

IMPULSIVE CONSENSUS FOR COMPLEX DYNAMICAL NETWORKS WITH NONIDENTICAL NODES AND COUPLING TIME-DELAYS*

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Abstract. This paper investigates the problem of global consensus between a complex dynamical network (CDN) and a known goal signal by designing an impulsive consensus control scheme. The dynamical network is complex with respect to the uncertainties, nonidentical nodes, and coupling time-delays. The goal signal can be a measurable vector function or a solution of a dynamical system. By utilizing the Lyapunov function and Lyapunov–Krasovskii functional methods, robust global exponential stability criteria are derived for the error system, under which global exponential impulsive consensus is achieved for the CDN. These criteria are expressed in terms of linear matrix inequalities (LMIs) and algebraic inequalities. Thus, the impulsive controller can be easily designed by solving the derived inequalities. Meanwhile, the estimations of the consensus rate for global exponential consensus are also obtained. Two examples with numerical simulations are worked out for illustration.

Key words. complex dynamical networks, synchronization, impulsive consensus, global exponential impulsive consensus, robust global exponential stability, consensus rate, time-delays, Lyapunov–Krasovskii function

AMS subject classifications. 93C30, 34K34, 34K45, 93D15, 93C65, 93C95

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1. Introduction. Synchronization of chaotic systems and its potential applications to secure communication has been an active research area since the 1990s. Numerous methods have been developed for chaos synchronization (see, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 49]). Recently, the more general problem of synchronization of complex dynamical networks (CDNs) has been studied in the literature (see, for example, [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]). A CDN consists of coupled nodes, which are usually dynamical systems. Several approaches have been proposed for synchronization of a CDN; for example, feedback control synchronization (see, for example, [22, 23]), adaptive synchronization [20], synchronization based on the invariance principle [10], the state-observer-based approach [26], and the impulsive synchronization scheme [31, 35, 54], etc.

It has been noticed that the synchronization of CDNs studied in the literature has limitations such as the following: (i) the coupled nodes have the same dynamics; (ii) by using the linearization technique and matrix eigenvalue method, most synchronization criteria are local; and (iii) uncertainty and time-delays which are common in practical CDNs have not been studied fully, although there are published results [8, 15, 20, 35] which study the robust synchronization problem with respect to uncertainties, and some results [21, 23, 24, 53] deal with a single constant time-delay.

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Uncertainties often occur due to parameter errors, modeling mismatches, measurement errors, approximations, channel noises, etc. And time-delays commonly occur due to the congestion of the network traffic and the fact that the switching and spreading speed of the hardware and circuit implementation is finite. Moreover, the time-delays presented in many real synchronization schemes are difficult to know a priori, are in the form of multiple time-delays, and are time-varying.

The CDNs with nonidentical nodes represent more general and practical networks than the models typically studied in the literature. For example, in an ecological species network, even if all the nodes belong to the same species, different individuals may have different characteristics and dynamic behaviors, and thus this species network cannot be modeled as a network with identical nodes. Moreover, some abrupt events (impulses) such as flu pandemic, war, etc., will lead to changes of species population and dynamic behavior in the network. For another example, here we consider military systems. Suppose one wants to use a missile network to attack the enemy's aircraft carrier in the ocean. Obviously, it needs many missiles from different military bases, and the target is composed of the static carrier and the aircraft (which might not be static) on it. We can look at all the missiles as nodes in the network, and the nodes from different bases will have different flying trajectories. Hence, this missile network is a CDN with nonidentical nodes. Moreover, it is necessary to adjust (via impulsive adjustment control) the trajectories of all missiles (nodes) before the network hits the target (achieves the consensus). So it is important to study the dynamical properties for CDNs with nonidentical nodes. However, to the best of our knowledge, no literature has been published for the consensus control problem of CDNs with nonidentical nodes. Obviously, allowing different dynamics of nodes in a CDN brings difficulties in achieving consensus. Without control, it has been shown recently how to quantify the ultimate bounds on the synchronization errors as related to the diversity of the node dynamics [42]. If uncertainty and time-delays occur simultaneously in a CDN with nonidentical nodes, and consensus is to be achieved to a known goal function, it will be much more difficult to use previous synchronization control schemes, especially for the global consensus problem. Hence, there is a need to study new consensus control schemes which can achieve the objective.

In this paper, we propose an impulsive consensus control scheme for the consensus problem between CDNs and the known signal. In this control scheme, the control signal is designed to be input into the CDN as follows: at impulsive instances, the impulse signal is input into the nodes, and at other times, the signal containing the goal signal is input into the nodes. Hence, this control scheme is a type of impulsive hybrid control scheme. Impulsive control arises naturally from a wide variety of applications, such as drug administration, spacecraft control, inspection processes in operations research, and native forest ecosystems management, just to name a few. Based on the stability theory of impulsive systems (see [28, 29, 30, 34, 36, 37, 38, 39, 40] and references therein), the impulsive control method (see [31, 32, 33, 34, 35, 46, 47, 48] and references therein) provides a greater prospect for solving many problems that are basically defined by continuous dynamical systems, but on which only discrete-time actions are exercised. An essential benefit of the impulsive control approach may be derived from the fact that such controls are typically simpler to implement and cheaper. In [9, 11], impulsive control was introduced to synchronize chaotic systems. Recently, impulsive synchronization for CDNs was also reported in [35]. Theory and experiments have proved that the impulsive synchronization scheme for chaotic systems or CDNs (with identical nodes) has good robustness against uncertainties and can achieve global synchronization.

The aim of this paper is to study the global exponential impulsive consensus (GEIC) problem for CDNs and a known goal function. The model of CDNs consists of the nonidentical nodes, uncertainties, and coupling time-delays. Here, the “uncertainties” means the uncertain parameters, which take values in some intervals. The exponential consensus scheme has an obvious advantage over other consensus schemes in that the consensus speed and consensus time can be estimated easily. By utilizing the Lyapunov function and Lyapunov–Krasovskii functional [41, 42] methods, robust global exponential stability results for delay error systems shall be established, and then we shall derive several criteria under which the GEIC is achieved for the uncertain CDN with nonidentical nodes and coupling time-delays. These criteria are expressed in terms of LMIs (linear matrix inequalities) and algebraic inequalities. Thus, the conditions of consensus are easy to test. Moreover, the solutions of the LMIs and algebraic inequalities give rise directly to impulsive controllers to achieve GEIC.

The organization of this paper is as follows. In sections 2 and 3, we define some notation and give some preliminaries. In section 4, impulsive consensus criteria are established. The impulsive consensus control is designed in section 5. Two representative examples are given in section 6. Section 7 concludes the paper.

2. Notation. We list the notation and symbols used in this paper as follows:

Let \mathbb{R}^n denote the n -dimensional Euclidean space, and let $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{Z} = \{0, 1, 2, \dots\}$, and let $\|\cdot\|$ be the Euclidean norm. Let I be the identity matrix. Matrix $X > (\geq, <, \leq) 0$ means that X is a symmetric positive definite (positive semidefinite, negative definite, negative semidefinite) matrix. Denote by $\lambda_{\max}(\cdot)$ ($\lambda_{\min}(\cdot)$) the maximal (minimal) eigenvalue of matrix (\cdot) . For a positive real number $\tau > 0$ and any $\varphi \in C([t_0 - \tau, t_0], \mathbb{R}^n)$, we define $\|\varphi\|_\tau \triangleq \sup_{t_0 - \tau \leq s \leq t_0} \{\|\varphi(s)\|\}$.

For a sequence $\{t_k, k \in \mathbb{Z}\}$ satisfying $0 \leq t_0 < t_1 < \dots < t_k < t_{k+1} < \dots$, let $\Delta_k \triangleq t_{k+1} - t_k, k \in \mathbb{Z}$, $\Delta_{\sup} \triangleq \sup_{k \in \mathbb{Z}} \{\Delta_k\}$, $\Delta_{\inf} \triangleq \inf_{k \in \mathbb{Z}} \{\Delta_k\}$, $S[t_0, t] \triangleq \max\{k : t_0 \leq t_k \leq t\}$ for any $t \geq t_0$.

Denote $\Sigma^* \triangleq \{\Sigma \in \mathbb{R}^{n^2 \times n^2} : \Sigma = \text{diag}(\epsilon_{11}, \dots, \epsilon_{n^2 n^2}), |\epsilon_{ij}| \leq 1; i, j = 1, 2, \dots, n\}$, and define $N[U, V] = \{(s_{ij}) \in \mathbb{R}^{n \times n} : u_{ij} \leq s_{ij} \leq v_{ij}, i, j = 1, 2, \dots, n\}$ as the interval matrix for known matrices $U = (u_{ij})_{n \times n}$ and $V = (v_{ij})_{n \times n}$.

The following abbreviations are used in the paper:

- GAS: globally asymptotically stable.
- RGAS: robustly globally asymptotically stable.
- GES: globally exponentially stable.
- RGES: robustly globally exponentially stable.
- GEIC: global exponential impulsive consensus.
- GIC: global impulsive consensus.

3. Preliminaries. Consider the uncertain CDN with N nonidentical nodes and coupling time-delays,

$$(3.1) \quad S_i : \quad \dot{x}_i = f_i(t, x_i) + g_i(x_1(t - h_i), \dots, x_N(t - h_i)), \quad i = 1, 2, \dots, N,$$

where $x = (x_1^T, x_2^T, \dots, x_n^T)^T \in \mathbb{R}^{nN}$; $x_i = (x_{i1}, x_{i2}, \dots, x_{in})^T \in \mathbb{R}^n$ represents the state of the i th node S_i ; $f_i : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth nonlinear vector function; $g_i : \mathbb{R}^{nN} \rightarrow \mathbb{R}^n$ is a smooth nonlinear vector function representing the coupling of the i th node with other nodes; and h_i represents the coupling time-delay of the signal transmitted from the network to the i th node, where h_i satisfies $0 \leq h_i \leq \tau$ for some constant $\tau > 0$ and $i = 1, 2, \dots, N$. We call the system (3.1) an uncertain CDN based

on the fact that there are uncertainties in functions f_i, g_i , $i = 1, 2, \dots, N$. In this paper, we use the following assumption.

Assumption 3.1. Assume that $f_i(t, x_i) = A_i x_i + \tilde{f}_i(t, x_i)$, where $A_i \in \mathbb{R}^{n \times n}$ is an interval matrix with $A_i \in N[A_{i_1}, A_{i_2}]$ for known matrices A_{i_1} and A_{i_2} , and functions \tilde{f}_i , $i = 1, \dots, N$, satisfy

$$\|\tilde{f}_i(t, s_1) - \tilde{f}_i(t, s_2)\| \leq L_i \|s_1 - s_2\|, \quad i = 1, \dots, N,$$

for some positive constants $L_i > 0$, $i = 1, \dots, N$, and for all $t \in \mathbb{R}_+$.

It should be noted that when the network (3.1) achieves consensus, namely, when the states $x_i(t) \rightarrow s(t)$, $i = 1, 2, \dots, N$, as $t \rightarrow \infty$, the coupling terms should vanish, i.e., $g_i(s, s, \dots, s) = 0$, $i = 1, 2, \dots, N$. Thus, we give the following assumption on function g_i .

