepsilon) \pi(t), \quad (16)$$

where

$$\pi(t) = \int_{t-\tau}^t e^{A(t-s)} BB^T P(\varepsilon) e^{A\tau} x(s - \tau) ds.$$

The initial condition ($t \in [-\tau, \tau]$) $\bar{\psi}_\tau(\theta) = \bar{\psi}(\tau + \theta), \forall \theta \in [-2\tau, 0]$ for system (16) can be defined as follows (pp. 132 in Hale (1977)):

$$\bar{\psi}_\tau(\theta) = \begin{cases} \psi(\tau + \theta), \forall \theta \in [-2\tau, -\tau], \\ x(\tau + \theta), \forall \theta \in [-\tau, 0], \end{cases} \quad (17)$$

where $x(t), \forall t \in [0, \tau]$ is given in the first equation in (14). Denote

$$\bar{\Omega}_{n,2\tau} \triangleq \{ \bar{\psi}_\tau(\theta), \forall \theta \in [-2\tau, 0] \mid \psi \in \Omega \} \subset \mathcal{C}_{n,2\tau}.$$

The solution to the time-delay system (13) for $t \geq 0$ with initial condition $\psi \in \Omega$ coincides with the solution to the time-delay system (16) with initial condition $\bar{\psi}_\tau \in \bar{\Omega}$ for all $t \geq \tau$.

By using ARE (10) and Lemma 11, the time derivative of $V_1(x(t)) = x^T(t) P(\varepsilon) x(t)$ along the trajectories of system (16) satisfies (inequality (28) in Zhou et al. (2010))

$$\dot{V}_1(x) \leq -\varepsilon x^T(t) P(\varepsilon) x(t) + n\varepsilon \pi^T(t) P(\varepsilon) \pi(t). \quad (18)$$

According to (31) in Zhou et al. (2010), we have

$$\pi^T(t) P(\varepsilon) \pi(t) \leq (n\varepsilon)^2 \tau \int_{t-\tau}^t e^{\omega\varepsilon(t-s+\tau)} V_1(x(s - \tau)) ds,$$

where $\omega = n - 1$. Notice that $e^{\omega\varepsilon(t-s+\tau)} \leq e^{2\tau\omega\varepsilon}, \forall s \in [t - \tau, t]$. Therefore, the above inequality can be continued as

$$\begin{aligned} & \pi^T(t) P(\varepsilon) \pi(t) \\ & \leq (n\varepsilon)^2 \tau e^{2\tau\omega\varepsilon} \int_{t-\tau}^t x^T(s - \tau) P(\varepsilon) x(s - \tau) ds \\ & = (n\varepsilon)^2 \tau e^{2\tau\omega\varepsilon} \int_{t-2\tau}^{t-\tau} x^T(s) P(\varepsilon) x(s) ds \\ & \leq (n\varepsilon)^2 \tau e^{2\tau\omega\varepsilon} \int_{t-2\tau}^t x^T(s) P(\varepsilon) x(s) ds, \end{aligned}$$

substituting of which into (18) gives

$$\begin{aligned} \dot{V}_1(x(t)) & \leq -\varepsilon x^T(t) P(\varepsilon) x(t) \\ & + e^{2\tau\omega\varepsilon} (n\varepsilon)^3 \tau \int_{t-2\tau}^t x^T(s) P(\varepsilon) x(s) ds. \end{aligned} \quad (19)$$

Choose another functional $V_2(x_t)$ as

$$V_2(x_t) = e^{2\tau\omega\varepsilon} (n\varepsilon)^3 \tau \int_0^{2\tau} \left(\int_{t-s}^t x^T(l) P(\varepsilon) x(l) dl \right) ds,$$

whose derivative is equal to

$$\begin{aligned} \dot{V}_2(x_t) & = 2e^{2\tau\omega\varepsilon} (n\varepsilon)^3 \tau^2 x^T(t) P(\varepsilon) x(t) \\ & - e^{2\tau\omega\varepsilon} (n\varepsilon)^3 \tau \int_{t-2\tau}^t x^T(s) P(\varepsilon) x(s) ds. \end{aligned} \quad (20)$$