Assumption 3.2. Assume that $g_i(x_1, \dots, x_N) = \sum_{j=1}^N B_{ij} x_j + \tilde{g}_i(x_1, \dots, x_N)$, where $B_{ij} \in \mathbb{R}^{n \times n}$ with $B_{ij} \in N[B_{ij_1}, B_{ij_2}]$ for known matrices B_{ij_1} and B_{ij_2} , and functions \tilde{g}_i satisfy $\tilde{g}_i(s, s, \dots, s) = 0$, $i = 1, \dots, N$, respectively, for some nonnegative constants $M_{ij} \geq 0$, $i, j = 1, \dots, N$,

$$(3.2) \quad \sum_{j=1}^N B_{ij} = 0, \quad i = 1, 2, \dots, N;$$

$$(3.3) \quad \|\tilde{g}_i(x_1, \dots, x_N) - \tilde{g}_i(y_1, \dots, y_N)\| \leq \sum_{j=1}^N M_{ij} \|x_j - y_j\|.$$

Problem formulation. Let $s(t)$ be a given measurable smooth vector function satisfying $s(t) \in \mathbb{R}^n$ for any $t \in \mathbb{R}_+$. The aim of this paper is to design an impulsive hybrid control scheme for CDN (3.1) such that consensus among the node states $x_i(t)$ ($i = 1, 2, \dots, N$) and $s(t)$ can be achieved.

Consider the uncertain CDN (3.1) under impulsive consensus control,

$$(3.4) \quad \dot{x}_i = f_i(t, x_i) + g_i(x_1(t - h_i), \dots, x_N(t - h_i)) + u_i(t, x_i, s), \quad i = 1, \dots, N,$$

where $\{u_i(t, x_i, s), i = 1, 2, \dots, N\}$ is the *impulsive hybrid controller* designed as (see Figure 3.1)

$$u_i(t, x_i, s) = \dot{s} - f_i(t, s) + \sum_{k=1}^{\infty} \delta(t - t_k) (C_{i_k} - I)(x_i(t) - s(t)),$$

where $C_{i_k} \in \mathbb{R}^{n \times n}$, $k \in \mathbb{Z}$, are *impulsive control gain matrices* to be designed; $\{t_k, k \in \mathbb{Z}\}$ are the *impulsive instances* satisfying $0 \leq t_0 < t_1 < t_2 < \dots$, with $\Delta_{\text{sup}} < \infty$ and $\lim_{k \rightarrow \infty} t_k = \infty$; and $\delta(\cdot)$ is the Dirac delta function.

Figure 3.1 depicts the entire impulsive control scheme for consensus (“impulsive consensus scheme” for short) between the known signal $s(t)$ and CDN (3.1) with coupling time-delays, where S_i stands for the i th node, S generates the objective vector function $s(t)$, and g_i is the delay network coupling of the i th node, $i = 1, 2, \dots, N$. In this control scheme, the control signal is input into the CDN as follows: at impulsive instance t_k , the impulse signal $(C_{i_k} - I)(x_i(t_k) - s(t_k))$ is input into the node S_i , and at other nonimpulse times $t \neq t_k$, the signal $\dot{s} - f_i(t, s)$ containing the goal signal is input into the node S_i of the CDN.

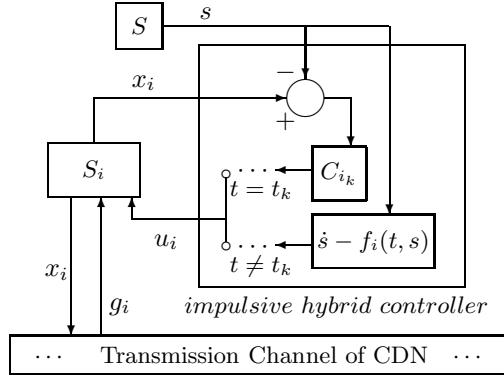


FIG. 3.1. Impulsive consensus scheme of S_i .

By the property of the Dirac delta function $\delta(\cdot)$ and assuming that $x(t^+) = x(t)$, (3.4) is equivalent to the following system:

$$(3.5) \quad \begin{cases} \dot{x}_i = f_i(t, x_i) + g_i(x_1(t - h_i), \dots, x_N(t - h_i)) + \dot{s} - f_i(t, s), & t \in [t_k, t_{k+1}), \\ \Delta x_i(t_k) = (C_{i_k} - I)(x_i(t_k) - s(t_k)), & t = t_k, k \in \mathbb{Z}, i = 1, 2, \dots, N, \end{cases}$$

where $\Delta x_i(t_k) = x_i(t_k) - x_i(t_k^-)$, $k \in \mathbb{Z}$, $i = 1, 2, \dots, N$.

Define the consensus errors as $e_i(t) := x_i(t) - s(t)$. Then, by Assumptions 3.1 and 3.2, one has an error dynamical system of the form

$$(3.6) \quad \begin{cases} \dot{e}_i = A_i e_i + \tilde{f}_i(t, x_i, s) + \sum_{j=1}^N B_{ij} e_j(t - h_i) + \hat{g}_i(x(t - h_i), s), & t \in [t_k, t_{k+1}), \\ \Delta e_i = (C_{i_k} - I)e_i(t), & t = t_k, k \in \mathbb{Z}, i = 1, 2, \dots, N, \end{cases}$$

where $\tilde{f}_i(t, x_i, s) = \hat{f}_i(t, x_i) - \tilde{f}_i(t, s)$ and $\hat{g}_i(x(t - h_i), s) = \tilde{g}_i(x_1(t - h_i), \dots, x_N(t - h_i)) - \tilde{g}_i(s(t - h_i), \dots, s(t - h_i))$, $i = 1, 2, \dots, N$.

Remark 3.1. It should be noted that system (3.1) represents a more general CDN than that considered in the literature in the following sense:

1. The nodes in (3.1) are nonidentical with uncertainties; i.e., functions f_i ($i = 1, 2, \dots, N$) can be different and uncertain.
2. If function $s(t)$ satisfies $\dot{s} = f_i(t, s)$, $i = 1, 2, \dots, N$, i.e., $s(t)$ is the common solution of every single node, then the impulsive consensus scheme is to make all states of the nonidentical nodes approach the same solution $s(t)$.
3. If all f_i are the same, i.e., $f_i = f$, $i = 1, 2, \dots, N$, and $s(t)$ is a solution of system $\dot{y} = f(t, y)$, then the consensus problem is the synchronization problem discussed in the literature; see, for example, [20, 21, 22, 23, 24, 25, 26, 27].

4. In the above mentioned literature on CDNs, the coupling coefficient matrices have the special form $B_{ij} = cC_{ij}\Gamma$, where the constant $c > 0$ denotes the coupling strength, $\Gamma = \text{diag}\{r_1, r_2, \dots, r_n\}$, and $C = (C_{ij})_{N \times N}$ are the coupling matrices of network nodes. And only a single time-delay τ , i.e., $h_i = \tau$, $i = 1, \dots, N$, is considered.

Remark 3.2. In the above impulsive consensus scheme, since the target s is known, there are no couplings in u_i between node S_i and other nodes, $i = 1, 2, \dots, N$. If one wants to design the control scheme with couplings in it (that is, a *distributed control scheme*), then we can revise u_i as \hat{u}_i :

$$\hat{u}_i(t, x, s) = \dot{s} - f_i(t, s) + \sum_{k=1}^{\infty} \delta(t - t_k) \left[(C_{i_k} - I)(x_i(t) - s(t)) + \sum_{j=1, j \neq i}^N C_{ij_k}(x_j(t) - s(t)) \right].$$

Clearly, if the consensus target s is not known a priori, but equations $\dot{s} = f_i(t, s)$, $i = 1, 2, \dots, N$, have a common solution s , the control \hat{u}_i with couplings is changed to \tilde{u}_i : $\tilde{u}_i = \sum_{k=1}^{\infty} \sum_{j=1}^N \delta(t - t_k) C_{ij_k} (x_j(t) - s(t))$. For these two distributed control schemes with couplings in them, the consensus issue between x_i , $i = 1, 2, \dots, N$, and s can be studied similarly as the case for u_i used in this paper.

It should be noted that if s is not specified and $\dot{s} = f_i(t, s)$, $i = 1, 2, \dots, N$, do not have any common solution, then, for the consensus issue of CDN (3.1), the distributed control scheme is necessary. In this case, letting $\bar{x}(t) = \frac{1}{N} \sum_{i=1}^N x_i(t)$, we can design the *impulsive distributed control* \bar{u}_i as

$$\bar{u}_i(t, x, \bar{x}) = \sum_{k=1}^{\infty} \delta(t - t_k) \left[(C_{ii_k} - I)(x_i(t) - \bar{x}(t)) + \sum_{j=1, j \neq i}^N C_{ij_k} (x_j(t) - \bar{x}(t)) \right].$$

Then the aim is to achieve consensus between x_i , $i = 1, 2, \dots, N$, and \bar{x} . This case is beyond the scope of this paper and is the subject of current work on impulsive consensus.

By [36], for any $X \in N[X_1, X_2]$, X can be formulated as

$$(3.7) \quad X = X_0 + \Delta X = X_0 + E\Sigma F,$$

where $X_0 = \frac{1}{2}(X_1 + X_2)$, $H = \frac{1}{2}(X_2 - X_1) = (h_{ij})_{n \times n}$, $EE^T = \text{diag}\{\sum_{j=1}^n h_{1j}, \dots, \sum_{j=1}^n h_{nj}\}$, $F^T F = \text{diag}\{\sum_{j=1}^n h_{j1}, \dots, \sum_{j=1}^n h_{jn}\}$, $\Sigma \in \Sigma^*$.

Assumption 3.3. For interval matrices A_i, B_{ij} in CDN (3.1), there exist known matrices E, F_{A_i}, F_{ij} such that for any $\Sigma \in \Sigma^*$, $i, j = 1, 2, \dots, N$,

$$(3.8) \quad [\Delta A_i \quad \Delta B_{ij}] = E\Sigma [F_{A_i} \quad F_{ij}].$$

DEFINITION 3.1. The error system (3.6) is said to be *RGAS* if, for any initial condition, $\phi \in C[[t_0 - \tau, t_0], \mathbb{R}^{nN}]$, any $A_i \in N[A_{i1}, A_{i2}]$, $B_{ij} \in N[B_{ij1}, B_{ij2}]$, and for any time-delays h_j with $0 \leq h_j \leq \tau$, the trivial solution of (3.6) is *GAS*.

DEFINITION 3.2. The error system (3.6) is said to be *RGES* with decay rate α if, for any initial condition, $\phi \in C[[t_0 - \tau, t_0], \mathbb{R}^{nN}]$, any $A_i \in N[A_{i1}, A_{i2}]$, $B_{ij} \in N[B_{ij1}, B_{ij2}]$, and for any time-delays h_j with $0 \leq h_j \leq \tau$, the trivial solution of (3.6) is *GES*; i.e., there exist two positive numbers $\alpha > 0, K > 0$ such that

$$(3.9) \quad \|e(t)\| \leq K \|\phi\|_{\tau} e^{-\alpha(t-t_0)}, \quad t \geq t_0,$$

where $\phi(t) = (\phi_1^T(t), \dots, \phi_N^T(t))^T \in \mathbb{R}^{nN}$, $\phi_i(t) \in \mathbb{R}^n$, and $\|\phi\|_{\tau}^2 = \sum_{i=1}^N \|\phi_i\|_{\tau}^2$.

DEFINITION 3.3. The impulsive consensus scheme is said to achieve *GEIC* at rate α if, for any initial condition ϕ , the error system (3.6) is *RGES* with decay rate α . If the system (3.6) is *RGAS*, then we say the network (3.1) can achieve *GIC*.

Remark 3.3. In the *GEIC* scheme, the consensus speed or consensus time can be estimated by using the decay rate, while in the *GIC* scheme this estimation is not presented.

LEMMA 3.1 (see [43]). Let $E \in \mathbb{R}^{n \times n^2}$, $F \in \mathbb{R}^{n^2 \times n}$, and $Y \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then, for any $\Sigma \in \Sigma^*$,

$$(3.10) \quad Y + E\Sigma F + F^T \Sigma E^T < 0$$

holds if and only if there exists a positive constant $\epsilon > 0$ such that

$$(3.11) \quad Y + \epsilon E E^T + \epsilon^{-1} F^T F < 0.$$

4. GEIC properties of CDNs. In this section, two types of GEIC properties of CDN (3.1) will be investigated. The first aims to show how impulses affect the consensus property, and the second how impulses can contribute to the achievement of consensus.

By Assumption 3.3, we denote $A_i = A_{i0} + E\Sigma F_{A_i}$ and $B_{ij} = B_{ij0} + E\Sigma F_{ij}$, where $\Sigma \in \Sigma^*$.

THEOREM 4.1. *Suppose that Assumptions 3.1–3.3 hold and that there exist positive definite matrices $P_i \in \mathbb{R}^{n \times n}$ and constants $\epsilon_{ij} > 0$, $\epsilon_i > 0$, $\alpha_i < 0$, $i, j = 1, 2, \dots, N$, such that*

(i) *there exist some positive constants $\nu_i > 0, \mu_i > 0$, satisfying*

$$(4.1) \quad \nu_i I \leq P_i \leq \mu_i I, \quad i = 1, 2, \dots, N;$$

(ii) *for $i = 1, 2, \dots, N$, the following LMIs hold:*

$$(4.2) \quad \begin{pmatrix} \Psi_i(A_{i0}) - \alpha_i P_i & P_i B_{i1_0} & \cdots & P_i B_{iN_0} & P_i E & F_{A_i}^T \\ B_{i1_0}^T P_i & -\epsilon_{i1} I & \cdots & 0 & 0 & F_{i1}^T \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ B_{iN_0}^T P_i & 0 & \cdots & -\epsilon_{iN} I & 0 & F_{iN}^T \\ E^T P_i & 0 & \cdots & 0 & -\epsilon_i^{-1} I & 0 \\ F_{A_i} & F_{i1} & \cdots & F_{iN} & 0 & -\epsilon_i I \end{pmatrix} < 0,$$

where $\Psi_i(A_{i0}) = P_i A_{i0} + A_{i0}^T P_i + 2L_i \sqrt{\frac{\mu_i}{\nu_i}} P_i + \sum_{j=1}^N M_{ij} \epsilon_{ij}^{-1} \|P_i\| I$, and α_i, ϵ_{ij} satisfy

$$(4.3) \quad -\max_{1 \leq i \leq N} \{\alpha_i\} > \sum_{i=1}^N \max_{1 \leq j \leq N} \left\{ \frac{\hat{\epsilon}_{ij}}{\nu_j} \right\},$$

where $\hat{\epsilon}_{ij} = \epsilon_{ij} + M_{ij} \epsilon_{ij} \|P_i\|$;

(iii) *there exist positive constants $\delta > 1, \sigma > 0$ satisfying*

$$(4.4) \quad \frac{\ln \beta_k}{\sigma} < \tau \delta \leq \Delta_{\text{inf}}, \quad k \in \mathbb{Z},$$

$$(4.5) \quad \sigma + a + b e^{\sigma \tau} = 0,$$

where $\beta_k = \max_{1 \leq i \leq N} \{\lambda_{\max}(P_i^{-1} C_{ik}^T P_i C_{ik})\}$, $a = \max_{1 \leq i \leq N} \{\alpha_i\}$, and $b = \sum_{i=1}^N \max_{1 \leq j \leq N} \left\{ \frac{\hat{\epsilon}_{ij}}{\nu_j} \right\}$. Then, the error system (3.6) is RGES with decay rate $\alpha \triangleq \frac{1}{2} \left\{ \sigma - \frac{\ln M}{\delta \tau} \right\}$, and hence the CDN (3.1) can achieve GEIC with $s(t)$ at rate α , where $M = \sup_{k \in \mathbb{Z}} \{\beta_k, e^{\sigma \tau}\}$.

Proof. Let the Lyapunov function candidate be $V(e) = \sum_{i=1}^N e_i^T P_i e_i$. For $t \in [t_k, t_{k+1})$, taking the Dini derivative of $V(e(t))$ along the trajectory of system (3.6), we get

$$(4.6) \quad D^+ V(e(t)) = \sum_{i=1}^N \left\{ 2e_i^T(t) P_i \left(A_i e_i(t) + \hat{f}_i + \sum_{j=1}^N B_{ij} e_j(t - h_i) + \hat{g}_i(x(t - h_i), s) \right) \right\}.$$

By condition (i) and Assumptions 3.1–3.3, we get

$$(4.7) \quad 2e_i^T P_i \hat{f}_i \leq 2\sqrt{e_i^T P_i e_i} \sqrt{e_i^T \hat{f}_i^T P_i \hat{f}_i} \leq 2L_i \sqrt{\frac{\mu_i}{\nu_i}} e_i^T P_i e_i,$$

$$(4.8) \quad 2e_i^T P_i \hat{g}_i(x(t - h_i), s) \leq \sum_{j=1}^N M_{ij} \epsilon_{ij}^{-1} \|P_i\| e_i^T e_i + \sum_{j=1}^N M_{ij} \epsilon_{ij} \|P_i\| e_j^T(t - h_i) e_j(t - h_i),$$

where $\epsilon_{ij} > 0$ ($i, j = 1, 2, \dots, N$) are some positive constants.

It follows from (4.6)–(4.8) that, for $t \in [t_k, t_{k+1})$,

$$(4.9) \quad D^+V(e(t)) \leq \sum_{i=1}^N \begin{pmatrix} e_i(t) \\ e_1(t - h_i) \\ \vdots \\ e_N(t - h_i) \end{pmatrix}^T \begin{pmatrix} \Psi_i(A_i) & P_i B_{i1} & \cdots & P_i B_{iN} \\ B_{i1}^T P_i & -\epsilon_{i1} I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{iN}^T P_i & 0 & \cdots & -\epsilon_{iN} I \end{pmatrix} \begin{pmatrix} e_i(t) \\ e_1(t - h_i) \\ \vdots \\ e_N(t - h_i) \end{pmatrix} + \sum_{i=1}^N \sum_{j=1}^N \hat{\epsilon}_{ij} e_j^T(t - h_i) e_j(t - h_i),$$

where $\Psi_i(A_i) = P_i A_i + A_i^T P_i + 2L_i \sqrt{\frac{\mu_i}{\nu_i}} P_i + \sum_{j=1}^N M_{ij} \epsilon_{ij}^{-1} \|P_i\| I$ and $\hat{\epsilon}_{ij} = \epsilon_{ij} + M_{ij} \epsilon_{ij} \|P_i\|$.

By Assumption 3.3, for $i = 1, 2, \dots, N$, we get

$$(4.10) \quad \Lambda_i \triangleq \begin{pmatrix} \Psi_i(A_i) - \alpha_i P_i & P_i B_{i1} & \cdots & P_i B_{iN} \\ B_{i1}^T P_i & -\epsilon_{i1} I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{iN}^T P_i & 0 & \cdots & -\epsilon_{iN} I \end{pmatrix} = \begin{pmatrix} \Psi_i(A_{i_0}) - \alpha_i P_i & P_i B_{i1_0} & \cdots & P_i B_{iN_0} \\ B_{i1_0}^T P_i & -\epsilon_{i1} I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{iN_0}^T P_i & 0 & \cdots & -\epsilon_{iN} I \end{pmatrix} + \begin{pmatrix} P_i E \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Sigma \begin{pmatrix} F_{A_i}^T \\ F_{i1}^T \\ \vdots \\ F_{iN}^T \end{pmatrix}^T + \begin{pmatrix} F_{A_i}^T \\ F_{i1}^T \\ \vdots \\ F_{iN}^T \end{pmatrix} \Sigma^T \begin{pmatrix} P_i E \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T.$$

By Lemma 3.1, for every $i = 1, 2, \dots, N$, we get that $\Lambda_i < 0$ holds if and only if there exists positive constant $\varepsilon_i > 0$ such that

$$(4.11) \quad \Lambda_i + \varepsilon_i \begin{pmatrix} P_i E \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} P_i E \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T + \varepsilon_i^{-1} \begin{pmatrix} F_{A_i}^T \\ F_{i1}^T \\ \vdots \\ F_{iN}^T \end{pmatrix} \begin{pmatrix} F_{A_i}^T \\ F_{i1}^T \\ \vdots \\ F_{iN}^T \end{pmatrix}^T < 0.$$

By the Schur complement theorem [45], for $i = 1, 2, \dots, N$, the inequality (4.11) is equivalent to (4.2). Thus, for $t \in [t_k, t_{k+1})$, by condition (ii), we get that

$$(4.12) \quad \begin{aligned} D^+V(e(t)) &\leq \sum_{i=1}^N \alpha_i e_i^T P_i e_i + \sum_{i=1}^N \sum_{j=1}^N \hat{\epsilon}_{ij} e_j^T (t - h_i) e_j (t - h_i) \\ &\leq \max_{1 \leq i \leq N} \{\alpha_i\} \cdot V(e(t)) + \sum_{i=1}^N \max_{1 \leq j \leq N} \left\{ \frac{\hat{\epsilon}_{ij}}{\nu_j} \right\} \cdot \bar{V}(t), \end{aligned}$$

where $\bar{V}(t) = \max_{-\tau \leq \theta \leq 0} \{V(e(t + \theta))\}$.

Hence, by the Halanay lemma and condition (iii), there exists a constant $\sigma > 0$ such that

$$(4.13) \quad V(e(t)) \leq \bar{V}(t_k) e^{-\sigma(t-t_k)}, \quad t \in [t_k, t_{k+1}),$$

where $\bar{V}(t_k) = \sup_{t_k - \tau \leq s \leq t_k} \{V(s)\}$ and $\sigma > 0$ satisfies (4.5).

When $t = t_k$, by condition (iii), we have

$$(4.14) \quad V(e(t_k)) = \sum_{i=1}^N e_i^T(t_k^-) C_{i_k}^T P_i C_{i_k} e_i(t_k^-) \leq \beta_k V(x(t_k^-)),$$

where $\beta_k = \max_{1 \leq i \leq N} \{\lambda_{\max}(P_i^{-1} C_{i_k}^T P_i C_{i_k})\}$, $k \in \mathbb{Z}$.