Now choose the Lyapunov-Krasovskii functional $V(x_t)$ as

$$V(x_t) \triangleq V_1(x(t)) + V_2(x_t). \quad (21)$$

Then it follows from (19) and (20) that the time derivative of $V(x_t)$ along the trajectories of system (16) satisfies

$$\begin{aligned} \dot{V}(x_t) & \leq -\varepsilon x^T(t) P(\varepsilon) x(t) \\ & + 2e^{2\tau\omega\varepsilon} (n\varepsilon)^3 \tau^2 x^T(t) P(\varepsilon) x(t) \\ & = -\varepsilon (1 - 2e^{2\tau\omega\varepsilon} n^3 \varepsilon^2 \tau^2) x^T(t) P(\varepsilon) x(t). \end{aligned} \quad (22)$$

Let $\varepsilon_1^* = \varepsilon_1^*(\tau) > 0$ be such that

$$1 - 2e^{2\tau\omega\varepsilon} n^3 \varepsilon^2 \tau^2 \geq \frac{1}{2}, \quad \forall \varepsilon \in [0, \varepsilon_1^*].$$

Such an ε_1^* clearly exists. Then inequality (22) implies

$$\dot{V}(x_t) \leq -\frac{1}{2} \varepsilon x^T(t) P(\varepsilon) x(t), \quad \forall \varepsilon \in (0, \varepsilon_1^*]. \quad (23)$$

The global stability of the closed-loop system then follows from the Lyapunov stability theorem.

In view of (12) and by using Lemma 11, we get

$$\begin{aligned} \|u(t)\|^2 & = x^T(t) e^{A^T \tau} P(\varepsilon) BB^T P(\varepsilon) e^{A\tau} x(t) \\ & \leq n\varepsilon x^T(t) e^{A^T \tau} P(\varepsilon) e^{A\tau} x(t) \\ & \leq n\varepsilon e^{\omega\varepsilon\tau} x^T(t) P(\varepsilon) x(t). \end{aligned}$$

With this, the inequality in (23) implies that

$$\dot{V}(x_t) \leq -\frac{1}{2n\varepsilon e^{\omega\varepsilon\tau}} \|u(t)\|^2. \quad (24)$$

Integrating both sides of (24) from 0 to ∞ gives

$$\lim_{t \rightarrow \infty} V(x_t) - V(x_\tau) \leq -\frac{1}{2n\varepsilon e^{\omega\varepsilon\tau}} \int_\tau^\infty \|u(t)\|^2 dt, \quad (25)$$

where $x_\tau = \bar{\psi}_\tau(\theta)$ is the initial state. Clearly, $V(x_t)$ in (21) satisfies $\lim_{\|x(t)\| \rightarrow 0} V(x_t) = 0$. Therefore, it follows from $\lim_{t \rightarrow \infty} \|x(t)\| = 0$ that $\lim_{t \rightarrow \infty} V(x_t) = 0$ from which the inequality in (25) can be written as

$$\int_\tau^\infty \|u(t)\|^2 dt \leq 2n\varepsilon e^{\omega\varepsilon\tau} V(x_\tau). \quad (26)$$

On the other hand, we can compute

$$\begin{aligned} \int_{-\tau}^0 \|u(t)\|^2 dt & = \int_{-\tau}^0 \|B^T P(\varepsilon) e^{A\tau} \psi(t)\|^2 dt \\ & \leq \tau \sup_{\psi \in \Omega} \{ \|\psi\|_c \} \|B^T\| \|e^{A\tau}\| \|P(\varepsilon)\|, \end{aligned}$$

and

$$\begin{aligned} \int_0^\tau \|u(t)\|^2 dt & = \int_0^\tau \|B^T P(\varepsilon) e^{A\tau} x(t)\|^2 dt \\ & \leq \tau \|B^T\| \|e^{A\tau}\| \|P(\varepsilon)\| \|\psi\|_c. \end{aligned} \quad (27)$$

As τ is finite and Ω is bounded, it follows from (15), (26), (27) and $\lim_{\varepsilon \rightarrow 0^+} P(\varepsilon) = 0$ that there exists an $\varepsilon^* \in (0, \varepsilon_1^*]$ such that

$$\begin{aligned} \|u\|_{L_2} &= \int_{-\tau}^{\infty} \|u(t)\|^2 dt \\ &= \int_{\tau}^{\infty} \|u(t)\|^2 dt + \int_0^{\tau} \|u(t)\|^2 dt + \int_{-\tau}^0 \|u(t)\|^2 dt \\ &\leq 1, \forall \varepsilon \in (0, \varepsilon^*], \quad \forall \psi \in \Omega. \end{aligned}$$

The proof is ended. ■

Remark 13. By combining the proof of Theorem 12 and the proof of Theorem 4 in Zhou et al. (2010), we can also show that Problem 3 is solvable by output feedback. The details are omitted for brevity.

Remark 14. The result in Theorem 12 can be interpreted in the “ L_2 vanishment” framework. In fact, it follows from the proof of this theorem that the time-delay system

$$\begin{cases} \dot{x}(t) = Ax(t) - BB^T P(\varepsilon) e^{A\tau} x(t - \tau), \\ y(t - \tau) \triangleq u(t - \tau) = -B^T P(\varepsilon) e^{A\tau} x(t - \tau), t \geq 0 \end{cases}$$

is globally asymptotically stable for all $\varepsilon \in (0, \varepsilon^*]$ and

$$\sup_{\psi \in \Omega} \{\|\psi\|_c\} \leq D < \infty \Rightarrow \lim_{\varepsilon \rightarrow 0^+} \|y(t)\|_{L_2} = 0,$$

which is similar to the L_2 vanishment defined in Definition 7.

Remark 15. The semi-global stabilization of system (2) with input magnitude saturation has been solved in Zhou et al. (2010). The above theorem shows that the semi-global stabilization of system (2) with energy constraints can also be achieved which supplements the results in Zhou et al. (2010). We also noticed that we have provided in this paper an elegant proof of the global stability for the closed-loop system by choosing the special Lyapunov-Krasovskii function (21) in the absence of energy (magnitude) constraints which is different from that in Zhou et al. (2010) where the Razumikhin Stability Theorem is utilized.

3.2 Delay-Independent Feedback

In the above subsection, the delay in the system is assumed to be exactly known and constant. But sometimes the delay is time-varying and not exactly known. For this reason, in this subsection, we reconsider system (2) with $\tau = \tau(t) : [0, \infty) \rightarrow \mathbf{R}^+$ being a continuous function of time. Since the bound on $\tau(t)$ can be *arbitrarily large yet finite*, as shown in Zhou et al. (2009), system (2) can be (globally) stabilized provided all the poles of A are zero and (A, B) is stabilizable. A delay-dependent controller was designed in Zhou et al. (2009). In this subsection, we will further show that such controller can also solve Problem 3.

Theorem 16. Assume that (A, B) satisfied Assumption 6 and, moreover, all the eigenvalues of $A \in \mathbf{R}^{n \times n}$ are zero. Let $P(\varepsilon)$ be the unique positive definite solution of the parametric ARE (10). Then there exists an $\varepsilon^* > 0$ such that the following state feedback law

$$u(t) = -B^T P(\varepsilon) x(t), \quad \varepsilon \in (0, \varepsilon^*], \quad (28)$$

solves Problem 3 for all values of delay satisfying

$$0 \leq \tau(t) \leq \bar{\tau}, \quad 0 \leq t < \infty, \quad (29)$$

where $\bar{\tau}$ is arbitrarily large and bounded scalar.

The proof of Theorem 16 is omitted due to space limitation.

4. A NUMERICAL EXAMPLE

Consider a linear system

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & -2\omega & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t - \tau), \quad (30)$$

which is a linearized model of the relative motion in the orbit plane (the in-plane motion) of a spacecraft with respect to another in a circular orbit around the Earth (Ichikawa (2008)), and the scalar τ represents the actuator delay induced by signal transformation (Polites (1999)). The positive scalar ω is the orbit rate (angular velocity) of the satellite in a circular motion. Notice that the Jordan canonical form of A is

$$A_J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{bmatrix},$$

and (A, B) is controllable. Therefore, it follows from Lemma 2 that (A, B) is NCVE. Notice that A is *not* Lyapunov stable. As (A, B) is NCVE, under the condition that the value of τ is exactly known, it follows from Theorem 12 that system (30) can be semi-globally stabilized with bounded energy by linear state feedback. According to Lemma 11, by solving the unique positive definite solution $P(\varepsilon)$ to ARE (10), the state feedback controller is given by