Denote $\mu = \max_{1 \leq i \leq N} \{\mu_i\}$ and $M = \sup_{k \in \mathbb{Z}} \{\beta_k, e^{\sigma\tau}\}$. We show, by induction, that

$$(4.15) \quad V(e(t)) \leq \mu M^{k-1} \|\phi\|_\tau^2 e^{-\sigma(t-t_0)}, \quad t \in [t_{k-1}, t_k), \quad k \in \mathbb{N}.$$

When $k = 1$, since for all $t \in [t_0 - \tau, t_0]$, $\|e_i(t)\| \leq \|\phi_i\|_\tau = \sup_{t_0 - \tau \leq t \leq t_0} \|\phi_i(t)\|$, we get for all $t \in [t_0 - \tau, t_0]$, $V(e(t)) \leq \max_{1 \leq i \leq N} \{\lambda_{\max}(P_i)\} \cdot \sum_{i=1}^N \|e_i(t)\|^2 \leq \mu \cdot \|\phi\|_\tau^2$, where $\|\phi\|_\tau^2 = \sum_{i=1}^N \|\phi_i\|_\tau^2$. Hence, we have $\bar{V}(t_0) \leq \mu \cdot \|\phi\|_\tau^2$. Thus, by (4.13), we get, for $t \in [t_0, t_1)$,

$$(4.16) \quad V(e(t)) \leq \bar{V}(t_0) e^{-\sigma(t-t_0)} \leq \mu \cdot \|\phi\|_\tau^2 e^{-\sigma(t-t_0)} = M^0 \mu \cdot \|\phi\|_\tau^2 e^{-\sigma(t-t_0)}.$$

Thus, (4.15) holds for $k = 1$.

Now, assume (4.15) holds for $k \leq m$, where $m > 1$. Then, we show that (4.15) holds for $k = m + 1$. By (4.14), (4.4), and the induction assumption, we have

$$(4.17) \quad \begin{aligned} V(e(t_m)) &\leq \beta_m V(e(t_m^-)) \leq \mu \beta_m M^{m-1} \|\phi\|_\tau^2 e^{-\sigma(t_m - t_0)} \\ &\leq \mu M^m \|\phi\|_\tau^2 e^{-\sigma(t_m - t_0)}. \end{aligned}$$

Hence, by condition (iii) and (4.17), for $k = m + 1$, $t \in [t_m, t_{m+1})$ we get

$$(4.18) \quad \begin{aligned} V(e(t)) &\leq \bar{V}(t_m) \cdot e^{-\sigma(t-t_m)} = \max \left\{ \sup_{t_m - \tau \leq t < t_m} \{V(e(t))\}, V(e(t_m)) \right\} e^{-\sigma(t-t_m)} \\ &\leq \max \left\{ \mu M^{m-1} \|\phi\|_\tau^2 e^{-\sigma(t_m - \tau - t_0)}, \mu M^m \|\phi\|_\tau^2 e^{-\sigma(t_m - t_0)} \right\} e^{-\sigma(t-t_m)} \\ &\leq \mu M^m \|\phi\|_\tau^2 e^{-\sigma(t-t_0)}. \end{aligned}$$

Therefore, by the induction principle, we see that (4.15) holds for all $k \in N$.

Lastly, we show that

$$(4.19) \quad \|e(t)\| \leq K \|\phi\|_\tau e^{-\alpha(t-t_0)}, \quad t \geq t_0,$$

where $\alpha = \frac{1}{2}\{\sigma - \frac{\ln M}{\delta\tau}\} > 0, K = \sqrt{\frac{L}{\nu}}, \nu = \min_{1 \leq i \leq N}\{\nu_i\}$.

Since $\delta\tau \leq \Delta_{inf} = \inf_{k \in N}\{t_{k+1} - t_k\}$, we have $k \leq \frac{t_k - t_0}{\delta\tau}$, which implies $M^k \leq e^{\frac{\ln M}{\delta\tau}(t_k - t_0)}$. Thus, for $t \in [t_k, t_{k+1})$, we get

$$(4.20) \quad \|e(t)\|^2 \leq \frac{V(e(t))}{\nu} \leq \frac{\mu}{\nu} \|\phi\|_\tau^2 M^{k-1} e^{-\sigma(t-t_0)} \leq \frac{\mu}{\nu} \|\phi\|_\tau^2 e^{-(\sigma - \frac{\ln M}{\delta\tau})(t-t_0)}.$$

Hence, by (4.20), the error system (3.6) is RGES with decay rate $\alpha = \frac{1}{2}\{\sigma - \frac{\ln M}{\delta\tau}\}$. Thus, the CDN (3.1) can achieve GEIC with the given state $s(t)$ at rate α . \square

Remark 4.1.

1. By (4.4), in condition (iii) of Theorem 4.1, it is permitted that the maximal impulsive gain $\beta \triangleq \sup_{k \in \mathbb{Z}}\{\beta_k\} \geq e^{\sigma\tau} > 1$. This means that under the hybrid control scheme, the CDN not only can achieve GEIC but also is robust to some extent with respect to impulses existing in the control input signals. In this case, there is a requirement on *minimal dwell time* Δ_{inf} (see (4.4)) such that it has sufficient active time between two consecutive impulses to guarantee the GEIC property of the whole CDN. Moreover, if we choose the impulsive distributed controllers $\hat{u}_i(t, x, s)$ proposed (see Remark 3.2), we can derive a similar result. Here, the impulsive gain β_k in Theorem 4.1 is chosen as $\beta_k = \max_{1 \leq i \leq N}\{\lambda_{\max}(P_i^{-1} \sum_{j=1}^N \beta_{jk} P_j)\}$, where $\beta_{jk} > 0$ satisfies the following matrix inequality for $j = 1, 2, \dots, N$:

$$(4.21) \quad \begin{pmatrix} C_{j1k}^T P_j C_{j1k} - \beta_{jk} P_j & C_{j1k}^T P_j C_{j2k} & \cdots & C_{j1k} P_j C_{jNk} \\ C_{j2k}^T P_j C_{j1k} & C_{j2k}^T P_j C_{j2k} - \beta_{jk} P_j & \cdots & C_{j2k}^T P_j C_{j2k} \\ \vdots & \vdots & \ddots & \vdots \\ C_{jNk}^T P_j C_{j1k} & C_{jNk}^T P_j C_{j2k} & \cdots & C_{jNk}^T P_j C_{jNk} - \beta_{jk} P_j \end{pmatrix} \leq 0.$$

2. By condition (ii) of Theorem 4.1 and parameters $\alpha_i < 0, i = 1, 2, \dots, N$, LMIs (4.2) imply that positive definite matrices $P_i, i = 1, 2, \dots, N$, satisfy

$$(4.22) \quad A_{i_0}^T P_i + P_i A_{i_0} < 0, \quad i = 1, 2, \dots, N.$$

A necessary condition for (4.22) to hold is that matrix A_{i_0} must be stable, i.e., all the real parts of eigenvalues of A_{i_0} are negative. For Hopfield neural networks (HNNs) and cell neural networks (see [50, 51, 52]), it is true. However, for some dynamical networks such as when the nodes in the network are chaotic systems, inequalities in (4.22), and thus condition (ii) of Theorem 4.1, cannot be satisfied. Hence, it is necessary to investigate the more general case, where A_{i_0} is allowed to be unstable.

THEOREM 4.2. *Let Assumptions 3.1–3.3 be satisfied. Suppose $\Delta_{sup} < \infty$ and that there exist positive definite matrices $P_i \in \mathbb{R}^{n \times n}$ satisfying (4.1) and constants $\epsilon_{ij} > 0, \epsilon_i > 0, \hat{\alpha}_i > 0, i, j = 1, 2, \dots, N$, such that*

(i) for $i = 1, 2, \dots, N$, the following LMIs hold:

$$(4.23) \quad \begin{pmatrix} \Psi_i(A_{i_0}) - \hat{\alpha}_i P_i & P_i B_{i1_0} & \cdots & P_i B_{iN_0} & P_i E & F_{A_i}^T \\ B_{i1_0}^T P_i & -\epsilon_{i1} I & \cdots & 0 & 0 & F_{i1}^T \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ B_{iN_0}^T P_i & 0 & \cdots & -\epsilon_{iN} I & 0 & F_{iN}^T \\ E^T P_i & 0 & \cdots & 0 & -\epsilon_i^{-1} I & 0 \\ F_{A_i} & F_{i1} & \cdots & F_{iN} & 0 & -\epsilon_i I \end{pmatrix} < 0,$$

where $\Psi_i(A_{i_0}) = P_i A_{i_0} + A_{i_0}^T P_i + 2L_i \sqrt{\frac{\mu_i}{\nu_i}} P_i + \sum_{j=1}^N M_{ij} \epsilon_{ij}^{-1} \|P_i\| I$;

(ii) for any $k \in \mathbb{Z}$,

$$(4.24) \quad \beta_k \triangleq \max_{1 \leq i \leq N} \{ \lambda_{\max}(P_i^{-1} C_{ik}^T P_i C_{ik}) \} < 1;$$

(iii) there exists a positive integer $m \geq 1$ such that $t_{k-m} \leq t_k - \tau < t_{k+1-m}$ for any $k \geq m, k \in \mathbb{Z}$, and the discrete system

$$(4.25) \quad z(k+1) = J_k(m)z(k), \quad k \in \mathbb{Z},$$

is GES with decay rate $\sigma > 0$, where

$$(4.26) \quad J_k(m) \triangleq \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \alpha_{k+1-m} & \alpha_{k+2-m} & \alpha_{k+3-m} & \cdots & \alpha_{k-1} & \tilde{\alpha}_{k-1} \end{pmatrix},$$

where $\tilde{\alpha}_{k-1} = \beta_k e^{p\Delta_{k-1}} + \alpha_{k-1}$, $\alpha_{k-j} = p_2 \Delta_{k-j} e^{p\Delta_{k-j}}$, $j = 1, 2, \dots, m$, and $p = p_1 + p_2$, $p_1 = \max_{1 \leq i \leq N} \{ \hat{\alpha}_i \}$, $p_2 = \max_{1 \leq i \leq N} \{ \sum_{j=1}^N \frac{\hat{\epsilon}_{ji}}{\nu_i} \}$, where $\hat{\epsilon}_{ij} = \epsilon_{ij} + M_{ij} \epsilon_{ij} \|P_i\|$;

(iv) there exists a constant $\mathcal{T}_0 \geq 0$ such that the average dwell time \mathcal{T}_a satisfies

$$(4.27) \quad S[t_0, t] \geq -\mathcal{T}_0 + \frac{t - t_0}{\mathcal{T}_a}, \quad t \geq t_0.$$

Then, the error system (3.6) is RGES with decay rate $\frac{\sigma}{2\mathcal{T}_a}$, and hence CDN (3.1) can achieve GEIC with $s(t)$ at rate $\frac{\sigma}{2\mathcal{T}_a}$.

Proof. Let the Lyapunov–Krasovskii functional candidate V be $V(e(t)) = V_1(t) + V_2(t)$, where for some positive constants $\lambda_{ij} > 0$, $i, j = 1, 2, \dots, N$,

$$V_1(t) = \sum_{i=1}^N e_i^T(t) P_i e_i(t), \quad V_2(t) = \sum_{i=1}^N \sum_{j=1}^N \lambda_{ij} \int_{t-h_j}^t e_i^T(s) e_i(s) ds.$$

Denote $\mu = \max_{1 \leq i \leq N} \{ \mu_i \}$, $\nu = \min_{1 \leq i \leq N} \{ \nu_i \}$. Then, for any $t \geq t_0 - \tau$, we have

$$(4.28) \quad \nu \|e(t)\|^2 \leq V(e(t)) \leq \mu \|e(t)\|^2 + \tau \sum_{i=1}^N \sum_{j=1}^N \lambda_{ij} \|e(t)\|_{\tau}^2.$$

Similarly to the proof of Theorem 4.1, by (4.1) and condition (i), for $t \in [t_k, t_{k+1})$, we have

$$(4.29) \quad D^+ V_1(t) \leq \max_{1 \leq i \leq N} \{ \hat{\alpha}_i \} \sum_{i=1}^N e_i^T P_i e_i + \sum_{i=1}^N \sum_{j=1}^N \hat{\epsilon}_{ji} e_i^T(t - h_j) e_i(t - h_j),$$

where $\hat{\epsilon}_{ij} = \epsilon_{ij} + M_{ij}\epsilon_{ij}\|P_i\|$. And for $t \in [t_k, t_{k+1})$, we get

$$(4.30) \quad D^+V_2(t) \leq \max_{1 \leq i \leq N} \left\{ \sum_{j=1}^N \frac{\lambda_{ij}}{v_i} \right\} \sum_{i=1}^N e_i^T(t) P_i e_i(t) - \sum_{i=1}^N \sum_{j=1}^N \lambda_{ij} e_i^T(t - h_j) e_i(t - h_j).$$

Thus, letting $\lambda_{ij} = \hat{\epsilon}_{ji}$ for $i, j = 1, 2, \dots, N$, for any $t \in [t_k, t_{k+1})$, yields from (4.29)–(4.30)

$$(4.31) \quad D^+V(e(t)) = D^+V_1(t) + D^+V_2(t) \leq (p_1 + p_2)V_1(t) \leq pV(e(t)),$$

where $p = p_1 + p_2 > 0$, $p_1 = \max_{1 \leq i \leq N} \{\hat{\alpha}_i\}$, and $p_2 = \max_{1 \leq i \leq N} \left\{ \sum_{j=1}^N \frac{\hat{\epsilon}_{ji}}{v_i} \right\}$.

Hence, by (4.31), we get that

$$(4.32) \quad V(e(t)) \leq V(e(t_k))e^{p(t-t_k)}, \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{Z}.$$

Moreover, when $t = t_k$, we obtain that

$$(4.33) \quad V_1(t_k) \leq \beta_k V_1(t_k^-), \quad k \in \mathbb{Z}.$$

By condition (iii), there exists $m \geq 1, m \in \mathbb{Z}$, such that $t_{k-m} \leq t_k - \tau < t_{k+1-m}$ for any $k \in \mathbb{Z}$. Then, for $t = t_k$, by the definition of $V_2(t)$, we get

$$(4.34) \quad \begin{aligned} V_2(t_k) &\leq p_2 \int_{t_k-\tau}^{t_k} \sum_{i=1}^N e_i^T(s) P_i e_i^T(s) ds \leq p_2 \int_{t_{k-m}}^{t_k} V_1(s) ds \\ &= p_2 \sum_{j=1}^m \int_{t_{k-j}}^{t_{k-j+1}} V_1(s) ds = p_2 \sum_{j=1}^m \Delta_{k-j} V_1(\hat{t}_{k-j+1}) \end{aligned}$$

for some $\hat{t}_{k-j+1} \in [t_{k-j}, t_{k-j+1})$, $j = 1, 2, \dots, m$.

Then, by (4.32) and (4.34), we get

$$(4.35) \quad V_2(t_k) \leq p_2 \sum_{j=1}^m \Delta_{k-j} e^{p\Delta_{k-j}} V(e(t_{k-j})).$$

Thus, it follows from (4.33) and (4.35) that

$$(4.36) \quad \begin{aligned} V(e(t_k)) &\leq (\beta_k + p_2 \Delta_{k-1}) e^{p\Delta_{k-1}} V(e(t_{k-1})) + p_2 \sum_{j=2}^m \Delta_{k-j} e^{p\Delta_{k-j}} V(e(t_{k-j})) \\ &\triangleq \tilde{\alpha}_{k-1} V(e(t_{k-1})) + \sum_{j=1}^{m-1} \alpha_{k-j-1} V(e(t_{k-j-1})), \end{aligned}$$

where $\tilde{\alpha}_{k-1} = \beta_k e^{p\Delta_{k-1}} + \alpha_{k-1}$ and $\alpha_{k-j} = p_2 \Delta_{k-j} e^{p\Delta_{k-j}}$, $j = 1, 2, \dots, m$.

For $k \in \mathbb{Z}$, let

$$(4.37) \quad \begin{cases} w_1(k) = V(e(t_{k+1})), \\ w_2(k) = V(e(t_{k+2})), \\ \vdots \\ w_m(k) = V(e(t_{k+m})), \end{cases}$$

and $W(k) = (w_1(k), w_2(k), \dots, w_m(k))^T$. Then, by (4.36)–(4.37), it follows that

$$(4.38) \quad \begin{pmatrix} w_1(k+1-m) \\ w_2(k+1-m) \\ \vdots \\ w_m(k+1-m) \end{pmatrix} \leq J_k(m) \begin{pmatrix} w_1(k-m) \\ w_2(k-m) \\ \vdots \\ w_m(k-m) \end{pmatrix},$$

where $J_k(m)$ is defined by (4.26).

Thus, by (4.38), we get

$$(4.39) \quad W(k-m+1) \leq J_k(m)W(k-m), \quad k \in \mathbb{Z}.$$

Let the comparison system be

$$(4.40) \quad \begin{cases} z(k+1) = J_k(m)z(k), & k \geq m-1, \\ z(m-1) = W(-1). \end{cases}$$

Then, by the comparison principle [45], we get

$$(4.41) \quad W(k-m) \leq z(k), \quad k \geq m-1.$$

Thus, by condition (iii), i.e., the discrete system (4.25) is GES with decay rate $\sigma > 0$, there exists constant $K > 0$ such that

$$(4.42) \quad \|W(k-m)\| \leq Ke^{-\sigma(k-m+1)}\|W(-1)\|, \quad k \geq m-1, \quad k \in \mathbb{Z},$$

where $\|W(-1)\|^2 = \sum_{j=0}^{m-1} V(e(t_j))^2$ and $\|W(k-m)\|^2 = \sum_{j=1}^m V(e(t_{j+k-m}))^2$.

Let $\bar{\lambda} = \sum_{i=1}^N \sum_{j=1}^N \hat{\epsilon}_{ji}$; then, for $j = 1, 2, \dots, m-1$, we get

$$(4.43) \quad \begin{aligned} V_2(t_j) &= \sum_{i=1}^N \sum_{k=1}^N \lambda_{ik} \int_{t_j-h_k}^{t_j} e_i^T(s)e_i(s)ds \leq \sum_{i=1}^N \sum_{k=1}^N \lambda_{ik} \int_{t_0-\tau}^{t_j} e_i^T(s)e_i(s)ds \\ &= \sum_{i=1}^N \sum_{k=1}^N \lambda_{ik} \left[\int_{t_0-\tau}^{t_0} e_i^T(s)e_i(s)ds + \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} e_i^T(s)e_i(s)ds \right] \\ &\leq \bar{\lambda}\tau\|\phi\|_\tau^2 + p_2 \sum_{l=0}^{j-1} V(e(t_l))e^{p\Delta_{l-1}}. \end{aligned}$$

Thus, for $j = 1, 2, \dots, m-1$, we get

$$(4.44) \quad \begin{aligned} V(e(t_j)) &\leq \bar{\lambda}\tau\|\phi\|_\tau^2 + \beta_j e^{p\Delta_{j-1}}V(e(t_{j-1})) + p_2 \sum_{l=0}^{j-1} V(e(t_l))e^{p\Delta_{l-1}} \\ &= \bar{\lambda}\tau\|\phi\|_\tau^2 + \tilde{\alpha}_{j-1}V(e(t_{j-1})) + \sum_{l=0}^{j-2} \alpha_l V(e(t_l)). \end{aligned}$$

Moreover, it follows from (4.28) and $\lambda_{ij} = \hat{\epsilon}_{ij}$ that

$$(4.45) \quad V(e(t_0)) \leq (\mu + \bar{\lambda}\tau)\|\phi\|_\tau^2.$$

Thus, from (4.44)–(4.45), by the induction method, there exist positive constants M_j , which are dependent on $\tau, \bar{\lambda}, \beta_j, p_2, p, \Delta_j, j = 1, 2, \dots, m-1$, such that

$$(4.46) \quad V(e(t_j)) \leq M_j\|\phi\|_\tau^2, \quad j = 1, 2, \dots, m-1.$$

Hence, we get

$$(4.47) \quad \|W(-1)\|^2 = \sum_{j=0}^{m-1} V(e(t_j))^2 \leq \left((\mu + \bar{\lambda}\tau)^2 + \sum_{j=1}^{m-1} M_j^2 \right) \|\phi\|_\tau^4.$$

By (4.42) and (4.47), for any $k \in \mathbb{Z}$, it follows that

$$(4.48) \quad V(e(t_k)) \leq \|W(k-m)\| \leq \sqrt{(\mu + \bar{\lambda}\tau)^2 + \sum_{j=1}^{m-1} M_j^2} K e^{-\sigma(k-m+1)} \|\phi\|_\tau^2.$$

Thus, by (4.28), (4.32), and (4.48), for any $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}$, we get

$$(4.49) \quad \|e(t)\|^2 \leq \frac{1}{\nu} V(e(t)) \leq \frac{\beta_k}{\nu} e^{p(t-t_k)} V(e(t_k)) \leq \tilde{K}^2 \|\phi\|_\tau^2 e^{-\sigma k},$$

where $\tilde{K}^2 = \frac{\sqrt{(\mu + \bar{\lambda}\tau)^2 + \sum_{j=1}^{m-1} M_j^2} \cdot K e^{\sigma(m-1) + p\Delta_{\text{sup}}}}{\nu}$.

For any $t \in [t_k, t_{k+1})$, we have $S[t_0, t] = k$, which yields by condition (iv) that $k \geq -\mathcal{T}_0 + \frac{t-t_0}{\mathcal{T}_a}$ for any $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}$. Thus, by (4.49), we get

$$(4.50) \quad \|e(t)\| \leq \tilde{K} e^{\frac{\sigma\mathcal{T}_0}{2}} \|\phi\|_\tau e^{-\frac{\sigma}{2\mathcal{T}_a}(t-t_0)}, \quad t \geq t_0,$$

which yields all the results of this theorem. The proof is thus complete. \square

Remark 4.2.

1. By condition (i) of Theorem 4.2 and parameters $\hat{\alpha}_i > 0$, we get that matrices $A_{i_0}^T P_i + P_i A_{i_0}$ may be nonnegative definite, i.e., $A_{i_0}^T P_i + P_i A_{i_0} \geq 0$, $i = 1, 2, \dots, N$. Thus, matrices A_{i_0} , $i = 1, 2, \dots, N$, are allowed to be unstable. In this case, the impulsive control is key in stabilizing the error system and thus in the CDN achieving GEIC. It requires the impulsive gains $\beta_k < 1$, $k \in \mathbb{Z}$, to offset the increment of V led by the unstable dynamics during intervals (t_k, t_{k+1}) , $k \in \mathbb{Z}$. Clearly, it should have enough numbers of impulse to enlarge the decrement of V at these impulsive instances. Hence, there is a requirement on *minimal impulse numbers* to provide sufficient impulsive control signals. The inequality (4.27) gives the estimation of the minimal impulse times in time interval $[t_0, t]$, $t \in \mathbb{R}_+$. Obviously, it also needs $\Delta_{\text{sup}} < \infty$; otherwise, from some time on, there is no impulsive control signal, and the error system is changed to an unstable continuous-time system.