$$u(t) = -B^T P(\varepsilon) e^{A\tau} x(t), \quad (31)$$

where $e^{A\tau}$ is given by

$$\begin{bmatrix} 4 - 3 \cos(\omega\tau) & 0 & \frac{1}{\omega} \sin(\omega\tau) & \frac{2}{\omega} (1 - \cos(\omega\tau)) \\ 6 \sin(\omega\tau) - 6\omega\tau & 1 & \frac{2}{\omega} (\cos(\omega\tau) - 1) & -3\tau + \frac{4}{\omega} \sin(\omega\tau) \\ 3\omega \sin(\omega\tau) & 0 & \cos(\omega\tau) & 2 \sin(\omega\tau) \\ (6 \cos(\omega\tau) - 6)\omega & 0 & -2 \sin(\omega\tau) & -3 + 4 \cos(\omega\tau) \end{bmatrix}$$

For simulation, we choose $\omega = 1$ and the initial condition as $x(\theta) = [2 \ -1 \ 2 \ -1]^T, \theta \in [-\tau, 0]$. For different low gain parameters ε , the L_2 norms of the input signals are plotted in Fig. 1 from which we clearly see that $\|u\|_{L_2}$ approaches to zero as ε does, namely, the L_2 semi-global stabilization problem for system (30) can be solved. Specially, for $\varepsilon = 0.15$ and $\varepsilon = 0.1$, the state evaluation and control signals are recorded in Fig 2.

5. CONCLUSIONS

This paper has considered semi-global stabilization of input delayed linear systems subject to control energy constraints. The parametric Lyapunov equation based L_2 low gain feedback is adopted to solve the problem. Two classes of linear feedback laws, one delay-dependent and the other delay-independent, were proposed. A system that is the linearized model of the relative motion in the orbit plane (the in-plane motion) of a spacecraft with respect to another in a circular orbit around the Earth, was used to illustrate the effectiveness of the proposed approach.

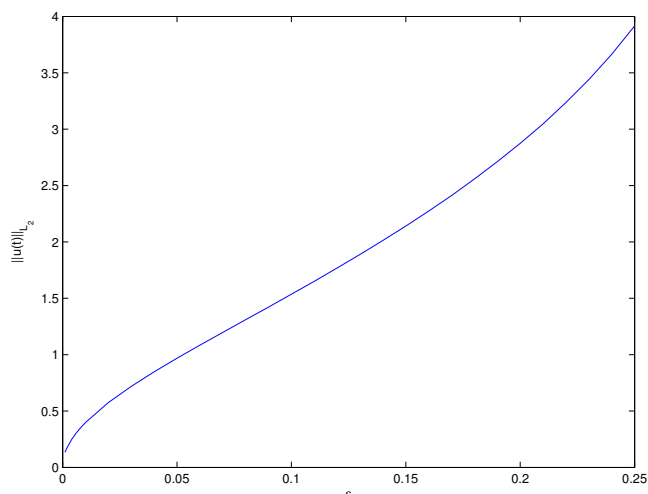


Fig. 1. L_2 norm of the control signal (31) for different values of ε .

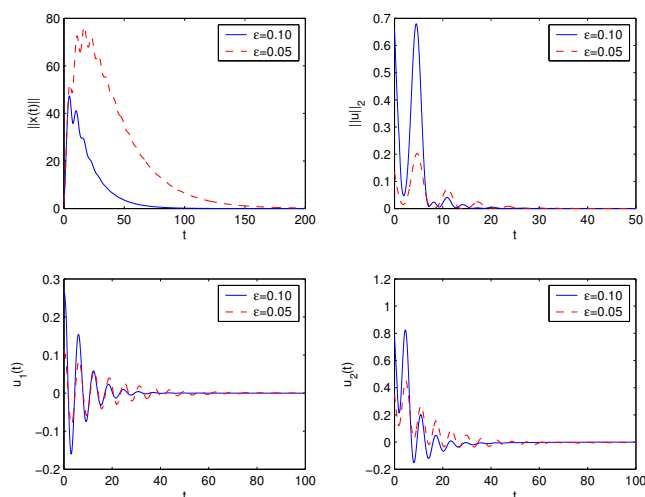


Fig. 2. State evolution and control signals of the time delay system (30) for different values of ε .

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