2. It should be noted that the results of Theorems 4.1 and 4.2 are based on the solvability of LMIs in (4.2) and (4.23), respectively. For Theorem 4.1, as discussed in Remark 4.1.2, a necessity for the solvability for LMIs in (4.2) is that midpoint matrices A_{i_0} , $i = 1, 2, \dots, N$, are Hurwitz. In addition, it also depends on how stable these midpoint matrices are and how big the matrix intervals $N[A_{i_1}, A_{i_2}]$, $i = 1, 2, \dots, N$, are as well. That is, the solvability of LMIs in (4.2) relates to the size of negative real parts of eigenvalues of midpoint matrices and the size of these matrix intervals. But for Theorem 4.2, there exists no solvability issue for LMIs in (4.23). Since for midpoint matrices A_{i_0} (stable or unstable), and for any given symmetric matrices $Q_i \geq 0$, the matrix inequality $P_i A_{i_0} + A_{i_0}^T P_i + Q_i - \hat{\alpha}_i P_i < 0$ is always solvable for some $P_i > 0$ and some constants $\hat{\alpha}_i > 0$, $i = 1, 2, \dots, N$, and thus by the Schur complement theorem it is easy to see that LMIs in (4.23) are solvable.

In the following, by using Theorem 4.2, we give some simple GEIC criteria for CDN (3.1).

COROLLARY 4.1. *Let Assumptions 3.1–3.3 be satisfied. Suppose that $\tau \leq t_k - t_{k-1}$ for any $k \in \mathbb{Z}$, i.e., $m = 1$ in Theorem 4.2, and that there exist matrices $P_i > 0$ and constants $\epsilon_{ij} > 0, \epsilon_i > 0, \hat{\alpha}_i > 0, i, j = 1, \dots, N$, such that (4.1) and (i), (ii), and (iv) of Theorem 4.2 hold, while condition (iii) of Theorem 4.2 is replaced by the following:*

(iii*) *There exists a positive constant σ satisfying $\sigma > 0$ such that*

$$(4.51) \quad \ln(\beta_k + p_2\tau) + p\Delta_{k-1} \leq -\sigma, \quad k \geq 1, \quad k \in \mathbb{Z},$$

where $\beta_k = \max_{1 \leq i \leq N} \{\lambda_{\max}(P_i^{-1}C_{ik}^T P_i C_{ik})\} < 1$ and $p = p_1 + p_2, p_1 = \max_{1 \leq i \leq N} \{\hat{\alpha}_i\}, p_2 = \max_{1 \leq i \leq N} \{\frac{1}{\nu_i} \sum_{j=1}^N \hat{\epsilon}_{ji}\}, \hat{\epsilon}_{ij} = \epsilon_{ij} + M_{ij} \epsilon_{ij} \|P_i\|$.

Then, system (3.6) is RGES with decay rate $\frac{\sigma}{2T_a}$, and thus CDN (3.1) achieves GEIC with $s(t)$.

Proof. Take the same Lyapunov–Krasovskii functional V as in Theorem 4.2, that is, $V(e(t)) = V_1(t) + V_2(t)$ as follows:

$$V_1(t) = \sum_{i=1}^N e_i^T(t) P_i e_i(t), \quad V_2(t) = \sum_{i=1}^N \sum_{j=1}^N \lambda_{ij} \int_{t-h_j}^t e_i^T(s) e_i(s) ds,$$

where $\lambda_{ij} = \hat{\epsilon}_{ji}$ for $i, j = 1, 2, \dots, N$. By the proof of Theorem 4.2, we obtain that (4.28)–(4.33) hold. For $t = t_k$, by the definition of $V_2(t)$, we get

$$(4.52) \quad V_2(t_k) \leq p_2 \int_{t_k-\tau}^{t_k} \sum_{i=1}^N e_i^T(s) P_i e_i^T(s) ds = p_2\tau V_1(t_k + \theta_k)$$

for some $\theta_k \in [-\tau, 0]$. Thus, it follows from $\tau \leq t_k - t_{k-1}$ and (4.32), (4.33), and (4.52) that

$$(4.53) \quad \begin{aligned} V(e(t_k)) &\leq \beta_k V_1(t_k^-) + p_2\tau V_1(t_k + \theta_k) \\ &\leq \beta_k V(e(t_{k-1})) e^{p(t_k - t_{k-1})} + p_2\tau V(e(t_{k-1})) e^{p(t_k + \theta_k - t_{k-1})} \\ &\leq e^{\ln(\beta_k + p_2\tau) + p(t_k - t_{k-1})} V(e(t_{k-1})), \quad k \in \mathbb{Z}. \end{aligned}$$

From condition (iii*) and (4.53), we get that for any $k \in \mathbb{Z}$,

$$(4.54) \quad V(e(t_k)) \leq e^{-\sigma} V(e(t_{k-1})) \leq \dots \leq e^{-\sigma k} V(e(t_0)).$$

Thus, for any $t \in [t_k, t_{k+1}), k \in \mathbb{Z}$, by (4.54), condition (iii*), and inequality (4.27), we have

$$(4.55) \quad V(e(t)) \leq e^{p(t-t_k)} e^{\sigma T_0 - \sigma(t_k - t_0)} V(e(t_0)) \leq e^{\sigma T_0 + |p - \frac{\sigma}{T_a}| e^{-\frac{\sigma}{T_a}(t-t_0)}} V(e(t_0)).$$

By (4.28) and (4.55), for any initial condition $e(s) = \phi(s), s \in [-\tau, 0]$, and any $t \in [t_k, t_{k+1}), k \in \mathbb{Z}$, we get

$$(4.56) \quad \|e(t)\| \leq \sqrt{\frac{\mu + \tau \sum_{i=1}^N \sum_{j=1}^N \epsilon_{ji}}{\nu}} \cdot \sqrt{e^{\sigma T_0 + |p - \frac{\sigma}{T_a}|}} \cdot \|\phi\|_{\tau} e^{-\frac{\sigma}{2T_a}(t-t_0)}.$$

Hence, it follows from (4.56) that all statements of this corollary are true. □

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Remark 4.3.

1. From Corollary 4.1, if $\tau \leq t_k - t_{k-1}$ for any $k \in \mathbb{Z}$, we get estimations of *maximal time-delay* τ^* and *maximal dwell time* Δ_{sup} between two consecutive impulses as

$$(4.57) \quad \tau^* \leq \sup_{k \geq 1, k \in \mathbb{Z}} \left\{ \frac{e^{-p\Delta_{k-1}-\sigma} - \beta_k}{p} \right\}, \quad \Delta_{\text{sup}} \leq \sup_{k \geq 1, k \in \mathbb{Z}} \left\{ \frac{-\sigma - \ln(\beta_k + p_2\tau)}{p} \right\}.$$

2. From Corollary 4.1, one can see how robust these impulse control signals are with respect to the unstable midpoint matrices A_{i_0} and the uncertainties in matrices A_i , $i = 1, 2, \dots, N$. The parameter $p = p_1 + p_2$, where p_1 is from unstable A_{i_0} , and p_2 is from uncertainties in matrices A_i , $i = 1, 2, \dots, N$, reflects the instability level in (3.6). By (4.51), for smaller τ , if we take the impulsive gains β_k as

$$(4.58) \quad 0 < \beta_k < e^{-p\Delta_{k-1}-\sigma} - p_2\tau, \quad k \in \mathbb{Z},$$

then system (3.6) is RGES, and thus CDN (3.1) achieves GEIC at rate $\frac{\sigma}{2\mathcal{T}_a}$.

COROLLARY 4.2. *Assume all conditions of Theorem 4.2 hold except condition (iii), which is replaced by the following:*

(iii**) *There exists a positive integer $m > 1$ such that $t_{k-m} \leq t_k - \tau < t_{k+1-m}$ for any $k \geq m, k \in \mathbb{Z}$, and the matrix $J(m)$ satisfies the spectral radius condition for some $\sigma > 0$,*

$$(4.59) \quad \rho(J(m)) < e^{-\sigma},$$

where

$$(4.60) \quad J(m) \triangleq \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ a & a & a & \cdots & a & a+b \end{pmatrix},$$

and $a = p_2\Delta_{\text{sup}}e^{p\Delta_{\text{sup}}}$, $b = \beta e^{p\Delta_{\text{sup}}}$, and $\beta = \sup_{k \in \mathbb{Z}} \{ \lambda_{\max}(P_i^{-1}C_{i_k}^T P_i C_{i_k}) \}$, $p = p_1 + p_2$, $p_1 = \max_{1 \leq i \leq N} \{ \hat{\alpha}_i \}$, $p_2 = \max_{1 \leq i \leq N} \{ \frac{1}{\nu_i} \sum_{j=1}^N \hat{\epsilon}_{ji} \}$, where $\hat{\epsilon}_{ij} = \epsilon_{ij} + M_{ij}\epsilon_{ij}\|P_i\|$.

Then, system (3.6) is RGES with decay rate $\frac{\sigma}{2\mathcal{T}_a}$, and hence CDN (3.1) achieves GEIC at rate $\frac{\sigma}{2\mathcal{T}_a}$.

Proof. In (4.36) of Theorem 4.2, since $\alpha_{k-m+j} \leq a$ ($j = 1, 2, \dots, m-1$), and $\tilde{\alpha}_k \leq a+b$, then, by (4.36), we get

$$(4.61) \quad V(e(t_k)) \leq (a+b)V(e(t_{k-1})) + a \sum_{j=1}^{m-1} V(e(t_{k-j+1})).$$

Thus, we derive the comparison system

$$(4.62) \quad z(k+1) = J(m)z(k), \quad k \geq m-1, k \in \mathbb{Z}.$$

For the discrete system (4.62), it is well known that (4.62) is GES if and only if the spectral radius of matrix $J(m)$ satisfies $\rho(J(m)) < 1$. Thus, from $\rho(J(m)) < e^{-\sigma}$ and by the similar proof of Theorem 4.2, we conclude that this corollary is true. \square

COROLLARY 4.3. *Assume that all conditions of Theorem 4.2 hold except condition (iii), which is replaced by the following:*

(iii^{***}) *There exists a positive integer $m > 1$ such that $t_{k-m} \leq t_k - \tau < t_{k+1-m}$ for any $k \geq m, k \in \mathbb{Z}$, and there exists some positive constant $0 < \gamma < 1$ such that every root λ_j ($j = 1, 2, \dots, m$) of the characteristic polynomial*

$$(4.63) \quad F_k(\lambda) \triangleq \lambda^m - \tilde{\alpha}_{k-1}\lambda^{m-1} - \alpha_{k-1}\lambda^{m-2} - \dots - \alpha_{k+2-m}\lambda - \alpha_{k+1-m}$$

satisfies that $|\lambda_j| \leq \gamma < 1, j = 1, 2, \dots, m$, where $\tilde{\alpha}_{k-1}, \alpha_j, j = k - 1, k - 2, \dots, k + 1 - m$, are the same as in Theorem 4.2.

Then system (3.6) is RGAS, and hence CDN (3.1) can achieve GIC with the given state $s(t)$.

Proof. It is easy to get that $F_k(\lambda)$ is the characteristic polynomial of matrix $J_k(m)$, i.e.,

$$(4.64) \quad F_k(\lambda) = |\lambda I - J_k(m)|.$$

Then, every root of λ_j of $F_k(\lambda)$ is the characteristic root of $J_k(m)$. Hence, if every root λ_j ($j = 1, 2, \dots, m$) lies strictly in the unit circle and satisfies $|\lambda_j| \leq e^{-\sigma} < 1$ for some $\sigma > 0$, then the spectral radius of every matrix $J_k(m)$ satisfies $\rho(J_k(m)) \leq \gamma < 1$. Thus, the comparison system $z(k + 1) = J_k(m)z(k)$ is GAS, and hence we conclude that system (3.6) is RGAS, and thus CDN (3.1) achieves GIC with $s(t)$. \square

5. Impulsive consensus control design. In this section, by using Theorem 4.2, Corollaries 4.1–4.3, and Remark 4.3, we design impulsive control gain matrices $\{C_{i_k}, k \in \mathbb{Z}\}$ for the CDN (3.1) such that GEIC can be achieved by the impulsive consensus control.

THEOREM 5.1. *Let Assumptions 3.1–3.3 be satisfied and let $\Delta_{\text{sup}} < \infty$. Assume that (4.1) and conditions (i) and (iv) of Theorem 4.2 still hold, while (ii)–(iii) are changed to (ii')–(iii') as follows:*

(ii') *There exist constants $0 < \tilde{\beta}_i < 1, i = 1, 2, \dots, N$, such that the following LMIs hold:*

$$(5.1) \quad \begin{pmatrix} \Omega_{1_i} & \Omega_{2_i} & Y_i^T \\ \Omega_{2_i}^T & -I & 0 \\ Y_i & 0 & -P_i \end{pmatrix} \leq 0, \quad i = 1, 2, \dots, N,$$

where $\Omega_{1_i} = P_i + Y_i^T + Y_i - \tilde{\beta}_i P_i, \Omega_{2_i} = P_i E + Y_i^T E, i = 1, 2, \dots, N;$

(iii') *constants $\beta_k \triangleq \max_{1 \leq i \leq N} \{\tilde{\beta}_i\}, k \in \mathbb{Z}$, satisfy condition (ii) of Theorem 4.2.*

Then, under impulsive control gain matrices $\{C_{i_k} = I + P_i^{-1}Y_i, i = 1, 2, \dots, N, k \in \mathbb{Z}\}$, the error system (3.6) is RGES with decay rate $\frac{\sigma}{2T_a}$, and hence the CDN (3.1) achieves GEIC with $s(t)$ at rate $\frac{\sigma}{2T_a}$.

Proof. Let $C_{i_k} = I + P_i^{-1}Y_i$; then, by using the Schur complement theorem, the LMI (5.1) is equivalent to

$$(5.2) \quad C_{i_k}^T P_i C_{i_k} - \tilde{\beta}_i P_i + \Omega_2^T \Omega_2 \leq 0,$$

which yields that $C_{i_k}^T P_i C_{i_k} - \tilde{\beta}_i P_i \leq 0$. Thus, we get that

$$(5.3) \quad V_1(t_k) = \sum_{i=1}^N e_i^T(t_k^-) C_{i_k}^T P_i C_{i_k} e_i(t_k^-) \leq \sum_{i=1}^N \tilde{\beta}_i V_1(t_k^-) \leq \beta_k V_1(t_k^-), \quad k \in \mathbb{Z}.$$

It is not hard to see that the result of this theorem can be derived by using Theorem 4.2. The proof is thus complete. \square

COROLLARY 5.1. *Assume that (4.1) and conditions (i) and (iv) of Theorem 4.2 and condition (ii') of Theorem 5.1 hold. Then, under impulsive control gain matrices $\{C_{i_k} = I + P_i^{-1}Y_i, i = 1, 2, \dots, N, k \in \mathbb{Z}\}$, condition (iii*) of Corollary 4.1 or (iii**) of Corollary 4.2 implies that CDN (3.1) can achieve GEIC at rate $\frac{\sigma}{27\tau_a}$, where \mathcal{T}_a satisfies (4.27), and (iii***) of Corollary 4.3 implies that the error system (3.6) is RGAS, and thus CDN (3.1) can achieve GIC.*

Proof. The results can be derived from Theorems 4.2 and 5.1 and Corollaries 4.1–4.3. \square

6. Examples and numerical simulations. In this section, two representative examples are given for illustration.

Example 6.1. Consider CDN (3.1), where $x_i = (x_{i1}, x_{i2}, x_{i3})^T, i = 1, 2, \dots, 10$, $f_i(t, x_i) = A_i x_i + \tilde{f}_i(t, x_i)$, in which $A_i \in N[\underline{A}, \bar{A}]$, where

$$\underline{A} = \begin{pmatrix} 0.3570 & -13.2633 & -0.6431 \\ 11.4195 & -0.1531 & -0.8700 \\ -0.2161 & -0.2204 & -0.0191 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 1.3570 & -12.2633 & 0.3569 \\ 12.4195 & 0.8469 & 0.1300 \\ 0.7839 & 0.7796 & 0.9809 \end{pmatrix},$$

$\tilde{f}_i(t, x_i) = (\sin t^2, -\sin x_{i2} - \cos t, \sin x_{i3} + 2 \sin(t-1))^T$, and $g_i(x_1(t-h_i), \dots, x_N(t-h_i)) = \sum_{j=1}^N B_{ij} x_j(t-h_i) = B_{ii} x_i(t-h_i) + B_{i,i+1} x_{i+1}(t-h_i) + B_{i,i+2} x_{i+2}(t-h_i)$, where $B_{ij} = B_{ij_1} = B_{ij_2}, i, j = 1, 2, \dots, N$, and

$$B_{ii} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.2 \\ 0 & 0 & -0.5 \end{pmatrix}, \quad B_{i,i+1} = \begin{pmatrix} -1.0 & -0.3 & 0 \\ 0 & 0.25 & -0.1 \\ 0 & 0 & 1.0 \end{pmatrix}, \quad B_{i,i+2} = \begin{pmatrix} 0.5 & -0.2 & 0 \\ 0 & -0.75 & -0.1 \\ 0 & 0 & -0.5 \end{pmatrix}$$

while $B_{N-1,N+1} = B_{N-1,1}, B_{N-1,N+2} = B_{N-1,2}, B_{N,N+1} = B_{N1}, B_{N,N+2} = B_{N2}$.

The goal function $s(t)$ is $s(t) = (s_1(t), s_2(t), s_3(t))^T = (\sin t, 2, e^{-2t})^T$.

In the following, we investigate the consensus between CDN (3.1) and the goal $s(t)$. It is easy to get

$$A_{i_0} = \begin{pmatrix} 0.8570 & -12.7633 & -0.1431 \\ 11.9195 & 0.3469 & -0.3700 \\ 0.2839 & 0.2796 & 0.4809 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

and $F^T = E$.

Letting $\epsilon_{ij} = 1, \epsilon_i = 1, \nu_i = 1, \mu_i = 2, \Delta_k = 0.1, h_i = 0.02, i = 1, 2, \dots, 10, k \in \mathbb{Z}$, and solving the LMIs in (ii) of Theorem 4.2, for $i = 1, 2, \dots, 10$, we get $\hat{\alpha}_i = 0.12$ and

$$P_i = \begin{pmatrix} 1.0650 & -0.0364 & 0.0005 \\ -0.0364 & 1.0811 & -0.0135 \\ 0.0005 & -0.0135 & 1.0567 \end{pmatrix}.$$

Thus, we have $p_1 = \max_{1 \leq i \leq 10} \{\alpha_i\} = 0.12, p_2 = \max_{1 \leq i \leq 10} \{\frac{1}{\nu_i} \sum_{j=1}^{10} \epsilon_{ji}\} = 10$, and letting $\sigma = 0.1160$, then, by (4.51), we obtain $\beta_k < 0.12154$. Thus, the conditions of Corollary 4.1 are satisfied, and the error system is RGES with decay rate 0.5800. Hence, the network can achieve GEIC with $s(t)$ at rate 0.5800.

In simulation, without loss of generality, initial conditions were set as $x_i(t) = 0$ whenever $t < 0$, and $x_i(0) \neq 0, i = 1, 2, \dots, N$. The matrices $A_i \in N[\underline{A}, \bar{A}]$ are set as $A_1 = A_2 = A_3 = \tilde{A}_1, A_4 = A_5 = A_6 = \tilde{A}_2, A_7 = A_8 = \tilde{A}_3$, and $A_9 = A_{10} = \tilde{A}_4$, where $\tilde{A}_2 = A_{i_0}, \tilde{A}_3 = \underline{A}, \tilde{A}_4 = \bar{A}$, and

$$\tilde{A}_1 = \underline{A} + \text{Rand}(3, 3) = \begin{pmatrix} -8.4496 & -11.7216 & 0.7353 \\ 13.1017 & -9.0022 & 0.9900 \\ 1.0867 & 1.4775 & -8.1654 \end{pmatrix},$$

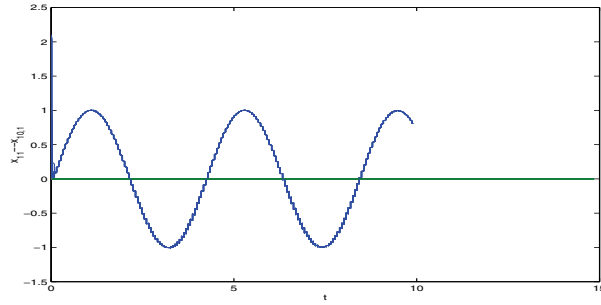


FIG. 6.1. Consensus properties of $x_{k1}(t), k = 1, 2, \dots, 10$, with $s_1 = \sin(t)$.

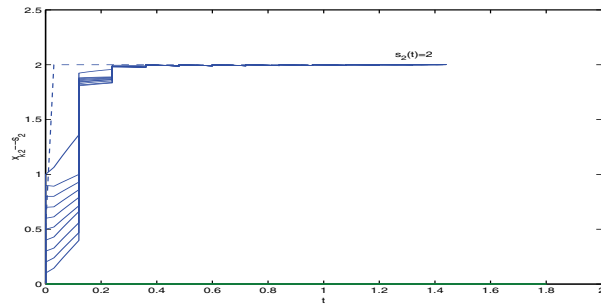


FIG. 6.2. Consensus properties of $x_{k2}(t), k = 1, 2, \dots, 10$, and the goal $s_2(t)$.

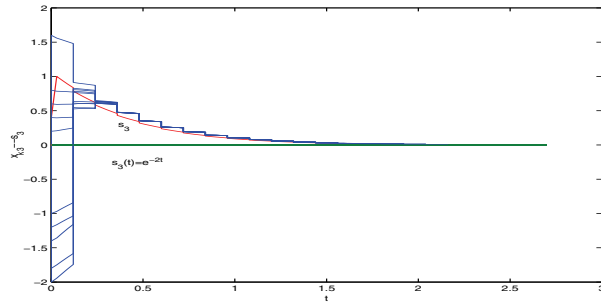


FIG. 6.3. Consensus properties of $x_{k3}(t), k = 1, 2, \dots, 10$, and the goal $s_3(t)$.

where matrix $\text{Rand}(3, 3) = (r_{ij})$ is a 3×3 random matrix with entries r_{ij} satisfying $0 \leq r_{ij} \leq 1$. From $\beta_k < 0.12154$, in simulation, under Theorem 5.1, we design the impulsive matrices C_{i_k} as $C_{i_k} = 0.12I$.

In Figures 6.1–6.3, one can see the consensus properties of the goal coordinate s_j of $s(t)$ and substates $x_{ij}, i = 1, 2, \dots, 10, j = 1, 2, 3$, of the CDN.

Example 6.2. Use the chaotic Colpitts’ oscillator as nodes of CDN. The Colpitts’ oscillator is described by

$$(6.1) \quad \dot{y} = A_0y + \varphi(y),$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & \alpha & 0 \\ -\sigma & -\gamma\sigma & -\sigma \\ a_1\beta & \beta & 0 \end{pmatrix}, \quad \varphi(y) = (0, 0, a_3\beta y_1^3)^T,$$

in which $\alpha, \beta, \sigma, a_1, \gamma, a_3 \in \mathbb{R}$. It is known that with parameters $\alpha = 2.4, \beta = 2.2, \sigma = 1, \gamma = 0.252, a_1 = 1$, and $a_3 = -0.2$, the oscillator (6.1) is chaotic.

Suppose that the CDN (3.1) with nonidentical nodes and time-delays is given by

$$(6.2) \quad \dot{x}_i = A_i x_i + \varphi(x_i) + \sum_{j=1}^N B_{ij} x_j(t - h_i), \quad i = 1, 2, \dots, N = 10,$$

where $x_i = (x_{i1}, x_{i2}, x_{i3})^T$, B_{ij} ($i, j = 1, 2, \dots, N$), are the same as those in Example 6.1, and $A_i \in N[\underline{A}, \bar{A}]$ with

$$\underline{A} = \begin{pmatrix} -0.5 & \alpha - 0.5 & -0.5 \\ -\sigma - 0.5 & -\gamma\sigma - 0.5 & -\sigma - 0.5 \\ a_1\beta - 0.5 & \beta - 0.5 & -0.5 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 0.5 & \alpha + 0.5 & 0.5 \\ -\sigma + 0.5 & -\gamma\sigma + 0.5 & -\sigma + 0.5 \\ a_1\beta + 0.5 & \beta + 0.5 & 0.5 \end{pmatrix}.$$

Suppose the goal vector function $s(t)$ is the solution of the Lorenz system

$$(6.3) \quad \dot{s} = Ls + \tilde{\varphi}(s),$$

where

$$s = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \quad L = \begin{pmatrix} -b_1 & b_1 & 0 \\ b_2 & -1 & 0 \\ 0 & 0 & -b_3 \end{pmatrix}, \quad \varphi(s) = (0, -s_1 s_3, s_1 s_2)^T,$$

in which $b_1, b_2, b_3 \in \mathbb{R}$. It is well known that under $b_1 = 10, b_2 = 28, b_3 = \frac{8}{3}$, Lorenz system (6.3) is chaotic.

It is easy to show for any matrix $A \in N[\underline{A}, \bar{A}]$, A is not a stable matrix. Hence, condition (i) of Theorem 4.1 is not satisfied. In the following, by using Theorem 4.2, we design the impulsive controllers $\{u_{i_k}, t_k\} = \{K_{i_k} x_i, t_k\}$ such that

$$(6.4) \quad \begin{cases} \dot{x}_i = A_i x_i + \varphi(x_i) + \sum_{j=1}^N B_{ij} x_j(t - h_i) + \dot{s} - [A_i s + \varphi(s)], & t \in [t_k, t_{k+1}), \\ \Delta x_i(t) = u_{i_k}(t) = K_{i_k}(x_i(t) - s(t)), & t = t_k, k \in \mathbb{Z}, \end{cases}$$

can achieve GEIC.

By [44], we get that $\|\frac{\partial \varphi(y)}{\partial y}\| \leq 5.28$, which implies that $L_i = 5.28, i = 1, \dots, 10$. By choosing $\epsilon_{ij} = 1, \epsilon_i = 1, \nu_i = 1, \mu_i = 2$, and solving the LMIs in (4.23) of Theorem 4.2, for $i = 1, 2, \dots, 10$, we get $\hat{\alpha}_i = 6, p_1 = 6, p_2 = 10$, and

$$P_i = \begin{pmatrix} 1.5437 & 0.3646 & 0.1371 \\ 0.3646 & 1.4298 & 0.1290 \\ 0.1371 & 0.1290 & 1.1520 \end{pmatrix}.$$

Then, by solving the LMIs in (4.23) of Theorem 5.1, we get that $\beta_k = 0.01$, and

$$Y_i = \begin{pmatrix} -1.5328 & -0.3546 & -0.1330 \\ -0.3546 & -1.4215 & -0.1264 \\ -0.1330 & -0.1264 & -1.1502 \end{pmatrix}, \quad K_{i_k} - I = P_i^{-1} Y_i = \begin{pmatrix} -0.9944 & 0.0054 & 0.0024 \\ 0.0054 & -0.9957 & 0.0012 \\ 0.0023 & 0.0011 & -1.0004 \end{pmatrix}.$$

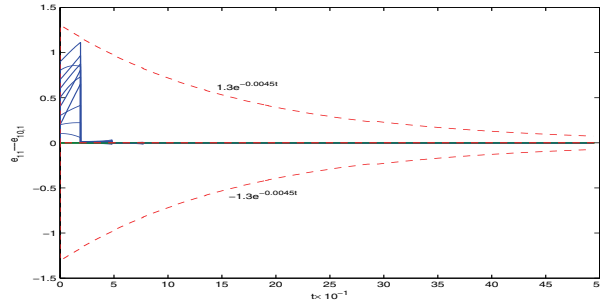


FIG. 6.4. Exponential stability properties of errors $e_{k1}(t), k = 1, 2, \dots, 10$.

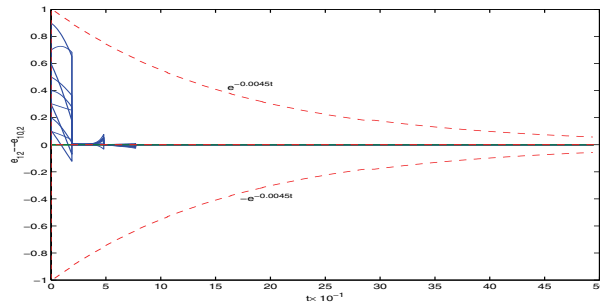


FIG. 6.5. Exponential stability properties of errors $e_{k2}(t), k = 1, 2, \dots, 10$.

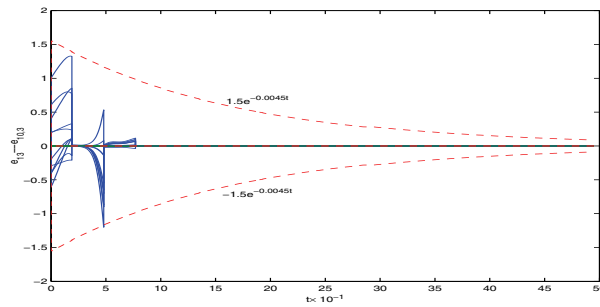


FIG. 6.6. Exponential stability properties of errors $e_{k3}(t), k = 1, 2, \dots, 10$.

Moreover, there exists $\sigma = 0.001$ such that (4.51) in Corollary 4.1 holds. By Remark 4.3, we get $\tau^* < 0.0619$ and $\Delta_{\text{sup}} \leq 0.1107$.

Thus, we can design the impulsive controllers $\{K_{i_k}x_i, t_k\}$ as follows.

Case 1. If $\tau \leq \Delta_k, k \in \mathbb{Z}$, then let $t_0 = 0, \Delta_k = 0.1, k \in \mathbb{Z}$, and $K_{i_k}, i = 1, 2, \dots, 10$, be chosen as above. Therefore, by Theorem 5.1, the impulsive controllers $\{K_{i_k}e_i, t_k\}$ can achieve GEIC between the CDN and $s(t)$. And the consensus rate is 0.005.

Case 2. If there is $m > 1$ such that $t_{k-m} \leq t_k - \tau < t_{k+1-m}, k \in \mathbb{Z}$, then, by Corollary 4.2, we design the impulsive controllers. For example, let $h_i = 0.05$ and $m = 2$; then by Corollary 4.2, the impulsive instances can be set as $t_0 = 0, \Delta_k = 0.03, k \in \mathbb{Z}$. Therefore, by Corollary 4.2, $\{K_{i_k}e_i, t_k\}$ can achieve GEIC between the CDN and $s(t)$. And the consensus rate is 0.0045.

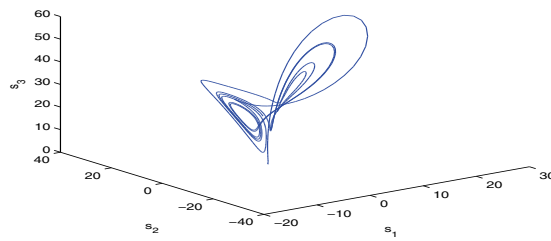


FIG. 6.7. The phase figure of the Lorenz system.

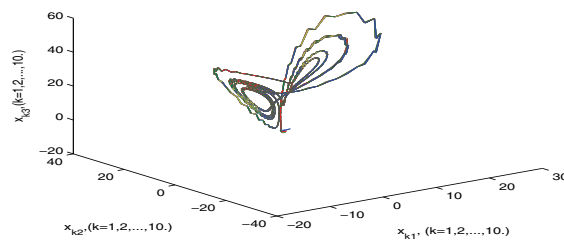


FIG. 6.8. The phase figure of the network under impulsive control.

In simulation, let $\tau = h_i = 0.05$, $\Delta_k = 0.03$, $k \in \mathbb{Z}$, and, without loss of generality, initial conditions $x_i(t) = 0$ whenever $t < 0$, and $x_i(0) \neq 0$, $i = 1, 2, \dots, N$. The matrices $A_i \in N[\underline{A}, \bar{A}]$ are set as $A_1 = A_2 = A_3 = \tilde{A}_1$, $A_4 = A_5 = A_6 = \tilde{A}_2$, $A_7 = A_8 = \tilde{A}_3$, and $A_9 = A_{10} = \tilde{A}_4$, where $\tilde{A}_2 = A_{i_0} = A_0$, $\tilde{A}_3 = \underline{A}$, $\tilde{A}_4 = \bar{A}$, and

$$\tilde{A}_1 = \underline{A} + \text{Rand}(3, 3) = \begin{pmatrix} 0.4501 & 2.3860 & -0.0435 \\ -1.2689 & 0.1393 & -1.4815 \\ 2.3068 & 2.4621 & 0.3214 \end{pmatrix}.$$

In Figures 6.4–6.6, one can see the exponential stability properties (at rate 0.0045) of errors between s_j of the Lorenz system (6.3) and node states x_{ij} , $i = 1, 2, \dots, 10$, $j = 1, 2, 3$, of the CDN with $h_1 = h_2 = h_3 = 0.05$. In Figures 6.7 and 6.8, one can see the whole consensus properties of Lorenz system (6.3) and the CDN (6.2) with coupling time-delay $h_i = 0.05$, $i = 1, 2, 3$.

7. Conclusions. In this paper, the global consensus problem has been studied between complex dynamical networks (CDNs) and a known goal signal. The CDNs are more general than the usual models with respect to the uncertainties, nonidentical nodes, and coupling time-delays. We referred to the consensus scheme as an impulsive consensus scheme. By using the methods of Lyapunov functions and Lyapunov–Krasovskii functionals, as well as linear matrix inequalities (LMIs), criteria for achieving global exponential impulsive consensus (GEIC) have been derived. The impulsive consensus controllers so designed are effective in handling the uncertainties of parameters, coupling time-delays, and, more important, in synchronizing a known goal signal and a CDN with nonidentical nodes. The consensus rates of GEIC were also estimated. Finally, two examples have been discussed to illustrate the theoretical results obtained in this paper.

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